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WEAK AND STRONG C'-COMPACTNESS IN NON-ARCHIMEDEAN BANACH SPACES

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ABSTRACT

Throughout K is a non-archimedean complete valued field with dense valuation $|\,.\,|$. An absolutely convex set A of a K-Banach space E is called (weakly) c'-compact if $\max_{x\in A} p(x)$ exists for each (weakly) continuous seminorm p on E.

Assuming the continuum hypothesis, we shall prove that, if **K** *has the cardinality of the continuum, in a strongly polar* **K**-*Banach space, each weakly c'-compact set is c'-compact.*

INTRODUCTION

(For unexplained terms, see below and [2], [3] and [6]). It was proved in 1986 ([5], theorem 2.7) that each weakly c'-compact set is c'-compact if E is a **K**-Banach space with a base. Further progress came about in 1989 ([1], theorem 5.2.13) when the same conclusion could be drawn for an arbitrary Banach space over a spherically complete **K**. However, if **K** is not spherically complete, the closed unit ball of (the polar space) ℓ^∞ is weakly c'-compact but not c'-compact. ([5], example p. 9). So, quite naturally the following problem arises :

> Let **K** be not spherically complete. Is every weakly c'-compact set in a strongly polar **K**-Banach space necessarily c'-compact?

In this note we give a partial solution (as stated in the abstract). The general problem remains open.

PRELIMINARIES

We assume that **K** is not spherically complete and that **K** has the cardinality of the continuum, for example $\mathbf{K} = \mathbf{C}_p$, the completion of the algebraic closure of $\mathbf{Q}_p$. The residue class field of **K** is **k** and the canonical map $\{ \lambda \in \mathbf{K} \mid |\lambda| \leq 1 \} \to \mathbf{k}$ is written $\lambda \to \overline{\lambda}$.

Let E be a **K**-Banach space. Its dual is E', the absolutely convex hull of a set $S \subset E$ is denoted by coS, the closure of coS by $\overline{\text{co}}S$ and its **K**-linear span by [S].

Recall that E is called *strongly polar* if every continuous seminorm p is polar.

(I.e. $p = \sup \{ |f| \mid f \in E', |f| \leq p \}$). We shall need the following results which are proved in [3]. Subspaces and images under continuous linear maps of strongly polar spaces are strongly polar. In a strongly polar space E every continuous linear function defined on a linear subspace can be extended to an element of E' and for every closed linear subspace D and $x \in E \setminus D$, there exists an $f \in E'$ that vanishes on D but $f(x) \neq 0$. Spaces of countable type are strongly polar.

1. TWO IMPLICATIONS OF THE CARDINALITY OF K

1.1.THEOREM: *If $c_0(I)$ is strongly polar, then I is at most countable.*

Proof: Suppose that I is uncountable. Then, using the continuum hypothesis, we have $\# \ell^\infty = \# K \leq \# I$. Hence, there exists a surjection of I onto the unit ball of ℓ^∞ which extends to a continuous linear surjection $c_0(I) \to \ell^\infty$. Now ℓ^∞ is not strongly polar so neither is $c_0(I)$.

1.2.COROLLARY: Let $t \in (0, 1]$. Any t-orthogonal set in a strongly polar space is at most countable.

2.WEAK AND STRONG C'-COMPACTNESS

2.1.PRELIMINARIES: Let E be a **K**-Banach space with norm $\|.\|$. For a closed and absolutely convex subset A of E, we put $A^i = \{ \lambda.a \mid \lambda \in \mathbf{K}, |\lambda| < 1, a \in A \}$. Then A^i and $\overline{A^i}$ are absolutely convex. The quotient $V_A = A / \overline{A^i}$ is, in a natural way, a **k**-vector space. Let $\pi : A \to V_A$ denote the quotient map.

The formula $\|\pi(x)\| = \inf \{ \|x - a\| \mid a \in \overline{A^i} \}$ defines a norm on V_A for which it becomes a **k**-Banach space ([2], proposition 3.2).

Any **k**-Banach space has, for each $t \in (0,1)$, a t-orthogonal base ([2], proposition 3.5).

2.2.LEMMA: Let $A \subset E$ be closed, bounded and absolutely convex. Let $t \in (0,1)$ and let $(e_i)_{i \in I}$ be a family in A such that $(\pi(e_i))_{i \in I}$ is t-orthogonal and such that $\alpha = \inf_{i \in I} \|\pi(e_i)\| > 0$.

Then $(e_i)_{i \in I}$ is t'-orthogonal for some $t' \in (0,t]$.

Proof: Put $\beta = \sup_{i \in I} \|e_i\|$. Now, let $J \subset I$ be finite and put $x = \sum_{i \in J} \lambda_i.e_i$ where $\lambda_i \in \mathbf{K}^*$ for each $i \in J$. It is no restriction to assume that $\max_{i \in J} |\lambda_i| = 1$. Then we have the following :

$$\|x\| \geq \|\pi(x)\| = \|\sum_{i \in J} \overline{\lambda_i}.\pi(e_i)\| \geq t.\max_{i \in J} \|\pi(e_i)\| \geq t.\alpha = t.\alpha.\beta^{-1}.\sup_{i \in I} \|e_i\| \geq t.\alpha.\beta^{-1}.\max_{i \in J} \|\lambda_i.e_i\|.$$

It suffices to choose $t' = t.\alpha.\beta^{-1}$ to complete the proof.

2.3.REMARK: In the proof of lemma 2.2, the condition on the cardinality of **K** is redundant. (See also [2], lemma 3.11).

2.4.LEMMA: Let A be a closed, bounded and absolutely convex subset of a strongly polar **K**-Banach space E. Then V_A is of countable type.

Proof: Let $(\pi(e_i))_{i\in I}$ be a t-orthogonal base of V_A for some $t \in (0,1)$.

For each $n \in \mathbf{N}_o$, put $I_n = \{\, i \in I \mid \|e_i\| \geq \frac{1}{n} \,\}$. By lemma 2.2, $(e_i)_{i\in I_n}$ is t'-orthogonal for some $t' \in (0,t]$ and by corollary 1.2, the set I_n is at most countable. It follows that $I = \bigcup_{n\in\mathbf{N}_0} I_n$ is countable, hence, V_A is of countable type.

2.5.THEOREM: Let A be an absolutely convex, weakly c'-compact subset of a strongly polar **K**-Banach space E. Then A is c'-compact.

Proof: We may assume that A is closed ([4], proposition 1.2). Weak c'-compactness implies weak boundedness, hence norm boundedness ([3], corollary 7.7). Choose $t \in (0,1)$ and let $(e_n)_{n\in\mathbf{N}_0} \subset A$ be such that $(\pi(e_n))_{n\in\mathbf{N}_0}$ is a t-orthogonal base of V_A. (Lemma 2.4). We may assume that $\|e_n\| \leq t^{-1}.\|\pi(e_n)\|$ for each $n \in \mathbf{N}_o$. Put $B = \overline{co}(e_n \mid n \in \mathbf{N}_o)$.

Then, as $B \subset A$, obviously $\overline{[B]} \subset \overline{[A]}$. Now, if this inclusion were strict, we could find (by strong polarness) an $f \in E'$ that vanishes on $\overline{[B]}$ but not on $\overline{[A]}$. By weak c'-compactness, $\alpha = \max_{x\in A} |f(x)|$ exists and is non-zero. Clearly $|f| < \alpha$ on A^i and thus $|f| < \alpha$ on $\overline{A^i}$.

On the other hand, it is not difficult to see that $A \subset B + \overline{A^i}$ (recall that $\operatorname{Ker} \pi = \overline{A^i}$).

Hence, it follows that $|f| < \alpha$ on A (since f vanishes on B) and this is a contradiction.

So, $\overline{[A]} = \overline{[\{e_n \mid n \in \mathbf{N}_o\}]}$. Now, again by strong polarness, the weak topology of $\overline{[A]}$ is the restriction to $\overline{[A]}$ of the weak topology of E. Hence, A is a closed, weakly c'-compact subset of a **K**-Banach space ($\overline{[A]}$) of countable type.

On the other hand, since $\overline{[A]}$ is of countable type, it has a base. Now simply apply [5], theorem 2.7 to conclude that A is c'-compact in $\overline{[A]}$ and thus in E.

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