Abstract. For a closed subspace $D$ of $\ell^\infty$ over a non-archimedean valued base field we study in this paper the property

1. There exists a continuous linear projection $P$ from $\ell^\infty$ onto $D$ with $\|P\| \leq 1$ ($D$ is orthocomplemented in $\ell^\infty$)

as related to the properties 2,3,4 below.

2. For every continuous linear functional $f \in D'$ there exists a continuous linear extension $\tilde{f} \in (\ell^\infty)'$ with $\|\tilde{f}\| = \|f\|$ ($D$ has the Hahn-Banach property in $\ell^\infty$).

3. The canonical quotient map $\pi_E : E \to E/D$ is strict, i.e. for each $z \in E/D$ there exists $x \in E$ with $\pi_E(x) = z$ and $\|x\| = \|z\|$ ($D$ is strict in $\ell^\infty$).

4. $D$ is weakly closed in $\ell^\infty$.

Also, certain duality arguments allow us to obtain several descriptions of the orthocomplemented subspaces of $c_0$. In particular it is shown (Theorem 4.3) that, if $K$ is not spherically complete, a closed hyperplane $H$ in $c_0$ having the Hahn-Banach property in $c_0$ is orthocomplemented.

1. PRELIMINARIES. Throughout $K$ is a non-archimedean valued field that is complete with respect to the metric induced by the non-trivial valuation $|\cdot|$. Also, $(E, \|\cdot\|)$ will be a (non-archimedean) Banach space over $K$.

For a Banach space $F$ over $K$ and a continuous linear map $T$ from $E$ into $F$, the kernel of $T$ is the set

$$\text{Ker } T = \{ x \in E : Tx = 0 \}.$$ 

Also, the norm of $T$ is given by

$$\|T\| = \sup \left\{ \frac{\|Tx\|}{\|x\|} : x \in E \setminus \{0\} \right\}$$ 

When there exists a linear isometry from $E$ onto $F$ we say that $E$ and $F$ are isometrically isomorphic and we write $E \simeq F$.

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The dual space $E'$ of $E$ consisting of all the continuous linear maps from $E$ to $K$ is again a Banach space. We set

$$J_E(x)(x') = x'(x) \quad (x \in E, x' \in E').$$

$E$ is called reflexive is $J_E$ is an isometry from $E$ onto $E''$.

For a closed subspace $D$ of $E$ we say that

a) $D$ has the HB-property (resp. HB$^+$-property) in $E$ if for every $f \in D'$ (resp. for every $\varepsilon > 0$ and for every $f \in D'$) there exists a continuous linear map $\hat{f} \in E'$ extending $f$ such that $\|\hat{f}\| = \|f\|$ (resp. $\|\hat{f}\| \leq (1 + \varepsilon)\|f\|$).

b) $D$ is strict in $E$ if the quotient map $\pi_E : E \to E/D$ is strict (i.e. for every $z \in E/D$ there exists an $x \in E$ for which $\pi_E(x) = z$ and $\|x\| = \|z\|$).

c) $D$ is orthocomplemented in $E$ if there exists a closed subspace $G$ of $E$ such that $D \cap G = \{0\}$, $E = D + G$ and

$$\|x + y\| = \max(\|x\|, \|y\|) \quad (x \in D, y \in G)$$

(such a $G$ is called an orthogonal complement of $D$ in $E$).

It is not difficult to prove the two following Propositions which include some elementary (but useful) descriptions for the orthocomplemented and the strict subspaces of an arbitrary Banach space.

**Proposition 1.1.** For a closed subspace $D$ of $E$ the following are equivalent.

i) $D$ is orthocomplemented in $E$.

ii) There exists a continuous linear isometry $\varphi : E/D \to E$ such that $\pi_E \circ \varphi$ is the identity on $E/D$.

iii) There exists a continuous linear projection $P$ from $E$ onto $D$ with $\|P\| = 1$ (This $P$ is called an orthoprojection from $E$ onto $D$).

**Proposition 1.2.** For a closed subspace $D$ of $E$ the following properties are equivalent:

i) $D$ is strict in $E$.

ii) There exists a (non-necessarily linear) map $\varphi : E/D \to E$ such that $\|\varphi(x)\| = \|x\|$ for all $x \in E/D$ and $\pi_E \circ \varphi$ is the identity on $E/D$.

iii) For each $x \in E$, $D$ is orthocomplemented in $D + Kx$.

Clearly, $D$ is orthocomplemented in $E$ if $D$ has the HB-property and $D$ is strict in $E$.

If $E'$ separates the points of $E$ then $D$ is orthocomplemented in $E$ if $D$ is weakly closed in $E$.

Most of what we are about to do concerns converses of the above implications when $E = \ell^\infty$ or $c_0$. Firstly we consider (co)finite-dimensional subspaces (sections 3,4) and
later on arbitrary closed subspaces of $\ell^\infty$ and $c_0$ (section 5). We assume that $K$ is not spherically complete, since if $K$ is spherically complete every closed subspace of $E$ is weakly closed and has the HB-property in $E$ ([3], Theorems 4.2, 4.7) and also every finite-dimensional subspace of $E$ is orthocomplemented ([7], Lemma 4.35). The basic machinery to our purpose is included in section 2.

The following problem arises in a natural way in this paper (see Problem 4 in section 5):

**Problem.** Suppose $K$ is not spherically complete. Let $D$ be a weakly closed subspace of $\ell^\infty$ such that $D$ is strict and has the HB-property in $\ell^\infty$. Does it follow that $D$ is orthocomplemented in $\ell^\infty$?

In fact we do not know the answer of this problem for any infinite-dimensional Banach space $E$ (instead of $\ell^\infty$) over a non-spherically complete field $K$.

However, if $K$ is spherically complete, the situation is completely different. Indeed, suppose that $|K| = [0, \infty)$. By a standard construction we can make a strict quotient map $\pi : c_0(I) \to \ell^\infty$ if $I$ has adequate cardinal. Now, $D = \text{Ker } \pi$ is a weakly closed subspace which is strict and has the HB-property in $c_0(I)$. If $D$ were orthocomplemented then $\ell^\infty$ would be isometrically isomorphic to a closed subspace of $c_0(I)$ and so $\ell^\infty$ has an orthogonal base: a contradiction ([7], Corollary 5.18).

For some other unexplained concepts and notations that we will use in the sequel, we refer to [3] and [7].

2. **GENERAL FACTS**

In this section we include some general results which will be useful in the rest of the paper.

First, we are going to see (Propositions 2.1 - 2.7) that strictness and the HB-property behave sometimes as "opposites" of one another.

**Proposition 2.1.** Let $D$ be a closed subspace of $E$.

i) If $D$ is strict in $E$ and $E/D \simeq c_0(I; s)$ for some set $I$ and $s : I \to (0, +\infty)$, then $D$ is orthocomplemented in $E$.

ii) If $D$ has the HB-property in $E$ and $D \simeq \ell^\infty(I; s)$ for some set $I$ and some $s : I \to (0, +\infty)$, then $D$ is orthocomplemented in $E$ (compare Theorem 1.2 of [5]).

**Proof.**

i) Let $\{u_i : i \in I\}$ be an orthogonal base of $E/D$. By strictness, there exists $\{z_i : i \in I\} \subset E$ such that $\pi_E(z_i) = u_i$ and $\|z_i\| = \|u_i\|$ for all $i \in I$. A standard argument shows that $\varphi : E/D \to E$ given by the formula $\sum_{i \in I} \lambda_i u_i \to$
\[ \sum_{i \in I} \lambda_i z_i \] is a linear isometry for which \( \pi_E \circ \varphi \) is the identity on \( E/D \). Hence, \( D \) is orthocomplemented.

ii) For each \( i \in I \) the coordinate function \( f_i \in D' \) given by \( f_i(x) = x_i \) \( (x = (x_i)_{i \in I} \in \ell^\infty(I, s)) \) has norm \( s(i)^{-1} \). By the HB-property, \( f_i \) extends to an \( \tilde{f}_i \in E' \) with \( \| \tilde{f}_i \| = s(i)^{-1} \). Then, \( P : E \to D; x \to (\tilde{f}_i(x))_{i \in I} \) is an orthoprojection from \( E \) onto \( D \).

As a special case we obtain

**Corollary 2.2.** If \( D \) is a closed hyperplane (resp. a one-dimensional subspace) in \( E \), then \( D \) is strict (resp. \( D \) has the HB-property) in \( E \) iff \( D \) is orthocomplemented in \( E \).

**Remarks 2.3.**

1.- Observe that if \( D \) is a closed hyperplane of \( E \), there is an \( f \in E' - \{0\} \) such that \( D = \text{Ker} f \). Then, \( D \) is orthocomplemented iff \( \| f \| = \max \{ \| f(x) \| : x \in E \setminus \{0\} \} \). In fact, if \( a \in E \) one can easily see that \( K a \) is an orthogonal complement of \( D \) iff \( \| f \| = \frac{1}{\| f(a) \|} \).

2.- If \( K \) is spherically complete the finite (co)dimensional version of the above Corollary 2.2 holds.

Indeed, observe that if \( \dim E/D < \infty \), then \( E/D \) has an orthogonal base ([7], Lemma 5.5). Also, every finite-dimensional subspace of \( E \) is orthocomplemented ([7], Lemma 4.35).

3.- But, for non-spherically complete fields \( K \) the generalization in Remark 2 does not hold. In fact, let \( \pi : c_0 \to K^2_0 \) be a strict surjection ([6], 2.3, Remark 1). Then, \( \text{Ker} \pi \) is a strict two-codimensional subspace of \( c_0 \) that cannot be orthocomplemented since \( K^2_0 \) has no orthogonal base ([7], p.68).

On the other hand, the adjoint of \( \pi \) is an isometry \( \pi' : (K^2_0)' \to \ell^\infty \) and by construction \( \text{Im} \pi' \) has the HB-property in \( \ell^\infty \). But it will follow from Theorem 3.3 that it is not orthocomplemented in \( \ell^\infty \).

However we do have the following related statement.

**Proposition 2.4.**

i) If \( D \) is a closed subspace of \( E \) of finite codimension and if all hyperplanes \( H \) containing \( D \) are strict (orthocomplemented) in \( E \), then \( D \) is orthocomplemented in \( E \).

ii) If \( D \) is a finite-dimensional subspace of \( E \) and if every one-dimensional subspace of \( D \) has the HB-property (is orthocomplemented) in \( E \), then \( D \) is orthocomplemented in \( E \).
Proof.
i) For a proof by induction with respect to the codimension of $D$ it suffices to show that, for closed subspaces $D_1, D_2$ of finite codimension, containing $D$ from

\[ D_1 \subset D_2, \dim D_2/D_1 = 1 \text{ and } \]

it follows that $D_1$ is orthocomplemented in $E$.
To see that, let $P$ be an orthoprojection from $E$ onto $D_2$. Then, $\dim \ker P = \operatorname{codim} D_1 - 1$ and so $D_1 + \ker P$ is a closed hyperplane of $E$. There is an orthoprojection $Q$ from $E$ onto $D_1 + \ker P$. Hence, $PQ$ is an orthoprojection from $E$ onto $D_1$.

ii) Almost identical to the proof of Lemma 4.35,iii) of [7].

The next two Propositions stress the duality between strictness and the HB-property.

**Proposition 2.5.** For a closed subspace $D$ of $E$ and its polar $D^0$ we have
i) If $D$ is orthocomplemented in $E$, then $D^0$ is orthocomplemented in $E'$.
ii) If $D$ has the HB-property in $E$, then $D^0$ is strict in $E'$.
iii) If $D$ is strict in $E$ and $E/D$ is reflexive, then $D^0$ has the HB-property in $E'$.

Proof.
i) If $S$ is an orthogonal complement of $D$ in $E$, then $S^0$ is an orthogonal complement of $D^0$ in $E'$.

Hence, its kernel, $D^0$, is strict in $E'$.

ii) The quotient map $\pi_E : E \rightarrow E/D$ has an isometrical adjoint $\pi'_E : (E/D)' \rightarrow E'$ for which $\pi'_E((E/D)') = D^0$. Hence, to show that $D^0$ has the HB-property in $E'$ it suffices to prove that for any $\varphi \in (E/D)'$ there exists a $\hat{\varphi} \in E''$ such $\|\hat{\varphi}\| = \|\varphi\|$ and $\hat{\varphi} \circ \pi'_E = \varphi$. By the reflexivity of $E/D$, there is a $z \in E/D$ such that $\varphi = J_{E/D}(z)$ and $\|z\| = \|\varphi\|$. Also, by strictness there is an $x \in E$ with $\pi_E(x) = z$ and $\|x\| = \|z\|$. Then, $\hat{\varphi} = J_E(x)$ satisfies the required conditions.

Now, we consider the converse of Proposition 2.5.

**Proposition 2.6.** Let $D$ be a closed subspace of $E$.
i) Let $D^0$ be orthocomplemented (resp. $D^0$ have the HB-property in $E$). If in addition $E$ is reflexive and $D$ is weakly closed then $D$ is orthocomplemented (resp. $D$ is strict) in $E$.

ii) If $D^0$ is strict in $E'$ and $D$ has the HB$^+$-property in $E$, then $D$ has the HB-property in $E$.
Proof.

i) By the previous Proposition the bipolar of $D$, $D^{00}$, is orthocomplemented (strict) in $E''$. By reflexivity and weak closedness $D$ is orthocomplemented (strict) in $E$.

ii) Let $i' : E' \to D'$ be the adjoint map of the canonical inclusion $i : D \to E$ and let $\rho : D' \to E'/D^0$ the natural map making the diagram

\[
\begin{array}{ccc}
E' & \xrightarrow{i'} & D' \\
\pi_{E'} & \searrow & \downarrow \rho \\
E'/D^0
\end{array}
\]

commute. It follows easily from the HB+ property of $D$ that $\rho$ is an isometrical isomorphism. Now, $\pi_{E'}$ is strict. Hence, so is $i'$, i.e. $D$ has the HB-property.

Although in the above results the HB-property and strictness seem dual properties, sometimes they have similar behaviour. This is the case in the next few propositions.

Observe that if $D$ is a closed subspace of $E$ and $S$ is a closed subspace of $D$, then we have in a natural way the following commutative diagram

\[
\begin{array}{ccc}
D & \xrightarrow{i_1} & E \\
\downarrow \pi_D & & \downarrow \pi_E \\
D/S & \xrightarrow{i_2} & E/S
\end{array}
\]

where $i_1$, $\pi_E$, $\pi_D$ are the obvious maps and $i_2$ makes the diagram commute.

**Proposition 2.7.** Let $D$ be a closed subspace of $E$ and let $S$ be a closed subspace of $D$. If $D$ is strict (resp. has the HB-property, is orthocomplemented) in $E$, then $i_2(D/S)$ is strict (resp. has the HB-property, is orthocomplemented) in $E/S$.

**Proof.** Suppose that $D$ is strict. Let $x \in E$. There is a $d \in D$ such that

\[\|x - i_1(d)\| \leq \|x - i_1(d')\|\] (for all $d' \in D$).

Now, for all $s' \in S$, $d' \in D$, we have

\[\|\pi_E(x) - i_2\pi_D(d)\| = \|\pi_E(x) - \pi_E(i_1(d))\|\]

\[\leq \|x - i_1(d)\| \leq \|x - i_1(d') - s'\|\]

Hence, $\|\pi_E(x) - i_2\pi_D(d)\| \leq \|\pi_E(x) - i_2\pi_D(d')\|$ for all $d' \in D$ and we see that the distance of $\pi_E(x)$ to $i_2(D/S)$ is attained, which means that $i_2(D/S)$ is strict in $E/S$.  

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Now, assume that $D$ has the HB-property and let $f \in (D/S)'$. Then $f \circ \pi_D \in D'$ so by assumption there is a $g \in E'$ such that \[ \|g\| = \|f \circ \pi_D\| = \|f\| \] and $g \circ i_1 = f \circ \pi_D$. Since $S \subset \ker g$ there is a unique $\tilde{f} \in (E/S)'$ such that $\tilde{f} \circ \pi_E = g$ (see the diagram).

\[
\begin{array}{c}
D \xrightarrow{i_1} E \\
\pi_D \downarrow \quad g \quad \pi_E \\
D/S \xrightarrow{\tilde{f}} E/S
\end{array}
\]

One verifies without pain that then also $\tilde{f} \circ i_2 = f$ and that \[ \|\tilde{f}\| = \|f\| \].

Finally, suppose that $D$ is orthocomplemented and let $P : E \to D$ be an orthoprojection from $E$ onto $D$. Since $S \subset \ker(\pi_D \circ P)$, there is a unique continuous linear map $Q : E/S \to D/S$ such that $Q \circ \pi_E = \pi_D \circ P$ and $\|Q\| \leq 1$. Also, $Q \circ i_2 \pi_D(x) = \pi_D(x)$ for all $x \in D$. So, since $\pi_D$ is surjective, we conclude that $Q \circ i_2$ is the identity on $D/S$, which implies that $i_2(D/S)$ is orthocomplemented in $E/S$.

A partial converse of Proposition 2.7 is the following.

**Proposition 2.8.** Let $D$ be a closed subspace of $E$. If for each closed subspace $S$ of $D$ with $\dim D/S = 1$ we have that $i_2(D/S)$ has the HB-property in $E/S$, then $D$ has the HB-property in $E$.

**Proof.** Let $f \in D' \setminus \{0\}$ and let $S = \ker f$. Then $f = \rho_1 \circ \pi_D$ where $\rho_1 : D/S \to K$ is a similarity (i.e. there exists a nonzero real number $c$ such that $|\rho_1(z)| = c\|z\|$ for all $z \in D/S$). By assumption and Corollary 2.2, there is an orthoprojection $\rho_2 : E/S \to D/S$ such that $\rho_2 \circ i_2$ is the identity on $D/S$. Now set $\tilde{f} = \rho_1 \cdot \rho_2 \circ \pi_E$. Then, $\|\tilde{f}\| = \|f\|$ and $\tilde{f} \circ i_1 = f$, and we are done.

**Remark 2.9.** Putting together Propositions 2.7 and 2.8 we derive that a closed subspace $D$ of $E$ has the HB-property in $E$ iff for every closed hyperplane $S$ of $D$, $i_2(D/S)$ has the HB-property in $E/S$. (Compare with Theorem 2.3 of [1]).

Observe that if $S, D$ are closed subspaces of $E$ with $S \subset D$, then the formula

\[ \pi_{E/D}(\pi_1(x)) = \pi_2 \circ \pi_E(x) \quad (x \in E) \]

defines an isometrical isomorphism $\pi_{E/D} : E/D \to (E/S)/(D/S)$ making the diagram

\[
\begin{array}{cccc}
D & \xrightarrow{i_1} & E & \xrightarrow{\pi_1} & E/D \\
\downarrow \pi_D & & \downarrow \pi_E & & \downarrow \pi_{E/D}
\end{array}
\]

\[
\begin{array}{cccc}
D/S & \xrightarrow{i_2} & E/S & \xrightarrow{\pi_2} & (E/S)/(D/S)
\end{array}
\]
Proposition 2.10 Let $S \subset D$ be closed subspaces of $E$. If $S$ is strict (resp. has the HB-property, is orthocomplemented) in $E$ and $D/S$ is strict (resp. has the HB-property, is orthocomplemented) in $E/S$, then $D$ is strict (resp. has the HB-property, is orthocomplemented) in $E$.

Proof.

a) Strictness: Let $z \in E/D$. Then, in the diagram (I), $\pi_{E/D}(z)$ admits a $y \in E/S$ such that $\pi_2(y) = \pi_{E/D}(z)$ and $\|y\| = \|\pi_{E/D}(z)\| = \|z\|$. Also, there is an $z \in E$ with $\pi_E(x) = y$ and $\|z\| = \|y\|$. Then, $\pi_1(x) = z$ and $\|x\| = \|y\| = \|z\|$. Hence, $D$ is strict in $E$.

b) HB-property: Let $f \in D'$ and let $g \in E'$ be such that the restrictions $g|S$ and $f|S$ coincide and $\|g\| = \|f|S\|$. Now consider $h = f - g|D \in D'$. Since $h = 0$ on $S$ there is a $h_1 \in (D/S)'$ with $h = h_1 \circ \pi_D$ and $\|h_1\| = \|h\|$. By assumption $h_1$ extends to a $h_2 \in (E/S)'$ (i.e. $h_2 \circ i_2 = h_1$) with $\|h_2\| = \|h_1\|$ (see the diagram).

Now set $j = h_2 \circ \pi_E$. We have that $\|j\| \leq \|f\|$ and $j \circ i_1 = h$. Then, $\tilde{f} = j + g$ is a continuous linear extension of $f$ with $\|\tilde{f}\| = \|f\|$ and we are done.

c) Orthocomplementation: By using diagram (I), there is by assumption a $\rho_2 : (E/S)/(D/S) \to E/S$ such that $\pi_2 \circ \rho_2$ is the identity and also a $\rho_1 : E/S \to E$ such that $\pi_E \circ \rho_1$ is the identity, $\rho_1$ and $\rho_2$ being linear isometries. Now define $\tau : E/D \to E$ by $\tau = \rho_1 \circ \rho_2 \circ \pi_{E/D}$. We have that $\tau$ is a linear isometry for which $\pi_{E/D} \circ (\pi_1 \circ \tau) = \pi_{E/D}$, and so $\pi_1 \circ \tau$ is the identity.

3. FINITE-(CO)DIMENSIONAL ORTHOCOMPLEMENTED SUBSPACES OF $\ell^\infty$

As we have already announced in the Preliminaries,

FROM NOW ON IN THIS PAPER (EXCEPT IN 3.2) WE ASSUME THAT $K$ IS NOT SPHERICALLY COMPLETE.
The results given in §2 can be applied now to obtain several descriptions of the finite-(co)dimensional subspaces of $\ell^\infty$ that have an orthogonal complement.

For subspaces of finite codimension the situation is satisfactory.

**Proposition 3.1.** Every closed finite-codimensional subspace of $\ell^\infty$ is orthocomplemented.

**Proof.** By reflexivity the map $D \to D^\circ$ is a bijection between the set of all finite-dimensional subspaces of $c_0$ and the set of all finite-codimensional subspaces of $\ell^\infty$. Since every finite-dimensional subspace of $c_0$ is orthocomplemented, we can apply Propositions 2.5 and 2.6 to derive our conclusion.

**Remark 3.2.** If $K$ is spherically complete the conclusion above no longer holds.

Indeed, suppose that the valuation on $K$ is dense. Let $X$ be a maximal orthogonal subset of $\ell^\infty$ and let $H$ be a closed hyperplane of $\ell^\infty$ containing $X$. Then $H$ is not orthocomplemented in $\ell^\infty$.

The pictures changes when we consider finite-dimensional subspaces of $\ell^\infty$.

**Theorem 3.3.** For a finite-dimensional subspace $D$ of $\ell^\infty$, the following properties are equivalent.

1) $D$ is orthocomplemented in $\ell^\infty$.
2) Every one-dimensional subspace of $D$ is orthocomplemented (has the H.B-property) in $\ell^\infty$.
3) For each $x = (x_n) \in D$, $\max_n |x_n|$ exists.

**Proof.** i) $\Rightarrow$ ii): By Proposition 2.5, there exists an orthogonal complement $S$ of $D^\circ$ in $c_0$. Then, $D \simeq S'$ in a natural way, and since $S$ is finite-dimensional, there is an $n \in \mathbb{N}$ such that $D \simeq K^n$. So, every one-dimensional subspace of $D$ is orthocomplemented in $D$ (and hence in $\ell^\infty$, by i)).

ii) $\Rightarrow$ i): It follows from Proposition 2.4 ii).

ii) $\iff$ iii): Let $f \in c_0'$. By Propositions 2.5 and 2.6 we have that $Kf$ is orthocomplemented in $c_0'$ iff $\text{Ker } f$ is orthocomplemented in $\ell^\infty$, and this happens iff $\|f\| = \max \{|f(x)| : \|x\| \leq 1\}$ (Remark 2.4.1). So, we conclude that $Kf$ is orthocomplemented in $c_0'$ iff $\|f\| = \max |f(e_n)|$ (where $e_1, e_2, \ldots$ is the canonical base of $c_0$). This is precisely ii) $\iff$ iii) (Recall that $c_0' \simeq \ell^\infty$, [7]. Exercise 3.Q.i))

For one-dimensional subspaces we prove the following curious Theorem, which will be useful in the sequel.

**Theorem 3.4.** A one-dimensional subspace of $\ell^\infty$ is strict iff it is orthocomplemented.
Proof. Clearly the orthocomplementation property implies strictness (see the Preliminaries).
Now suppose that $D = Kx$ ($x = (x_1, x_2, \ldots) \in \ell^\infty, x \neq 0$). If $D$ is not orthocomplemented then $|x_n| < \|x\|$ for all $n$ (Theorem 3.3). We are going to prove that there exists a $y \in \ell^\infty$ such that the linear hull $[x, y]$ of $\{x, y\}$ has no orthogonal base and by Proposition 1 we are done.

Let $K = B_0$ and let $B_1 \supset B_2 \supset \ldots$ be bounded discs in $K$ whose intersection is empty. For each $n \in \mathbb{N}$ let $r_n = \text{diam } B_n$ (the diameter of $B_n$). Define a function $\varphi : K \to [0, +\infty)$ by the formula

$$\varphi(\lambda) = \lim_{n \to \infty} \text{dist}(\lambda, B_n) \quad (\lambda \in K).$$

Then $\inf\{\varphi(\lambda) : \lambda \in K\} = d$, where $d = \lim_{n \to \infty} r_n > 0$, but $d$ is not attained (observe that $d \neq r_n$ for each $n \in \mathbb{N}$). We shall construct $c_1, c_2, \ldots \in K$ such that $||y - \lambda x|| = \varphi(\lambda)||x|| \quad (\lambda \in K)$

with $y := (c_1 x_1, c_2 x_2, \ldots)$ (Then, $\text{dist}(y, Kx)$ is not attained and it follows easily that $[x, y]$ has no orthogonal base).

Let $n \in \mathbb{N}$. If $x_n = 0$ we set $c_n = 0$. Now let $x_n \neq 0$. Then, we may choose an $k(n) \in \mathbb{N}$ for which

$$r_{k(n)} \leq \frac{||x||d}{|x_n|} \quad (II)$$

and take $c_n \in B_{k(n)} \setminus B_{k(n)+1}$.

Now let $\lambda \in K$. First we prove that $||y - \lambda x|| \leq \varphi(\lambda)||x||$, i.e. that, for each $n \in \mathbb{N}$, $|c_n - \lambda| |x_n| \leq \varphi(\lambda)||x||$. This is obvious when $x_n = 0$, so let $x_n \neq 0$. There is a unique $m \in \{0, 1, 2, \ldots\}$ such that $\lambda \in B_m \setminus B_{m+1}$. We distinguish two cases.

a) $m \geq k(n)$. Then $c_n \in B_{k(n)}$ and $\lambda \in B_m \subset B_{k(n)}$. Hence, by (II) we obtain

$$|c_n - \lambda| |x_n| \leq r_{k(n)} |x_n| \leq ||x|| \varphi(\lambda).$$

b) $m < k(n)$. Then $c_n \in B_{k(n)} \subset B_{m+1}$ while $\lambda \not\in B_{m+1}$ so that $|c_n - \lambda| = \varphi(\lambda)$ and

$$|c_n - \lambda| |x_n| = \varphi(\lambda) |x_n| \leq \varphi(\lambda)||x||.$$

To finish, we prove that $||y - \lambda x|| \geq \varphi(\lambda)||x||$. Let $\varepsilon > 0$. Without loss we can assume $\varepsilon < r_m - d$. From our assumption on $x$ it follows that $J := \{n \in \mathbb{N} : ||x||d < |x_n|(d+\varepsilon)\}$ is infinite. If $n \in J$, then by (II)

$$r_{k(n)} < d + \varepsilon < r_m$$
so that \( k(n) > m \). Thus we are in case b) of above, so \( |c_n - \lambda| |x_n| > \frac{\alpha}{2 + \epsilon} \varphi(\lambda) \|x\| \) and we are done.

**Remark 3.5.** Taking into account Corollary 2.2 and Theorem 3.4, for a one-dimensional subspace \( D \) of \( \ell^\infty \) one verifies

- \( D \) is orthocomplemented if and only if \( D \) is strict.
- \( D \) has the HB-property if and only if \( D \) is orthocomplemented.

We know that the implication

\( D \) has the HB-property \( \Rightarrow \) \( D \) is orthocomplemented

does not hold for every finite-dimensional subspace \( D \) of \( \ell^\infty \). Next we will see (Corollary 3.7) that the implication

\( D \) is strict \( \Rightarrow \) \( D \) has the HB-property

holds for every finite-dimensional (in fact for every weakly closed subspace) \( D \) of \( \ell^\infty \).

This will be a consequence of the following result.

**Theorem 3.6.** (Compare Theorem 2.3 of \([5]\)). Let \( M \) be a closed subspace of \( \ell^\infty \). The following are equivalent.

i) \( M \) is weakly closed in \( \ell^\infty \).

ii) \( \ell^\infty /M \cong K^n \) for some \( n \in \mathbb{N} \) or \( \ell^\infty /M \cong \ell^\infty \).

iii) \( \ell^\infty /M \) is reflexive.

iv) For every (for some) closed subspace \( S \) of \( M \) with \( \dim M/S = 1 \), \( S \) is weakly closed in \( \ell^\infty \).

**Proof.** The implications ii) \( \Rightarrow \) iii) and iii) \( \Rightarrow \) i) are obvious.

i) \( \Rightarrow \) ii): For a closed subspace \( D \) of \( c_0 \) the adjoint of the inclusion map \( D \rightarrow c_0 \) is a quotient map, so \( D' \cong c_0 /D^0 \). By applying this for \( D := M^0 \) and by using \( M^0 = M \) we obtain \( (M^0)' \cong c_0 /M^0 \cong \ell^\infty /M \). Since \( M^0 \) is a closed subspace of \( c_0 \), we have that \( M^0 \cong K^n \) for some \( n \in \mathbb{N} \) (and so \( \ell^\infty /M \cong K^n \)) or \( M^0 \cong c_0 \) (and so \( \ell^\infty /M \cong \ell^\infty \)).

i) \( \Rightarrow \) iv): If \( S \) is a closed subspace of \( M \) with \( \dim M/S = 1 \), then \( S \) is weakly closed in \( M \). By (c) \( \Rightarrow \) (h) in Theorem 2.3 of \([5]\), it follows that \( S \) is also weakly closed in \( \ell^\infty \).

iv) \( \Rightarrow \) i): Let \( S \) be a closed subspace of \( M \) as in iv). Since \( (\ell^\infty /S)' \) separates the points of \( \ell^\infty /S \) and \( \dim M/S = 1 \), we have that \( ((\ell^\infty /S)/(M/S))' \) separates also the points of \( (\ell^\infty /S)/(M/S) \) which is isometrically isomorphic to \( \ell^\infty /M \) (see diagram (I)). Hence, \( M \) is weakly closed in \( \ell^\infty \).

**Corollary 3.7.** If \( D \) is a weakly closed subspace of \( \ell^\infty \) and \( D \) is strict in \( \ell^\infty \), then \( D \) has the HB-property in \( \ell^\infty \).

**Proof.** Let \( S \) be a closed subspace of \( D \) with \( \dim D/S = 1 \). It suffices to prove that \( t_2(D/S) \) has the HB-property in \( \ell^\infty /S \) (Proposition 2.8).
By strictness and Proposition 2.7, \( i_2(D/S) \) is a one-dimensional and strict subspace of \( \ell^\infty/S \). But \( \ell^\infty/S \cong K^n \) for some \( n \) or \( \ell^\infty/S \cong \ell^\infty \) (Theorem 3.6). Now, the conclusion follows by Theorem 3.4.

**Remark 3.8.** Looking at Theorem 3.4 and Corollary 3.7 the following question arises in a natural way.

**Problem 1.** Is every finite-dimensional and strict subspace of \( \ell^\infty \) orthocomplemented in \( \ell^\infty \)?

Observe that this problem is equivalent to each one of the following questions.

**Problem 2.** Let \( D \) be a finite-dimensional strict subspace of \( \ell^\infty \). Is there any one-dimensional subspace \( Kx (x \in D \setminus \{0\}) \) of \( D \) that is strict (orthocomplemented) in \( \ell^\infty \), i.e. \( \|x\| = \max_n |x_n| \)?

**Problem 3.** Let \( D \) be a finite-dimensional strict subspace of \( \ell^\infty \), \( \dim D \geq 2 \). Is there any closed subspace \( G \) of \( D \) with \( 0 \subsetneq G \subsetneq D \) such that \( G \) is strict (orthocomplemented) in \( \ell^\infty \)?

Indeed, it follows by Theorems 3.3 and 3.4 that if Problem 1 has an affirmative answer then so has Problem 2. Also, it is obvious to pass from Problem 2 to Problem 3. Finally, suppose that Problem 3 has an affirmative answer. We prove by induction that Problem 1 has also an affirmative answer. Let \( D \) be a \( n \)-dimensional strict subspace of \( \ell^\infty \). We may assume that \( n \geq 2 \) (Theorem 3.4). Let \( 0 \subsetneq G \subsetneq D \) be such that \( G \) is strict (and hence orthocomplemented, by the induction hypothesis) in \( \ell^\infty \). Since \( D/G \) is strict in \( \ell^\infty/G \) (Proposition 2.7) and \( \ell^\infty/G \cong \ell^\infty \) (Theorem 3.6) it follows by the induction hypothesis that \( D/G \) is orthocomplemented in \( \ell^\infty/G \). Now the orthocomplementation of \( D \) follows from Proposition 2.10.

4. **FINITE-(CO)DIMENSIONAL ORTHOCOMPLEMENTED SUBSPACES OF \( c_0 \)**

It is well known that every finite-dimensional subspace of \( c_0 \) is orthocomplemented (see [7]).

We now translate the results we have found in the above section about orthocomplemented finite-dimensional subspaces of \( \ell^\infty \) into statements about finite-codimensional subspaces of \( c_0 \). The next lemma, which is a direct consequence of Propositions 2.5 and 2.6, contains the key to do that.
Lemma 4.1. Let $D$ be a closed subspace of $c_0$ (resp. a weakly closed subspace of $\ell^\infty$). Then,

$D \begin{cases} 
  \text{is orthocomplemented} \\
  \text{is strict} \\
  \text{has the HB-property}
\end{cases} \text{ in } c_0 \text{ (resp. in } \ell^\infty) \quad \iff 
D^0 \begin{cases} 
  \text{is orthocomplemented} \\
  \text{has the HB-property} \\
  \text{is strict}
\end{cases} \text{ in } \ell^\infty \text{ (resp. in } c_0),

(observe that every weakly closed subspace of $\ell^\infty$ has the HB$^+$-property, [5], Theorem 2.3).

Theorem 3.3 admits the following "dual":

**Theorem 4.2.** Let $S$ be a closed subspace of $c_0$ with finite codimension. Then the following properties are equivalent

i) $S$ is orthocomplemented in $c_0$.

ii) Every hyperplane containing $S$ is orthocomplemented (strict) in $c_0$.

iii) If $f \in c_0$ and $f = 0$ on $S$, then $\|f\| = \max_{n} |f(e_n)|$ (where $e_1, e_2, \ldots$ is the canonical base of $c_0$).

Analogously, Theorem 3.4 converts into the following result for closed hyperplanes of $c_0$.

**Theorem 4.3.** A closed hyperplane in $c_0$ has the HB-property in $c_0$ iff it is orthocomplemented in $c_0$.

In the same line, from Corollary 3.7 we deduce

**Corollary 4.4.** Every closed subspace of $c_0$ with the HB-property in $c_0$, is strict in $c_0$.

Finally, Problems 1-3 of the previous section give rise to the following equivalent questions.

Let $S$ be a closed subspace of $c_0$ that has finite codimension and the HB-property in $c_0$.

**Problem I.** Is $S$ orthocomplemented in $c_0$?

**Problem II.** Is there any closed hyperplane $H$ in $c_0$ with $H \supset S$ such that $H$ has the HB-property (is orthocomplemented) in $c_0$?
Problem III. If $2 \leq \text{codim } S$, is there a closed subspace $T$ of $c_0$ with $S \nsubseteq T \nsubseteq c_0$ such that $T$ has the HB-property (is orthocomplemented) in $c_0$?

5. Some Consequences and Remarks

Next we shall apply the results proved in the previous sections to study orthocomplementation for arbitrary closed subspaces of $\ell^\infty$ and $c_0$.

Theorem 5.1. Let $D$ be a closed subspace of $\ell^\infty$. Then the following are equivalent.

i) $D$ is orthocomplemented in $\ell^\infty$.

ii) $D \simeq K^n$ for some $n \in \mathbb{N}$ or $D \simeq \ell^\infty$ and $D$ is strict (has the HB-property) in $\ell^\infty$.

iii) $D$ is weakly closed and strict (has the HB-property) in $\ell^\infty$ and $D'$ has an orthogonal base.

iv) $D$ is weakly closed and for every closed subspace $F$ of $D$ with $\dim D/F < \infty$, $D/F$ is orthocomplemented in $\ell^\infty/F$.

v) $D$ is strict and there exists a closed subspace $F$ of $D$ with $\dim D/F = 1$ such that $F$ is orthocomplemented in $\ell^\infty$.

vi) There exists a closed subspace $F$ of $D$ with $\dim D/F = 1$ such that $F$ is orthocomplemented in $\ell^\infty$ and $D/F$ is orthocomplemented (strict) in $\ell^\infty/F$.

Proof. i)$\Rightarrow$ii): Clearly $D$ is strict and weakly closed. By Corollary 3.7, $D$ has the HB-property in $\ell^\infty$.

Also, $D'$ is isometrically isomorphic to a closed subspace of $c_0$ and so $D' \simeq K^n$ (for some $n \in \mathbb{N}$) or $D' \simeq c_0$. Since $D$ is reflexive ([5], Lemma 2.2) we derive that $D \simeq K^n$ or $D \simeq \ell^\infty$.

ii)$\Rightarrow$iii): Follows from Theorem 2.3 of [5] and Corollary 3.7.

iii)$\Rightarrow$i): By reflexivity of $D$ ([5], Lemma 2.2), $D \simeq \ell^\infty(I;s)$ for some set $I$ and some $s : I \to (0, +\infty)$. Now, apply Proposition 2.1.

i)$\Rightarrow$iv): Follows from Proposition 2.7.

iv)$\Rightarrow$iii): By Proposition 2.8, $D$ has the HB-property in $\ell^\infty$.

On the other hand, since $D' \simeq c_0/D^0$ is of countable type, it is enough to see that every finite-dimensional subspace $G$ of $c_0/D^0$ has an orthogonal base. Let $\pi_0 : c_0 \to c_0/D^0$ be the canonical surjection. There is a finite-dimensional subspace $M$ of $c_0$ with $\pi_0(M) = G$. Since $D^0 + M$ is weakly closed in $c_0$ ([7], Lemma 3.14 and [3], Theorem 4.7), there exists a weakly closed subspace $S$ of $\ell^\infty$ such that $D^0 + M = S^0$.

By assumption and Proposition 2.7 we conclude that $D^0$ is orthocomplemented in $S^0$ (observe that $(\ell^\infty/S)' \simeq S^0$ and under this isometry $(D/S)^0$ maps onto $D^0$). Then, there is a closed subspace $M_1$ of $c_0$ which is an orthogonal complement of $D^0$ in $S^0$. In
particular, \( D^0 + M = D^0 + M_1 \). So, \( \pi_0(M_1) = G \). But \( M_1 \), being a subspace of \( c_0 \), has an orthogonal base. Hence, so has \( G \).

i)\(\Rightarrow\)v): Clearly \( D \) is strict in \( \ell^\infty \).

Now, let \( F \) be a closed subspace of \( D \) with \( \dim D/F = 1 \). By i)\(\Rightarrow\)ii) and Proposition 3.1 it follows that \( F \) is orthocomplemented in \( D \) (and hence in \( \ell^\infty \)).

v)\(\Rightarrow\)vi): Let \( F \) be a closed subspace of \( D \) with \( \dim D/F = 1 \). By strictness of \( D \) and Proposition 2.7 it follows that \( D/F \) is strict in \( \ell^\infty/F \). Since \( F \) is weakly closed in \( \ell^\infty \), we can apply Theorem 3.4 and Theorem 3.6 i)\(\Rightarrow\)ii) to conclude that \( D/F \) is orthocomplemented in \( \ell^\infty/F \).

vi)\(\Rightarrow\)i): Follows by Proposition 2.10.

Recall that an absolutely convex set \( A \) of a locally convex space over \( K \) is called:

a) \( c' \)-compact: if for each neighbourhood \( U \) of 0 there exists a finite set \( B \subseteq A \) such that \( A \subseteq U + \text{co}B \) (where \( \text{co}B \) is the absolutely convex hull of \( B \)).

b) \( KM' \)-compactoid: if it is complete and there exists a compact set \( X \subseteq A \) such that \( A \) is the closed absolutely convex hull of \( X \) (for the general properties of such sets see [4]).

By using Proposition 2.3 of [2] and a proof similar to the one given for (d) \(\iff\) (i) in Theorem 2.3 of [5], it is not difficult to obtain the following.

**Theorem 5.2.** Let \( D \) be a closed subspace of \( \ell^\infty \). Then, properties i) - vi) of Theorem 5.1 are equivalent to

- vii) \( D \) is strict (has the HB-property) in \( \ell^\infty \) and \( B_D = \{ x \in D : \|x\| \leq 1 \} \) is weakly \( KM' \)-compactoid in \( \ell^\infty \).

- viii) \( D \) is strict (has the HB-property) in \( \ell^\infty \) and \( B_D \) is weakly closed and weakly \( c' \)-compact in \( \ell^\infty \).

As in section 4, we can now dualize Theorems 5.1 and 5.2 to describe the orthocomplemented subspaces of \( c_0 \).

Observe that as a direct consequence of Propositions 2.7 and 2.8, we have

**Lemma 5.3.** Let \( D \) be a weakly closed subspace of \( \ell^\infty \) and let \( F \) be a closed subspace of \( D \) with \( \dim D/F < \infty \) (so, \( F \) is weakly closed, Theorem 3.6). Then, \( D/F \) is orthocomplemented (resp. is strict, has the HB-property) in \( \ell^\infty/F \) iff \( D^0 \) is orthocomplemented (resp. has the HB-property, is strict) in \( F^0 \).

Then, putting together Lemmas 4.1 and 5.3 we have that Theorems 5.1 and 5.2 convert into the following descriptions of the orthocomplemented subspaces of \( c_0 \).

**Theorem 5.4.** For a closed subspace \( S \) of \( c_0 \) the following properties are equivalent.
i) $S$ is orthocomplemented in $c_0$.

ii) $c_0/S \cong K^n$ for some $n \in \mathbb{N}$ or $c_0/S \cong c_0$ and $S$ has the HB-property (is strict) in $c_0$.

iii) $S$ has the HB-property (is strict) in $c_0$ and $c_0/S$ has an orthogonal base.

iv) $S$ is orthocomplemented in any closed subspace $T$ of $c_0$ with $T \supset S$ and $\dim T/S < \infty$.

v) $S$ has the HB-property in $c_0$ and there exists a closed subspace $T$ of $c_0$ with $T \supset S$ and $\dim T/S = 1$ such that $T$ is orthocomplemented in $c_0$.

vi) There exists a closed subspace $T$ of $c_0$ with $T \supset S$ and $\dim T/S = 1$ such that $S$ is orthocomplemented in $T$ and $T$ is orthocomplemented in $c_0$.

vii) $S$ has the HB-property (is strict) in $c_0$ and $B(c_0/S)'$ is weakly-$*$ KM-compactoid in $(c_0/S)'$.

viii) $S$ has the HB-property (is strict) in $c_0$ and $B(c_0/S)'$ is weakly-$*$ $c'$-compact in $(c_0/S)'$.

**Remarks 5.5.**

1. There is a closed subspace $D$ of $\ell^\infty$ with $D \cong \ell^\infty$ (and hence $D$ is weakly closed [5], Theorem 2.3) such that $D$ is not orthocomplemented in $\ell^\infty$.

**Example:** Choose $\lambda_1, \lambda_2, \ldots$ in $K$ with $0 < |\lambda_1| < |\lambda_2| < \ldots \uparrow 1$. There are $z_1, z_2, \ldots$ in $c_0$ with $|\lambda_1| \leq \|z_i\| < 1$ for all $i$ such that every $x \in c_0$ with $\|x\| < 1$ can be written as $x = \sum_{i=1}^{\infty} \mu_i z_i$ where $|\mu_i| \leq 1$ for all $i$ and $\mu_i \to 0$. Now, the map $T : c_0 \to c_0$ given by $T(\sum_{i=1}^{\infty} \lambda_i e_i) = \sum_{i=1}^{\infty} \lambda_i z_i$ is a continuous linear function mapping $\{x \in c_0 : \|x\| \leq 1\}$ onto $\{x \in c_0 : \|x\| < 1\}$: if $x \in c_0$ is such that $\|Tx\| = 1$, then $\|x\| > 1$. So $T$ (and hence $\ker T$) is not strict. Thus, $D = (\ker T)'^0$ satisfies the required conditions (Lemma 4.1).

2. There exists a closed subspace $D$ of $\ell^\infty$ such that $D \cong K$ (hence $D$ is weakly closed) and such that $D$ is not orthocomplemented in $\ell^\infty$.

**Example:** We know (Remark 2.3.3) that there exists a linear isometry $i$ from $K^2_\ell$ into $\ell^\infty$ (Recall that $K^2_\ell \cong (K^2_\ell)'$). Since $K^2_\ell$ does not contain non-trivially mutually orthogonal elements, we derive that every one-dimensional subspace $D$ of $K^2_\ell$ satisfies our requirements.

3. There exists a closed subspace $D$ of $\ell^\infty$ with the HB-property in $\ell^\infty$ such that $D'$ has an orthogonal base but $D$ is not orthocomplemented in $\ell^\infty$.

**Example:** Take for $D$ the closed subspace of $\ell^\infty$ constructed in [7], 4.J (observe that since $D$ is not reflexive, it is not orthocomplemented in $\ell^\infty$).

4. Looking at Theorem 5.1 and the above Remark the following question arises in a natural way.
Problem. Can we without harm remove the weak closedness of $D$ in property iii) (when $D$ is strict) or in property iv) of Theorem 5.1?

5. There is a weakly closed subspace $D$ of $\ell^\infty$ such that $D^\prime$ has an orthogonal base but $D$ is not orthocomplemented in $\ell^\infty$.

Example: Take $D = H^0$, where $H$ is a closed hyperplane of $c_0$ which is not orthocomplemented in $c_0$ and apply Lemma 4.1.

6. There is a finite-dimensional (and hence weakly closed) subspace $D$ of $\ell^\infty$ such that $D$ has the HB-property in $\ell^\infty$ but is not orthocomplemented in $\ell^\infty$.

Example: See Remark 2.3.3.

7. Finally observe that Problems 1-3 appearing in Remark 3.8 are equivalent to

Problem 4. Let $D$ be a weakly closed subspace of $\ell^\infty$ such that $D$ is strict and has the HB-property in $\ell^\infty$. Does it follow that $D$ is orthocomplemented in $\ell^\infty$?

Indeed, clearly if Problem 4 has an affirmative answer then so has Problem 1 (recall that every finite-dimensional and strict subspace of $\ell^\infty$ has the HB-property in $\ell^\infty$, Corollary 3.7).

Conversely, assume Problem 1 has an affirmative answer and let $D$ be a weakly closed subspace of $\ell^\infty$ such that $D$ is strict. Let $F$ be a closed subspace of $D$ with $\dim D/F < \infty$. By Theorem 5.1 i) $\iff$ iv) it is enough to prove that $D/F$ is orthocomplemented in $\ell^\infty/F$. For that observe that it follows from Proposition 2.7 that $D/F$ is a one-dimensional and strict subspace of $\ell^\infty/F$. But $F$ is weakly closed in $\ell^\infty$ and so $\ell^\infty/F \simeq K^n$ (for some $n$) or $\ell^\infty/F \simeq \ell^\infty$ (Theorem 3.6). By assumption $D/F$ is orthocomplemented in $\ell^\infty/F$ and we are done.

REFERENCES


