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NON-ARCHIMEDEAN t-FRAMES AND FM-SPACES

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ABSTRACT. We generalize the notion of t-orthogonality in p-adic Banach spaces by introducing t-frames (§2). This we use to prove that a Fréchet-Montel (FM-)space is of countable type (Theorem 3.1), the non-archimedean counterpart of a well known theorem in functional analysis over $\mathbb{R}$ or $\mathbb{C}$ ([6], p. 231). We obtain several characterizations of FM-spaces (Theorem 3.3) and characterize the nuclear spaces among them (§4).

1. **Preliminaries.** Throughout this paper K is a non-archimedean non-trivially valued complete field with valuation $|\,.\,|$. For the basic notions and properties concerning normed and locally convex spaces over K we refer to [11] and [7]. However we recall the following.

1. Let E be a K-vector space. Let $X \subset E$. The absolutely convex hull of X is denoted by $\mathrm{co}\, X$, its linear hull by $[X]$. For a (non-archimedean) seminorm p on E we denote by E_p the vector space $E/\operatorname{Ker} p$ and by $\pi_p: E \longrightarrow E_p$ the canonical surjection. The formula $\|\pi_p(x)\| = p(x)$ defines a norm on E_p.

2. Let $(E, \|\cdot\|)$ be a normed space over K. For $r > 0$ we write $B(0,r) := \{x \in E : \|x\| \leq r\}$. Let $a \in E$, $X \subset E$. Then $\operatorname{dist}(a,X) := \inf\{\|a-x\| : x \in X\}$. For $n \in N$ and $x_1,\ldots,x_n \in E$ we consider $\operatorname{Vol}(x_1,\ldots,x_n) := \|x_1\| \cdot \operatorname{dist}(x_2,[x_1]) \cdot \operatorname{dist}(x_3,[x_1,x_2]) \cdots \operatorname{dist}(x_n,[x_1,\ldots,x_{n-1}])$. For properties of this Volume Function (in particular, its symmetry), we refer to [10]. A linear continuous map $E \longrightarrow F$, where F is a normed space, is said to be *compact* if it sends the unit ball of E into a compactoid set (see below).

3. Now let E be a Hausdorff locally convex space over K. A subset X of E is called *compactoid* if for every zero-neighbourhood U in E there exists a finite set S of E such that $X \subset \mathrm{co}\, S + U$. E is said to be of *countable type* if for each continuous seminorm p the normed space E_p is of countable type (Recall that a normed space is called *of countable type* if it is the closed linear hull of a countable set). E is called *nuclear* if for every continuous seminorm p on E there exists a continuous seminorm q on E with $p \leq q$, and such that Φ_{pq} is compact, where Φ_{pq} is the unique map making the diagram

$$\begin{array}{ccc} & E & \\ {}^{\pi_q}\swarrow & & \searrow{}^{\pi_p} \\ E_q & \xrightarrow[\Phi_{pq}]{} & E_p \end{array}$$

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commute. E is called *Montel* if it is polar, polarly barrelled and if each closed bounded subset is a complete compactoid. A Fréchet space which is Montel is called an FM-*space*.

The closure of a set $X \subset E$ is denoted by $\bar{X}$.

2. **t-frames in p-adic Banach spaces.** Throughout §2 E is a normed space over K. We introduce a concept which generalizes the notion of t-orthogonality and it allows us to prove one of the main Theorems in the paper (Theorem 3.1).

DEFINITION 2.1. Let $t \in (0, 1]$, and let $X \subset E$ be a subset not containing 0. We call X a t-frame if for every $n \in N$ and distinct $x_1, \dots, x_n \in X$ we have $\mathrm{Vol}(x_1, \dots, x_n) \geq t^{n-1} \cdot \|x_1\| \cdot \cdots \cdot \|x_n\|$.

We make the following simple observations. Let $t \in (0, 1]$.

1. *Any t-orthogonal set in E is a t-frame.* (Let $\{e_i : i \in I\}$ be a t-orthogonal set in E, let $i_1, \dots, i_n$ be n distinct elements of I. Then, by the definition of the Volume Function and by t-orthogonality,

$$\begin{aligned}\mathrm{Vol}(e_{i_1}, \dots, e_{1_n}) &= \|e_{i_1}\| \cdot \mathrm{dist}(e_{i_2}, [e_{i_1}]) \cdot \cdots \cdot \mathrm{dist}(e_{i_n}, [e_{i_1}, \dots, e_{i_{n-1}}]) \\ &\geq \|e_{i_1}\| \cdot t \cdot \|e_{i_2}\| \cdot \cdots \cdot t \cdot \|e_{i_n}\| = t^{n-1} \cdot \|e_{i_1}\| \cdots \|e_{i_n}\|).\end{aligned}$$ ∎

2. *Every t-frame in E is a linearly independent set.*
3. *Every subset of a t-frame is itself a t-frame.*
4. *Every t-frame in E can be extended to a maximal t-frame.*

By a t-frame sequence we shall mean a sequence $x_1, x_2, \dots$ in E such that $\{x_1, x_2, \dots\}$ is a t-frame.

PROPOSITION 2.2 (COMPARE [8], THEOREM 2). *A bounded subset X of E is a compactoid if and only if for every $t \in (0, 1]$ every t-frame sequence in X tends to 0.*

PROOF. Suppose X is a compactoid. Suppose, for some $t \in (0, 1]$, and some $\alpha > 0$, X contains a t-frame sequence $x_1, x_2, \dots$ for which $\|x_n\| \geq \alpha$ for all n. Then, for each $n \in N$,

$$\mathrm{Vol}(x_1, \dots, x_n) \geq t^{n-1} \cdot \|x_1\| \cdot \cdots \cdot \|x_n\| \geq \alpha^n t^{n-1}$$

implying $\lim_{n\to\infty} \inf \sqrt[n]{\mathrm{Vol}(x_1, \dots, x_n)} \geq \alpha t > 0$ conflicting the compactoidity of X ([8], §2). This proves one half of the statement. The other half is obvious. ∎

The following two Propositions are crucial for Theorem 2.5.

PROPOSITION 2.3. *Let $0 < t < 1$; let X be a maximal t-frame in E. Then $\overline{[X]} = E$.*

PROOF. Let $D := \overline{[X]}$. If $D \neq E$ then we can find a nonzero $a \in E$ with $\mathrm{dist}(a, D) \geq t \cdot \|a\|$ ([11], Lemma 3.14, here we use that $t \neq 1$). So we shall prove that $\mathrm{dist}(a, D) < t \cdot \|a\|$ for every $a \in E - D$. By maximality $\{a\} \cup X$ is no longer a t-frame, yielding the existence of a $k \in N$ and distinct $x_1, \dots, x_k \in X$ such that

$$\mathrm{Vol}(a, x_1, \dots, x_k) < t^k \cdot \|a\| \cdot \|x_1\| \cdot \cdots \cdot \|x_k\|.$$

On the other hand we have

$$\begin{aligned}\mathrm{Vol}(a,x_1,\ldots,x_k) &= \mathrm{dist}(a,[x_1,\ldots,x_k])\cdot \mathrm{Vol}(x_1,\ldots,x_k)\\ &\geq \mathrm{dist}(a,D)\cdot t^{k-1}\cdot \|x_1\|\cdots\|x_k\|.\end{aligned}$$

So $\mathrm{dist}(a,D) < t\cdot\|a\|$. ∎

REMARK. We now can easily find examples of t-frames X that are s-orthogonal for no $s\in(0,1]$: Let $0<t<1$, let E have no base, choose for X a maximal t-frame (Observe that the clause $t\neq 1$ is essential!).

PROPOSITION 2.4. *Every uncountable subset of c_0 contains an infinite compactoid.*

PROOF. Let X be an uncountable subset of c_0; it has a bounded uncountable subset Y. Let $e_1,e_2,\ldots$ be the standard basis of c_0. We have $B(0,1)+[e_1,e_2,\ldots]=c_0$ so there exists an $n_1\in N$ such that

$$Y_1 := Y\cap\left(B(0,1)+[e_1,e_2,\ldots e_{n_1}]\right)$$

is uncountable. In its turn, there exists an $n_2\in N$ such that

$$Y_2 := Y_1\cap\left(B(0,1/2)+[e_1,e_2,\ldots,e_{n_2}]\right)$$

is uncountable. We obtain uncountable sets $Y_1\supset Y_2\supset\cdots$ such that $Y_n\subset B(0,1/n)+D_n$ for each n where D_n is a finite-dimensional space. Choose distinct $x_1,x_2,\ldots$ where $x_n\in Y_n$ for each n, and set $Z:=\{x_1,x_2,\ldots\}$. Then Z is infinite, bounded, in X. Also, for each $n\in N$ we have

$$Z\subset\{x_1,\ldots,x_{n-1}\}\cup Y_n\subset[x_1,\ldots,x_{n-1}]+B(0,1/n)+D_n\subset B(0,1/n)+\hat{D}_n$$

where $\hat{D}_n$ is a finite-dimensional space. It follows that Z is a compactoid. ∎

THEOREM 2.5. *The following assertions about the normed space E are equivalent.*

(i) E is of countable type.

(ii) For every $t\in(0,1)$, every t-frame in E is countable.

(iii) For some $t\in(0,1)$, every t-frame in E is countable.

PROOF. (i) $\Rightarrow$ (ii). We may assume $E=c_0$. Let X be a t-frame in E. For each $n\in N$ set $X_n:=\{x\in X:\|x\|\geq 1/n\}$. If, for some n, X_n were uncountable it would contain an infinite compactoid $\{x_1,x_2,\ldots\}$ by Proposition 2.4. Then from Proposition 2.2 $\lim_{k\to\infty}x_k=0$, a contradiction.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i). Let X be a maximal t-frame in E. By assumption X is countable. By Proposition 2.3, $E=\overline{[X]}$ is of countable type. ∎

REMARK. The question if Theorem 2.5 remains true when we consider in (i) and (ii) t-orthogonal sets instead t-frames is an open problem in non-archimedean analysis ([11], p. 199).

3. **Characterizations of FM-spaces among F-spaces.** *From now on in this paper E is a polar Hausdorff locally convex space over K.*

It is proved in [6], Theorem 11.6.2, that a Fréchet Montel space over $\mathbb{R}$ or $\mathbb{C}$ is separable. It does not simply carry over the non-archimedean case because K may be not locally compact; so we have to deal with compactoids (§1.3) rather than compact sets. This modification is obstructing the classical proof which is essentially based upon separability. It is here where the t-frames of §2 come to the rescue as will be demonstrated in the following theorem (for other applications of t-frames in p-adic analysis, see [9], p. 51–57).

THEOREM 3.1. *An* FM*-space is of countable type.*

PROOF. Let the topology of the FM-space E be defined by the sequence of seminorms $p_1 \leq p_2 \leq \cdots$. Set $U_n = \{x \in E : p_n(x) \leq 1\}$. Choose $\lambda \in K$, $|\lambda| > 1$.

It suffices to show that $E_1 := E_{p_1}$ is of countable type. Let X be a t-frame in $(E_1, \|\cdot\|_1)$ for some $t \in (0,1)$; we show (Theorem 2.5) that X is countable. Suppose not. We may assume that $\inf\{\|x\|_1 : x \in X\} > 0$. Choose an $A_1 \subset E$ such that $\pi_{p_1}(A_1) = X$. Since $E = \bigcup_n \lambda^n U_2$ there exists an n_2 such that $A_2 := A_1 \cap \lambda^{n_2} U_2$ is uncountable. Inductively we arrive at uncountable sets $A_1 \supset A_2 \supset \cdots$ such that A_n is p_n-bounded for each $n \geq 2$. Choose distinct $a_1, a_2, \ldots$ with $a_n \in A_n$ for each n. Then $\{a_1, a_2, \ldots\}$ is bounded in E. As E is Montel, it is a compactoid. By Proposition 2.2, $\lim_{n\to\infty} \pi_{p_1}(a_n) = 0$ conflicting $\inf\{\|x\|_1 : x \in X\} > 0$. ∎

LEMMA 3.2. *Every bounded subset B of a Fréchet space E, is compactoid for the topology of uniform convergence on the $\beta(E',E)$-compactoid subsets of E' (where $\beta(E',E)$ denotes the strong topology on E' with respect to the dual pair $\langle E, E'\rangle$).*

PROOF. Consider the canonical map $J_E\colon E \to E'' = \big(E', \beta(E',E)\big)'$. It is easy to see that the set $J_E(B)$ is equicontinuous on $\big(E', \beta(E',E)\big)$. By [7] Lemma 10.6 we have that on $J_E(B)$ the topology $\tau_{\beta c}$ (on E'') of the uniform convergence on the $\beta(E',E)$-compactoid subsets of E', coincides with the weak topology $\sigma(E'', E')$. Hence $J_E(B)$ is $\tau_{\beta c}$-compactoid in E''. Since J_E is an homeomorphism from E onto a subspace of E'' ([7], Lemmas 9.2, 9.3) we are done. ∎

THEOREM 3.3. *For a Fréchet space E, the following properties are equivalent.*

(i) *E is an* FM*-space.*

(ii) *Every bounded subset of E is compactoid.*

(iii) *In E every weakly convergent sequence is convergent and $\big(E', \beta(E',E)\big)$ is of countable type.*

(iv) *In E' every $\sigma(E',E)$-convergent sequence is $\beta(E',E)$-convergent and E is of countable type.*

(v) *Both E and $\big(E', \beta(E',E)\big)$ are of countable type.*

(vi) *$\big(E', \beta(E',E)\big)$ is nuclear.*

(vii) *$\big(E', \beta(E',E)\big)$ is Montel.*

(viii) *Every $\sigma(E', E)$-bounded subset of E' is $\beta(E', E)$-compactoid.*

PROOF. The implications (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii), (i) $\Rightarrow$ (vi) $\Rightarrow$ (viii) and (i) $\Rightarrow$ (vii) $\Rightarrow$ (viii) are known (see [7]) or easy. Also, from Theorem 3.1 we can easily prove (i) $\Rightarrow$ (iv) and (i) $\Rightarrow$ (v).

Now we prove (viii) $\Rightarrow$ (ii): Since E is a polar Fréchet space, its topology τ is the topology of uniform convergence on the $\sigma(E', E)$-bounded subsets of E'. By (viii) these subsets are $\beta(E', E)$-compactoid. Now apply Lemma 3.2.

The implication (v) $\Rightarrow$ (iii) follows from [7] Proposition 4.11.

Finally, for the proof of (iv) $\Rightarrow$ (ii) observe that the topology on a polar Fréchet space of countable type is the topology of uniform convergence on the $\sigma(E', E)$-null sequences in E' (see [4], Theorem 3.2). By (iv) these sequences are $\beta(E', E)$-convergent. Now apply Lemma 3.2. ∎

REMARK. It is known that a Fréchet space E over $\mathbb{R}$ over $\mathbb{C}$ is nuclear if and only if $\big(E', \beta(E', E)\big)$ is nuclear ([6], p. 491).

In the non-archimedean case the situation is essentially different. Indeed, in 4.1 we will give an example of an FM-space which is not nuclear (while its strong dual is by (i) $\Leftrightarrow$ (vi)). To do that we need some preliminary concepts and results.

DEFINITION 3.4. Let $A = (a_i^k)$ be a matrix of strictly positive real numbers such that $a_i^{k+1} > a_i^k$ for all i and all k. Then the corresponding Köthe sequence space $K(A)$ is defined by

$$K(A) = \{\alpha = (\alpha_i) : \lim_i |\alpha_i| \cdot a_i^k = 0 \text{ for all } k\}.$$

On $K(A)$ we consider the sequence of norms (p_k), where

$$p_k(\alpha) = \max_i |\alpha_i| \cdot a_i^k, \quad k = 1, 2, \ldots; \quad \alpha \in K(A).$$

It is known that $K(A)$ is a polar Fréchet space of countable type. For the importance of this class of spaces and for their further properties we refer to [3].

We then have:

PROPOSITION 3.5. *Let $\Lambda = K(A)$ be a Köthe space and let Λ^* the corresponding Köthe dual space. Then the following properties are equivalent:*

(i) *Λ is an* FM-*space.*

(ii) *$\big(\Lambda^*, \beta(\Lambda^*, \Lambda)\big)$ is of countable type.*

(iii) *$\big(\Lambda^*, \beta(\Lambda^*, \Lambda)\big)$ is nuclear.*

(iv) *$\big(\Lambda^*, \beta(\Lambda^*, \Lambda)\big)$ is Montel.*

(v) *The unit vectors $e_1, e_2, \ldots$ form a Schauder basis for $\Lambda^*, \beta(\Lambda^*, \Lambda)$.*

(vi) *$n(\Lambda^*, \Lambda) = \beta(\Lambda^*, \Lambda)$ (where $n(\Lambda^*, \Lambda)$ is the natural topology on Λ^*).*

(vii) *No subspace of Λ is isomorphic (linearly homeomorphic) to c_0.*

(viii) *The sequence of coordinate projections (P_i), where $P_i : \Lambda \longrightarrow \Lambda : \alpha = (\alpha_i) \longrightarrow \alpha_i e_i$, converges to the zero-map uniformly on every bounded subset of Λ.*

(ix) *The sequence of sections-maps* (S_n), *where* $S_n: \Lambda \longrightarrow \Lambda : \alpha = (\alpha_i) \longrightarrow (\alpha_1, \alpha_2, \ldots, \alpha_n, 0, 0, \ldots)$ *converges to the identity map* Id *uniformly on every bounded subset of* Λ.

PROOF. We only have to prove (i) $\Rightarrow$ (v) $\Rightarrow$ (vi), (vii) $\Rightarrow$ (viii) and (ix) $\Rightarrow$ (i). The other implications are easy.

(i) $\Rightarrow$ (v): The unit vectors $e_1, e_2, \ldots$ form a Schauder basis for $(\Lambda^*, \sigma(\Lambda^*, \Lambda))$. Then, apply (i) $\Rightarrow$ (iv) in 3.3.

(v) $\Rightarrow$ (vi): By [4], p. 21 it suffices to prove that $\beta(\Lambda^*, \Lambda)$ is compatible with the duality (Λ^*, Λ) and this is done as in [1], Proposition 20.

(vii) $\Rightarrow$ (viii): Suppose Λ contains a bounded subset D on which (P_i) does not converge uniformly to the zero-map. We show that Λ contains a subspace isomorphic to c_0.

From the assumption it follows that there exist $\varepsilon > 0$, $k \in N$ and an increasing sequence of indices (i_n) such that, for all n, there exists $\alpha^n = (\alpha_i^n) \in D$ with $|\alpha_{i_n}^n| \cdot a_{i_n}^k > \varepsilon$, $n = 1, 2, \ldots$. We put $z_{i_n} = \alpha_{i_n}^n \cdot e_{i_n}$, $n = 1, 2, \ldots$. Then, the sequence (z_{i_n}) is bounded in Λ.

Now we can define a linear map

$$T: c_0 \longrightarrow \Lambda : \sigma = (\sigma_n) \longrightarrow \sum_n \sigma_n z_{i_n}.$$

We prove that T is an isomorphism from c_0 into Λ. It is easy to see that T is injective and continuous. Also, $T: c_0 \longrightarrow \operatorname{Im} T$ is open.

Indeed, for $\sigma = (\sigma_n) \in c_0$, we have $p_k\big(T(\sigma)\big) = \max_{n=1}^{\infty} |\sigma_n \alpha_{i_n}^n| \cdot a_{i_n}^k \geq \varepsilon \cdot \|\sigma\|_{c_0}$.

(ix) $\Rightarrow$ (i): We prove that Id: $\Lambda \longrightarrow \Lambda$ transforms bounded subsets into compactoid subsets. Observe that (ix) means that $\lim_n S_n = \text{Id}$ in $L_{cl}(\Lambda, \Lambda)$. Then apply Proposition 4 in [2]. ∎

The next corollary is for later use.

COROLLARY 3.6. *If for every* $k \in N$ *and every subsequence* (i_n) *of the indices there exists* $h > k$ *such that the sequence* $(a_{i_n}^h / a_{i_n}^k)_n$ *is bounded, then* $K(A)$ *is an* FM-*space.*

PROOF. An analysis of the proof of (vii) $\Rightarrow$ (viii) shows that if $K(A)$ is not an FM-space, there exist a subsequence of the indices (i_n) and elements η_{i_n} in K, $n = 1, 2, \ldots$ such that the linear map $T: c_0 \longrightarrow \operatorname{Im} T : (\sigma_n) \longrightarrow (\sigma_n \eta_{i_n})$ is an isomorphism of c_0 into Λ.

Consider now in c_0 the subspace c_{00} generated by the unit vectors $e_1, e_2, \ldots$. Then c_{00} is isomorphic to the subspace F of $K(A)$ generated by $e_{i_1}, e_{i_2}, \ldots$. Therefore the topology induced by $K(A)$ on F is normable. This means that there exists k such that for all $h > k$ there exists $t_h > 0$ with $p_h(\delta) \leq t_h \cdot p_k(\delta)$ for all $\delta \in K(A)$. In particular, for $\delta = e_{i_n}$,

$n = 1, 2, \ldots$, we have that there is a k such that for all $h > k$, there exists $t_h > 0$ with $a^h_{i_n} \leq t_h \cdot a^k_{i_n}$ for all n, and we are done. ∎

4. **Characterizations of nuclear spaces among FM-spaces.** We start this section with the construction of an FM-space which is not nuclear.

EXAMPLE 4.1. For $k = 1, 2, \ldots$, consider the infinite matrix

$$A^k = (a^k_{ij}) = \begin{pmatrix} 1^k & \cdots & 2^k & \cdots & j^k & \cdots \\ 1^k & \cdots & 2^k & \cdots & j^k & \cdots \\ \vdots & & \vdots & & & \cdots \\ (k+1)^k & \cdots & (k+1)^k & \cdots & (k+1)^k & \cdots \\ (k+2)^k & \cdots & (k+2)^k & \cdots & (k+2)^k & \cdots \\ \vdots & & \vdots & & & \cdots \end{pmatrix} \begin{array}{l} \\ \\ \\ \to (k+1) \\ \\ \\ \end{array}$$

We can think of A^k as a sequence for some order, $k = 1, 2, \ldots$ (we fix the same order for all k). We then consider the Köthe space

$$K(A) = \{\beta = (\beta_{ij}) : \lim_{i,j} |\beta_{ij}| \cdot a^k_{ij} = 0, k = 1, 2, \ldots\}$$

equipped with the sequence of norms (p_k) where $p_k(\beta) = \max_{i,j} |\beta_{ij}| \cdot a^k_{ij}$.

We first show that $K(A)$ is not nuclear. If $k > 1$, then the sequence (a^1_{ij}/a^k_{ij}) contains a constant sequence. Then by [3] Proposition 3.5 the conclusion follows.

We now apply Corollary 3.6 in order to prove that $K(A)$ is an FM-space.

Choose k and any subsequence of the indices $(i_n, j_m)_{n,m}$. We consider the corresponding elements $a^k_{i_n j_m}$ of A^k. There are several possibilities.

a) The subsequence $(a^k_{i_n j_m})_{n,m}$ contains an infinite number of elements of some row of A^k.

If this row is between the rows $1, \ldots, k$, take $h = k + 1$. Then the sequence of the quotients $(a^h_{i_n j_m}/a^k_{i_n j_m})_{n,m}$ is unbounded.

If this row is the $(k + r)$-th row for some $r \geq 1$, then take $h = k + r$.

b) The subsequence $(a^k_{i_n j_m})_{n,m}$ consists of finitely many elements of an infinite number of rows. Consider then a subsequence with one element in an infinite number of rows below the kth row. Such a subsequence looks like

$$(k + l_1)^k, (k + l_2)^k, (k + l_3)^k, \ldots$$

with $(l_n)_n$ increasing to infinity. Take now $h = k + 1$. ∎

Finally we investigate what the situation exactly is.

DEFINITION 4.2. A locally convex space X is said to be *quasinormable* if for every zero-neighbourhood U in X there exists a zero-neighbourhood V in X, $V \subset U$, such that on U^o the topology $\beta(X', X)$ coincides with norm topology of X'_{V^o}.

DEFINITION 4.3. Let X be a locally convex space. A sequence $(a_n) \subset X'$ is said to be *locally convergent to zero* if there exists a zero-neighbourhood U in X such that $(a_n) \subset X'_{U^o}$ and $\lim_n \|a_n\|_{U^o} = 0$.

THEOREM 4.4. *For an* FM-*space* E *the following properties are equivalent.*

(i) E *is nuclear.*

(ii) E *is quasinormable.*

(iii) Every $\beta(E', E)$-*convergent sequence in* E' *is locally convergent.*

PROOF. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) follow by [2], Proposition 14 and [5], 5.2 respectively.

(iii) $\Rightarrow$ (i) Since E is of countable type (Theorem 3.1) its topology can be described by the $\sigma(E', E)$-null sequences on E' ([4], Theorem 3.2). By Theorem 3.3 (i) $\Rightarrow$ (iv) these sequences are null-sequences in $\beta(E', E)$ and by (iii) they are locally convergent to zero. The conclusion then follows from [5], 4.6.i). ∎

COROLLARY 4.5. *The Köthe space in 4.1 is also an example of an* FM-*space which is not quasinormable.*

REFERENCES

1. N. De Grande-De Kimpe, *Perfect locally K-convex sequence spaces*, Proc. Kon. Ned. Akad. v. Wet. A **74**(1971), 471–482.
2. ———, *On spaces of operators between locally K-convex spaces*, Proc. Kon. Ned. Akad. v. Wet. A **75**(1972), 113–129.
3. ———, *Non-archimedean Fréchet spaces generalizing spaces of analytic functions*, Proc. Kon. Ned. Akad. v. Wet. A **85**(1982), 423–439.
4. ———, *Non-archimedean topologies of countable type and associated operators*, Proc. Kon. Ned. Akad. v. Wet. A **90**(1987), 15–28.
5. ———, *Nuclear topologies on non-archimedean locally convex spaces*, Proc. Kon. Ned. Akad. v. Wet. A **90**(1987), 279–292.
6. H. Jarchow, *Locally convex spaces*, Teubner, Stuttgart, (1981).
7. W. H. Schikhof, *Locally convex spaces over non-spherically complete fields I-II*, Bull Soc. Math. Belgique, (B) **XXXVIII** (1986), 187–224.
8. ———, *P-adic nonconvex compactoids*, Proc. Kon Ned. Akad. v. Wet. A **92**(1989), 339–342.
9. W. H. Schikhof and A. C. M. van Rooij, *Seven papers on p-adic analysis*, Report 9125, Math. Inst. Katholieke Universiteit, Nijmegen, (1991).

10. A. C. M. Van Rooij, *Notes on p-adic Banach spaces*, Reports 7633 and 7725, Math. Inst. Katholieke Universiteit, Nijmegen (1976, 1977).

11. ———, *Non-archimedean Functional Analysis*, Marcel Dekker, New York, (1978).

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