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Non-reflexive and non-spherically complete subspaces of the $p$-adic space $l^\infty$

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ABSTRACT

By forming tensor products we construct natural examples of non-reflexive (Section 2) and non-spherically complete (Section 3) closed subspaces of the non-archimedean space $l^\infty$. Also, we study (Section 4) conditions under which two spherically complete Banach spaces are isomorphic; as an application we describe the spherical completion of the subspaces of $l^\infty$ constructed in the paper.

1. PRELIMINARIES

Throughout this paper $K$ is a non-archimedean valued field that is complete under the metric induced by the non-trivial valuation $|\cdot|$, and $E,F$ are pseudo-reflexive ([4], p. 60) non-archimedean Banach spaces over $K$. By $E \cong F$ we mean that $E$ and $F$ are isomorphic, i.e., there is a linear isometry from $E$ onto $F$. We will denote the completed tensor product of $E$ and $F$ in the sense of [4], p. 123 by $E \hat{\otimes} F$.

A Banach space is called spherically complete if every sequence of closed balls

$$B(a_1, r_1) \supset B(a_2, r_2) \supset \cdots$$

for which $r_1 > r_2 > \cdots$ has a non-empty intersection. By $E^\nu$ we will denote the spherical completion of a Banach space $E$ (see [4], p. 148).

A subset $B$ of $E$ is called compactoid if for every $r > 0$ there exists a finite set $S$...
in $E$ such that $B \subset \text{co} S + B(0, r)$, where $\text{co} S$ denotes the absolutely convex hull of $S$.

$L(E, F)$ will denote the Banach space of all continuous linear maps from $E$ into $F$, endowed with the usual norm. The topological dual space of $E$ is $E' = L(E, K)$. $E$ is called reflexive if the canonical linear map $J_E$ from $E$ into $E''$ is a surjective isometry.

By $C(E, F)$ we will denote the closed subspace of $L(E, F)$ consisting of all compact linear maps from $E$ into $F$ (i.e., the maps $T \in L(E, F)$ for which the image of the closed unit ball of $E$ is a compactoid subset of $F$).

We say that $F$ is a strict quotient of $E$ if there exists a $T \in L(E, F)$ such that $\|T\| \leq 1$ and for every $y \in F$ there is $x \in E$ for which $Tx = y$ and $\|x\| = \|y\|$. For unexplained terms and background we refer to [4].

2. NON-REFLEXIVE SUBSPACES OF $l^\infty$

It is well known that, if $K$ is spherically complete, no infinite-dimensional Banach space over $K$ is reflexive ([4], 4.16). However, non-spherically complete fields have a more satisfactory behaviour with regard to reflexivity. In fact, suppose that $K$ is not spherically complete. Then $c_0$ and $l^\infty$ are reflexive spaces (more generally, the spaces $c_0(N,s)$ and $l^\infty(N,s)$ are reflexive, for every function $s : N \to (0, \infty)$, [4], 4.22.ii). Also, every quotient and every closed subspace of $c_0(N,s)$ is reflexive ([2], 9.9). But quotients and closed subspaces of $l^\infty$ need not be reflexive. For quotients this is easily seen: $l^\infty/c_0$ is a non-reflexive quotient of $l^\infty$ since its dual is trivial ([4], 4.15). The construction of a non-reflexive closed subspace of $l^\infty$ is more laborious and was given in [4], 4.J.

In this section we are going to show that, by taking tensor products, we can construct in a simple way, natural examples of non-reflexive closed subspaces of $l^\infty$ when $K$ is not spherically complete. To do that we need some preliminary machinery.

Recall ([4], 4.34) that the maps

$U_{EF} : E' \hat{\otimes} F' \to (E \hat{\otimes} F)'$

$V_{EF} : E \hat{\otimes} F \to (E' \hat{\otimes} F')'$

given by

$U_{EF}(g \otimes h)(x \otimes y) = g(x)h(y)$

$V_{EF}(x \otimes y)(g \otimes h) = g(x)h(y)$

$(x \in E, g \in E', y \in F, h \in F')$

are linear isometries. We need two facts.

**Lemma 2.1.** $U_{EF}$ is surjective iff $L(E, F') = C(E, F')$.

**Proof.** By 4.41 of [4]

$E' \hat{\otimes} F' \simeq C(E, F')$
and by 4.27 of [4] 
\[(E \hat{\otimes} F)' \simeq L(E, F').\]
It is very easy to see that, under these identifications, the map $U_{EF}$ converts into the canonical inclusion of $C(E, F')$ into $L(E, F')$. □

**Lemma 2.2.** The diagram

\[
\begin{array}{ccc}
(E \hat{\otimes} F)'' & \xrightarrow{U_{EF}'} & (E' \hat{\otimes} F')' \\
\downarrow J_{E \hat{\otimes} F} & & \downarrow J_{E' \hat{\otimes} F'} \\
E \hat{\otimes} F & \xrightarrow{V_{EF}} & E'' \hat{\otimes} F''
\end{array}
\]
is commutative.

**Proof.** Direct verification. □

By using these lemmas we now can prove:

**Theorem 2.3.** Let $K$ be not spherically complete, let $G := l^\infty \hat{\otimes} l^\infty$. Then,

(i) $G$ is isomorphic to a closed subspace of $l^\infty$.

(ii) $G' \simeq c_0$.

(iii) $G$ is not reflexive.

**Proof.** (i) Since $(c_0)' \simeq l^\infty$ ([4], 3.Q.ii), the map $U_{c_0c_0}$ yields a linear isometry from $l^\infty \hat{\otimes} l^\infty$ into a closed subspace of $(c_0 \hat{\otimes} c_0)'$. Now the conclusion follows from the fact that $c_0 \hat{\otimes} c_0 \simeq c_0$ ([4], 4.R.ii).

(ii) We have $(l^\infty)' \simeq c_0$ ([4], 4.17) and $L(l^\infty, c_0) = C(l^\infty, c_0)$ ([4], 5.19) so, by 2.1, $U_{EF}$ is surjective when $E := l^\infty$, $F := l^\infty$. Hence,

$G' \simeq (l^\infty)' \hat{\otimes} (l^\infty)' \simeq c_0 \hat{\otimes} c_0 \simeq c_0$.

(iii) Suppose $l^\infty \hat{\otimes} l^\infty$ were reflexive; we derive a contradiction. In the diagram of 2.2 (again with $E = F = l^\infty$) the maps $J_{E \hat{\otimes} F}$ and $U_{EF}'$ are bijections hence so is $V_{EF}$. Then $U_{EF}'$ is surjective hence, by 2.1, $L(E', F'') = C(E', F'')$, i.e. $L(c_0, l^\infty) = C(c_0, l^\infty)$, a contradiction. □

**Remark 2.4.** (1) From 2.3(iii) we conclude that if $K$ is not spherically complete, then $l^\infty \hat{\otimes} l^\infty$ is not isomorphic to $l^\infty$. (The same conclusion holds when $K$ is spherically complete and the valuation on $K$ is dense, see 3.2.)

(2) The following slight extension of 2.3 will be needed in 4.2. Suppose that $K$ is not spherically complete. Let $s : N \to \{s_1, s_2, \ldots\} \subset (0, \infty)$ be such that \(m \in N : s(m) = s_n\) is an infinite set for all $n \in N$. Then, $H := l^\infty \hat{\otimes} l^\infty(N, s)$ is isomorphic to a non-reflexive closed subspace of $l^\infty(N, s)$ and $H' \simeq c_0(N, 1/s)$.

Indeed, observe that if $\{e_i : i \in N\}$ is the canonical orthogonal base of $c_0$ and $c_0(N, 1/s)$, then $\{e_i \otimes e_j : i, j \in N\}$ is an orthogonal base of $c_0 \hat{\otimes} c_0(N, 1/s)$ ([4], 4.30), and so
\[ c_0 \otimes c_0(N, 1/s) \simeq c_0(N, 1/s). \]

The rest follows, by 3.Q and 4.22 of [4], like in the proof of 2.3.

3. NON-SPHERICALLY COMPLETE SUBSPACES OF \( l^\infty \)

As it is well known ([4], 4.A), \( l^\infty \) is spherically complete if and only if \( K \) is spherically complete.

In this section we study when the space \( l^\infty \otimes l^\infty \) considered in 2.3 is spherically complete. To do that, recall that if \( E \) is a Banach space, then \( l^\infty \otimes E \) is isomorphic to the Banach space of all compactoid sequences on \( E \), endowed with the supremum norm ([4], 4.R.v).

**Theorem 3.1.** Let \( E \) be a Banach space over \( K \) containing an orthogonal sequence \( v_1, v_2, \ldots \) such that \( \|v_1\| > \|v_2\| > \cdots \) and \( \lim_n \|v_n\| = 1 \). Then, \( l^\infty \otimes E \) is not spherically complete. In particular, \( l^\infty \otimes l^\infty \) is not spherically complete if the valuation on \( K \) is dense.

**Proof.** Suppose \( l^\infty \otimes E \) is spherically complete. For each \( n = 1, 2, \ldots \) let \( f_n \) be a compactoid sequence on \( E \) defined by \( f_n(m) = v_m \) if \( m \leq n \) and \( f_n(m) = 0 \) if \( m > n \). Since

\[ B(f_n, \|v_{n+1}\|) \supseteq B(f_{n+1}, \|v_{n+2}\|) \quad \text{for all } n, \]

we derive the existence of a compactoid sequence \( f = (x_1, x_2, \ldots) \) in \( E \) such that \( \|f - f_n\|_u \leq \|v_{n+1}\| \) for all \( n \) (where \( \| \cdot \|_u \) denotes the supremum norm). Given \( i = 1, 2, \ldots \), we have that

\[ \|x_i - v_i\| \leq \|f - f_n\|_u \leq \|v_{n+1}\| \quad \text{for all } n \geq i \]

and so \( \|x_i - v_i\| < \|v_i\| \). Hence, \( x_1, x_2, \ldots \) is an orthogonal sequence in \( E \) ([4], 5.B) and \( \|x_i\| = \|v_i\| > 1 \) for all \( i \), which implies that \( \{x_1, x_2, \ldots\} \) is not compactoid in \( E \) ([3], 2.2), a contradiction. \( \Box \)

**Remark 3.2.** From 3.1 we conclude that if the valuation on \( K \) is dense and \( K \) is spherically complete, then \( l^\infty \otimes l^\infty \) is not isomorphic to \( l^\infty \) (compare 2.4.1).

For discretely valued fields the situation is completely different. In fact, we have:

**Proposition 3.3.** Suppose that the valuation on \( K \) is discrete. Then \( l^\infty \otimes l^\infty \) is isomorphic to \( l^\infty \) and is, in particular, spherically complete.

**Proof.** Observe that if \( K \) is discretely valued, \( l^\infty \) has an orthonormal base ([4], 5.16) and so \( l^\infty \simeq c_0(I) \) for some infinite set \( I \). From 4.R.ii of [4] we derive that \( l^\infty \otimes l^\infty \simeq l^\infty \). \( \Box \)
4. ISOMETRIES BETWEEN SPHERICALLY COMPLETE BANACH SPACES

With an eye on 3.1 and 3.3, the following question arises in a natural way. Describe the spherical completion of \( l^\infty \otimes l^\infty \) when the valuation on \( K \) is dense. The key to the answer (Corollary 4.4) is given by the next theorem.

**Theorem 4.1.** Let \( E, F \) be spherically complete Banach spaces. Then the following are equivalent.

(i) \( E \cong F \).

(ii) There exist linear isometries from \( E \) into \( F \) and from \( F \) into \( E \).

(iii) \( E \) is isomorphic to an orthocomplemented subspace of \( F \) and \( F \) is isomorphic to an orthocomplemented subspace of \( E \).

(iv) There exist strict quotient maps from \( E \) onto \( F \) and from \( F \) onto \( E \).

**Proof.** Clearly (i) \( \Rightarrow \) (ii) and (iii) \( \Rightarrow \) (iv). Also, (ii) \( \Rightarrow \) (iii) follows directly from [4], 4.7.

(iv) \( \Rightarrow \) (i): let \( \Gamma = \{|\lambda| : \lambda \in K, \lambda \neq 0\} \) be the value group of \( K \) and let \( H \) be a system of representatives of \( (0, \infty)/\Gamma \). By the remark following 5.2 of [4] we know that for every \( h \in H \), all the maximal orthogonal subsets of \( \{x \in E : \|x\| \in h\Gamma\} \) (resp. of \( \{y \in F : \|y\| \in h\Gamma\} \)) have the same cardinality. We denote by \( N_h(E) \) (resp. \( N_h(F) \)) a set with this cardinality.

Let \( X, Y \) be maximal orthogonal systems in \( E - \{0\} \) and \( F - \{0\} \) respectively, and let \( [X] \) and \( [Y] \) be the corresponding closed linear hulls. By 4.7 of [4] we have that \( E = [X]^\vee \) and \( F = [Y]^\vee \). So,

\[
E \cong \left( \bigoplus_{h \in H} c_0(N_h(E), h) \right)^\vee
\]

and analogously

\[
F \cong \left( \bigoplus_{h \in H} c_0(N_h(F), h) \right)^\vee.
\]

From (iv) it follows that for each \( h \in H \), the sets \( N_h(E) \) and \( N_h(F) \) have the same cardinality. Hence, \( c_0(N_h(E), h) \cong c_0(N_h(F), h) \) for each \( h \in H \), which proves that \( E \cong F \). \( \square \)

**Remark 4.2.** Theorem 4.1 does not remain true when spherical completeness is dropped.

**Example.** Let \( s_1, s_2, \ldots \) be a sequence in \( (0, \infty) \) such that \( s_1 > s_2 > \cdots \) and \( \lim_{n \to \infty} s_n = 1 \). Make \( s : N \to \{s_1, s_2, \ldots\} \subset (0, \infty) \) such that \( \{m \in N : s(m) = s_n\} \) is an infinite set for all \( n \in N \).

Take \( E = l^\infty(N, s) \) and \( F = l^\infty \otimes l^\infty(N, s) \). \( E \) is, in a natural way, isometrically embedded in \( F \). Also, there is a linear isometry from \( F \) into \( E \) (see 2.4.2). But \( E \) is not isomorphic to \( F \). (Apply 2.4.2 when \( K \) is not spherically complete and 3.1, for \( E = l^\infty(N, s) \), when \( K \) is spherically complete.)

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However, for arbitrary Banach spaces we do have the following.

**Corollary 4.3.** If there exists linear isometries from $E$ into $F$ and from $F$ into $E$, then $E^\vee \simeq F^\vee$.

**Proof.** It follows from 4.1(i) $\Leftrightarrow$ (ii) and [4], 4.42. □

This is enough material to prove our main result of this section.

**Corollary 4.4.** $(l^\infty \hat{\otimes} l^\infty)^\vee \simeq (l^\infty)^\vee$. In particular, if $K$ is spherically complete then $(l^\infty \hat{\otimes} l^\infty)^\vee \simeq l^\infty$.

**Proof.** We have an obvious embedding from $l^\infty$ into $l^\infty \hat{\otimes} l^\infty$ and an embedding from $l^\infty \hat{\otimes} l^\infty$ into $l^\infty$ (see 2.3(i)). Now apply 4.3. □

**Remark 4.5.** (1) It follows from 4.41 of [4] that $l^\infty \hat{\otimes} E$ is isomorphic to $C(c_0, E)$. Now suppose that $E$ is an infinite-dimensional spherically complete space (hence, $K$ is spherically complete ([4], 4.3) and $E$ contains an infinite orthogonal sequence ([4], 5.5)). Since $L(c_0, E)$ is also spherically complete ([4], 4.5), we can apply 4.2 of [4] and 3.1 to conclude that $C(c_0, E)$ is not orthocomplemented in $L(c_0, E)$ when the valuation on $K$ is dense (compare [1], 3.3).

(2) Observe that $M := l^\infty \hat{\otimes} c_0$ is also isomorphic to a closed subspace of $l^\infty$. In contrast to 2.3 we have that if $K$ is not spherically complete, then $M$ is reflexive ([4], 4.R.1 and 4.22.ii). Also, $M$ is spherically complete if and only if the valuation on $K$ is discrete (see 3.1 and 3.3).

(3) For $n \in \mathbb{N}$, $n \geq 2$, let $G_n = l^\infty \hat{\otimes} \cdots \hat{\otimes} l^\infty$. By induction, we can easily see that the results proved in this paper for $G_2 = l^\infty \hat{\otimes} l^\infty$ are also true for $G_n$. More concretely, we have:

(i) $G_n$ is isomorphic to a closed subspace of $l^\infty$. If $K$ is not spherically complete, $G_n$ is a non-reflexive space for which $G_n^\prime \simeq c_0$ (see 2.3).

(ii) The valuation on $K$ is discrete $\Leftrightarrow G_n \simeq l^\infty$ $\Leftrightarrow G_n$ is spherically complete (see 3.1 and 3.3).

(iii) $(G_n)^\vee \simeq (l^\infty)^\vee$ (see 4.4).

These facts lead us to the following question.

**Problem.** Suppose that the valuation on $K$ is dense. Are $G_m$ and $G_n$ isomorphic for all (or some) $m \neq n$, $m, n > 1$?

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**REFERENCES**


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