p-ADIC ORTHOCOMPLEMENTED SUBSPACES IN ℓ^∞

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Abstract. For a closed subspace D of ℓ^∞ over a non-archimedean valued base field we study in this paper the property

1. There exists a continuous linear projection P from ℓ^∞ onto D with $\|P\| \leq 1$ (D is *orthocomplemented* in ℓ^∞)

   as related to the properties 2,3,4 below.

2. For every continuous linear functional $f \in D'$ there exists a continuous linear extension $\tilde{f} \in (\ell^\infty)'$ with $\|\tilde{f}\| = \|f\|$ (D has the *Hahn-Banach property* in ℓ^∞).
3. The canonical quotient map $\pi_E : E \to E/D$ is strict, i.e. for each $z \in E/D$ there exists $x \in E$ with $\pi_E(x) = z$ and $\|x\| = \|z\|$ (D is *strict* in ℓ^∞).
4. D is *weakly closed* in ℓ^∞.

Also, certain duality arguments allow us to obtain several descriptions of the orthocomplemented subspaces of c_0. In particular it is shown (Theorem 4.3) that, if K is not spherically complete, a closed hyperplane H in c_0 having the Hahn-Banach property in c_0 is orthocomplemented.

1. *PRELIMINARIES*. Throughout K is a non-archimedean valued field that is complete with respect to the metric induced by the non-trivial valuation $|\cdot|$. Also, $(E, \|\cdot\|)$ will be a (non-archimedean) Banach space over K.

For a Banach space F over K and a continuous linear map T from E into F, the kernel of T is the set

$$\text{Ker } T = \{x \in E : Tx = 0\}.$$

Also, the norm of T is given by

$$\|T\| = \sup\left\{\frac{\|Tx\|}{\|x\|} : x \in E\backslash\{0\}\right\}$$

When there exists a linear isometry from E onto F we say that E and F are isometrically isomorphic and we write $E \simeq F$.

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The dual space E' of E consisting of all the continuous linear maps from E to K is again a Banach space. We set

$$J_E(x)(x') = x'(x) \quad (x \in E, x' \in E').$$

E is called *reflexive* is J_E is an isometry from E onto E''.

For a closed subspace D of E we say that

a) D has the *HB-property* (resp. HB$^+$-property) in E if for every $f \in D'$ (resp. for every $\varepsilon > 0$ and for every $f \in D'$) there exists a continuous linear map $\tilde{f} \in E'$ extending f such that $\|\tilde{f}\| = \|f\|$ (resp. $\|\tilde{f}\| \leq (1+\varepsilon)\|f\|$).

b) D is *strict* in E if the quotient map $\pi_E : E \to E/D$ is strict (i.e. for every $z \in E/D$ there exists an $x \in E$ for which $\pi_E(x) = z$ and $\|x\| = \|z\|$).

c) D is *orthocomplemented* in E if there exists a closed subspace G of E such that $D \cap G = \{0\}$, $E = D + G$ and

$$\|x + y\| = \max(\|x\|, \|y\|) \quad (x \in D, y \in G)$$

(such a G is called an orthogonal complement of D in E).

It is not difficult to prove the two following Propositions which include some elementary (but useful) descriptions for the orthocomplemented and the strict subspaces of an arbitrary Banach space.

Proposition 1.1. *For a closed subspace D of E the following are equivalent.*

i) *D is orthocomplemented in E.*

ii) *There exists a continuous linear isometry $\varphi : E/D \to E$ such that $\pi_E \circ \varphi$ is the identity on E/D.*

iii) *There exists a continuous linear projection P from E onto D with $\|P\| = 1$ (This P is called an orthoprojection from E onto D).*

Proposition 1.2. *For a closed subspace D of E the following properties are equivalent:*

i) *D is strict in E.*

ii) *There exists a (non-necessarily linear) map $\varphi : E/D \to E$ such that $\|\varphi(x)\| = \|x\|$ for all $x \in E/D$ and $\pi_E \circ \varphi$ is the identity on E/D.*

iii) *For each $x \in E$, D is orthocomplemented in $D + Kx$.*

Clearly, D is orthocomplemented in $E \Rightarrow D$ has the HB-property and D is strict in E.

If E' separates the points of E then D is orthocomplemented in $E \Rightarrow D$ is weakly closed in E.

Most of what we are about to do concerns converses of the above implications when $E = \ell^\infty$ or c_0. Firstly we consider (co)finite-dimensional subspaces (sections 3,4) and later on arbitrary closed subspaces of ℓ^∞ and c_0 (section 5). We assume that K is not spherically complete, since if K is spherically complete every closed subspace of E is weakly closed and has the HB-property in E ([3], Theorems 4.2, 4.7) and also every finite-dimensional subspace of E is orthocomplemented ([7], Lemma 4.35). The basic machinery to our purpose is included in section 2.

The following problem arises in a natural way in this paper (see Problem 4 in section 5):

Problem. Suppose K is not spherically complete. Let D be a weakly closed subspace of ℓ^∞ such that D is strict and has the HB-property in ℓ^∞. Does it follow that D is orthocomplemented in ℓ^∞?

In fact we do not know the answer of this problem for any infinite-dimensional Banach space E (instead of ℓ^∞) over a non-spherically complete field K.

However, if K is spherically complete, the situation is completely different. Indeed, suppose that $|K| = [0, \infty)$. By a standard construction we can make a strict quotient map $\pi : c_0(I) \to \ell^\infty$ if I has adequate cardinal. Now, $D = \text{Ker } \pi$ is a weakly closed subspace which is strict and has the HB-property in $c_0(I)$. If D were orthocomplemented then ℓ^∞ would be isometrically isomorphic to a closed subspace of $c_0(I)$ and so ℓ^∞ has an orthogonal base: a contradiction ([7], Corollary 5.18).

For some other unexplained concepts and notations that we will use in the sequel, we refer to [3] and [7].

2. *GENERAL FACTS*

In this section we include some general results which will be useful in the rest of the paper.

First, we are going to see (Propositions 2.1 - 2.7) that strictness and the HB-property behave sometimes as "opposites" of one another.

Proposition 2.1. *Let D be a closed subspace of E.*

i) *If D is strict in E and $E/D \simeq c_0(I; s)$ for some set I and $s : I \to (0, +\infty)$, then D is orthocomplemented in E.*
ii) *If D has the* HB*-property in E and $D \simeq \ell^\infty(I; s)$ for some set I and some $s : I \to (0, +\infty)$, then D is orthocomplemented in E (compare Theorem* 1.2 of [5]).

Proof.

i) Let $\{u_i : i \in I\}$ be an orthogonal base of E/D. By strictness, there exists $\{z_i : i \in I\} \subset E$ such that $\pi_E(z_i) = u_i$ and $\|z_i\| = \|u_i\|$ for all $i \in I$. A standard argument shows that $\varphi : E/D \to E$ given by the formula $\sum_{i \in I} \lambda_i u_i \to$

$\sum_{i \in I} \lambda_i z_i$ is a linear isometry for which $\pi_E \circ \varphi$ is the identity on E/D. Hence, D is orthocomplemented.

ii) For each $i \in I$ the coordinate function $f_i \in D'$ given by $f_i(x) = x_i$ $\left(x = (x_i)_{i \in I} \in \ell^\infty(I; s)\right)$ has norm $s(i)^{-1}$. By the HB-property, f_i extends to an $\tilde{f}_i \in E'$ with $\|\tilde{f}_i\| = s(i)^{-1}$. Then, $P : E \to D$; $x \to \left(\tilde{f}_i(x)\right)_{i \in I}$ is an orthoprojection from E onto D.

As a special case we obtain

Corollary 2.2. *If D is a closed hyperplane (resp. a one-dimensional subspace) in E, then D is strict (resp. D has the* HB*-property) in E iff D is orthocomplemented in E.*

Remarks 2.3.

1.- Observe that if D is a closed hyperplane of E, there is an $f \in E' - \{0\}$ such that $D = \operatorname{Ker} f$. Then, D is orthocomplemented iff $\|f\| = \max\left\{\frac{|f(x)|}{\|x\|} : x \in E\backslash\{0\}\right\}$. In fact, if $a \in E$ one can easily see that Ka is an orthogonal complement of D iff $\|f\| = \frac{|f(a)|}{\|a\|}$.

2.- If K is spherically complete the finite (co)dimensional version of the above Corollary 2.2 holds.
Indeed, observe that if $\dim E/D < \infty$, then E/D has an orthogonal base ([7], Lemma 5.5). Also, every finite-dimensional subspace of E is orthocomplemented ([7], Lemma 4.35).

3.- But, for non-spherically complete fields K the generalization in Remark 2 does not hold. In fact, let $\pi : c_0 \to K_\nu^2$ be a strict surjection ([6], 2.3, Remark 1). Then, $\operatorname{Ker}\pi$ is a strict two-codimensional subspace of c_0 that cannot be orthocomplemented since K_ν^2 has no orthogonal base ([7], p.68).
On the other hand, the adjoint of π is an isometry $\pi' : (K_\nu^2)' \to \ell^\infty$ and by construction $\operatorname{Im}\pi'$ has the HB-property in ℓ^∞. But it will follow from Theorem 3.3 that it is not orthocomplemented in ℓ^∞.

However we do have the following related statement.

Proposition 2.4.

i) *If D is a closed subspace of E of finite codimension and if all hyperplanes H containing D are strict (orthocomplemented) in E, then D is orthocomplemented in E.*

ii) *If D is a finite-dimensional subspace of E and if every one-dimensional subspace of D has the* HB*-property (is orthocomplemented) in E, then D is orthocomplemented in E.*

Proof.

i) For a proof by induction with respect to the codimension of D it suffices to show that, for closed subspaces D_1, D_2 of finite codimension, containing D from

$$D_1 \subset D_2, \dim D_2/D_1 = 1 \text{ and}$$
$$D_2 \text{ is orthocomplemented in } E,$$

it follows that D_1 is orthocomplemented in E.
To see that, let P be an orthoprojection from E onto D_2. Then, dim Ker P = codim $D_1 - 1$ and so $D_1 +$ Ker P is a closed hyperplane of E. There is an orthoprojection Q from E onto $D_1 +$ Ker P. Hence, PQ is an orthoprojection from E onto D_1.

ii) Almost identical to the proof of Lemma 4.35,iii) of [7].

The next two Propositions stress the duality between strictness and the HB-property.

Proposition 2.5. *For a closed subspace D of E and its polar D^0 we have*

i) *If D is orthocomplemented in E, then D^0 is orthocomplemented in E'.*

ii) *If D has the HB-property in E, then D^0 is strict in E'.*

iii) *If D is strict in E and E/D is reflexive, then D^0 has the HB-property in E'.*

Proof.

i) If S is an orthogonal complement of D in E, then S^0 is an orthogonal complement of D^0 in E'.

ii) If $i : D \hookrightarrow E$ is the canonical inclusion then its adjoint $i' : E' \to D'$ is a strict map. Hence, its kernel, D^0, is strict in E'.

iii) The quotient map $\pi_E : E \to E/D$ has an isometrical adjoint $\pi'_E : (E/D)' \to E'$ for which $\pi'_E((E/D)') = D^0$. Hence, to show that D^0 has the HB-property in E' it suffices to prove that for any $\varphi \in (E/D)''$ there exists a $\tilde{\varphi} \in E''$ such $\|\tilde{\varphi}\| = \|\varphi\|$ and $\tilde{\varphi} \circ \pi'_E = \varphi$. By the reflexivity of E/D, there is a $z \in E/D$ such that $\varphi = J_{E/D}(z)$ and $\|z\| = \|\varphi\|$. Also, by strictness there is an $x \in E$ with $\pi_E(x) = z$ and $\|x\| = \|z\|$. Then, $\tilde{\varphi} = J_E(x)$ satisfies the required conditions.

Now, we consider the converse of Proposition 2.5.

Proposition 2.6. *Let D be a closed subspace of E.*

i) *Let D^0 be orthocomplemented (resp. D^0 have the* HB*-property in E). If in addition E is reflexive and D is weakly closed then D is orthocomplemented (resp. D is strict) in E.*

ii) *If D^0 is strict in E' and D has the* HB$^+$*-property in E, then D has the* HB*-property in E.*

Proof.

i) By the previous Proposition the bipolar of D, D^{00}, is orthocomplemented (strict) in E''. By reflexivity and weak closedness D is orthocomplemented (strict) in E.

ii) Let $i' : E' \to D'$ be the adjoint map of the canonical inclusion $i : D \to E$ and let $\rho : D' \to E'/D^0$ the natural map making the diagram

$$\begin{array}{ccc} E' & \xrightarrow{i'} & D' \\ & \pi_{E'}\searrow \quad \swarrow\rho & \\ & E'/D^0 & \end{array}$$

commute. It follows easily from the HB^+-property of D that ρ is an isometrical isomorphism. Now, $\pi_{E'}$ is strict. Hence, so is i', i.e. D has the HB-property.

Although in the above results the HB-property and strictness seem dual properties, sometimes they have similar behaviour. This is the case in the next few propositions.

Observe that if D is a closed subspace of E and S is a closed subspace of D, then we have in a natural way the following commutative diagram

$$\begin{array}{ccc} D & \xrightarrow{i_1} & E \\ \downarrow{\scriptstyle \pi_D} & & \downarrow{\scriptstyle \pi_E} \\ D/S & \xrightarrow[i_2]{} & E/S \end{array}$$

where i_1, π_E, π_D are the obvious maps and i_2 makes the diagram commute.

Proposition 2.7. *Let D be a closed subspace of E and let S be a closed subspace of D. If D is strict (resp. has the HB-property, is orthocomplemented) in E, then $i_2(D/S)$ is strict (resp. has the HB-property, is orthocomplemented) in E/S.*

Proof. Suppose that D is strict. Let $x \in E$. There is a $d \in D$ such that

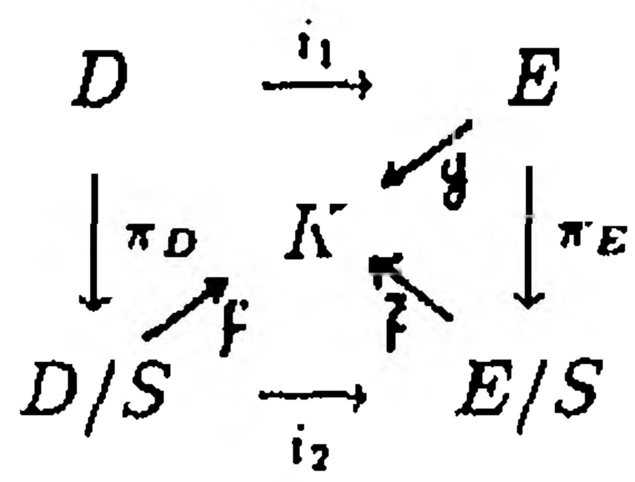
$$\|x - i_1(d)\| \leq \|x - i_1(d')\| \quad (d' \in D).$$

Now, for all $s' \in S$, $d' \in D$, we have

$$\begin{aligned} &\|\pi_E(x) - i_2\pi_D(d)\| = \|\pi_E(x) - \pi_E(i_1(d))\| \\ &\leq \|x - i_1(d)\| \leq \|x - i_1(d') - s'\| \end{aligned}$$

Hence, $\|\pi_E(x) - i_2\pi_D(d)\| \leq \|\pi_E(x) - i_2\pi_D(d')\|$ for all $d' \in D$ and we see that the distance of $\pi_E(x)$ to $i_2(D/S)$ is attained, which means that $i_2(D/S)$ is strict in E/S.

Now, assume that D has the HB-property and let $f \in (D/S)'$. Then $f \circ \pi_D \in D'$ so by assumption there is a $g \in E'$ such that $\|g\| = \|f \circ \pi_D\| = \|f\|$ and $g \circ i_1 = f \circ \pi_D$. Since $S \subset \operatorname{Ker} g$ there is a unique $\tilde{f} \in (E/S)'$ such that $\tilde{f} \circ \pi_E = g$ (see the diagram).



One verifies without pain that then also $\tilde{f} \circ i_2 = f$ and that $\|\tilde{f}\| = \|f\|$.

Finally, suppose that D is orthocomplemented and let $P : E \to D$ be an orthoprojection from E onto D. Since $S \subset \operatorname{Ker}(\pi_D \circ P)$, there is a unique continuous linear map $Q : E/S \to D/S$ such that $Q \circ \pi_E = \pi_D \circ P$ and $\|Q\| \leq 1$. Also, $Q \circ i_2 \pi_D(x) = \pi_D(x)$ for all $x \in D$. So, since π_D is surjective, we conclude that $Q \circ i_2$ is the identity on D/S, which implies that $i_2(D/S)$ is orthocomplemented in E/S.

A partial converse of Proposition 2.7 is the following.

Proposition 2.8. *Let D be a closed subspace of E. If for each closed subspace S of D with* $\dim D/S = 1$ *we have that $i_2(D/S)$ has the HB-property in E/S, then D has the HB-property in E.*

Proof. Let $f \in D' \backslash \{0\}$ and let $S = \operatorname{Ker} f$. Then $f = \rho_1 \circ \pi_D$ where $\rho_1 : D/S \to K$ is a similarity (i.e. there exists a nonzero real number c such that $|\rho_1(z)| = c\|z\|$ for all $z \in D/S$). By assumption and Corollary 2.2, there is an orthoprojection $\rho_2 : E/S \to D/S$ such that $\rho_2 \circ i_2$ is the identity on D/S. Now set $\tilde{f} = \rho_1 \cdot \rho_2 \circ \pi_E$. Then, $\|\tilde{f}\| = \|f\|$ and $\tilde{f} \circ i_1 = f$, and we are done.

Remark 2.9. Putting together Propositions 2.7 and 2.8 we derive that a closed subspace D of E has the HB-property in E iff for every closed hyperplane S of D, $i_2(D/S)$ has the HB-property in E/S. (Compare with Theorem 2.3 of [1]).

Observe that if S, D are closed subspaces of E with $S \subset D$, then the formula

$$\pi_{E/D}(\pi_1(x)) = \pi_2 \circ \pi_E(x) \quad (x \in E)$$

defines an isometrical isomorphism $\pi_{E/D} : E/D \to (E/S)/(D/S)$ making the diagram

$$\begin{array}{ccccc} D & \xrightarrow{i_1} & E & \xrightarrow{\pi_1} & E/D \\ \downarrow{\scriptstyle \pi_D} & & \downarrow{\scriptstyle \pi_E} & & \downarrow{\scriptstyle \pi_{E/D}} \\ D/S & \xrightarrow{i_2} & E/S & \xrightarrow{\pi_2} & (E/S)/(D/S) \end{array} \qquad (I)$$

commute, where all the maps appearing in the diagram are the natural ones.

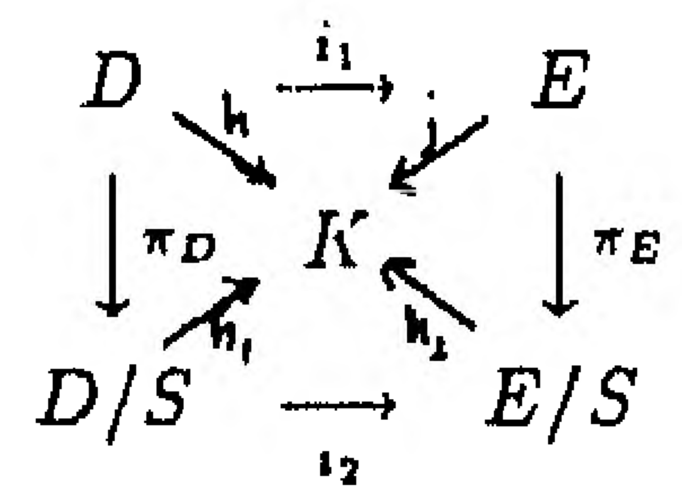
Then, we have:

***Proposition 2.10** Let $S \subset D$ be closed subspaces of E. If S is strict (resp. has the HB-property, is orthocomplemented) in E and D/S is strict (resp. has the HB-property, is orthocomplemented) in E/S, then D is strict (resp. has the HB-property, is orthocomplemented) in E.*

Proof.

a) *Strictness*: Let $z \in E/D$. Then, in the diagram (I), $\pi_{E/D}(z)$ admits a $y \in E/S$ such that $\pi_2(y) = \pi_{E/D}(z)$ and $\|y\| = \|\pi_{E/D}(z)\| = \|z\|$. Also, there is an $x \in E$ with $\pi_E(x) = y$ and $\|x\| = \|y\|$. Then, $\pi_1(x) = z$ and $\|x\| = \|y\| = \|z\|$. Hence, D is strict in E.

b) *HB-property*: Let $f \in D'$ and let $g \in E'$ be such that the restrictions $g|S$ and $f|S$ coincide and $\|g\| = \|f|S\|$. Now consider $h = f - g|D \in D'$. Since $h = 0$ on S there is a $h_1 \in (D/S)'$ with $h = h_1 \circ \pi_D$ and $\|h_1\| = \|h\|$. By assumption h_1 extends to a $h_2 \in (E/S)'$ (I.e. $h_2 \circ i_2 = h_1$) with $\|h_2\| = \|h_1\|$ (see the diagram).



Now set $j = h_2 \circ \pi_E$. We have that $\|j\| \le \|f\|$ and $j \circ i_1 = h$. Then, $\tilde{f} = j + g$ is a continuous linear extension of f with $\|\tilde{f}\| = \|f\|$ and we are done.

c) *Orthocomplementation*: By using diagram (I), there is by assumption a $\rho_2 : (E/S)/(D/S) \to E/S$ such that $\pi_2 \circ \rho_2$ is the identity and also a $\rho_1 : E/S \to E$ such that $\pi_E \circ \rho_1$ is the identity, ρ_1 and ρ_2 being linear isometries. Now define $\tau : E/D \to E$ by $\tau = \rho_1 \circ \rho_2 \circ \pi_{E/D}$. We have that τ is a linear isometry for which $\pi_{E/D} \circ (\pi_1 \circ \tau) = \pi_{E/D}$, and so $\pi_1 \circ \tau$ is the identity.

3. *FINITE-(CO)DIMENSIONAL ORTHOCOMPLEMENTED SUBSPACES OF ℓ^∞*

As we have already announced in the Preliminaries,

FROM NOW ON IN THIS PAPER (EXCEPT IN 3.2) WE ASSUME THAT K IS NOT SPHERICALLY COMPLETE.

The results given in §2 can be applied now to obtain several descriptions of the finite-(co)dimensional subspaces of ℓ^∞ that have an orthogonal complement.

For subspaces of finite codimension the situation is satisfactory.

Proposition 3.1. *Every closed finite-codimensional subspace of ℓ^∞ is orthocomplemented.*

Proof. By reflexivity the map $D \to D^0$ is a bijection between the set of all finite-dimensional subspaces of c_0 and the set of all finite-codimensional subspaces of ℓ^∞. Since every finite-dimensional subspace of c_0 is orthocomplemented, we can apply Propositions 2.5 and 2.6 to derive our conclusion.

Remark 3.2. If K is spherically complete the conclusion above no longer holds.

Indeed, suppose that the valuation on K is dense. Let X be a maximal orthogonal subset of ℓ^∞ and let H be a closed hyperplane of ℓ^∞ containing X. Then H is not orthocomplemented in ℓ^∞.

The pictures changes when we consider finite-dimensional subspaces of ℓ^∞.

Theorem 3.3. *For a finite-dimensional subspace D of ℓ^∞, the following properties are equivalent.*

i) *D is orthocomplemented in ℓ^∞.*

ii) *Every one-dimensional subspace of D is orthocomplemented (has the H.B-property) in ℓ^∞.*

iii) *For each $x = (x_n) \in D$, $\max_n |x_n|$ exists.*

Proof. i) $\Rightarrow$ ii): By Proposition 2.5, there exists an orthogonal complement S of D^0 in c_0. Then, $D \simeq S'$ in a natural way, and since S is finite-dimensional, there is an $n \in \mathbb{N}$ such that $D \simeq K^n$. So, every one-dimensional subspace of D is orthocomplemented in D (and hence in ℓ^∞, by i)).

ii) $\Rightarrow$ i): It follows from Proposition 2.4 ii).

ii) $\Longleftrightarrow$ iii): Let $f \in c_0'$. By Propositions 2.5 and 2.6 we have that Kf is orthocomplemented in c_0' iff Ker f is orthocomplemented in ℓ^∞, and this happens iff $\|f\| = \max\{|f(x)| : \|x\| \leq 1\}$ (Remark 2.4.1). So, we conclude that Kf is orthocomplmented in c_0' iff $\|f\| = \max_n |f(e_n)|$ (where $e_1, e_2, \ldots$ is the canonical base of c_0). This is precisely ii) $\Longleftrightarrow$ iii) (Recall that $c_0' \simeq \ell^\infty$, [7]. Exercise 3.Q.i))

For one-dimensional subspaces we prove the following curious Theorem, which will be useful in the sequel.

Theorem 3.4. *A one-dimensional subspace of ℓ^∞ is strict iff it is orthocomplemented.*

Proof. Clearly the orthocomplementation property implies strictness (see the Preliminaries).
Now suppose that $D = Kx$ ($x = (x_1, x_2, \ldots) \in \ell^\infty, x \neq 0$). If D is not orthocomplemented then $|x_n| < \|x\|$ for all n (Theorem 3.3). We are going to prove that there exists a $y \in \ell^\infty$ such that the linear hull $[x, y]$ of $\{x, y\}$ has no orthogonal base and by Proposition 1.2 we are done.

Let $K = B_0$ and let $B_1 \supset B_2 \supset \ldots$ be bounded discs in K whose intersection is empty. For each $n \in \mathbf{N}$ let $r_n = \operatorname{diam} B_n$ (the diameter of B_n). Define a function $\varphi : K \to [0, +\infty)$ by the formula

$$\varphi(\lambda) = \lim_{n\to\infty} \operatorname{dist}(\lambda, B_n) \quad (\lambda \in K).$$

Then $\inf\{\varphi(\lambda) : \lambda \in K\} = d$, where $d = \lim_{n\to\infty} r_n > 0$, but d is not attained (observe that $d \neq r_n$ for each $n \in \mathbf{N}$). We shall construct $c_1, c_2, \ldots \in K$ such that

$$\|y - \lambda x\| = \varphi(\lambda)\|x\| \quad (\lambda \in K)$$

with $y := (c_1x_1, c_2x_2, \ldots)$ (Then, $\operatorname{dist}(y, Kx)$ is not attained and it follows easily that $[x, y]$ has no orthogonal base).

Let $n \in \mathbf{N}$. If $x_n = 0$ we set $c_n = 0$. Now let $x_n \neq 0$. Then, we may choose a $k(n) \in \mathbf{N}$ for which

$$r_{k(n)} \leq \frac{\|x\|d}{|x_n|} \tag{II}$$

and take $c_n \in B_{k(n)} \backslash B_{k(n)+1}$.
Now let $\lambda \in K$. First we prove that $\|y - \lambda x\| \leq \varphi(\lambda)\|x\|$, i.e. that, for each $n \in \mathbf{N}$, $|c_n - \lambda|\,|x_n| \leq \varphi(\lambda)\|x\|$. This is obvious when $x_n = 0$, so let $x_n \neq 0$. There is a unique $m \in \{0, 1, 2, \ldots\}$ such that $\lambda \in B_m \backslash B_{m+1}$. We distinguish two cases.

a) $m \geq k(n)$. Then $c_n \in B_{k(n)}$ and $\lambda \in B_m \subset B_{k(n)}$. Hence, by (II) we obtain

$$|c_n - \lambda|\,|x_n| \leq r_{k(n)}|x_n| \leq \|x\|\varphi(\lambda).$$

b) $m < k(n)$. Then $c_n \in B_{k(n)} \subset B_{m+1}$ while $\lambda \notin B_{m+1}$ so that $|c_n - \lambda| = \varphi(\lambda)$ and

$$|c_n - \lambda|\,|x_n| = \varphi(\lambda)|x_n| \leq \varphi(\lambda)\|x\|.$$

To finish, we prove that $\|y - \lambda x\| \geq \varphi(\lambda)\|x\|$. Let $\varepsilon > 0$. Without loss we can assume $\varepsilon < r_m - d$. From our assumption on x it follows that $J := \{n \in \mathbf{N} : \|x\|d < |x_n|(d+\varepsilon)\}$ is infinite. If $n \in J$, then by (II)

$$r_{k(n)} < d + \varepsilon < r_m$$

so that $k(n) > m$. Thus we are in case b) of above, so $|c_n - \lambda|\,|x_n| > \frac{d}{d+\epsilon}\varphi(\lambda)\|x\|$ and we are done.

Remark 3.5. Taking into account Corollary 2.2 and Theorem 3.4, for a one-dimensional subspace D of ℓ^∞ one verifies

D is orthocomplemented $\iff$ D is strict $\iff$ D has the HB-property.

We know that the implication

D has the HB-property $\Rightarrow$ D is orthocomplemented

does not hold for every finite-dimensional subspace D of ℓ^∞. Next we will see (Corollary 3.7) that the implication

D is strict $\Rightarrow$ D has the HB-property

holds for every finite-dimensional (in fact for every weakly closed subspace) D of ℓ^∞.

This will be a consequence of the following result.

Theorem 3.6. *(Compare Theorem 2.3 of [5]). Let M be a closed subspace of ℓ^∞. The following are equivalent.*

i) *M is weakly closed in ℓ^∞.*

ii) *$\ell^\infty/M \simeq K^n$ for some $n \in \mathbb{N}$ or $\ell^\infty/M \simeq \ell^\infty$.*

iii) *ℓ^∞/M is reflexive.*

iv) *For every (for some) closed subspace S of M with dim $M/S = 1$, S is weakly closed in ℓ^∞.*

Proof. The implications ii) $\Rightarrow$ iii) and iii) $\Rightarrow$ i) are obvious.

i)$\Rightarrow$ii): For a closed subspace D of c_0 the adjoint of the inclusion map $D \to c_0$ is a quotient map, so $D' \simeq c_0'/D^0$. By applying this for $D := M^0$ and by using $M^{00} = M$ we obtain $(M^0)' \simeq c_0'/M^{00} \simeq \ell^\infty/M$. Since M^0 is a closed subspace of c_0, we have that $M^0 \simeq K^n$ for some $n \in \mathbb{N}$ (and so $\ell^\infty/M \simeq K^n$) or $M^0 \simeq c_0$ (and so $\ell^\infty/M \simeq \ell^\infty$).

i)$\Rightarrow$iv): If S is a closed subspace of M with dim $M/S = 1$, then S is weakly closed in M. By (c)$\Rightarrow$(h) in Theorem 2.3 of [5], it follows that S is also weakly closed in ℓ^∞.

iv)$\Rightarrow$i): Let S be a closed subspace of M as in iv). Since $(\ell^\infty/S)'$ separates the points of ℓ^∞/S and dim $M/S = 1$, we have that $\left((\ell^\infty/S)/(M/S)\right)'$ separates also the points of $(\ell^\infty/S)/(M/S)$ which is isometrically isomorphic to ℓ^∞/M (see diagram (I)). Hence, M is weakly closed in ℓ^∞.

Corollary 3.7. *If D is a weakly closed subspace of ℓ^∞ and D is strict in ℓ^∞, then D has the HB-property in ℓ^∞.*

Proof. Let S be a closed subspace of D with dim $D/S = 1$. It suffices to prove that $i_2(D/S)$ has the HB-property in ℓ^∞/S (Proposition 2.8).

By strictness and Proposition 2.7, $i_2(D/S)$ is a one-dimensional and strict subspace of ℓ^∞/S. But $\ell^\infty/S \simeq K^n$ for some n or $\ell^\infty/S \simeq \ell^\infty$ (Theorem 3.6). Now, the conclusion follows by Theorem 3.4.

Remark 3.8. Looking at Theorem 3.4 and Corollary 3.7 the following question arises in a natural way.

Problem 1. Is every finite-dimensional and strict subspace of ℓ^∞ orthocomplemented in ℓ^∞?

Observe that this problem is equivalent to each one of the following questions.

Problem 2. Let D be a finite-dimensional strict subspace of ℓ^∞. Is there any one-dimensional subspace Kx ($x \in D\backslash\{0\}$) of D that is strict (orthocomplemented) in ℓ^∞, i.e. $\|x\| = \max_n |x_n|$?

Problem 3. Let D be a finite-dimensional strict subspace of ℓ^∞, dim $D \geq 2$. Is there any closed subspace G of D with $0 \subsetneqq G \subsetneqq D$ such that G is strict (orthocomplemented) in ℓ^∞?

Indeed, it follows by Theorems 3.3 and 3,4 that if Problem 1 has an affirmative answer then so has Problem 2. Also, it is obvious to pass from Problem 2 to Problem 3. Finally, suppose that Problem 3 has an affirmative answer. We prove by induction that Problem 1 has also an affirmative answer. Let D be a n-dimensional strict subspace of ℓ^∞. We may assume that $n \geq 2$ (Theorem 3.4). Let $0 \subsetneqq G \subsetneqq D$ be such that G is strict (and hence orthocomplemented, by the induction hypothesis) in ℓ^∞. Since D/G is strict in ℓ^∞/G (Proposition 2.7) and $\ell^\infty/G \simeq \ell^\infty$ (Theorem 3.6) it follows by the induction hypothesis that D/G is orthocomplemented in ℓ^∞/G. Now the orthocomplementation of D follows from Proposition 2.10.

4. FINITE-(CO)DIMENSIONAL ORTHOCOMPLEMENTED SUBSPACES OF c_0

It is well known that every finite-dimensional subspace of c_0 is orthocomplemented (see [7]).

We now translate the results we have found in the above section about orthocomplemented finite-dimensional subspaces of ℓ^∞ into statements about finite-codimensional subspaces of c_0. The next lemma, which is a direct consequence of Propositions 2.5 and 2.6, contains the key to do that.

Lemma 4.1. *Let D be a closed subspace of c_0 (resp. a weakly closed subspace of ℓ^∞). Then,*

$$D \left\{ \begin{array}{l} \text{is orthocomplemented} \\ \text{is strict} \\ \text{has the HB-property} \end{array} \right\} \text{ in } c_0 \text{ (resp. in } \ell^\infty)$$

$$\text{iff } D^0 \left\{ \begin{array}{l} \text{is orthocomplemented} \\ \text{has the HB-property} \\ \text{is strict} \end{array} \right\} \text{ in } \ell^\infty \text{(resp. in } c_0),$$

(observe that every weakly closed subspace of ℓ^∞ has the HB$^+$-property, [5], Theorem 2.3).

Theorem 3.3 admits the following "dual":

Theorem 4.2. *Let S be a closed subspace of c_0 with finite codimension. Then the following properties are equivalent*

i) *S is orthocomplemented in c_0.*

ii) *Every hyperplane containing S is orthocomplemented (strict) in c_0.*

iii) *If $f \in c_0'$ and $f = 0$ on S, then $\|f\| = \max_n |f(e_n)|$ (where $e_1, e_2, \dots$ is the canonical base of c_0).*

Analogously, Theorem 3.4 converts into the following result for closed hyperplanes of c_0.

Theorem 4.3. *A closed hyperplane in c_0 has the HB-property in c_0 iff it is orthocomplemented in c_0.*

In the same line, from Corollary 3.7 we deduce

Corollary 4.4. *Every closed subspace of c_0 with the HB-property in c_0, is strict in c_0.*

Finally, Problems 1-3 of the previous section give rise to the following equivalent questions.

Let S be a closed subspace of c_0 that has finite codimension and the HB-property in c_0.

Problem I. Is S orthocomplemented in c_0?

Problem II. Is there any closed hyperplane H in c_0 with $H \supset S$ such that H has the HB-property (is orthocomplemented) in c_0?

Problem III. If $2 \leq \operatorname{codim} S$, is there a closed subspace T of c_0 with $S \subsetneqq T \subsetneqq c_0$ such that T has the HB-property (is orthocomplemented) in c_0?

5. SOME CONSEQUENCES AND REMARKS

Next we shall apply the results proved in the previous sections to study orthocomplementation for arbitrary closed subspaces of ℓ^∞ and c_0.

Theorem 5.1. *Let D be a closed subspace of ℓ^∞. Then the following are equivalent.*

i) *D is orthocomplemented in ℓ^∞.*

ii) *$D \simeq K^n$ for some $n \in \mathbb{N}$ or $D \simeq \ell^\infty$ and D is strict (has the HB-property) in ℓ^∞.*

iii) *D is weakly closed and strict (has the HB-property) in ℓ^∞ and D' has an orthogonal base.*

iv) *D is weakly closed and for every closed subspace F of D with $\dim D/F < \infty$, D/F is orthocomplemented in ℓ^∞/F.*

v) *D is strict and there exists a closed subspace F of D with $\dim D/F = 1$ such that F is orthocomplemented in ℓ^∞.*

vi) *There exists a closed subspace F of D with $\dim D/F = 1$ such that F is orthocomplemented in ℓ^∞ and D/F is orthocomplemented (strict) in ℓ^∞/F.*

Proof. i)⇒ii): Clearly D is strict and weakly closed. By Corollary 3.7, D has the HB-property in ℓ^∞.

Also, D' is isometrically isomorphic to a closed subspace of c_0 and so $D' \simeq K^n$ (for some $n \in \mathbb{N}$) or $D' \simeq c_0$. Since D is reflexive ([5], Lemma 2.2) we derive that $D \simeq K^n$ or $D \simeq \ell^\infty$.

ii)⇒iii): Follows from Theorem 2.3 of [5] and Corollary 3.7.

iii)⇒i): By reflexivity of D ([5], Lemma 2.2), $D \simeq \ell^\infty(I; s)$ for some set I and some $s : I \to (0, +\infty)$. Now, apply Proposition 2.1.

i)⇒iv): Follows from Proposition 2.7.

iv⇒iii): By Proposition 2.8, D has the HB-property in ℓ^∞.

On the other hand, since $D' \simeq c_0/D^0$ is of countable type, it is enough to see that every finite-dimensional subspace G of c_0/D^0 has an orthogonal base. Let $\pi_0 : c_0 \to c_0/D^0$ be the canonical surjection. There is a finite-dimensional subspace M of c_0 with $\pi_0(M) = G$. Since $D^0 + M$ is weakly closed in c_0 ([7], Lemma 3.14 and [3], Theorem 4.7), there exists a weakly closed subspace S of ℓ^∞ such that $D^0 + M = S^0$. By assumption and Proposition 2.7 we conclude that D^0 is orthocomplemented in S^0 (observe that $(\ell^\infty/S)' \simeq S^0$ and under this isometry $(D/S)^0$ maps onto D^0). Then, there is a closed subspace M_1 of c_0 which is an orthogonal complement of D^0 in S^0. In particular, $D^0 + M = D^0 + M_1$. So, $\pi_0(M_1) = G$. But M_1, being a subspace of c_0, has an orthogonal base. Hence, so has G.

i)$\Rightarrow$v): Clearly D is strict in ℓ^∞.

Now, let F be a closed subspace of D with $\dim D/F = 1$. By i)$\Rightarrow$ii) and Proposition 3.1 it follows that F is orthocomplemented in D (and hence in ℓ^∞).

v)$\Rightarrow$vi): Let F be a closed subspace of D with $\dim D/F = 1$. By strictness of D and Proposition 2.7 it follows that D/F is strict in ℓ^∞/F. Since F is weakly closed in ℓ^∞, we can apply Theorem 3.4 and Theorem 3.6 i)$\Rightarrow$ii) to conclude that D/F is orthocomplemented in ℓ^∞/F.

vi)$\Rightarrow$i): Follows by Proposition 2.10.

Recall that an absolutely convex set A of a locally convex space over K is called:

a) *c'-compact*: if for each neighbourhood U of 0 there exists a finite set $B \subset A$ such that $A \subset U + \mathrm{co}B$ (where $\mathrm{co}B$ is the absolutely convex hull of B).

b) *KM-compactoid*: if it is complete and there exists a compact set $X \subset A$ such that A is the closed absolutely convex hull of X (for the general properties of such sets see [4]).

By using Proposition 2.3 of [2] and a proof similar to the one given for (d) $\Longleftrightarrow$ (i) in Theorem 2.3 of [5], it is not difficult to obtain the following.

Theorem 5.2. *Let D be a closed subspace of ℓ^∞. Then, properties* i) - vi) *of Theorem* 5.1 *are equivalent to*

vii) *D is strict (has the HB-property) in ℓ^∞ and $B_D = \{x \in D : \|x\| \leq 1\}$ is weakly KM-compactoid in ℓ^∞.*

viii) *D is strict (has the HB-property) in ℓ^∞ and B_D is weakly closed and weakly c'-compact in ℓ^∞.*

As in section 4, we can now dualize Theorems 5.1 and 5.2 to describe the orthocomplemented subspaces of c_0.

Observe that as a direct consequence of Propositions 2.7 and 2.8, we have

Lemma 5.3. *Let D be a weakly closed subspace of ℓ^∞ and let F be a closed subspace of D with* $\dim D/F < \infty$ *(so, F is weakly closed, Theorem* 3.6*). Then, D/F is orthocomplemented (resp. is strict, has the HB-property) in ℓ^∞/F iff D^0 is orthocomplemented (resp. has the HB-property, is strict) in F^0.*

Then, putting together Lemmas 4.1 and 5.3 we have that Theorems 5.1 and 5.2 convert into the following descriptions of the orthocomplemented subspaces of c_0.

Theorem 5.4. *For a closed subspace S of c_0 the following properties are equivalent.*

i) *S is orthocomplemented in c_0.*

ii) *$c_0/S \simeq K^n$ for some $n \in \mathbf{N}$ or $c_0/S \simeq c_0$ and S has the HB-property (is strict) in c_0.*

iii) *S has the HB-property (is strict) in c_0 and c_0/S has an orthogonal base.*

iv) *S is orthocomplemented in any closed subspace T of c_0 with $T \supset S$ and* $\dim T/S < \infty$.

v) *S has the HB-property in c_0 and there exists a closed subspace T of c_0 with $T \supset S$ and* $\dim T/S = 1$ *such that T is orthocomplemented in c_0.*

vi) *There exists a closed subspace T of c_0 with $T \supset S$ and* $\dim T/S = 1$ *such that S is orthocomplemented in T and T is orthocomplemented in c_0.*

vii) *S has the HB-property (is strict) in c_0 and $B_{(c_0/S)'}$ is weakly-$*$ KM-compactoid in $(c_0/S)'$.*

viii) *S has the HB-property (is strict) in c_0 and $B_{(c_0/S)'}$ is weakly-$*$ c'-compact in $(c_0/S)'$.*

Remarks 5.5.

1. *There is a closed subspace D of ℓ^∞ with $D \simeq \ell^\infty$ (and hence D is weakly closed* [5], *Theorem* 2.3) *such that D is not orthocomplemented in ℓ^∞.*

Example: Choose $\lambda_1, \lambda_2, \ldots$ in K with $0 < |\lambda_1| < |\lambda_2| < \ldots \uparrow 1$. There are $z_1, z_2, \ldots$ in c_0 with $|\lambda_1| \le \|z_i\| < 1$ for all i such that every $x \in c_0$ with $\|x\| < 1$ can be written as $x = \sum_{i=1}^{\infty} \mu_i z_i$ where $|\mu_i| \le 1$ for all i and $\mu_i \to 0$. Now, the map $T : c_0 \to c_0$ given by $T(\sum_{i=1}^{\infty} \lambda_i e_i) = \sum_{i=1}^{\infty} \lambda_i z_i$ is a continuous linear function mapping $\{x \in c_0 : \|x\| \le 1\}$ onto $\{x \in c_0 : \|x\| < 1\}$: if $x \in c_0$ is such that $\|Tx\| = 1$, then $\|x\| > 1$. So T (and hence Ker T) is not strict. Thus, $D = (\text{Ker } T)^0$ satisfies the required conditions (Lemma 4.1).

2. *There exists a closed subspace D of ℓ^∞ such that $D \simeq K$ (hence D is weakly closed) and such that D is not orthocomplemented in ℓ^∞.*

Example: We know (Remark 2.3.3) that there exists a linear isometry i from K_ν^2 into ℓ^∞ (Recall that $K_\nu^2 \simeq (K_\nu^2)'$). Since K_ν^2 does not contain non-trivially mutually orthogonal elements, we derive that every one-dimensional subspace D of K_ν^2 satisfies our requirements.

3. *There exists a closed subspace D of ℓ^∞ with the HB-property in ℓ^∞ such that D' has an orthogonal base but D is not orthocomplemented in ℓ^∞.*

Example: Take for D the closed subspace of ℓ^∞ constructed in [7], 4.J (observe that since D is not reflexive, it is not orthocomplemented in ℓ^∞).

4. Looking at Theorem 5.1 and the above Remark the following question arises in a natural way.

Problem. Can we without harm remove the weak closedness of D in property iii) (when D is strict) or in property iv) of Theorem 5.1?

5. *There is a weakly closed subspace D of ℓ^∞ such that D' has an orthogonal base but D is not orthocomplemented in ℓ^∞.*

Example: Take $D = H^0$, where H is a closed hyperplane of c_0 which is not orthocomplemented in c_0 and apply Lemma 4.1.

6. *There is a finite-dimensional (and hence weakly closed) subspace D of ℓ^∞ such that D has the HB-property in ℓ^∞ but is not orthocomplemented in ℓ^∞.*

Example: See Remark 2.3.3.

7. Finally observe that Problems 1-3 appearing in Remark 3.8 are equivalent to

Problem 4. Let D be a weakly closed subspace of ℓ^∞ such that D is strict and has the HB-property in ℓ^∞. Does it follow that D is orthocomplemented in ℓ^∞?

Indeed, clearly if Problem 4 has an affirmative answer then so has Problem 1 (recall that every finite-dimensional and strict subspace of ℓ^∞ has the HB-property in ℓ^∞, Corollary 3.7).

Conversely, assume Problem 1 has an affirmative answer and let D be a weakly closed subspace of ℓ^∞ such that D is strict. Let F be a closed subspace of D with $\dim D/F < \infty$. By Theorem 5.1 i) $\Longleftrightarrow$ iv) it is enough to prove that D/F is orthocomplemented in ℓ^∞/F. For that observe that it follows from Proposition 2.7 that D/F is a one-dimensional and strict subspace of ℓ^∞/F. But F is weakly closed in ℓ^∞ and so $\ell^\infty/F \simeq K^n$ (for some n) or $\ell^\infty/F \simeq \ell^\infty$ (Theorem 3.6). By assumption D/F is orthocomplemented in ℓ^∞/F and we are done.

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