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Non-reflexive and non-spherically complete subspaces of the \( p \)-adic space \( l^\infty \)

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ABSTRACT

By forming tensor products we construct natural examples of non-reflexive (Section 2) and non-spherically complete (Section 3) closed subspaces of the non-archimedean space \( l^\infty \). Also, we study (Section 4) conditions under which two spherically complete Banach spaces are isomorphic; as an application we describe the spherical completion of the subspaces of \( l^\infty \) constructed in the paper.

1. PRELIMINARIES

Throughout this paper \( K \) is a non-archimedean valued field that is complete under the metric induced by the non-trivial valuation \( | \cdot | \), and \( E, F \) are pseudo-reflexive ([4], p. 60) non-archimedean Banach spaces over \( K \). By \( E \simeq F \) we mean that \( E \) and \( F \) are isomorphic, i.e., there is a linear isometry from \( E \) onto \( F \). We will denote the completed tensor product of \( E \) and \( F \) in the sense of [4], p. 123 by \( E \hat{\otimes} F \).

A Banach space is called spherically complete if every sequence of closed balls

\[
B(a_1, r_1) \supset B(a_2, r_2) \supset \cdots
\]

for which \( r_1 > r_2 > \cdots \) has a non-empty intersection. By \( E^\gamma \) we will denote the spherical completion of a Banach space \( E \) (see [4], p. 148).

A subset \( B \) of \( E \) is called compactoid if for every \( r > 0 \) there exists a finite set \( S \)
in $E$ such that $B \subset \text{co } S + B(0, r)$, where $\text{co } S$ denotes the absolutely convex hull of $S$.

$L(E, F)$ will denote the Banach space of all continuous linear maps from $E$ into $F$, endowed with the usual norm. The topological dual space of $E$ is $E' = L(E, K)$. $E$ is called reflexive if the canonical linear map $J_E$ from $E$ into $E''$ is a surjective isometry.

By $C(E, F)$ we will denote the closed subspace of $L(E, F)$ consisting of all compact linear maps from $E$ into $F$ (i.e., the maps $T \in L(E, F)$ for which the image of the closed unit ball of $E$ is a compactoid subset of $F$).

We say that $F$ is a strict quotient of $E$ if there exists a $T \in L(E, F)$ such that $\|T\| \leq 1$ and for every $y \in F$ there is $x \in E$ for which $Tx = y$ and $\|x\| = \|y\|$.

For unexplained terms and background we refer to [4].

2. NON-REFLEXIVE SUBSPACES OF $l^\infty$

It is well known that, if $K$ is spherically complete, no infinite-dimensional Banach space over $K$ is reflexive ([4], 4.16). However, non-spherically complete fields have a more satisfactory behaviour with regard to reflexivity. In fact, suppose that $K$ is not spherically complete. Then $c_0$ and $l^\infty$ are reflexive spaces (more generally, the spaces $c_0(N, s)$ and $l^\infty(N, s)$ are reflexive, for every function $s : N \to (0, \infty)$, [4], 4.22.ii). Also, every quotient and every closed subspace of $c_0(N, s)$ is reflexive ([2], 9.9). But quotients and closed subspaces of $l^\infty$ need not be reflexive. For quotients this is easily seen: $l^\infty/c_0$ is a non-reflexive quotient of $l^\infty$ since its dual is trivial ([4], 4.15). The construction of a non-reflexive closed subspace of $l^\infty$ is more laborious and was given in [4], 4.J.

In this section we are going to show that, by taking tensor products, we can construct in a simple way, natural examples of non-reflexive closed subspaces of $l^\infty$ when $K$ is not spherically complete. To do that we need some preliminary machinery.

Recall ([4], 4.34) that the maps

$$U_{EF} : E' \hat{\otimes} F' \to (E \hat{\otimes} F)'$$

$$V_{EF} : E \hat{\otimes} F \to (E' \hat{\otimes} F')'$$

given by

$$U_{EF}(g \otimes h)(x \otimes y) = g(x)h(y)$$

$$V_{EF}(x \otimes y)(g \otimes h) = g(x)h(y)$$

$$(x \in E, \ g \in E', \ y \in F, \ h \in F')$$

are linear isometries. We need two facts.

**Lemma 2.1.** $U_{EF}$ is surjective iff $L(E, F') = C(E, F')$.

**Proof.** By 4.41 of [4]

$$E' \hat{\otimes} F' \simeq C(E, F')$$

122
and by 4.27 of [4]

$$(E \hat{\otimes} F)' \simeq L(E, F').$$

It is very easy to see that, under these identifications, the map $U_{EF}$ converts into
the canonical inclusion of $C(E, F')$ into $L(E, F').$ \(\square\)

**Lemma 2.2.** The diagram

\[
\begin{array}{ccc}
(E \hat{\otimes} F)^{''} & \xrightarrow{U_{EF}'} & (E' \hat{\otimes} F')' \\
\downarrow J_{E \hat{\otimes} F} & & \downarrow V_{EF} \\
E \hat{\otimes} F & \xleftarrow{U_{EF}''} & E'' \hat{\otimes} F''
\end{array}
\]

is commutative.

**Proof.** Direct verification. \(\square\)

By using these lemmas we now can prove:

**Theorem 2.3.** Let $K$ be not spherically complete, let $G := l^\infty \hat{\otimes} l^\infty.$ Then,

(i) $G$ is isomorphic to a closed subspace of $l^\infty.$

(ii) $G' \simeq c_0.$

(iii) $G$ is not reflexive.

**Proof.** (i) Since $(c_0)' \simeq l^\infty$ ([4], 3.Q.ii), the map $U_{c_0c_0}$ yields a linear isometry from $l^\infty \hat{\otimes} l^\infty$ into a closed subspace of $(c_0 \hat{\otimes} c_0)'.$ Now the conclusion follows from the fact that $c_0 \hat{\otimes} c_0 \simeq c_0$ ([4], 4.R.ii).

(ii) We have $(l^\infty)' \simeq c_0$ ([4], 4.17) and $L(l^\infty, c_0) = C(l^\infty, c_0)$ ([4], 5.19) so, by 2.1, $U_{EF}$ is surjective when $E := l^\infty, F := l^\infty.$ Hence,

$$G' \simeq (l^\infty)' \hat{\otimes} (l^\infty)' \simeq c_0 \hat{\otimes} c_0 \simeq c_0.$$

(iii) Suppose $l^\infty \hat{\otimes} l^\infty$ were reflexive; we derive a contradiction. In the dia­
gram of 2.2 (again with $E = F = l^\infty$) the maps $J_{E \hat{\otimes} F}$ and $U_{EF}'$ are bijections
hence so is $V_{EF}.$ Then $U_{EF}'$ is surjective hence, by 2.1, $L(E', F'') = C(E', F''),$

i.e. $L(c_0, l^\infty) = C(c_0, l^\infty),$ a contradiction. \(\square\)

**Remark 2.4.** (1) From 2.3(iii) we conclude that if $K$ is not spherically complete,
then $l^\infty \hat{\otimes} l^\infty$ is not isomorphic to $l^\infty.$ (The same conclusion holds when $K$
is spherically complete and the valuation on $K$ is dense, see 3.2.)

(2) The following slight extension of 2.3 will be needed in 4.2. Suppose that $K$
is not spherically complete. Let $s : N \to \{s_1, s_2, \ldots\} \subset (0, \infty)$ be such that
$
\{m \in N : s(m) = s_n\}$
is an infinite set for all $n \in N.$ Then, $H := l^\infty \hat{\otimes} l^\infty(N, s)$ is
isomorphic to a non-reflexive closed subspace of $l^\infty(N, s)$ and $H' \simeq c_0(N, 1/s).$

Indeed, observe that if $\{e_i : i \in N\}$ is the canonical orthogonal base of $c_0$
and $c_0(N, 1/s),$ then $\{e_i \otimes e_j : i, j \in N\}$ is an orthogonal base of $c_0 \hat{\otimes} c_0(N, 1/s)$ ([4],
4.30), and so
The rest follows, by 3.9 and 4.22 of [4], like in the proof of 2.3.

3. NON-SPHERICALLY COMPLETE SUBSPACES OF \( l^\infty \)

As it is well known ([4], 4.A), \( l^\infty \) is spherically complete if and only if \( K \) is spherically complete.

In this section we study when the space \( l^\infty \hat{\otimes} l^\infty \) considered in 2.3 is spherically complete. To do that, recall that if \( E \) is a Banach space, then \( l^\infty \hat{\otimes} E \) is isomorphic to the Banach space of all compactoid sequences on \( E \), endowed with the supremum norm ([4], 4.R.v).

**Theorem 3.1.** Let \( E \) be a Banach space over \( K \) containing an orthogonal sequence \( v_1, v_2, \ldots \) such that \( \|v_1\| > \|v_2\| > \cdots \) and \( \lim_n \|v_n\| = 1 \). Then, \( l^\infty \hat{\otimes} E \) is not spherically complete. In particular, \( l^\infty \hat{\otimes} l^\infty \) is not spherically complete if the valuation on \( K \) is dense.

**Proof.** Suppose \( l^\infty \hat{\otimes} E \) is spherically complete. For each \( n = 1, 2, \ldots \) let \( f_n \) be a compactoid sequence on \( E \) defined by \( f_n(m) = v_m \) if \( m \leq n \) and \( f_n(m) = 0 \) if \( m > n \). Since

\[
B(f_n, \|v_{n+1}\|) \supset B(f_{n+1}, \|v_{n+2}\|) \quad \text{for all } n,
\]

we derive the existence of a compactoid sequence \( f = (x_1, x_2, \ldots) \) in \( E \) such that \( \|f - f_n\|_u \leq \|v_{n+1}\| \) for all \( n \) (where \( \| \cdot \|_u \) denotes the supremum norm). Given \( i = 1, 2, \ldots \), we have that

\[
\|x_i - v_i\| \leq \|f - f_n\|_u \leq \|v_{n+1}\| \quad \text{for all } n \geq i
\]

and so \( \|x_i - v_i\| < \|v_i\| \). Hence, \( x_1, x_2, \ldots \) is an orthogonal sequence in \( E \) ([4], 5.B) and \( \|x_i\| = \|v_i\| > 1 \) for all \( i \), which implies that \( \{x_1, x_2, \ldots\} \) is not compactoid in \( E \) ([3], 2.2), a contradiction. \( \square \)

**Remark 3.2.** From 3.1 we conclude that if the valuation on \( K \) is dense and \( K \) is spherically complete, then \( l^\infty \hat{\otimes} l^\infty \) is not isomorphic to \( l^\infty \) (compare 2.4.1).

For discretely valued fields the situation is completely different. In fact, we have:

**Proposition 3.3.** Suppose that the valuation on \( K \) is discrete. Then \( l^\infty \hat{\otimes} l^\infty \) is isomorphic to \( l^\infty \) and is, in particular, spherically complete.

**Proof.** Observe that if \( K \) is discretely valued, \( l^\infty \) has an orthonormal base ([4], 5.16) and so \( l^\infty \cong c_0(I) \) for some infinite set \( I \). From 4.R.ii of [4] we derive that \( l^\infty \hat{\otimes} l^\infty \cong l^\infty \). \( \square \)
4. ISOMETRIES BETWEEN SPHERICALLY COMPLETE BANACH SPACES

With an eye on 3.1 and 3.3, the following question arises in a natural way. Describe the spherical completion of $l^\infty \otimes l^\infty$ when the valuation on $K$ is dense. The key to the answer (Corollary 4.4) is given by the next theorem.

**Theorem 4.1.** Let $E, F$ be spherically complete Banach spaces. Then the following are equivalent.

(i) $E \simeq F$.

(ii) There exist linear isometries from $E$ into $F$ and from $F$ into $E$.

(iii) $E$ is isomorphic to an orthocomplemented subspace of $F$ and $F$ is isomorphic to an orthocomplemented subspace of $E$.

(iv) There exist strict quotient maps from $E$ onto $F$ and from $F$ onto $E$.

**Proof.** Clearly (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv). Also, (ii) $\Rightarrow$ (iii) follows directly from [4], 4.7.

(iv) $\Rightarrow$ (i): let $\Gamma = \{|\lambda| : \lambda \in K, \lambda \neq 0\}$ be the value group of $K$ and let $H$ be a system of representatives of $(0, \infty)/\Gamma$. By the remark following 5.2 of [4] we know that for every $h \in H$, all the maximal orthogonal subsets of \{ $x \in E: \|x\| \in h\Gamma$ \} (resp. \{ $y \in F: \|y\| \in h\Gamma$ \}) have the same cardinality. We denote by $N_h(E)$ (resp. $N_h(F)$) a set with this cardinality.

Let $X, Y$ be maximal orthogonal systems in $E - \{0\}$ and $F - \{0\}$ respectively, and let $[X]$ and $[Y]$ be the corresponding closed linear hulls. By 4.7 of [4] we have that $E = [X]^\vee$ and $F = [Y]^\vee$. So,

$$E \simeq \left( \bigoplus_{h \in H} c_0(N_h(E), h) \right)^\vee$$

and analogously

$$F \simeq \left( \bigoplus_{h \in H} c_0(N_h(F), h) \right)^\vee.$$

From (iv) it follows that for each $h \in H$, the sets $N_h(E)$ and $N_h(F)$ have the same cardinality. Hence, $c_0(N_h(E), h) \simeq c_0(N_h(F), h)$ for each $h \in H$, which proves that $E \simeq F.$ \qed

**Remark 4.2.** Theorem 4.1 does not remain true when spherical completeness is dropped.

**Example.** Let $s_1, s_2, \ldots$ be a sequence in $(0, \infty)$ such that $s_1 > s_2 > \cdots$ and $\lim_{n} s_n = 1$. Make $s: N \to \{s_1, s_2, \ldots\} \subset (0, \infty)$ such that $\{m \in N: s(m) = s_n\}$ is an infinite set for all $n \in N$.

Take $E = l^\infty(N, s)$ and $F = l^\infty \otimes l^\infty(N, s)$. $E$ is, in a natural way, isometrically embedded in $F$. Also, there is a linear isometry from $F$ into $E$ (see 2.4.2). But $E$ is not isomorphic to $F$. (Apply 2.4.2 when $K$ is not spherically complete and 3.1, for $E = l^\infty(N, s)$, when $K$ is spherically complete.)
However, for arbitrary Banach spaces we do have the following.

**Corollary 4.3.** If there exists linear isometries from $E$ into $F$ and from $F$ into $E$, then $E^\vee \simeq F^\vee$.

**Proof.** It follows from 4.1(i) $\Leftrightarrow$ (ii) and [4], 4.42. □

This is enough material to prove our main result of this section.

**Corollary 4.4.** $(l^\infty \hat{\otimes} l^\infty)^\vee \simeq (l^\infty)^\vee$. In particular, if $K$ is spherically complete then $(l^\infty \hat{\otimes} l^\infty)^\vee \simeq l^\infty$.

**Proof.** We have an obvious embedding from $l^\infty$ into $l^\infty \hat{\otimes} l^\infty$ and an embedding from $l^\infty \hat{\otimes} l^\infty$ into $l^\infty$ (see 2.3(i)). Now apply 4.3. □

**Remark 4.5.** (1) It follows from 4.41 of [4] that $l^\infty \hat{\otimes} E$ is isomorphic to $C(c_0, E)$. Now suppose that $E$ is an infinite-dimensional spherically complete space (hence, $K$ is spherically complete ([4], 4.3) and $E$ contains an infinite orthogonal sequence ([4], 5.5)). Since $L(c_0, E)$ is also spherically complete ([4], 4.5), we can apply 4.2 of [4] and 3.1 to conclude that $C(c_0, E)$ is not orthocomplemented in $L(c_0, E)$ when the valuation on $K$ is dense (compare [1], 3.3).

(2) Observe that $M := l^\infty \hat{\otimes} c_0$ is also isomorphic to a closed subspace of $l^\infty$. In contrast to 2.3 we have that if $K$ is not spherically complete, then $M$ is reflexive ([4], 4.R.1 and 4.22.ii). Also, $M$ is spherically complete if and only if the valuation on $K$ is discrete (see 3.1 and 3.3).

(3) For $n \in \mathbb{N}, n \geq 2$, let $G_n = l^\infty \hat{\otimes} \ldots \hat{\otimes} l^\infty$. By induction, we can easily see that the results proved in this paper for $G_2 = l^\infty \hat{\otimes} l^\infty$ are also true for $G_n$. More concretely, we have:

(i) $G_n$ is isomorphic to a closed subspace of $l^\infty$. If $K$ is not spherically complete, $G_n$ is a non-reflexive space for which $G_n^\prime \simeq c_0$ (see 2.3).

(ii) The valuation on $K$ is discrete $\Leftrightarrow G_n \simeq l^\infty \Leftrightarrow G_n$ is spherically complete (see 3.1 and 3.3).

(iii) $(G_n)^\vee \simeq (l^\infty)^\vee$ (see 4.4).

These facts lead us to the following question.

**Problem.** Suppose that the valuation on $K$ is dense. Are $G_m$ and $G_n$ isomorphic for all (or some) $m \neq n, m, n > 1$?

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