COMPACTIFICATION AND COMPACTOIDIFICATION

E. Beckenstein, L. Narici, and W. Schikhof

Abstract. After discussing some of the many ways to get the Banaschewski compactification $\beta_0T$ of an arbitrary ultraregular space $T$, we develop another construction of $\beta_0T$ in Th. 2.1. Using those ideas, we develop an analog of $\beta_0T$—what we call a compactoidification $\kappa T$ of an ultraregular space $T$ in Sec. 3; $\kappa T$ is, in essence, a complete absolutely convex compactoid 'superset' of $T$ to which continuous maps of $T$ with precompact range into any complete absolutely convex compactoid subset may be 'continuously extended.'

1991 Mathematics subject classification: 46S10, 54D35, 54C45

1 The Many Faces

For any topological spaces $X$ and $Y$, $C(X, Y)$ and $C^*(X, Y)$ denote the spaces of continuous maps of $X$ into $Y$ and the continuous maps of $X$ into $Y$ with relatively compact range, respectively. To say that a topological space $X$ is ultraregular or ultranormal means, respectively, that the clopen sets are a basis or disjoint closed subsets of $X$ may be separated by clopen sets. A synonym for ultraregular is 0-dimensional. We have a slight preference for the former in order to avoid confusion with other notions of dimension. Throughout the discussion, $T$ denotes at least a Hausdorff space. For an ultraregular space $E$ containing at least two points and ultraregular $T$, B. Banaschewski [2] discovered a compactification $\beta_0T$ of $T$ in which every $x \in C^*(T, E)$ may be continuously extended to $\beta_0x \in C(\beta_0T, E)$. $\beta_0T$ is nowadays usually called the Banaschewski compactification of $T$. It functions as the natural analog of the Stone-Čech compactification ($\beta_0T$ is $\beta T$ for ultranormal $T$) in non-Archimedean analysis. Like the Stone-Čech compactification, the Banaschewski compactification is a protean entity, assuming many different guises. We discuss some of them in this section and then develop a new one in Sec. 2.

1.1 As a completion

Let $E$ be an ultraregular space containing at least two points and let $T$ be ultraregular. Let $C^*(T, E)$ denote the weakest uniform structure on $T$ making each $x \in C^*(T, E)$ uniformly continuous into the compact space $\text{cl} x(T)$ equipped with its unique compatible uniform
structure. By [1], pp. 92-93, since $T$ is ultraregular, $C^* (T, E)$ is compatible with the topology on $T$ and $C^* (T, E)$ is a precompact uniform structure on $T$. Since $C^* (T, E)$ is precompact, its completion $\beta_T T$ is compact and is called the Banaschewski compactification of $T$. $\beta_T T$ is ultranormal ([2], p. 131, Satz 2 or [1], p. 93, Theorem 1)—hence ultraregular—and, by the usual process of extension by continuity function from a dense subspace to the whole space, each $x \in C^* (T, E)$ may be continuously extended to a unique continuous function $\beta_x x \in C^* (\beta_T T, E)$. $\beta_T T$ is unique in a sense we discuss in the context of $E$-compactifications (Th. 1.6). At this point the reader may find the notation $\beta_T T$ curious. Why $\beta_T T$ and not $\beta_E T$? As long as $E$ is ultraregular and contains at least two points ([1], p. 93, [8], pp. 240-243), the uniformity $C^* (T, E)$ does not depend on $E$! A fundamental system of entourages for $C^* (T, E)$, no matter what $E$ is, is defined by the sets

$$V_P = \bigcup \{ V \times V : V \in P \}$$

where $P$ is any finite open (therefore clopen) cover of $T$ by pairwise disjoint sets. The completion of $T$ with respect to this uniformity is the way Banaschewski obtained $\beta_T T$. The definition of $\beta_T T$ as the completion of $C^* (T, E)$ where $E$ is the discrete space of integers was first given in [7], though the idea of treating compactifications as completions is due to Nachbin. The connection with the Stone-Čech compactification is the following.

**Definition 1.1** Let $P$ be a finite clopen cover of a topological space $S$ by pairwise disjoint sets and let $V$ denote the uniformity generated by $V_P$. We say that $S$ is strongly ultraregular if $V = C^* (T, R)$.

**Theorem 1.2** ([8], pp. 251-2) (a) Every ultranormal $T_1$-space $S$ is strongly ultraregular.

(b) If a topological space $S$ is strongly ultraregular then $\beta_S S = \beta S$.

### 1.2 As an E-Compactification

Tihonov proved that a completely regular space $T$ may be characterized as one that is homeomorphic to a subspace of a product $[0, 1]^m$ of unit intervals. Even though his name is not associated with it, he created the first version of the Stone-Čech compactification $\beta T$ of $T$ by then taking the closure of $T$ in $[0, 1]^m$. Engelking and Mrówka [5] developed analogous notions of $E$-completely regular space $T$ and $E$-compactification $\beta_E T$. Let $S$ and $E$ be two topological spaces. $S$ is called $E$-completely regular if it is homeomorphic to a subspace of the $m$-fold topological product $E^m$ for some cardinal $m$. If $E = R$ or $[0, 1]$, this is the familiar notion of complete regularity. With 2 denoting the discrete space $\{0, 1\}$, it happens that

**Theorem 1.3** ([16], p. 17) A topological space $S$ is 2-completely regular if and only if it is an ultraregular $T_0$-space.

An $E$-compact space is one which is homeomorphic to a closed subspace of a topological product $E^m$ for some cardinal $m$. The 2-compact spaces are characterized as follows:

**Theorem 1.4** ([5], p.430, Example (iii)) A topological space $S$ is 2-compact if and only if it is compact and ultraregular.
An $E$-compactification $\beta_E T$ of an $E$-completely regular space $T$ is

1. an $E$-compact space which contains $T$ as a dense subset and
2. ("the $E$-extension property") each $x \in C(T, E)$ may be extended to $\beta_E x \in C(\beta_E T, E)$.

The following analogs of properties of the Stone-Čech compactification obtain for $E$-compactifications.

**Theorem 1.5** ([5], p. 433, Theorem 4, [16], pp. 25-27, 4.3 and 4.4). An $E$-completely regular (Hausdorff) space $T$ has a Hausdorff $E$-compactification $\beta_E T$ with the following properties:

(a) If $S$ is an $E$-compact space then every continuous function $x : T \to S$ has a continuous extension $\bar{x} : \beta_E T \to S$.

(b) The space $\beta_E T$ is unique in the sense that if $S$ is an $E$-compact space containing $T$ as a dense subset and such that every continuous $x : T \to E$ has a continuous extension to $S$, then $S$ is homeomorphic to $\beta_E T$ under a homeomorphism that is the identity on $T$.

(c) $T$ is $E$-compact if and only if $T = \beta_E T$.

How does this apply to $\beta_0 T$? Ultraregular spaces $T$ are $2$-completely regular by Th. 1.3. Since $\beta_0 T$ is compact and ultranormal, it follows that $\beta_0 T$ is $2$-compact by Th. 1.4. Therefore, by Th. 1.5(b) it follows that

**Theorem 1.6** UNIQUENESS OF $\beta_0 T$. $\beta_0 T$ is homeomorphic to $\beta_E T$ under a homeomorphism that is the identity on $T$, as would be any ultraregular compactification of an ultraregular $T$ with the $E$-extension property.

### 1.3 As a Space of Characters

Let $F$ be an ultraregular Hausdorff topological field so that $X = C^* (T, F)$ may be considered as an $F$-algebra. A character of $X$ is a nonzero algebra homomorphism from $X$ into $F$. Let the set $H$ of characters of $X$ be equipped with the weakest topology for which the maps $H \to F$, $h \mapsto h(x)$, are continuous for each $x \in C^* (T, F)$. For each $p \in \beta_0 T$, let $p^*$ denote the evaluation map at $p$, the map $C^* (T, F) \to F$, $x \mapsto \beta_0 x (p)$. It is trivial to verify that each $p^*$ is a character of $C^* (T, F)$. But more is true: You get all the characters of $C^* (T, F)$ this way. In fact, the map

$$A : \beta_0 T \to H$$

$$p \mapsto p^*$$

establishes a homeomorphism between $\beta_0 T$ and $H$. The details may be found in [1], Theorem 3 and [8], Theorem 8.15.
1.4 Characters Again

Once again $\beta_0 T$ is realized as a space of nonzero homomorphisms—ring homomorphisms this time—into the very simple (discrete) field 2 with 2 elements.

A commutative ring $X$ with identity in which each element is idempotent is called a Boolean ring. A subcollection $X$ of the set of subsets of a given set $T$ which is closed under union, intersection and set difference of any two of its members is called a ring of sets. Such a collection forms a ring in the usual algebraic sense if addition and multiplication are taken to be symmetric difference and intersection, respectively. If the sets in $X$ cover $T$ then $X$ is called a covering ring. Since $X$ must have a multiplicative identity (i.e., with respect to intersection) any covering ring must contain $T$ as an element. Any covering ring $X$ generates (in the sense that it is a subbase for) a ultraregular topology on $T$; the topology is ultraregular since the complement $T - A$ of any open set (member of $X$) must belong to $X$. In the converse direction, the class $Cl(T)$ of clopen subsets obviously constitutes a covering ring of any topological space $T$.

Let $X$ be a Boolean ring and endow $2^X$ with the product topology. The Stone space $S(X)$ of the Boolean ring $X$ is the subspace of $2^X$ of all nonzero ring homomorphisms of $X$ into 2. $S(X)$ is called the Stone space because of Stone's use of it in his remarkable characterization of compact ultraregular spaces.

The Stone Representation Theorem ([1], Theorem 4, [12], [4] p.227 or [6], pp. 77-80) If $T$ is a compact ultraregular space, then $T$ is homeomorphic to the Stone space of the Boolean ring $Cl(T)$ of clopen subsets of $T$. Conversely, the Stone space $S(X)$ of any Boolean ring $X$ is a compact ultraregular Hausdorff space and $X$ is ring-isomorphic to the Boolean ring $Cl(T)$ of clopen subsets of $S(X)$.

If $T$ is ultraregular then $\beta_0 T$ is the Stone space of $Cl(T)$. Indeed, the map $\beta : T \to S(Cl(T)), t \mapsto \beta t$, defined for $t \in T$ and $K \in Cl(T)$ by

$$\beta t(K) = \begin{cases} 1 & \text{if } t \in K \\ 0 & \text{if } t \notin K \end{cases}$$

is a homeomorphism of $T$ onto a dense subset of the compact ultraregular Hausdorff space $S(Cl(T))$.

1.5 As a Space of Measures

Let $T$ be ultraregular and let $Cl(T)$ be the ring (algebra, actually, since $T \in Cl(T)$) of clopen subsets of $T$, and let $F$ be an ultraregular Hausdorff topological field. A 0-1 measure on $T$ is a finitely additive set function $m : Cl(T) \to \{0,1\} \subset F$ satisfying the condition:

$$m(U) = 0 \quad \text{and} \quad U \supset V \in Cl(T) \implies m(V) = 0$$

in other words, that clopen subsets of sets of measure 0 also have measure 0. Measures $m_t$, 'concentrated at points $t \in T$' (also called 'purely atomic' or 'the point mass at t') which
are 1 on a clopen set $U$ if $t \in U$ and 0 otherwise are 0-1 measures on $T$. The weak clopen topology for the collection $M$ of all 0-1 measures on $T$ has as a neighborhood base $m_0 \in M$ sets of the form

$$V(m_0; S_1, \ldots, S_n) = \{m \in M : m(S_j) = m_0(S_j), j = 1, \ldots, n\}$$

where the $S_j$ are clopen sets and $n \in \mathbb{N}$. It is trivial to verify that the map $t \mapsto m_t$ is a homeomorphism of $T$ into $M$. Using the techniques of [1] one can demonstrate that $M$ is a compact ultranormal Hausdorff space to which any $tG(T, F)$ may be continuously extended. It follows that $\beta_0 T = M$ in the sense of Th. 1.6.

Last, let us mention that $\beta_0 T$ may also be realized as a Wallman compactification utilizing the lattice of clopen subsets of $T$.

2 A New Approach

A construction of $\beta_0 T$ using the methods of non-Archimedean functional analysis is presented in Theorem 2.1. The proof hinges on the fact that, for a local field $F$, if $U$ is a neighborhood of 0 in a locally $F$-convex space $X$ then its polar $U^\sigma$ is $\sigma(X', X)$-compact ([15], Th. 4.11). Note that $\sigma(X', X)$ is ultraregular since the seminorms $p_x(f) = |f(x)|$, $x \in X$, $f \in X'$, are non-Archimedean.

**Theorem 2.1** Let $F$ be a local field, let $T$ be ultraregular and let $C^*(T, F)$ denote the sup-normed space of all continuous $F$-valued functions on $T$ with relatively compact range. There is an ultranormal compactification $\beta_0 T$ of $T$ such that any $x \in C^*(T, F)$ may be continuously extended to a function $\beta_0 x \in C(\beta_0 T, F)$.

**Proof.** For $t \in T$, let $t^\ast$ denote the evaluation map $x \mapsto x(t)$ for any $x \in C^*(T, F)$. We note that each such $t^\ast$ is a continuous linear form (algebra homomorphism, actually) and is of norm one. Thus $T^\ast = \{t^\ast : t \in T\} \subseteq U$ where $U$ denotes the unit ball of the norm-dual $C^*(T, F)'$ of $C^*(T, F)$. Furthermore, the map $i : T \rightarrow C^*(T, F)', t \mapsto t^\ast$, embeds $T$ homeomorphically in $C^*(T, F)'$ endowed with its weak-* topology by the following argument. The map $i$ is obviously injective. If a net $t_s \rightarrow t \in T$ then $x(t_s) \rightarrow x(t)$ for any $x \in C^*(T, F)$; hence $t_s^\ast \rightarrow t^\ast$ and therefore $i$ is continuous. To see that $i$ is a homeomorphism onto $i(K)$, let $K$ be a closed subset of $T$. Since $T$ is ultraregular, if $t \notin K$ then there exists $x \in C^*(T, F)$ such that $x(t) = 0$ and $|x(K)| = r > 1$. Hence the polar $\{x\}^\circ$ of $\{x\}$ is a neighborhood of $t^\ast$ disjoint from $K^\ast$ and $K^\ast$ is a closed subset of $i(K)$. As $U$ is the polar of the unit ball of $C^*(T, F)$, it follows that $U$ is weak-*compact ([15], Th. 4.11). Therefore the closure $cT$ in $U$ of (the homeomorphic image of ) $T^\ast$ is compact in $C^*(T, F)'$ endowed with the weak-* topology. As to the continuous extendibility of $x \in C^*(T, F)$, consider the canonical image $Jx$ of $x$ in the second algebraic dual of $C^*(T, F)$, i.e., for any $f \in C^*(T, F)'$, $Jx(f) = f(x)$. Clearly $Jx$ is weak-*continuous on $C^*(T, F)'$; so, therefore, is its restriction $\beta_0 x = Jx |_{cT}$. Should this be called $c_T$ rather than $cT$? No topologically significant changes occur for different $F$'s: the compactness of the ultraregular space $cT$ and the fact that $T$ is $C^*$-embedded in $cT$ imply that $cT = \beta_0 T$ by Th. 1.6.
3 Compactoidification

In this section we construct a compactoidification \( \kappa T \) of an ultraregular space \( T \). \((F, |\cdot|)\) denotes a complete nontrivially ultravalued field throughout. As usual, we abbreviate 'F-convex' to 'convex.' A map \( f \) defined on an absolutely convex subset \( A \) of a vector space over \( F \) with values in some absolutely convex set in a vector space over \( F \) is called affine if
\[ f(ax + by) = af(x) + bf(y) \]
for all \( x, y \in A \) and all \( a, b \in F \) with \( |a| < 1 \) and \( |b| < 1 \).

Definition 3.1 A compactoidification of an ultraregular space \( T \) is a pair \((i, \kappa T)\) where \( \kappa T \) is a complete absolutely convex compactoid subset of some Hausdorff locally convex space \( E \) over \( F \) and \( i : T \to \kappa T \) is a continuous map with precompact range for which following extendibility property holds: For any complete absolutely convex compactoid subset \( A \) of some Hausdorff locally convex space \( E \) over \( F \) and any continuous map \( j : T \to A \) with precompact range, there exists a unique continuous affine map \( J : \kappa T \to A \) such that \( J \circ i = j \).

\[ \begin{array}{c}
\kappa T \\
i \\
\downarrow J \\
T \to A
\end{array} \]

Theorem 3.2 A compactoidification is unique in the following natural sense: if \((i_1, \kappa_1 T)\) and \((i_2, \kappa_2 T)\) are compactoidifications of \( T \) then there exists a unique affine homeomorphism \( J_1 : \kappa_1 T \to \kappa_2 T \) such that \( J_1 \circ i_1 = i_2 \). Moreover, the map \( i \) must be injective.

Proof. By definition, there exist unique continuous affine maps \( J_1 \) and \( J_2 \) such that \( J_2 \circ i_1 = i_2 \) and \( J_1 \circ i_2 = i_1 \). Thus, \( J_1 \circ (J_2 \circ i_1) = J_1 \circ i_2 = i_1 \).

\[ \begin{array}{c}
\kappa_1 T \\
i_1 \\
\downarrow J_2 \\
T \to \kappa_2 T
\end{array} \]

Since the identity map \( I_1 : t \mapsto t \) of \( \kappa_1 T \) onto \( \kappa_1 T \) also satisfies \( I_1 \circ i_1 = i_1 \), it follows from the uniqueness that \( I_1 = J_1 \circ J_2 \). Similarly, \( I_2 = J_2 \circ J_1 \) where \( I_2 \) is the identity map of \( \kappa_2 T \) onto \( \kappa_2 T \). It follows that \( J_1 \) is a homeomorphism of \( \kappa_1 T \) onto \( \kappa_2 T \) and \( J_2 \) is its inverse. If \( i_1(t_1) = i_1(t_2) \) then \( i_2(t_1) = J_1 \circ i_1(t_1) = J_1 \circ i_1(t_2) = i_2(t_2) \) so if one of the maps \( i \) is 1-1, all such \( i \) must be. As shown in Theorem 3.3, there is an \( i \) that is 1-1.

In the notation of Sec. 2:

Theorem 3.3 Let \( T \) be ultraregular and let the continuous dual \( C^*(T, F)' \) of \( C^*(T, F) \) carry the weak-* topology. Then

(a) the closed absolutely convex hull \( \kappa T \) of \( T^* \) is the unit ball \( U \) of \( C^*(T, F)' \) and

(b) the pair \((i, \kappa T)\) is a compactoidification of \( T \).

Proof. Clearly the absolute convex hull \( B \) of \( T^* \) is contained in the unit ball \( U \) of \( C^*(T, F)' \). Since \( U \) is a complete compactoid by the \( p \)-adic Alaoglu theorem ([9], Prop.
3.1), so, therefore, is the closed absolutely convex hull $\kappa T$ of the compact set $\text{cl} \ T^*$.

It follows from [10], Prop. 1.3 that $B$ is edged (i.e., if the valuation of $F$ is dense then $\text{cl} \ B = \bigcap \{ a(\text{cl}B) : a \in F, |a| > 1 \}$) and therefore ([9], Th. 4.7) a polar set in $C^* (T, F)'$.

If $\text{cl} \ B \neq U$ there must exist $g \in C^* (T, F)'$ such that $|g| \leq 1$ on $B$ and $|g(f)| > 1$ for some $f \in U - \text{cl} \ B$. Since $g$ must be an evaluation map determined by some point $x \in C^* (T, F)$ by [9], Lemma 7.1, we have found an $x$ such that $|x(t)| = |t'(x)| \leq 1$ for all $t \in T$ but $|f(x)| > 1$. As this contradicts $\|f\| \leq 1$, the proof of (a) is complete.

(b) As in the proof of Th. 2.1, $i$ is a homeomorphism onto the precompact set $T^*$. To verify the extendibility requirement, let $A$ be a complete absolutely convex compactoid and let $j : T \to A$ be continuous with precompact range. We define the affine extension $J$ of $j$ on the absolutely convex hull $B$ of $T^*$ by taking $J \left( \sum_{i=1}^{n} a_i t_i^* \right) = \sum_{i=1}^{n} a_i j(t_i)$ for $a_i \in F, |a_i| \leq 1$, $i = 1, \ldots, n$. The definition makes sense because the $t_i^*$ are linearly independent for distinct $t_i$. Evidently $j = J \circ i$. To prove the continuity of $J$, let $s \to \mu_s = \sum_{i=1}^{n} a_i^* t_i^*$ be a net in $B$ convergent to $0$ in the weak-* topology. Let $[A]$ denote the linear span of $A$ and note that for any $f \in [A]'$, the map $f \circ j \in C^* (T, F)'$, since $j(T)$ is precompact. Thus,

$$f(J(\mu_s)) = f \left( \sum_{i=1}^{n} a_i^* j(t_i) \right) = \sum_{i=1}^{n} a_i^* f(j(t_i)) = \mu_s(f \circ j) \to 0$$

and we conclude that $J(\mu_s) \to 0$ in the weak topology of $[A]$. As $A$ is of countable type, hence a polar space, the weak topology coincides with the initial one on the compactoid $A$ ([9], Th. 5.12) so $J(\mu_s) \to 0$ in $A$. By continuity and ‘affinity,’ $J$ extends uniquely to a continuous affine map of $\text{cl} \ B = \kappa T$ into $A$, since $A$ is complete.

**References**


St. John’s University
Staten Island, NY 10301 USA
e-mail: beckenst at sjuvm.stjohns.edu

St. John’s University
Jamaica, NY 11439 USA
e-mail: naricil at sjuvm.stjohns.edu
fax: 718-380-0353

Matematisch Instituut
K. U. Nijmegen
Toernooiveld
6525 ED Nijmegen, The Netherlands
e-mail: schikhof at sci.kun.nl