Abstract. For a closed subspace $D$ of $\ell^\infty$ over a non-archimedean valued base field we study in this paper the property

1. There exists a continuous linear projection $P$ from $\ell^\infty$ onto $D$ with $\|P\| \leq 1$ ($D$ is orthocomplemented in $\ell^\infty$)

as related to the properties 2, 3, 4 below,

2. For every continuous linear functional $f \in D'$ there exists a continuous linear extension $\tilde{f} \in (\ell^\infty)'$ with $\|\tilde{f}\| = \|f\|$ ($D$ has the Hahn-Banach property in $\ell^\infty$).

3. The canonical quotient map $\pi_E : E \to E/D$ is strict, i.e. for each $z \in E/D$ there exists $x \in E$ with $\pi_E(x) = z$ and $\|x\| = \|z\|$ ($D$ is strict in $\ell^\infty$).

4. $D$ is weakly closed in $\ell^\infty$.

Also, certain duality arguments allow us to obtain several descriptions of the orthocomplemented subspaces of $c_0$. In particular it is shown (Theorem 4.3) that, if $K$ is not spherically complete, a closed hyperplane $H$ in $c_0$ having the Hahn-Banach property in $c_0$ is orthocomplemented.

1. PRELIMINARIES. Throughout $K$ is a non-archimedean valued field that is complete with respect to the metric induced by the non-trivial valuation $\cdot \cdot$. Also, $(E, \|\cdot\|)$ will be a (non-archimedean) Banach space over $K$.

For a Banach space $F$ over $K$ and a continuous linear map $T$ from $E$ into $F$, the kernel of $T$ is the set

$$\text{Ker } T = \{x \in E : Tx = 0\}.$$ 

Also, the norm of $T$ is given by

$$\|T\| = \sup \left\{ \frac{\|Tx\|}{\|x\|} : x \in E \setminus \{0\} \right\}$$

When there exists a linear isometry from $E$ onto $F$ we say that $E$ and $F$ are isometrically isomorphic and we write $E \simeq F$.

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The dual space $E'$ of $E$ consisting of all the continuous linear maps from $E$ to $K$ is again a Banach space. We set

$$J_E(x)(x') = x'(x) \quad (x \in E, x' \in E').$$

$E$ is called reflexive is $J_E$ is an isometry from $E$ onto $E''$.

For a closed subspace $D$ of $E$ we say that

a) $D$ has the HB-property (resp. HB$^+$-property) in $E$ if for every $f \in D'$ (resp. for every $\varepsilon > 0$ and for every $f \in D'$) there exists a continuous linear map $\tilde{f} \in E'$ extending $f$ such that $\|\tilde{f}\| = \|f\|$ (resp. $\|\tilde{f}\| \leq (1 + \varepsilon)\|f\|$).

b) $D$ is strict in $E$ if the quotient map $\pi_E : E \to E/D$ is strict (i.e. for every $z \in E/D$ there exists an $x \in E$ for which $\pi_E(x) = z$ and $\|x\| = \|z\|$).

c) $D$ is orthocomplemented in $E$ if there exists a closed subspace $G$ of $E$ such that $D \cap G = \{0\}$, $E = D + G$ and

$$\|x + y\| = \max(\|x\|, \|y\|) \quad (x \in D, y \in G)$$

(such a $G$ is called an orthogonal complement of $D$ in $E$).

It is not difficult to prove the two following Propositions which include some elementary (but useful) descriptions for the orthocomplemented and the strict subspaces of an arbitrary Banach space.

**Proposition 1.1.** For a closed subspace $D$ of $E$ the following are equivalent.

i) $D$ is orthocomplemented in $E$.

ii) There exists a continuous linear isometry $\varphi : E/D \to E$ such that $\pi_E \circ \varphi$ is the identity on $E/D$.

iii) There exists a continuous linear projection $P$ from $E$ onto $D$ with $\|P\| = 1$ (This $P$ is called an orthoprojection from $E$ onto $D$).

**Proposition 1.2.** For a closed subspace $D$ of $E$ the following properties are equivalent:

i) $D$ is strict in $E$.

ii) There exists a (non-necessarily linear) map $\varphi : E/D \to E$ such that $\|\varphi(x)\| = \|x\|$ for all $x \in E/D$ and $\pi_E \circ \varphi$ is the identity on $E/D$.

iii) For each $x \in E$, $D$ is orthocomplemented in $D + Kx$.

Clearly, $D$ is orthocomplemented in $E$ if $D$ has the HB-property and $D$ is strict in $E$.

If $E'$ separates the points of $E$ then $D$ is orthocomplemented in $E$ if $D$ is weakly closed in $E$.

Most of what we are about to do concerns converses of the above implications when $E = \ell^\infty$ or $c_0$. Firstly we consider (co)finite-dimensional subspaces (sections 3, 4) and
later on arbitrary closed subspaces of $\ell^\infty$ and $c_0$ (section 5). We assume that $K$ is not spherically complete, since if $K$ is spherically complete every closed subspace of $E$ is weakly closed and has the HB-property in $E$ ([3], Theorems 4.2, 4.7) and also every finite-dimensional subspace of $E$ is orthocomplemented ([7], Lemma 4.35). The basic machinery to our purpose is included in section 2.

The following problem arises in a natural way in this paper (see Problem 4 in section 5):

**Problem.** Suppose $K$ is not spherically complete. Let $D$ be a weakly closed subspace of $\ell^\infty$ such that $D$ is strict and has the HB-property in $\ell^\infty$. Does it follow that $D$ is orthocomplemented in $\ell^\infty$?

In fact we do not know the answer of this problem for any infinite-dimensional Banach space $E$ (instead of $\ell^\infty$) over a non-spherically complete field $K$.

However, if $K$ is spherically complete, the situation is completely different. Indeed, suppose that $|K| = [0, \infty)$. By a standard construction we can make a strict quotient map $\pi : c_0(I) \to \ell^\infty$ if $I$ has adequate cardinal. Now, $D = \text{Ker } \pi$ is a weakly closed subspace which is strict and has the HB-property in $c_0(I)$. If $D$ were orthocomplemented then $\ell^\infty$ would be isometrically isomorphic to a closed subspace of $c_0(I)$ and so $\ell^\infty$ has an orthogonal base: a contradiction ([7], Corollary 5.18).

For some other unexplained concepts and notations that we will use in the sequel, we refer to [3] and [7].

2. **GENERAL FACTS**

In this section we include some general results which will be useful in the rest of the paper.

First, we are going to see (Propositions 2.1 - 2.7) that strictness and the HB-property behave sometimes as "opposites" of one another.

**Proposition 2.1.** Let $D$ be a closed subspace of $E$.

i) If $D$ is strict in $E$ and $E/D \simeq c_0(I; s)$ for some set $I$ and $s : I \to (0, +\infty)$, then $D$ is orthocomplemented in $E$.

ii) If $D$ has the HB-property in $E$ and $D \simeq \ell^\infty(I; s)$ for some set $I$ and some $s : I \to (0, +\infty)$, then $D$ is orthocomplemented in $E$ (compare Theorem 1.2 of [5]).

**Proof.**

i) Let $\{u_i : i \in I\}$ be an orthogonal base of $E/D$. By strictness, there exists $\{z_i : i \in I\} \subset E$ such that $\pi_E(z_i) = u_i$ and $\|z_i\| = \|u_i\|$ for all $i \in I$. A standard argument shows that $\varphi : E/D \to E$ given by the formula $\sum_{i \in I} \lambda_i z_i \to$
\[ \sum_{i \in I} \lambda_i z_i \] is a linear isometry for which \( \pi_E \circ \varphi \) is the identity on \( E/D \). Hence, \( D \) is orthocomplemented.

ii) For each \( i \in I \) the coordinate function \( f_i \in D' \) given by \( f_i(x) = x_i \) (\( x = (x_i)_{i \in I} \in \ell^\infty(I, s) \)) has norm \( s(i)^{-1} \). By the HB-property, \( f_i \) extends to an \( \tilde{f}_i \in E' \) with \( \| \tilde{f}_i \| = s(i)^{-1} \). Then, \( P : E \to D; x \to (\tilde{f}_i(x))_{i \in I} \) is an orthoprojection from \( E \) onto \( D \).

As a special case we obtain

**Corollary 2.2.** If \( D \) is a closed hyperplane (resp. a one-dimensional subspace) in \( E \), then \( D \) is strict (resp. \( D \) has the HB-property) in \( E \) iff \( D \) is orthocomplemented in \( E \).

**Remarks 2.3.**

1.- Observe that if \( D \) is a closed hyperplane of \( E \), there is an \( f \in E' \setminus \{0\} \) such that \( D = \text{Ker} \ f \). Then, \( D \) is orthocomplemented iff \( \|f\| = \max \left\{ \frac{|f(x)|}{\|x\|} : x \in E \setminus \{0\} \right\} \).

In fact, if \( a \in E \) one can easily see that \( Ka \) is an orthogonal complement of \( D \) iff \( \|f\| = \frac{|f(a)|}{\|a\|} \).

2.- If \( K \) is spherically complete the finite (co)dimensional version of the above Corollary 2.2 holds.

Indeed, observe that if \( \dim E/D < \infty \), then \( E/D \) has an orthogonal base ([7], Lemma 5.5). Also, every finite-dimensional subspace of \( E \) is orthocomplemented ([7], Lemma 4.35).

3.- But, for non-spherically complete fields \( K \) the generalization in Remark 2 does not hold. In fact, let \( \pi : c_0 \to K_2^2 \) be a strict surjection ([6], 2.3, Remark 1). Then, \( \text{Ker} \pi \) is a strict two-codimensional subspace of \( c_0 \) that cannot be orthocomplemented since \( K_2^2 \) has no orthogonal base ([7], p.68).

On the other hand, the adjoint of \( \pi \) is an isometry \( \pi' : (K_2^2)' \to \ell^\infty \) and by construction \( \text{Im} \pi' \) has the HB-property in \( \ell^\infty \). But it will follow from Theorem 3.3 that it is not orthocomplemented in \( \ell^\infty \).

However we do have the following related statement.

**Proposition 2.4.**

i) If \( D \) is a closed subspace of \( E \) of finite codimension and if all hyperplanes \( H \) containing \( D \) are strict (orthocomplemented) in \( E \), then \( D \) is orthocomplemented in \( E \).

ii) If \( D \) is a finite-dimensional subspace of \( E \) and if every one-dimensional subspace of \( D \) has the HB-property (is orthocomplemented) in \( E \), then \( D \) is orthocomplemented in \( E \).
Proof.
i) For a proof by induction with respect to the codimension of \( D \) it suffices to show that, for closed subspaces \( D_1, D_2 \) of finite codimension, containing \( D \) from
\[
D_1 \subset D_2, \dim D_2/D_1 = 1 \text{ and } \quad D_2 \text{ is orthocomplemented in } E,
\]
it follows that \( D_1 \) is orthocomplemented in \( E \).
To see that, let \( P \) be an orthoprojection from \( E \) onto \( D_2 \). Then, \( \dim \ker P = \text{codim } D_1 - 1 \) and so \( D_1 + \ker P \) is a closed hyperplane of \( E \). There is an orthoprojection \( Q \) from \( E \) onto \( D_1 + \ker P \). Hence, \( PQ \) is an orthoprojection from \( E \) onto \( D_1 \).

ii) Almost identical to the proof of Lemma 4.35,iii) of [7].

The next two Propositions stress the duality between strictness and the HB-property.

Proposition 2.5. For a closed subspace \( D \) of \( E \) and its polar \( D^0 \) we have

i) If \( D \) is orthocomplemented in \( E \), then \( D^0 \) is orthocomplemented in \( E' \).

ii) If \( D \) has the HB-property in \( E \), then \( D^0 \) is strict in \( E' \).

iii) If \( D \) is strict in \( E \) and \( E/D \) is reflexive, then \( D^0 \) has the HB-property in \( E' \).

Proof.

i) If \( S \) is an orthogonal complement of \( D \) in \( E \), then \( S^0 \) is an orthogonal complement of \( D^0 \) in \( E' \).

ii) If \( i : D \hookrightarrow E \) is the canonical inclusion then its adjoint \( i' : E' \rightarrow D' \) is a strict map. Hence, its kernel, \( D^0 \), is strict in \( E' \).

iii) The quotient map \( \pi_E : E \rightarrow E/D \) has an isometrical adjoint \( \pi'_E : (E/D)' \rightarrow E' \) for which \( \pi'_E((E/D)') = D^0 \). Hence, to show that \( D^0 \) has the HB-property in \( E' \) it suffices to prove that for any \( \varphi \in (E/D)' \) there exists a \( \tilde{\varphi} \in E'' \) such \( ||\tilde{\varphi}|| = ||\varphi|| \) and \( \tilde{\varphi} \circ \pi'_E = \varphi \). By the reflexivity of \( E/D \), there is a \( z \in E/D \) such that \( \varphi = J_{E/D}(z) \) and \( ||z|| = ||\varphi|| \). Also, by strictness there is an \( x \in E \) with \( \pi_E(x) = z \) and \( ||x|| = ||z|| \). Then, \( \tilde{\varphi} = J_E(x) \) satisfies the required conditions.

Now, we consider the converse of Proposition 2.5.

Proposition 2.6. Let \( D \) be a closed subspace of \( E \).

i) Let \( D^0 \) be orthocomplemented (resp. \( D^0 \) have the HB-property in \( E \)). If in addition \( E \) is reflexive and \( D \) is weakly closed then \( D \) is orthocomplemented (resp. \( D \) is strict) in \( E \).

ii) If \( D^0 \) is strict in \( E' \) and \( D \) has the HB⁺-property in \( E \), then \( D \) has the HB-property in \( E \).
Proof.
i) By the previous Proposition the bipolar of \( D, D^{00} \), is orthocomplemented (strict) in \( E'' \). By reflexivity and weak closedness \( D \) is orthocomplemented (strict) in \( E \).

ii) Let \( i': E' \to D' \) be the adjoint map of the canonical inclusion \( i : D \to E \) and let \( \rho : D' \to E'/D^0 \) the natural map making the diagram

\[
\begin{array}{ccc}
E' & \xrightarrow{i'} & D' \\
\pi_{E'} \downarrow & & \downarrow \rho \\
E'/D^0
\end{array}
\]

commute. It follows easily from the HB\(^{+}\)-property of \( D \) that \( \rho \) is an isometrical isomorphism. Now, \( \pi_{E'} \) is strict. Hence, so is \( i' \), i.e. \( D \) has the HB-property.

Although in the above results the HB-property and strictness seem dual properties, sometimes they have similar behaviour. This is the case in the next few propositions.

Observe that if \( D \) is a closed subspace of \( E \) and \( S \) is a closed subspace of \( D \), then we have in a natural way the following commutative diagram

\[
\begin{array}{ccc}
D & \xrightarrow{i_1} & E \\
\downarrow \pi_D & & \downarrow \pi_E \\
D/S & \xrightarrow{i_2} & E/S
\end{array}
\]

where \( i_1, \pi_E, \pi_D \) are the obvious maps and \( i_2 \) makes the diagram commute.

**Proposition 2.7.** Let \( D \) be a closed subspace of \( E \) and let \( S \) be a closed subspace of \( D \). If \( D \) is strict (resp. has the HB-property, is orthocomplemented) in \( E \), then \( i_2(D/S) \) is strict (resp. has the HB-property, is orthocomplemented) in \( E/S \).

**Proof.** Suppose that \( D \) is strict. Let \( x \in E \). There is a \( d \in D \) such that

\[
\|x - i_1(d)\| \leq \|x - i_1(d')\| \quad (d' \in D).
\]

Now, for all \( s' \in S, d' \in D \), we have

\[
\|\pi_E(x) - i_2\pi_D(d)\| = \|\pi_E(x) - \pi_E(i_1(d))\|
\]

\[
\leq \|x - i_1(d)\| \leq \|x - i_1(d') - s'\|
\]

Hence, \( \|\pi_E(x) - i_2\pi_D(d)\| \leq \|\pi_E(x) - i_2\pi_D(d')\| \) for all \( d' \in D \) and we see that the distance of \( \pi_E(x) \) to \( i_2(D/S) \) is attained, which means that \( i_2(D/S) \) is strict in \( E/S \).
Now, assume that $D$ has the HB-property and let $f \in (D/S)'$. Then $f \circ \pi_D \in D'$ so by assumption there is a $g \in E'$ such that $\|g\| = \|f \circ \pi_D\| = \|f\|$ and $g \circ i_1 = f \circ \pi_D$. Since $S \subset \text{Ker} \ g$ there is a unique $\tilde{f} \in (E/S)'$ such that $\tilde{f} \circ \pi_E = g$ (see the diagram).

One verifies without pain that then also $\tilde{f} \circ i_2 = f$ and that $\|\tilde{f}\| = \|f\|$.

Finally, suppose that $D$ is orthocomplemented and let $P : E \to D$ be an orthoprojection from $E$ onto $D$. Since $S \subset \text{Ker}(\pi_D \circ P)$, there is a unique continuous linear map $Q : E/S \to D/S$ such that $Q \circ \pi_E = \pi_D \circ P$ and $\|Q\| \leq 1$. Also, $Q \circ i_2 \pi_D(x) = \pi_D(x)$ for all $x \in D$. So, since $\pi_D$ is surjective, we conclude that $Q \circ i_2$ is the identity on $D/S$, which implies that $i_2(D/S)$ is orthocomplemented in $E/S$.

A partial converse of Proposition 2.7 is the following.

**Proposition 2.8.** Let $D$ be a closed subspace of $E$. If for each closed subspace $S$ of $D$ with $\dim D/S = 1$ we have that $i_2(D/S)$ has the HB-property in $E/S$, then $D$ has the HB-property in $E$.

**Proof.** Let $f \in D' \setminus \{0\}$ and let $S = \text{Ker} \ f$. Then $f = \rho_1 \circ \pi_D$ where $\rho_1 : D/S \to K$ is a similarity (i.e. there exists a nonzero real number $c$ such that $|\rho_1(z)| = c\|z\|$ for all $z \in D/S$). By assumption and Corollary 2.2, there is an orthoprojection $\rho_2 : E/S \to D/S$ such that $\rho_2 \circ i_2$ is the identity on $D/S$. Now set $\tilde{f} = \rho_1 \cdot \rho_2 \circ \pi_E$. Then, $\|\tilde{f}\| = \|f\|$ and $\tilde{f} \circ i_1 = f$, and we are done.

**Remark 2.9.** Putting together Propositions 2.7 and 2.8 we derive that a closed subspace $D$ of $E$ has the HB-property in $E$ iff for every closed hyperplane $S$ of $D$, $i_2(D/S)$ has the HB-property in $E/S$. (Compare with Theorem 2.3 of [1]).

Observe that if $S, D$ are closed subspaces of $E$ with $S \subset D$, then the formula

$$\pi_{E/D}(\pi_1(x)) = \pi_2 \circ \pi_E(x) \quad (x \in E)$$

defines an isometrica! isomorphism $\pi_{E/D} : E/D \to (E/S)/(D/S)$ making the diagram

$$
\begin{array}{ccc}
D & \xrightarrow{i_1} & E \\
\downarrow \pi_D & & \downarrow \pi_E \\
D/S & \xrightarrow{i_2} & E/S \\
\end{array}
\quad (I)
\begin{array}{ccc}
& & \downarrow \pi_{E/D} \\
& & (E/S)/(D/S)
\end{array}
$$

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Proposition 2.10 Let $S \subset D$ be closed subspaces of $E$. If $S$ is strict (resp. has the HB-property, is orthocomplemented) in $E$ and $D/S$ is strict (resp. has the HB-property, is orthocomplemented) in $E/S$, then $D$ is strict (resp. has the HB-property, is orthocomplemented) in $E$.

Proof.

a) Strictness: Let $z \in E/D$. Then, in the diagram (I), $\pi_{E/D}(z)$ admits a $y \in E/S$ such that $\pi_2(y) = \pi_{E/D}(z)$ and $\|y\| = \|\pi_{E/D}(z)\| = \|z\|$. Also, there is an $x \in E$ with $\pi_E(x) = y$ and $\|x\| = \|y\|$. Then, $\pi_1(x) = z$ and $\|x\| = \|y\| = \|z\|$. Hence, $D$ is strict in $E$.

b) HB-property: Let $f \in D'$ and let $g \in E'$ be such that the restrictions $g|S$ and $f|S$ coincide and $\|g\| = \|f|S\|$. Now consider $h = f - g|D \in D'$. Since $h = 0$ on $S$ there is a $h_1 \in (D/S)'$ with $h = h_1 \circ \pi_D$ and $\|h_1\| = \|h\|$. By assumption $h_1$ extends to a $h_2 \in (E/S)'$ (I.e. $h_2 \circ i_2 = h_1$) with $\|h_2\| = \|h_1\|$ (see the diagram).

Now set $j = h_2 \circ \pi_E$. We have that $\|j\| \leq \|f\|$ and $j \circ i_1 = h$. Then, $\tilde{f} = j + g$ is a continuous linear extension of $f$ with $\|\tilde{f}\| = \|f\|$ and we are done.

c) Orthocomplementation: By using diagram (I), there is by assumption a $\rho_2 : (E/S)/(D/S) \to E/S$ such that $\pi_2 \circ \rho_2$ is the identity and also a $\rho_1 : E/S \to E$ such that $\pi_E \circ \rho_1$ is the identity, $\rho_1$ and $\rho_2$ being linear isometries. Now define $\tau : E/D \to E$ by $\tau = \rho_1 \circ \rho_2 \circ \pi_{E/D}$. We have that $\tau$ is a linear isometry for which $\pi_{E/D} \circ (\pi_1 \circ \tau) = \pi_{E/D}$, and so $\pi_1 \circ \tau$ is the identity.

3. FINITE-(CO)DIMENSIONAL ORTHOCOMPLEMENTED SUBSPACES OF $\ell^\infty$

As we have already announced in the Preliminaries,

FROM NOW ON IN THIS PAPER (EXCEPT IN 3.2) WE ASSUME THAT $K$ IS NOT SPHERICALLY COMPLETE.
The results given in §2 can be applied now to obtain several descriptions of the finite-(co)dimensional subspaces of $\ell^\infty$ that have an orthogonal complement.

For subspaces of finite codimension the situation is satisfactory.

**Proposition 3.1.** Every closed finite-codimensional subspace of $\ell^\infty$ is orthocomplemented.

**Proof.** By reflexivity the map $D \to D^0$ is a bijection between the set of all finite-dimensional subspaces of $c_0$ and the set of all finite-codimensional subspaces of $\ell^\infty$. Since every finite-dimensional subspace of $c_0$ is orthocomplemented, we can apply Propositions 2.5 and 2.6 to derive our conclusion.

**Remark 3.2.** If $K$ is spherically complete the conclusion above no longer holds.

Indeed, suppose that the valuation on $K$ is dense. Let $X$ be a maximal orthogonal subset of $\ell^\infty$ and let $H$ be a closed hyperplane of $\ell^\infty$ containing $X$. Then $H$ is not orthocomplemented in $\ell^\infty$.

The pictures changes when we consider finite-dimensional subspaces of $\ell^\infty$.

**Theorem 3.3.** For a finite-dimensional subspace $D$ of $\ell^\infty$, the following properties are equivalent.

i) $D$ is orthocomplemented in $\ell^\infty$.

ii) Every one-dimensional subspace of $D$ is orthocomplemented (has the H.B-property) in $\ell^\infty$.

iii) For each $x = (x_n) \in D$, $\max_n |x_n|$ exists.

**Proof.** i) $\Rightarrow$ ii): By Proposition 2.5, there exists an orthogonal complement $S$ of $D^0$ in $c_0$. Then, $D \simeq S'$ in a natural way, and since $S$ is finite-dimensional, there is an $n \in \mathbb{N}$ such that $D \simeq K^n$. So, every one-dimensional subspace of $D$ is orthocomplemented in $D$ (and hence in $\ell^\infty$, by i)).

ii) $\Rightarrow$ i): It follows from Proposition 2.4 ii).

ii) $\iff$ iii): Let $f \in c_0'$. By Propositions 2.5 and 2.6 we have that $Kf$ is orthocomplemented in $c_0'$ iff $\text{Ker } f$ is orthocomplemented in $\ell^\infty$, and this happens iff $\|f\| = \max \{ |f(x)| : \|x\| \leq 1 \}$ (Remark 2.4.1). So, we conclude that $Kf$ is orthocomplemented in $c_0'$ iff $\|f\| = \max |f(e_n)|$ (where $e_1, e_2, \ldots$ is the canonical base of $c_0$). This is precisely ii) $\iff$ iii) (Recall that $c_0' \simeq c_0^\infty$, [7]. Exercise 3.4.1))

For one-dimensional subspaces we prove the following curious Theorem, which will be useful in the sequel.

**Theorem 3.4.** A one-dimensional subspace of $\ell^\infty$ is strict iff it is orthocomplemented.
Proof. Clearly the orthocomplementation property implies strictness (see the Preliminaries).
Now suppose that $D = Kx$ ($x = (x_1, x_2, \ldots) \in \ell^\infty, x \neq 0$). If $D$ is not orthocomplemented then $|x_n| < \|x\|$ for all $n$ (Theorem 3.3). We are going to prove that there exists a $y \in \ell^\infty$ such that the linear hull $[x, y]$ of $\{x, y\}$ has no orthogonal base and by Proposition 1.2 we are done.

Let $K = B_0$ and let $B_1 \supset B_2 \supset \ldots$ be bounded discs in $K$ whose intersection is empty. For each $n \in \mathbb{N}$ let $r_n = \text{diam } B_n$ (the diameter of $B_n$). Define a function $\varphi : K \to [0, +\infty)$ by the formula

$$\varphi(\lambda) = \lim_{n \to \infty} \text{dist}(\lambda, B_n) \quad (\lambda \in K).$$

Then $\inf\{\varphi(\lambda) : \lambda \in K\} = d$, where $d = \lim_{n \to \infty} r_n > 0$, but $d$ is not attained (observe that $d \neq r_n$ for each $n \in \mathbb{N}$). We shall construct $c_1, c_2, \ldots \in K$ such that

$$\|y - \lambda x\| = \varphi(\lambda)\|x\| \quad (\lambda \in K)$$

with $y := (c_1 x_1, c_2 x_2, \ldots)$ (Then, $\text{dist}(y, Kx)$ is not attained and it follows easily that $[x, y]$ has no orthogonal base).

Let $n \in \mathbb{N}$. If $x_n = 0$ we set $c_n = 0$. Now let $x_n \neq 0$. Then, we may choose a $k(n) \in \mathbb{N}$ for which

$$r_{k(n)} \leq \frac{\|x\|d}{|x_n|} \quad (II)$$

and take $c_n \in B_{k(n)} \setminus B_{k(n)+1}$.

Now let $\lambda \in K$. First we prove that $\|y - \lambda x\| \leq \varphi(\lambda)\|x\|$, i.e. that, for each $n \in \mathbb{N}$, $|c_n - \lambda| |x_n| \leq \varphi(\lambda)\|x\|$. This is obvious when $x_n = 0$, so let $x_n \neq 0$. There is a unique $m \in \{0, 1, 2, \ldots\}$ such that $\lambda \in B_m \setminus B_{m+1}$. We distinguish two cases.

a) $m \geq k(n)$. Then $c_n \in B_{k(n)}$ and $\lambda \in B_m \subset B_{k(n)}$. Hence, by (II) we obtain

$$|c_n - \lambda| |x_n| \leq r_{k(n)} |x_n| \leq \|x\| \varphi(\lambda).$$

b) $m < k(n)$. Then $c_n \in B_{k(n)} \subset B_{m+1}$ while $\lambda \not\in B_{m+1}$ so that $|c_n - \lambda| = \varphi(\lambda)$ and

$$|c_n - \lambda| |x_n| = \varphi(\lambda) |x_n| \leq \varphi(\lambda)\|x\|.$$

To finish, we prove that $\|y - \lambda x\| \geq \varphi(\lambda)\|x\|$. Let $\varepsilon > 0$. Without loss we can assume $\varepsilon < r_m - d$. From our assumption on $x$ it follows that $J := \{n \in \mathbb{N} : \|x\|d < |x_n|(d+\varepsilon)\}$ is infinite. If $n \in J$, then by (II)

$$r_{k(n)} < d + \varepsilon < r_m$$

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so that \( k(n) > m \). Thus we are in case b) of above, so \( |c_n - \lambda| |x_n| > \frac{d}{2\pi} \varphi(\lambda) \|x\| \) and we are done.

**Remark 3.5.** Taking into account Corollary 2.2 and Theorem 3.4, for a one-dimensional subspace \( D \) of \( \ell^\infty \) one verifies

\( D \) is orthocomplemented \( \iff \) \( D \) is strict \( \iff \) \( D \) has the HB-property.

We know that the implication

\( D \) has the HB-property \( \Rightarrow \) \( D \) is orthocomplemented

does not hold for every finite-dimensional subspace \( D \) of \( \ell^\infty \). Next we will see (Corollary 3.7) that the implication

\( D \) is strict \( \Rightarrow \) \( D \) has the HB-property

holds for every finite-dimensional (in fact for every weakly closed subspace) \( D \) of \( \ell^\infty \).

This will be a consequence of the following result.

**Theorem 3.6.** (Compare Theorem 2.3 of [5]). Let \( M \) be a closed subspace of \( \ell^\infty \). The following are equivalent.

i) \( M \) is weakly closed in \( \ell^\infty \).

ii) \( \ell^\infty /M \simeq K^n \) for some \( n \in \mathbb{N} \) or \( \ell^\infty /M \simeq \ell^\infty \).

iii) \( \ell^\infty /M \) is reflexive.

iv) For every (for some) closed subspace \( S \) of \( M \) with \( \dim M/S = 1 \), \( S \) is weakly closed in \( \ell^\infty \).

**Proof.** The implications ii) \( \Rightarrow \) iii) and iii) \( \Rightarrow \) i) are obvious.

i)\(\Rightarrow\)ii): For a closed subspace \( D \) of \( c_0 \) the adjoint of the inclusion map \( D \rightarrow c_0 \) is a quotient map, so \( D' \simeq c_0/D^0 \). By applying this for \( D := M^0 \) and by using \( M^{00} = M \) we obtain \( (M^0)' \simeq c_0/M^{00} \simeq \ell^\infty /M \). Since \( M^0 \) is a closed subspace of \( c_0 \), we have that \( M^0 \simeq K^n \) for some \( n \in \mathbb{N} \) (and so \( \ell^\infty /M \simeq K^n \)) or \( M^0 \simeq c_0 \) (and so \( \ell^\infty /M \simeq \ell^\infty \)).

i)\(\Rightarrow\)iv): If \( S \) is a closed subspace of \( M \) with \( \dim M/S = 1 \), then \( S \) is weakly closed in \( M \). By (c)\(\Rightarrow\)(h) in Theorem 2.3 of [5], it follows that \( S \) is also weakly closed in \( \ell^\infty \).

iv)\(\Rightarrow\)i): Let \( S \) be a closed subspace of \( M \) as in iv). Since \( (\ell^\infty /S)' \) separates the points of \( \ell^\infty /S \) and \( \dim M/S = 1 \), we have that \( ((\ell^\infty /S)/(M/S))' \) separates also the points of \( (\ell^\infty /S)/(M/S) \) which is isometrically isomorphic to \( \ell^\infty /M \) (see diagram (I)). Hence, \( M \) is weakly closed in \( \ell^\infty \).

**Corollary 3.7.** If \( D \) is a weakly closed subspace of \( \ell^\infty \) and \( D \) is strict in \( \ell^\infty \), then \( D \) has the HB-property in \( \ell^\infty \).

**Proof.** Let \( S \) be a closed subspace of \( D \) with \( \dim D/S = 1 \). It suffices to prove that \( t_2(D/S) \) has the HB-property in \( \ell^\infty /S \) (Proposition 2.8).
By strictness and Proposition 2.7, $\iota_2(D/S)$ is a one-dimensional and strict subspace of $\ell^\infty/S$. But $\ell^\infty/S \simeq K^n$ for some $n$ or $\ell^\infty/S \simeq \ell^\infty$ (Theorem 3.6). Now, the conclusion follows by Theorem 3.4.

**Remark 3.8.** Looking at Theorem 3.4 and Corollary 3.7 the following question arises in a natural way.

**Problem 1.** Is every finite-dimensional and strict subspace of $\ell^\infty$ orthocomplemented in $\ell^\infty$?

Observe that this problem is equivalent to each one of the following questions.

**Problem 2.** Let $D$ be a finite-dimensional strict subspace of $\ell^\infty$. Is there any one-dimensional subspace $K \{x \in D \setminus \{0\}\}$ of $D$ that is strict (orthocomplemented) in $\ell^\infty$, i.e. $\|x\| = \max_n |x_n|$?

**Problem 3.** Let $D$ be a finite-dimensional strict subspace of $\ell^\infty$, $\dim D \geq 2$. Is there any closed subspace $G$ of $D$ with $0 \subseteq G \subsetneq D$ such that $G$ is strict (orthocomplemented) in $\ell^\infty$?

Indeed, it follows by Theorems 3.3 and 3.4 that if Problem 1 has an affirmative answer then so has Problem 2. Also, it is obvious to pass from Problem 2 to Problem 3. Finally, suppose that Problem 3 has an affirmative answer. We prove by induction that Problem 1 has also an affirmative answer. Let $D$ be a $n$-dimensional strict subspace of $\ell^\infty$. We may assume that $n \geq 2$ (Theorem 3.4). Let $0 \subseteq G \subsetneq D$ be such that $G$ is strict (and hence orthocomplemented, by the induction hypothesis) in $\ell^\infty$. Since $D/G$ is strict in $\ell^\infty/G$ (Proposition 2.7) and $\ell^\infty/G \simeq \ell^\infty$ (Theorem 3.6) it follows by the induction hypothesis that $D/G$ is orthocomplemented in $\ell^\infty/G$. Now the orthocomplementation of $D$ follows from Proposition 2.10.

4. **FINITE-(CO)DIMENSIONAL ORTHOCOMPLEMENTED SUBSPACES OF $c_0$**

It is well known that every finite-dimensional subspace of $c_0$ is orthocomplemented (see [7]).

We now translate the results we have found in the above section about orthocomplemented finite-dimensional subspaces of $\ell^\infty$ into statements about finite-codimensional subspaces of $c_0$. The next lemma, which is a direct consequence of Propositions 2.5 and 2.6, contains the key to do that.
Lemma 4.1. Let $D$ be a closed subspace of $c_0$ (resp. a weakly closed subspace of $\ell^\infty$). Then,

$$
D \begin{cases} 
\text{is orthocomplemented} \\
\text{is strict} \\
\text{has the HB-property}
\end{cases}
in c_0 \text{ (resp. in } \ell^\infty),
$$

iff $D^0 \begin{cases} 
\text{is orthocomplemented} \\
\text{has the HB-property} \\
\text{is strict}
\end{cases}
in \ell^\infty \text{ (resp. in } c_0),$

(observe that every weakly closed subspace of $\ell^\infty$ has the HB$^+$-property, [5], Theorem 2.3).

Theorem 3.3 admits the following "dual":

Theorem 4.2. Let $S$ be a closed subspace of $c_0$ with finite codimension. Then the following properties are equivalent

i) $S$ is orthocomplemented in $c_0$.

ii) Every hyperplane containing $S$ is orthocomplemented (strict) in $c_0$.

iii) If $f \in c_0$ and $f = 0$ on $S$, then $\|f\| = \max_n |f(e_n)|$ (where $e_1, e_2, \ldots$ is the canonical base of $c_0$).

Analogously, Theorem 3.4 converts into the following result for closed hyperplanes of $c_0$.

Theorem 4.3. A closed hyperplane in $c_0$ has the HB-property in $c_0$ iff it is orthocomplemented in $c_0$.

In the same line, from Corollary 3.7 we deduce

Corollary 4.4. Every closed subspace of $c_0$ with the HB-property in $c_0$, is strict in $c_0$.

Finally, Problems 1-3 of the previous section give rise to the following equivalent questions.

Let $S$ be a closed subspace of $c_0$ that has finite codimension and the HB-property in $c_0$.

Problem I. Is $S$ orthocomplemented in $c_0$?

Problem II. Is there any closed hyperplane $H$ in $c_0$ with $H \supset S$ such that $H$ has the HB-property (is orthocomplemented) in $c_0$?
Problem III. If $2 \leq \text{codim } S$, is there a closed subspace $T$ of $c_0$ with $S \subsetneq T \subsetneq c_0$ such that $T$ has the HB-property (is orthocomplemented) in $c_0$?

5. SOME CONSEQUENCES AND REMARKS

Next we shall apply the results proved in the previous sections to study orthocomplementation for arbitrary closed subspaces of $\ell^\infty$ and $c_0$.

Theorem 5.1. Let $D$ be a closed subspace of $\ell^\infty$. Then the following are equivalent.

i) $D$ is orthocomplemented in $\ell^\infty$.

ii) $D \simeq K^n$ for some $n \in \mathbb{N}$ or $D \simeq \ell^\infty$ and $D$ is strict (has the HB-property) in $\ell^\infty$.

iii) $D$ is weakly closed and strict (has the HB-property) in $\ell^\infty$ and $D'$ has an orthogonal base.

iv) $D$ is weakly closed and for every closed subspace $F$ of $D$ with $\dim D/F < \infty$, $D/F$ is orthocomplemented in $\ell^\infty/F$.

v) $D$ is strict and there exists a closed subspace $F$ of $D$ with $\dim D/F = 1$ such that $F$ is orthocomplemented in $\ell^\infty$.

vi) There exists a closed subspace $F$ of $D$ with $\dim D/F = 1$ such that $F$ is orthocomplemented in $\ell^\infty$ and $D/F$ is orthocomplemented (strict) in $\ell^\infty/F$.

Proof. i) $\Rightarrow$ ii): Clearly $D$ is strict and weakly closed. By Corollary 3.7, $D$ has the HB-property in $\ell^\infty$.

Also, $D'$ is isometrically isomorphic to a closed subspace of $c_0$ and so $D' \simeq K^n$ (for some $n \in \mathbb{N}$) or $D' \simeq c_0$. Since $D$ is reflexive ([5], Lemma 2.2) we derive that $D \simeq K^n$ or $D \simeq \ell^\infty$.

ii) $\Rightarrow$ iii): Follows from Theorem 2.3 of [5] and Corollary 3.7.

iii) $\Rightarrow$ i): By reflexivity of $D$ ([5], Lemma 2.2), $D \simeq \ell^\infty(I; s)$ for some set $I$ and some $s : I \to (0, +\infty)$. Now, apply Proposition 2.1.

i) $\Rightarrow$ iv): Follows from Proposition 2.7.

iv) $\Rightarrow$ iii): By Proposition 2.8, $D$ has the HB-property in $\ell^\infty$.

On the other hand, since $D' \simeq c_0/D^0$ is of countable type, it is enough to see that every finite-dimensional subspace $G$ of $c_0/D^0$ has an orthogonal base. Let $\pi_0 : c_0 \to c_0/D^0$ be the canonical surjection. There is a finite-dimensional subspace $M$ of $c_0$ with $\pi_0(M) = G$. Since $D^0 + M$ is weakly closed in $c_0$ ([7], Lemma 3.14 and [3], Theorem 4.7), there exists a weakly closed subspace $S$ of $\ell^\infty$ such that $D^0 + M = S^0$. By assumption and Proposition 2.7 we conclude that $D^0$ is orthocomplemented in $S^0$ (observe that $(\ell^\infty/S)' \simeq S^0$ and under this isometry $(D/S)^0$ maps onto $D^0$). Then, there is a closed subspace $M_1$ of $c_0$ which is an orthogonal complement of $D^0$ in $S^0$. In
particular, \( D^0 + M = D^0 + M_1 \). So, \( \pi_0(M_1) = G \). But \( M_1 \), being a subspace of \( c_0 \), has an orthogonal base. Hence, so has \( G \).

i) \( \Rightarrow \) v): Clearly \( D \) is strict in \( \ell^\infty \).

Now, let \( F \) be a closed subspace of \( D \) with \( \dim D/F = 1 \). By i) \( \Rightarrow \) ii) and Proposition 3.1 it follows that \( F \) is orthocomplemented in \( D \) (and hence in \( \ell^\infty \)).

v) \( \Rightarrow \) vi): Let \( F \) be a closed subspace of \( D \) with \( \dim D/F = 1 \). By strictness of \( D \) and Proposition 2.7 it follows that \( D/F \) is strict in \( \ell^\infty/F \). Since \( F \) is weakly closed in \( \ell^\infty \), we can apply Theorem 3.4 and Theorem 3.6 i) \( \Rightarrow \) ii) to conclude that \( D/F \) is orthocomplemented in \( \ell^\infty/F \).

vi) \( \Rightarrow \) i): Follows by Proposition 2.10.

Recall that an absolutely convex set \( A \) of a locally convex space over \( K \) is called:

a) \( c' \)-compact: if for each neighbourhood \( U \) of 0 there exists a finite set \( B \subset A \) such that \( A \subset U + \text{co}B \) (where \( \text{co}B \) is the absolutely convex hull of \( B \)).

b) \( KM \)-compactoid: if it is complete and there exists a compact set \( X \subset A \) such that \( A \) is the closed absolutely convex hull of \( X \) (for the general properties of such sets see [4]).

By using Proposition 2.3 of [2] and a proof similar to the one given for (d) \( \iff \) (i) in Theorem 2.3 of [5], it is not difficult to obtain the following.

**Theorem 5.2.** Let \( D \) be a closed subspace of \( \ell^\infty \). Then, properties i) - vi) of Theorem 5.1 are equivalent to

vii) \( D \) is strict (has the HB-property) in \( \ell^\infty \) and \( B_D = \{ x \in D : \|x\| \leq 1 \} \) is weakly \( KM \)-compactoid in \( \ell^\infty \).

viii) \( D \) is strict (has the HB-property) in \( \ell^\infty \) and \( B_D \) is weakly closed and weakly \( c' \)-compact in \( \ell^\infty \).

As in section 4, we can now dualize Theorems 5.1 and 5.2 to describe the orthocomplemented subspaces of \( c_0 \).

Observe that as a direct consequence of Propositions 2.7 and 2.8, we have

**Lemma 5.3.** Let \( D \) be a weakly closed subspace of \( \ell^\infty \) and let \( F \) be a closed subspace of \( D \) with \( \dim D/F < \infty \) (so, \( F \) is weakly closed, Theorem 3.6). Then, \( D/F \) is orthocomplemented (resp. is strict, has the HB-property) in \( \ell^\infty/F \) iff \( D^0 \) is orthocomplemented (resp. has the HB-property, is strict) in \( F^0 \).

Then, putting together Lemmas 4.1 and 5.3 we have that Theorems 5.1 and 5.2 convert into the following descriptions of the orthocomplemented subspaces of \( c_0 \).

**Theorem 5.4.** For a closed subspace \( S \) of \( c_0 \) the following properties are equivalent.
i) $S$ is orthocomplemented in $c_0$.

ii) $c_0/S \simeq K^n$ for some $n \in \mathbb{N}$ or $c_0/S \simeq c_0$ and $S$ has the HB-property (is strict) in $c_0$.

iii) $S$ has the HB-property (is strict) in $c_0$ and $c_0/S$ has an orthogonal base.

iv) $S$ is orthocomplemented in any closed subspace $T$ of $c_0$ with $T \supset S$ and $\dim T/S < \infty$.

v) $S$ has the HB-property in $c_0$ and there exists a closed subspace $T$ of $c_0$ with $T \supset S$ and $\dim T/S = 1$ such that $T$ is orthocomplemented in $c_0$.

vi) There exists a closed subspace $T$ of $c_0$ with $T \supset S$ and $\dim T/S = 1$ such that $S$ is orthocomplemented in $T$ and $T$ is orthocomplemented in $c_0$.

vii) $S$ has the HB-property (is strict) in $c_0$ and $B(c_0/S)'$ is weakly-\ast KM-compactoid in $(c_0/S)'$.

viii) $S$ has the HB-property (is strict) in $c_0$ and $B(c_0/S)'$ is weakly-\ast $c'$-compact in $(c_0/S)'$.

Remarks 5.5.

1. There is a closed subspace $D$ of $\ell^\infty$ with $D \simeq \ell^\infty$ (and hence $D$ is weakly closed [5], Theorem 2.3) such that $D$ is not orthocomplemented in $\ell^\infty$.

Example: Choose $\lambda_1, \lambda_2, \ldots$ in $K$ with $0 < |\lambda_1| < |\lambda_2| < \ldots \uparrow 1$. There are $z_1, z_2, \ldots$ in $c_0$ with $|\lambda_1| \leq \|z_i\| < 1$ for all $i$ such that every $x \in c_0$ with $\|x\| < 1$ can be written as $x = \sum_{i=1}^\infty \mu_i z_i$ where $|\mu_i| \leq 1$ for all $i$ and $\mu_i \to 0$. Now, the map $T: c_0 \to c_0$ given by $T(\sum_{i=1}^\infty \lambda_i e_i) = \sum_{i=1}^\infty \lambda_i z_i$ is a continuous linear function mapping $\{x \in c_0 : \|x\| \leq 1\}$ onto $\{x \in c_0 : \|x\| < 1\}$: if $x \in c_0$ is such that $\|Tx\| = 1$, then $\|x\| > 1$. So $T$ (and hence Ker $T$) is not strict. Thus, $D = (\text{Ker } T)^0$ satisfies the required conditions (Lemma 4.1).

2. There exists a closed subspace $D$ of $\ell^\infty$ such that $D \simeq K$ (hence $D$ is weakly closed) and such that $D$ is not orthocomplemented in $\ell^\infty$.

Example: We know (Remark 2.3.3) that there exists a linear isometry $i$ from $K_2^2$ into $\ell^\infty$ (Recall that $K_2^2 \simeq (K_2^2)'$). Since $K_2^2$ does not contain non-trivially mutually orthogonal elements, we derive that every one-dimensional subspace $D$ of $K_2^2$ satisfies our requirements.

3. There exists a closed subspace $D$ of $\ell^\infty$ with the HB-property in $\ell^\infty$ such that $D'$ has an orthogonal base but $D$ is not orthocomplemented in $\ell^\infty$.

Example: Take for $D$ the closed subspace of $\ell^\infty$ constructed in [7], 4.J (observe that since $D$ is not reflexive, it is not orthocomplemented in $\ell^\infty$).

4. Looking at Theorem 5.1 and the above Remark the following question arises in a natural way.
**Problem.** Can we without harm remove the weak closedness of $D$ in property iii) (when $D$ is strict) or in property iv) of Theorem 5.1?

5. *There is a weakly closed subspace $D$ of $\ell^\infty$ such that $D'$ has an orthogonal base but $D$ is not orthocomplemented in $\ell^\infty$.***

**Example:** Take $D = H^0$, where $H$ is a closed hyperplane of $c_0$ which is not orthocomplemented in $c_0$ and apply Lemma 4.1.

6. *There is a finite-dimensional (and hence weakly closed) subspace $D$ of $\ell^\infty$ such that $D$ has the HB-property in $\ell^\infty$ but is not orthocomplemented in $\ell^\infty$.***

**Example:** See Remark 2.3.3.

7. Finally observe that Problems 1-3 appearing in Remark 3.8 are equivalent to

**Problem 4.** Let $D$ be a weakly closed subspace of $\ell^\infty$ such that $D$ is strict and has the HB-property in $\ell^\infty$. Does it follow that $D$ is orthocomplemented in $\ell^\infty$?

Indeed, clearly if Problem 4 has an affirmative answer then so has Problem 1 (recall that every finite-dimensional and strict subspace of $\ell^\infty$ has the HB-property in $\ell^\infty$, Corollary 3.7).

Conversely, assume Problem 1 has an affirmative answer and let $D$ be a weakly closed subspace of $\ell^\infty$ such that $D$ is strict. Let $F$ be a closed subspace of $D$ with $\dim D/F < \infty$. By Theorem 5.1 i) $\iff$ iv) it is enough to prove that $D/F$ is orthocomplemented in $\ell^\infty/F$. For that observe that it follows from Proposition 2.7 that $D/F$ is a one-dimensional and strict subspace of $\ell^\infty/F$. But $F$ is weakly closed in $\ell^\infty$ and so $\ell^\infty/F \simeq K^n$ (for some $n$) or $\ell^\infty/F \simeq \ell^\infty$ (Theorem 3.6). By assumption $D/F$ is orthocomplemented in $\ell^\infty/F$ and we are done.

**REFERENCES**


