The Orlicz-Pettis property in \( p \)-adic analysis

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Received December 15, 1992. Revised June 16, 1993

**Abstract**

For a non-archimedean locally convex space \( E \) the property (O.P.): “every weakly convergent sequence in \( E \) is convergent” is studied. Examples are given (1.3, 2.4-2.7). If the scalar field \( K \) is spherically complete every \( E \) has (O.P.) (1.2). If not, the property (O.P.) is closely related to “\( E \) does not contain \( \ell^\infty \)” (3.2).

**Terminology**

Throughout \( K \) is a non-archimedean nontrivially valued field that is complete with respect to the metric induced by the valuation \(||\). For notations, definitions, ... we refer to [5] for normed spaces and to [6] for general locally convex spaces. However, we recall the following. Let \( E, F \) be \( K \)-linear spaces. The \( K \)-linear span of a set \( X \subset E \) is denoted \([X]\), the (algebraic) dual of \( E \) is \( E^* \). If \( p, q \) are (non-archimedean) seminorms on \( E, F \) respectively we denote by \( p \otimes q \) the seminorm

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1 Research partially supported by Grant DGICYT PS90-100

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$z \mapsto \inf \left\{ \max_{1 \leq i \leq n} p(x_i) q(y_i) : n \in \mathbb{N}, z = \sum_{i=1}^{n} x_i \otimes y_i : x_i \in E, y_i \in F \right\}$ on $E \otimes F$. A seminorm $p$ on $E$ is polar if $p = \sup \{ |f| : f \in E^*, |f| \leq p \}$.

Now let $E, F$ be locally convex spaces over $K$. The topological dual of $E$ is $E'$, the weak topology on $E$ is $\sigma(E, E')$. $E$ is called of countable type if for every continuous seminorm $p$ on $E$ the normed space $E / \ker p$ is of countable type. $E$ is called a polar space if the topology is generated by polar seminorms whereas $E$ is called strongly polar if every continuous seminorm is polar. On the tensor product $E \otimes F$ we always consider the topology generated by the seminorms $p \otimes q$ where $p, q$ are continuous seminorms on $E, F$ respectively.

From now on in this paper “locally convex” will mean “Hausdorff locally convex”.

1. (O.P.)-spaces

The classical Banach space $\ell^1$ (over $\mathbb{R}$ or $\mathbb{C}$) has the property that every weakly convergent sequence is norm convergent, which is known as the Orlicz-Pettis Theorem. In our non-archimedean theory we therefore define

**DEFINITION 1.1.** A locally convex space over $K$ is called Orlicz-Pettis space ((O.P.)-space) if every weakly convergent sequence is convergent.

We first consider some immediate examples. It was shown by Monna in [3] that $c_0$ is an (O.P.)-space (observe that in our case the dual of $c_0$ is no longer $\ell^1$ but $\ell^\infty$). By straightforward arguments one can prove that subspaces, products and locally convex direct sums of (O.P.)-spaces are again (O.P.)-spaces. ([4], Propositions 1.2, 1.4.) As every space of countable type is a subspace of some power of $c_0$ we obtain the (O.P.)-property for every space of countable type. Now let $E$ be any locally convex space and let $x_1, x_2, \ldots$ be a sequence in $E$ converging weakly to 0, and set $D = [x_1, x_2, \ldots]$. If also $x_n \to 0$ in $\sigma(D, D')$ then, as $D$ is of countable type, it would follow that $x_n \to 0$ in the original topology of $E$. Now $\sigma(E, E')|D = \sigma(D, D')$ as soon as every $f \in D'$ has an extension $\tilde{f} \in E'$. Such extensions exist certainly if $K$ is spherically complete, by Ingleton’s Theorem. Hence,

**Theorem 1.2**

*If $K$ is spherically complete every locally convex space over $K$ is an (O.P.)-space.*
Theorem 1.3

The following spaces are (O.P.)-spaces.

(i) Every strongly polar space.
(ii) Every space of countable type.
(iii) Every Banach space with a base.
(iv) Any $K$-vector space equipped with the strongest locally convex topology.

Proof. All the spaces indicated in (i), (ii), (iii) have the property that every continuous linear function on a subspace of countable type can be extended to a continuous linear function on the whole space ([6], Definition 3.5, Theorem 4.4 and [5], Corollary 3.18). To prove (iv) just observe that the space is linearly homeomorphic to the locally convex direct sum of a collection of onedimensional spaces. □

Because of Theorem 1.2 we assume from this point on in this paper that $K$ is not spherically complete.

To obtain a space which is not (O.P.), take $E := \ell^\infty$. In fact, since $(\ell^\infty)' \approx c_0$ (see [5], Theorem 4.17) it follows that $(1,0,0,...),(0,1,0,...),(0,0,1,0,...),...$ tends weakly to 0 but, of course, is not norm convergent. This example also tells us that the class of the (O.P.)-spaces is not closed for forming of quotients: Every Banach space (in particular $\ell^\infty$) is a quotient of a Banach space with a base. The space $\ell^\infty$ will play a key role in characterizing (O.P.)-spaces (Theorem 3.2).

The following observation will be used in the sequel several times. If $E$ is an (O.P.)-space then the weak topology $\sigma(E,E')$ is Hausdorff. Indeed, if $x \in E$ and $f(x) = 0$ for all $f \in E'$ then $0,x,0,x,...$ converges weakly, hence strongly, so $x = 0$. Obviously the converse does not hold (again, take $E := \ell^\infty$).

2. (O.P.)-spaces of continuous functions

To establish the (O.P.)-property for some spaces of (vector valued) continuous functions we first prove three structure theorems.

Theorem 2.1

If $E$ and $F$ are (O.P.)-spaces then so is $E \otimes F$. 
Proof. Let $z_1, z_2, \ldots$ be a sequence in $E \otimes F$ converging weakly to 0. Let $p$ resp. $q$ be a continuous seminorm on $E$ resp. $F$. We shall prove that $(p \otimes q)(z_n) \to 0$. Let $0 < t < 1$. By an obvious modification of [5], Theorem 4.30(ii) (see also [1], Lemma 2.1) for each $n \in \mathbb{N}$ there exists $x_1^n, \ldots, x_{m_n}^n, y_1^n, \ldots, y_{m_n}^n \in E$ such that

1. $z_n = \sum_{k=1}^{m_n} x_k^n \otimes y_k^n$,
2. $y_1^n, \ldots, y_{m_n}^n$ are $t$-orthogonal with respect to $q$,
3. $1 \leq q(y_k^n) \leq 2 (k \in \{1, \ldots, m_n\})$.

Now let $f \in E'$. The map $f \otimes 1 : E \otimes F \to F$ sends $z_1, z_2, \ldots$ into a weakly convergent, hence convergent, sequence in $F$, i.e.,

$$\lim_{n \to \infty} q \left( \sum_{k=1}^{m_n} f(x_k^n) y_k^n \right) = 0.$$ 

By $t$-orthogonality and (iii)

$$\lim_{n \to \infty} \max_{1 \leq k \leq m_n} |f(x_k^n)| = 0.$$

As the latter is true for every $f \in E'$ the sequence $x_1^1, x_2^1, \ldots, x_{m_1}^1, x_1^2, \ldots, x_{m_2}^2, \ldots$ converges weakly to 0 in the (O.P.)-space $E$, hence with respect to $p$ so that

$$(p \otimes q)(z_n) \leq \max_{1 \leq k \leq m_n} p(x_k^n)q(y_k^n) \leq 2 \max_{1 \leq k \leq m_n} p(x_k^n) \to 0 \quad \text{for } n \to \infty$$

implying $(p \otimes q)(z_n) \to 0$. □

**Theorem 2.2**

Let $(E, \tau)$ be a metrizable locally convex space and let $D$ be a dense subspace of $E$. If $D$ is an (O.P.)-space then so is $E$.

**Proof.** There is an invariant metric $d$ on $E$ inducing $\tau$. Let $x_1, x_2, \ldots$ be a sequence in $E$ such that $\lim_{n \to \infty} x_n = 0$ in $\sigma(E, E')$. For each $n \in \mathbb{N}$, choose a $y_n \in D$ with $d(x_n, y_n) < 1/n$. Then $x_n - y_n \overset{\tau}{\to} 0$ so $x_n - y_n \to 0$ in $\sigma(E, E')$ so that $y_n = x_n - (x_n - y_n) \to 0$ in $\sigma(E, E')$, hence in $\sigma(D, D')$. Now $D$ is an (O.P.)-space, so $y_n \overset{\tau}{\to} 0$. But then also $x_n = (x_n - y_n) + y_n \overset{\tau}{\to} 0$. □

**Problem.** Does the conclusion hold if we drop the metrizability condition?

**Theorem 2.3**

Every metrizable (O.P.)-space is polar.
Proof. Let $p_1, p_2, \ldots$ be seminorms defining the topology $\tau$ of a (metrizable) locally convex space $E$. For each $n \in \mathbb{N}$ define the seminorm $\tilde{p}_n$ by
\[
\tilde{p}_n = \sup \{|f| : f \in E', |f| \leq p_n\}.
\]
Then the topology $\tilde{\tau}$ induced by the $\tilde{p}_n$ is polar. From the first inclusion in
\[
(\ast) \quad \sigma(E, E') \subset \tilde{\tau} \subset \tau
\]
we obtain that $\tilde{\tau}$ is Hausdorff, so $(E, \tilde{\tau})$ is metrizable. Since $(E, \tau)$ is (O.P.) we conclude from $(\ast)$ that $\tau$ and $\tilde{\tau}$ have the same convergent sequences implying $\tau = \tilde{\tau}$ by metrizability. We see that $\tau$ is polar. □

Remark. There exist (nonmetrizable) (O.P.)-spaces that are not polar ([4], Proposition 4.1).

Now we turn to function spaces. For a Hausdorff zerodimensional topological space $X$ and a locally convex space $E$ over $K$ we define
\begin{align*}
PC(X, E) & : \text{The space of all continuous functions } f : X \to E \text{ for which } f(X) \text{ is precompact, endowed with the topology } \tau_u \text{ of uniform convergence,} \\
C(X, E) & : \text{The space of all continuous functions } X \to E, \text{ endowed with the topology } \tau_c \text{ of compact convergence,} \\
Cb(X, E) & : \text{The space of all bounded continuous functions } X \to E, \text{ endowed with the strict topology } \tau_\beta. \text{ This is the topology generated by all seminorms } f \mapsto \sup_{x \in X} |\phi(x)|p(f(x)) (f \in C_b(X, E)) \text{ where } \phi : X \to K \text{ is a bounded function vanishing at infinity and } p \text{ is a continuous seminorm on } E.
\end{align*}

In the sequel we shall often restrict to Fréchet spaces $E$ as we need the Theorems 2.2 and 2.3.

Theorem 2.4

Let $E$ be a Fréchet space. Then $PC(X, E)$ is an (O.P.)-space if and only if $E$ is an (O.P.)-space.

Proof. The constant functions form a subspace of $PC(X, E)$ that is linearly homeomorphic to $E$ which proves the “only if”. To complete the proof, let $E$ be an (O.P.)-space. Then, as $PC(X, K)$ has a base ([5], Cor. 5.23), we have by Theorems 1.3(iii) and 2.1 that $PC(X, K) \otimes E$ is a metrizable (O.P.)-space. It is easily seen that the map $T : PC(X, K) \otimes E \to PC(X, E)$, given by the formula
\[
T\left(\sum_{j=1}^{n} f_j \otimes a_j\right)(x) = \sum_{j=1}^{n} f_j(x)a_j \quad (x \in X)
\]
is a linear homeomorphism onto a dense subspace of $PC(X, E)$. Then $PC(X, E)$ is an (O.P.)-space by Theorem 2.2. □
Corollary 2.5

Let \( E \) be a Fréchet space. Then \( C(X, E) \) is an (O.P.)-space if and only if \( E \) is an (O.P.)-space.

Proof. We prove the “if”. Let \( E \) be an (O.P.)-space, let \( f_1, f_2, \ldots \) be a sequence in \( C(X, E) \) converging weakly to \( 0 \). Then, for every compact set \( H \) in \( X \), the sequence \( n \mapsto f_n|H \) converges weakly to \( 0 \) in \( PC(H, E) \). By Theorem 2.4 \( f_n \to 0 \) uniformly on \( H \) and the conclusion follows. \( \square \)

Corollary 2.6

Let \( E \) be a Fréchet space. Then \( C_b(X, E) \) is an (O.P.)-space if and only if \( E \) is an (O.P.)-space.

Proof. Again we prove the “if”. Let \( E \) be an (O.P.)-space, let \( f_1, f_2, \ldots \) converge weakly to \( 0 \) in \( C_b(X, E) \). Since \( \tau_c \subset \tau_\beta \) this sequence converges also weakly to zero in \( (C_b(X, E), \tau_c) \). The latter, being a subspace of \( (C(X, E), \tau_c) \) is (O.P.) by Corollary 2.5, so \( f_n \to 0 \) uniformly on compacts. Further, \( E \) is polar and so is \( (C_b(X, E), \tau_\beta) \) implying that \( \{f_1, f_2, \ldots\} \) is \( \tau_\beta \)-bounded. Now apply Proposition 2.11 and Corollary 2.12 of [2] to conclude that \( f_n \to 0 \) in \( \tau_\beta \). \( \square \)

The picture changes if we endow \( C_b(X, E) \) with the uniform topology \( \tau_u \):

Corollary 2.7

Let \( E \) be a Fréchet space. Then \( (C_b(X, E), \tau_u) \) is an (O.P.)-space \( \iff X \) is pseudocompact and \( E \) is an (O.P.)-space.

Proof. \( \Rightarrow \). If \( X \) is not pseudocompact we can find a countably infinite clopen partition \( X = \bigcup_n X_n \). Choose \( e \in E \) and define \( T: \ell^\infty \to C_b(X, E) \) by the formula

\[
T(\alpha_1, \alpha_2, \ldots)(x) = \alpha_n e \quad \text{if } n \in \mathbb{N}, \ x \in X_n.
\]

It is easily seen that \( T \) is a linear homeomorphism of \( \ell^\infty \) onto a subspace of \( (C_b(X, E), \tau_u) \) which yields a contradiction as \( \ell^\infty \) is not (O.P.). For the other conclusion, consider again the constant functions.

\( \Leftarrow \). One verifies that, if \( X \) is pseudocompact, then \( C_b(X, E) = PC(X, E) \). Now apply Theorem 2.4. \( \square \)
3. Fréchet (O.P.)-spaces

An (O.P.)-space cannot contain a copy of $\ell^\infty$ as $\ell^\infty$ itself is not (O.P.). This simple observation is the starting point of Theorem 3.2 that characterizes Fréchet (O.P.)-spaces. The key result is Proposition 3.1 in which we describe polar spaces that do not contain $\ell^\infty$.

**Proposition 3.1**

For a polar locally convex space $E$ the following are equivalent.

(a) $E$ does not contain a subspace linearly homeomorphic to $\ell^\infty$.

(b) Every continuous linear map $\ell^\infty \to E$ is compact.

(γ) $E$ does not contain a complemented subspace linearly homeomorphic to $\ell^\infty$.

**Proof.** (α) ⇒ (β). Let $T : \ell^\infty \to E$ be a noncompact continuous linear map; we derive a contradiction. Let $e_1, e_2, \ldots$ be the unit vectors of $\ell^\infty$. Then $\{Te_1, Te_2, \ldots\}$ is not a compactoid (otherwise the weak closure of the absolutely convex hull of $Te_1, Te_2, \ldots$ would be a compactoid ([6], Theorem 5.13) hence so would its subset $T(\{x \in \ell^\infty : \|x\| \leq 1\})$ implying compactness of $T$). So, there exists a continuous polar seminorm $p$ such that $\{Te_1, Te_2, \ldots\}$ is not a $p$-compactoid. By [7], Theorem 2 there exists a $t \in (0, 1)$ and a subsequence $z_1, z_2, \ldots$ of $Te_1, Te_2, \ldots$ that is $t$-orthogonal with respect to $p$ and such that $\inf_n p(z_n) > 0$. Without loss, assume $p(z_n) \geq 1$ for each $n$.

Now, inductively we shall construct a subsequence $u_1, u_2, \ldots$ of $z_1, z_2, \ldots$ and $f_1, f_2, \ldots \in E'$ such that $|f_n| \leq 2t^{-1}p$ for all $n$ and

$$|f_m(u_n)| = \begin{cases} 0 & \text{if } m > n \\ 1 & \text{if } m = n \end{cases} \quad |f_m(u_n)| \leq \frac{1}{2} \text{ if } m < n$$

To do that, observe that the function $h_1 : \lambda z_1 \mapsto \lambda (\lambda \in K)$ satisfies $|h_1| \leq p$. By polarity it can be extended to an $f_1 \in E'$ such that $|f_1| \leq 2p$. Set $u_1 := z_1$. Suppose $f_1, \ldots, f_{m-1}$ and $u_1, \ldots, u_{m-1}$ are chosen with the required properties. Since $Te_n \to 0$ weakly we have $z_n \to 0$ weakly. So we can find a $k$ (larger than the indexes with respect to $z$ of $u_1, \ldots, u_{m-1}$) such that $|f_1(z_n)| \leq 1/2, \ldots, |f_{m-1}(z_n)| \leq 1/2$ for $n \geq k$. Choose $u_m := z_k$. The function $h_m : \lambda_1 u_1 + \ldots + \lambda_m u_m \mapsto \lambda_m (\lambda_1, \ldots, \lambda_m \in K)$ satisfies $|h_m| \leq t^{-1}p$ so it can be extended to a function $f_m \in E'$ such that $|f_m| \leq 2t^{-1}p$. We see that $f_1, \ldots, f_m$ and $u_1, \ldots, u_m$ have the required properties.
Now we have that \( u_1, u_2, \ldots \) is a subsequence, say \( T e_{i_1}, T e_{i_2}, \ldots \) of \( T e_1, T e_2, \ldots \). Define a linear isometry \( \Omega : \ell^\infty \to \ell^\infty \) by the formula

\[
(\Omega(y_1, y_2, \ldots))_n = \begin{cases} 
0 & \text{if } n \not\in \{i_1, i_2, \ldots\} \\
y_{i_m} & \text{if } m \in \mathbb{N}, n = i_m
\end{cases}
\]

and set \( S := T \circ \Omega \). Then obviously \( S \) is continuous and \( S \) is described by the formula

\[
S(y_1, y_2, \ldots) = \sigma(E, E') - \sum_{n=1}^{\infty} y_n u_n.
\]

Finally let \( y = (y_1, y_2, \ldots) \in \ell^\infty, y \neq 0 \). There is an \( m \in \mathbb{N} \) such that \( |y_m| > \frac{1}{2} ||y|| \). We have \( p(Sy) \geq \frac{1}{2} t |f_m(Sy)| = \frac{1}{2} t |\sum_{n \geq m} y_n f_m(u_n)| \). If \( n > m \) we have \( |y_m f_m(u_n)| \leq \frac{1}{2} |y_n| \leq \frac{1}{2} ||y|| \) whereas \( |y_m f_m(u_m)| = |y_m| > \frac{1}{2} ||y|| \) so \( p(Sy) \geq \frac{1}{2} ||y|| \) implying that \( S \) is a linear homeomorphism from \( \ell^\infty \) onto \( S(\ell^\infty) \subset E \) which gives the desired contradiction.

\((\beta) \Rightarrow (\gamma)\) is obvious. The implication \((\gamma) \Rightarrow (\alpha)\) was proved in [8], Theorem 1.2 for (polar) Banach spaces \( E \). From here the step to locally convex \( E \) is easy ([4], Lemma 4.6). □

**Theorem 3.2**

For a Fréchet space \( E \) the following are equivalent.

\( (a) \) \( E \) is an \( (O.P.) \)-space.

\( (\beta) \) \( E \) is polar, weakly sequentially complete and \( E \) does not contain a subspace linearly homeomorphic to \( \ell^\infty \).

\( (\gamma) \) \( E \) is polar, weakly sequentially complete and \( E \) does not contain a complemented subspace linearly homeomorphic to \( \ell^\infty \).

\( (\delta) \) \( E \) is polar, weakly sequentially complete and every continuous linear map \( \ell^\infty \to E \) is compact.

**Proof.** The equivalence of \( (\beta), (\gamma), (\delta) \) follows from Proposition 3.1.

\( (a) \Rightarrow (\beta) \). Theorem 2.3 yields polarness of \( E \). Now let \( x_1, x_2, \ldots \) be a weakly Cauchy sequence. Then \( x_{n+1} - x_n \to 0 \) weakly, hence strongly. As \( E \) is complete, there is an \( x \in E \) with \( x_n \to x \) strongly, hence weakly. Thus, \( E \) is weakly sequentially complete. Obviously, \( E \) does not contain \( \ell^\infty \).
(δ) ⇒ (α). Let \( x_1, x_2, \ldots \) be a sequence in \( E \) tending weakly to 0. By weak sequential completeness the formula

\[
(\eta_1, \eta_2, \ldots) \mapsto T \sigma(E, E') - \lim_{n \to \infty} \sum_{i=1}^{n} \eta_i x_i
\]

defines a linear map \( T : \ell^\infty \to E \). \( E \) is polar and \( \{x_1, x_2, \ldots\} \) is weakly bounded hence bounded (\([6]\), Cor. 7.7) and it follows that \( T \) is continuous. By (δ), \( T \) is compact. Then \( \{x_1, x_2, \ldots\} \) is a compactoid on which the weak and strong topologies coincide (\([6]\), Theorem 5.12). Thus, \( x_n \to 0 \) strongly and the theorem is proved. □

**Problem.** Does there exist a Banach (or Fréchet) space that is polar, is not (O.P.), and does not contain \( \ell^\infty \)? In other words, may we drop the condition of weak sequential completeness in Theorem 3.2?

**References**