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NON-ARCHIMEDEAN \( t \)-FRAMES AND FM-SPACES

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Abstract. We generalize the notion of \( t \)-orthogonality in \( p \)-adic Banach spaces by introducing \( t \)-frames \((\S 2)\). This we use to prove that a Fréchet-Montel (FM-)space is of countable type (Theorem 3.1), the non-archimedean counterpart of a well known theorem in functional analysis over \( \mathbb{R} \) or \( \mathbb{C} \) ([6], p. 231). We obtain several characterizations of FM-spaces (Theorem 3.3) and characterize the nuclear spaces among them \((\S 4)\).

1. Preliminaries. Throughout this paper \( K \) is a non-archimedean non-trivially valued complete field with valuation \( | \cdot | \). For the basic notions and properties concerning normed and locally convex spaces over \( K \) we refer to [11] and [7]. However we recall the following.

1. Let \( E \) be a \( K \)-vector space. Let \( X \subset E \). The absolutely convex hull of \( X \) is denoted by \( \text{co} \, X \), its linear hull by \([X]\). For a (non-archimedean) seminorm \( p \) on \( E \) we denote by \( E_p \) the vector space \( E/ \text{Ker} \, p \) and by \( \pi_p \colon E \to E_p \) the canonical surjection. The formula \( \|\pi_p(x)\| = p(x) \) defines a norm on \( E_p \).

2. Let \((E, \| \cdot \|)\) be a normed space over \( K \). For \( r > 0 \) we write \( B(0, r) := \{x \in E : \|x\| \leq r\} \). Let \( a \in E, X \subset E \). Then \( \text{dist}(a, X) := \inf \{\|a - x\| : x \in X\} \). For \( n \in \mathbb{N} \) and \( x_1, \ldots, x_n \in E \) we consider \( \text{Vol}(x_1, \ldots, x_n) := \|x_1\| \cdot \text{dist}(x_2, [x_1]) \cdot \text{dist}(x_3, [x_1, x_2]) \cdots \text{dist}(x_n, [x_1, \ldots, x_{n-1}]) \). For properties of this Volume Function (in particular, its symmetry), we refer to [10]. A linear continuous map \( E \to F \), where \( F \) is a normed space, is said to be compact if it sends the unit ball of \( E \) into a compactoid set (see below).

3. Now let \( E \) be a Hausdorff locally convex space over \( K \). A subset \( X \) of \( E \) is called compactoid if for every zero-neighbourhood \( U \) in \( E \) there exists a finite set \( S \) of \( E \) such that \( X \subset \text{co} \, S + U \). \( E \) is said to be of countable type if for each continuous seminorm \( p \) the normed space \( E_p \) is of countable type (Recall that a normed space is called of countable type if it is the closed linear hull of a countable set). \( E \) is called nuclear if for every continuous seminorm \( p \) on \( E \) there exists a continuous seminorm \( q \) on \( E \) with \( p \leq q \), and such that \( \Phi_{pq} \) is compact, where \( \Phi_{pq} \) is the unique map making the diagram

\[
\begin{array}{ccc}
E_P & \xrightarrow{\pi_p} & E_p \\
\downarrow \Phi_{pq} & & \downarrow \pi_p \\
E_Q & \xleftarrow{\pi_q} & E_q
\end{array}
\]

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commute. $E$ is called *Montel* if it is polar, polarly barrelled and if each closed bounded subset is a complete compactoid. A Fréchet space which is Montel is called an *FM-space*.

The closure of a set $X \subseteq E$ is denoted by $\overline{X}$.

2. **$t$-frames in $p$-adic Banach spaces.** Throughout §2 $E$ is a normed space over $K$. We introduce a concept which generalizes the notion of $t$-orthogonality and it allows us to prove one of the main Theorems in the paper (Theorem 3.1).

**Definition 2.1.** Let $t \in (0, 1]$, and let $X \subseteq E$ be a subset not containing 0. We call $X$ a $t$-frame if for every $n \in N$ and distinct $x_1, \ldots, x_n \in X$ we have

$$\mbox{Vol}(x_1, \ldots, x_n) \geq t^{n-1} \cdot \|x_1\| \cdot \cdots \cdot \|x_n\|.$$  

We make the following simple observations. Let $t \in (0, 1]$.

1. Any $t$-orthogonal set in $E$ is a $t$-frame. (Let $\{e_i : i \in I\}$ be a $t$-orthogonal set in $E$, let $i_1, \ldots, i_n$ be $n$ distinct elements of $I$. Then, by the definition of the Volume Function and by $t$-orthogonality,

$$\mbox{Vol}(e_1, \ldots, e_n) = \|e_1\| \cdot \mbox{dist}(e_2, [e_1]) \cdot \cdots \cdot \mbox{dist}(e_n, [e_1, \ldots, e_{n-1}]) \geq \|e_1\| \cdot t \cdot \|e_2\| \cdot \cdots \cdot t \cdot \|e_n\| = t^{n-1} \cdot \|e_1\| \cdots \cdot \|e_n\|.$$

2. Every $t$-frame in $E$ is a linearly independent set.
3. Every subset of a $t$-frame is itself a $t$-frame.
4. Every $t$-frame in $E$ can be extended to a maximal $t$-frame.

By a $t$-frame sequence we shall mean a sequence $x_1, x_2, \ldots \in E$ such that $\{x_1, x_2, \ldots\}$ is a $t$-frame.

**Proposition 2.2 (Compare [8], Theorem 2).** A bounded subset $X$ of $E$ is a compactoid if and only if for every $t \in (0, 1]$ every $t$-frame sequence in $X$ tends to 0.

**Proof.** Suppose $X$ is a compactoid. Suppose, for some $t \in (0, 1]$, and some $\alpha > 0$, $X$ contains a $t$-frame sequence $x_1, x_2, \ldots$ for which $\|x_n\| \geq \alpha$ for all $n$. Then, for each $n \in N$,

$$\mbox{Vol}(x_1, \ldots, x_n) \geq t^{n-1} \cdot \|x_1\| \cdot \cdots \cdot \|x_n\| \geq \alpha^n \cdot t^{n-1},$$

implying $\lim_{n \to \infty} \inf \sqrt[n]{\mbox{Vol}(x_1, \ldots, x_n)} \geq \alpha > 0$ conflicting the compactoidity of $X$ ([8], §2). This proves one half of the statement. The other half is obvious.

The following two Propositions are crucial for Theorem 2.5.

**Proposition 2.3.** Let $0 < t < 1$; let $X$ be a maximal $t$-frame in $E$. Then $\overline{\{x\}} = E$.

**Proof.** Let $D := \overline{\{x\}}$. If $D \neq E$ then we can find a nonzero $a \in E$ with $\mbox{dist}(a, D) \geq t \cdot \|a\|$ ([11], Lemma 3.14, here we use that $t \neq 1$). So we shall prove that $\mbox{dist}(a, D) < t \cdot \|a\|$ for every $a \in E - D$. By maximality $\{a\} \cup X$ is no longer a $t$-frame, yielding the existence of a $k \in N$ and distinct $x_1, \ldots, x_k \in X$ such that

$$\mbox{Vol}(a, x_1, \ldots, x_k) < t^k \cdot \|a\| \cdot \|x_1\| \cdot \cdots \cdot \|x_k\|.$$
On the other hand we have
\[ \text{Vol}(a, x_1, \ldots, x_k) = \text{dist}(a, [x_1, \ldots, x_k]) \cdot \text{Vol}(x_1, \ldots, x_k) \]
\[ \geq \text{dist}(a, D) \cdot t^{k-1} \cdot \|x_1\| \cdots \|x_k\|. \]
So \( \text{dist}(a, D) < t \cdot \|a\|. \)

**Remark.** We now can easily find examples of \( t \)-frames \( X \) that are \( s \)-orthogonal for no \( s \in (0, 1) \): Let \( 0 < t < 1 \), let \( E \) have no base, choose for \( X \) a maximal \( t \)-frame (Observe that the clause \( t \neq 1 \) is essential!).

**Proposition 2.4.** Every uncountable subset of \( c_0 \) contains an infinite compactoid.

**Proof.** Let \( X \) be an uncountable subset of \( c_0 \); it has a bounded uncountable subset \( Y \). Let \( e_1, e_2, \ldots \) be the standard basis of \( c_0 \). We have \( B(0, 1) + [e_1, e_2, \ldots] = c_0 \) so there exists an \( n_1 \in N \) such that
\[ Y_1 := Y \cap (B(0, 1) + [e_1, e_2, \ldots, e_{n_1}]) \]
is uncountable. In its turn, there exists an \( n_2 \in N \) such that
\[ Y_2 := Y_1 \cap (B(0, 1/2) + [e_1, e_2, \ldots, e_{n_2}]) \]
is uncountable. We obtain uncountable sets \( Y_1 \supset Y_2 \supset \cdots \) such that \( Y_n \subset B(0, 1/n) + D_n \) for each \( n \) where \( D_n \) is a finite-dimensional space. Choose distinct \( x_1, x_2, \ldots \) where \( x_n \in Y_n \) for each \( n \), and set \( Z := \{x_1, x_2, \ldots\} \). Then \( Z \) is infinite, bounded, in \( X \). Also, for each \( n \in N \) we have
\[ Z \subset \{x_1, \ldots, x_{n-1}\} \cup Y_n \subset [x_1, \ldots, x_{n-1}] + B(0, 1/n) + D_n \subset B(0, 1/n) + \hat{D}_n \]
where \( \hat{D}_n \) is a finite-dimensional space. It follows that \( Z \) is a compactoid.

**Theorem 2.5.** The following assertions about the normed space \( E \) are equivalent.
(i) \( E \) is of countable type.
(ii) For every \( t \in (0, 1) \), every \( t \)-frame in \( E \) is countable.
(iii) For some \( t \in (0, 1) \), every \( t \)-frame in \( E \) is countable.

**Proof.** (i) \( \Rightarrow \) (ii). We may assume \( E = c_0 \). Let \( X \) be a \( t \)-frame in \( E \). For each \( n \in N \) set \( X_n := \{x \in X : \|x\| \geq 1/n\} \). If, for some \( n \), \( X_n \) were uncountable it would contain an infinite compactoid \( \{x_1, x_2, \ldots\} \) by Proposition 2.4. Then from Proposition 2.2 \( \lim_{k \to \infty} x_k = 0 \), a contradiction.
(ii) \( \Rightarrow \) (iii) is obvious.
(iii) \( \Rightarrow \) (i). Let \( X \) be a maximal \( t \)-frame in \( E \). By assumption \( X \) is countable. By Proposition 2.3, \( E = [X] \) is of countable type.

**Remark.** The question if Theorem 2.5 remains true when we consider in (i) and (ii) \( t \)-orthogonal sets instead \( t \)-frames is an open problem in non-archimedean analysis ([11], p. 199).
3. Characterizations of FM-spaces among F-spaces. From now on in this paper $E$ is a polar Hausdorff locally convex space over $K$.

It is proved in [6], Theorem 11.6.2, that a Fréchet Montel space over $\mathbb{R}$ or $\mathbb{C}$ is separable. It does not simply carry over the non-archimedean case because $K$ may be not locally compact; so we have to deal with compactoids ($\S$1.3) rather than compact sets. This modification is obstructing the classical proof which is essentially based upon separability. It is here where the $t$-frames of $\S$2 come to the rescue as will be demonstrated in the following theorem (for other applications of $t$-frames in $p$-adic analysis, see [9], p. 51–57).

**Theorem 3.1.** An FM-space is of countable type.

**Proof.** Let the topology of the FM-space $E$ be defined by the sequence of seminorms $p_1 \leq p_2 \leq \cdots$. Set $U_\lambda = \{ x \in E : p_\lambda(x) \leq 1 \}$. Choose $\lambda \in K$, $|\lambda| > 1$.

It suffices to show that $E_\lambda := E_{n_\lambda}$ is of countable type. Let $X$ be a $t$-frame in $(E_\lambda, \| \cdot \|_1)$ for some $t \in (0, 1)$; we show (Theorem 2.5) that $X$ is countable. Suppose not. We may assume that $\inf \{ \| x \|_1 : x \in X \} > 0$. Choose an $A_1 \subset E$ such that $\pi_{p_1}(A_1) = X$. Since $E = \bigcup_\lambda \lambda^n U_2$ there exists an $n_2$ such that $A_2 := A_1 \cap \lambda^{n_2} U_2$ is uncountable. Inductively we arrive at uncountable sets $A_1 \supset A_2 \supset \cdots$ such that $A_n$ is $p_{n_\lambda}$-bounded for each $n \geq 2$. Choose distinct $a_1, a_2, \ldots$ with $a_n \in A_n$ for each $n$. Then $\{ a_1, a_2, \ldots \}$ is bounded in $E$. As $E$ is Montel, it is a compactoid. By Proposition 2.2, $\lim_{n \to \infty} \pi_{p_n}(a_n) = 0$ conflicting $\inf \{ \| x \|_1 : x \in X \} > 0$.

**Lemma 3.2.** Every bounded subset $B$ of a Fréchet space $E$, is compactoid for the topology of uniform convergence on the $\beta(E', E)$-compactoid subsets of $E'$ (where $\beta(E', E)$ denotes the strong topology on $E'$ with respect to the dual pair $(E, E')$).

**Proof.** Consider the canonical map $J_E : E \to E'' = (E', \beta(E', E))^\prime$. It is easy to see that the set $J_E(B)$ is equicontinuous on $(E', \beta(E', E))$. By [7] Lemma 10.6 we have that on $J_E(B)$ the topology $\tau_{lk}$ (on $E''$) of the uniform convergence on the $\beta(E', E)$-compactoid subsets of $E'$, coincides with the weak topology $\sigma(E'', E')$. Hence $J_E(B)$ is $\tau_{lk}$-compactoid in $E''$. Since $J_E$ is an homeomorphism from $E$ onto a subspace of $E''$ ([7], Lemmas 9.2, 9.3) we are done.

**Theorem 3.3.** For a Fréchet space $E$, the following properties are equivalent.

(i) $E$ is an FM-space.

(ii) Every bounded subset of $E$ is compactoid.

(iii) In $E$ every weakly convergent sequence is convergent and $(E', \beta(E', E))$ is of countable type.

(iv) In $E'$ every $\sigma(E', E)$-convergent sequence is $\beta(E', E)$-convergent and $E$ is of countable type.

(v) Both $E$ and $(E', \beta(E', E))$ are of countable type.

(vi) $(E', \beta(E', E))$ is nuclear.

(vii) $(E', \beta(E', E))$ is Montel.
(viii) Every $\sigma(E', E)$-bounded subset of $E'$ is $\beta(E', E)$-compactoid.

**Proof.** The implications (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii), (i) $\Rightarrow$ (vi) $\Rightarrow$ (viii) and (i) $\Rightarrow$ (vii) $\Rightarrow$ (viii) are known (see [7]) or easy. Also, from Theorem 3.1 we can easily prove (i) $\Rightarrow$ (iv) and (i) $\Rightarrow$ (v).

Now we prove (viii) $\Rightarrow$ (ii): Since $E$ is a polar Fréchet space, its topology $\tau$ is the topology of uniform convergence on the $\sigma(E', E)$-bounded subsets of $E'$. By (viii) these subsets are $\beta(E', E)$-compactoid. Now apply Lemma 3.2.

The implication (v) $\Rightarrow$ (iii) follows from [7] Proposition 4.11.

Finally, for the proof of (iv) $\Rightarrow$ (ii) observe that the topology on a polar Fréchet space of countable type is the topology of uniform convergence on the $\sigma(E', E)$-null sequences in $E'$ (see [4], Theorem 3.2). By (iv) these sequences are $\beta(E', E)$-convergent. Now apply Lemma 3.2. ■

**Remark.** It is known that a Fréchet space $E$ over $\mathbb{R}$ over $\mathbb{C}$ is nuclear if and only if $(E', \beta(E', E))$ is nuclear ([6], p. 491).

In the non-archimedean case the situation is essentially different. Indeed, in 4.1 we will give an example of an FM-space which is not nuclear (while its strong dual is by (i) $\Leftrightarrow$ (vi)). To do that we need some preliminary concepts and results.

**Definition 3.4.** Let $A = (a_{ik})$ be a matrix of strictly positive real numbers such that $a_{i+1,k} > a_{ik}$ for all $i$ and all $k$. Then the corresponding Köthe sequence space $K(A)$ is defined by

$$K(A) = \{ \alpha = (\alpha_i) : \lim_i |\alpha_i| \cdot a_{ik} = 0 \text{ for all } k \}.$$ 

On $K(A)$ we consider the sequence of norms $(p_k)$, where

$$p_k(\alpha) = \max_i |\alpha_i| \cdot a_{ik}, \quad k = 1, 2, \ldots; \quad \alpha \in K(A).$$

It is known that $K(A)$ is a polar Fréchet space of countable type. For the importance of this class of spaces and for their further properties we refer to [3].

We then have:

**Proposition 3.5.** Let $\Lambda = K(A)$ be a Köthe space and let $\Lambda^*$ the corresponding Köthe dual space. Then the following properties are equivalent:

(i) $\Lambda$ is an FM-space.

(ii) $(\Lambda^*, \beta(\Lambda^*, \Lambda))$ is of countable type.

(iii) $(\Lambda^*, \beta(\Lambda^*, \Lambda))$ is nuclear.

(iv) $(\Lambda^*, \beta(\Lambda^*, \Lambda))$ is Montel.

(v) The unit vectors $e_1, e_2, \ldots$ form a Schauder basis for $\Lambda^*, \beta(\Lambda^*, \Lambda)$.

(vi) $n(\Lambda^*, \Lambda) = \beta(\Lambda^*, \Lambda)$ (where $n(\Lambda^*, \Lambda)$ is the natural topology on $\Lambda^*$).

(vii) No subspace of $\Lambda$ is isomorphic (linearly homeomorphic) to $c_0$.

(viii) The sequence of coordinate projections $(P_i)$, where $P_i : \Lambda \rightarrow \Lambda : \alpha = (\alpha_i) \rightarrow \alpha_i e_i$, converges to the zero-map uniformly on every bounded subset of $\Lambda$. 

(ix) The sequence of sections-maps \((S_n)\), where \(S_n: \Lambda \rightarrow \Lambda : \alpha = (\alpha_i) \rightarrow (\alpha_1, \alpha_2, \ldots, \alpha_n, 0, 0, \ldots)\) converges to the identity map \(\text{Id}\) uniformly on every bounded subset of \(\Lambda\).

PROOF. We only have to prove \((i) \Rightarrow (v) \Rightarrow (vi), (vii) \Rightarrow (viii)\) and \((ix) \Rightarrow (i)\). The other implications are easy.

(i) \Rightarrow (v): The unit vectors \(e_1, e_2, \ldots\) form a Schauder basis for \((\Lambda^*, \sigma(\Lambda^*, \Lambda))\). Then, apply (i) \(\Rightarrow (iv)\) in 3.3.

(v) \(\Rightarrow (vi)\): By [4], p. 21 it suffices to prove that \(\beta(\Lambda^*, \Lambda)\) is compatible with the duality \((\Lambda^*, \Lambda)\) and this is done as in [1], Proposition 20.

(vii) \(\Rightarrow (viii)\): Suppose \(\Lambda\) contains a bounded subset \(D\) on which \((P_i)\) does not converge uniformly to the zero-map. We show that \(\Lambda\) contains a subspace isomorphic to \(c_0\).

From the assumption it follows that there exist \(\varepsilon > 0, k \in \mathbb{N}\) and an increasing sequence of indices \((i_n)\) such that, for all \(n\), there exists \(\alpha^n = (\alpha^n_i) \in D\) with \(|\alpha^n_i| \cdot a^n_k > \varepsilon, n = 1, 2, \ldots\). We put \(z_{i_n} = \alpha^n_i \cdot e_{i_n}, n = 1, 2, \ldots\). Then, the sequence \((z_{i_n})\) is bounded in \(\Lambda\).

Now we can define a linear map

\[ T: c_0 \rightarrow \Lambda : \sigma = (\sigma_n) \rightarrow \sum_n \sigma_n z_{i_n}. \]

We prove that \(T\) is an isomorphism from \(c_0\) into \(\Lambda\). It is easy to see that \(T\) is injective and continuous. Also, \(T: c_0 \rightarrow \text{Im} \ T\) is open.

Indeed, for \(\sigma = (\sigma_n) \in c_0\), we have \(p_k(T(\sigma)) = \max_{n \geq k} |\sigma_n a^n_k| \cdot a^n_k \geq \varepsilon \cdot \|\sigma\|_{c_0}\).

(ix) \(\Rightarrow (i)\): We prove that \(\text{Id}: \Lambda \rightarrow \Lambda\) transforms bounded subsets into compactoid subsets. Observe that (ix) means that \(\lim_n S_n = \text{Id}\) in \(L_c(\Lambda, \Lambda)\). Then apply Proposition 4 in [2].

The next corollary is for later use.

**Corollary 3.6.** If for every \(k \in \mathbb{N}\) and every subsequence \((i_n)\) of the indices there exists \(h > k\) such that the sequence \((\alpha^n_k / a_k^k)\) is bounded, then \(\text{K}(\Lambda)\) is an FM-space.

**Proof:** An analysis of the proof of (vii) \(\Rightarrow (viii)\) shows that if \(\text{K}(\Lambda)\) is not an FM-space, there exist a subsequence of the indices \((i_n)\) and elements \(\eta_{i_n}\) in \(\Lambda, n = 1, 2, \ldots\) such that the linear map \(T: c_0 \rightarrow \text{Im} \ T: (\sigma_n) \rightarrow (\sigma_n \eta_{i_n})\) is an isomorphism of \(c_0\) into \(\Lambda\).

Consider now in \(c_0\) the subspace \(c_0K\) generated by the unit vectors \(e_1, e_2, \ldots\). Then \(c_0K\) is isomorphic to the subspace \(F\) of \(\text{K}(\Lambda)\) generated by \(e_1, e_2, \ldots\). Therefore the topology induced by \(\text{K}(\Lambda)\) on \(F\) is normable. This means that there exists \(k\) such that for all \(h > k\) there exists \(t_h > 0\) with \(p_h(\delta) \leq t_h \cdot p_k(\delta)\) for all \(\delta \in \text{K}(\Lambda)\). In particular, for \(\delta = e_{i_n}\),
4. Characterizations of nuclear spaces among FM-spaces. We start this section with the construction of an FM-space which is not nuclear.

**Example 4.1.** For \( k = 1, 2, \ldots \), consider the infinite matrix

\[
A^k = (\alpha^k_{ij}) = 
\begin{pmatrix}
1^k & \cdots & 2^k & \cdots & j^k & \cdots \\
1^k & \cdots & 2^k & \cdots & j^k & \cdots \\
\vdots & \ddots & \vdots & \ddots & \vdots & \cdots \\
(k+1)^k & \cdots & (k+1)^k & \cdots & (k+1)^k & \cdots \\
(k+2)^k & \cdots & (k+2)^k & \cdots & (k+2)^k & \cdots \\
\vdots & \ddots & \vdots & \ddots & \vdots & \cdots \\
\end{pmatrix} \rightarrow (k+1)
\]

We can think of \( A^k \) as a sequence for some order, \( k = 1, 2, \ldots \) (we fix the same order for all \( k \)). We then consider the Köthe space

\[ K(A) = \{ \beta = (\beta_{ij}) : \lim_{i,j} |\beta_{ij}| \cdot \alpha^k_{ij} = 0, k = 1, 2, \ldots \} \]
equipped with the sequence of norms \((p_k)\) where

\[ p_k(\beta) = \max_{i,j} |\beta_{ij}| \cdot \alpha^k_{ij}. \]

We first show that \( K(A) \) is not nuclear. If \( k > 1 \), then the sequence \((\alpha^k_{ij})\) contains a constant sequence. Then by [3] Proposition 3.5 the conclusion follows.

We now apply Corollary 3.6 in order to prove that \( K(A) \) is an FM-space.

Choose \( k \) and any subsequence of the indices \((l_m, j_m)_{m,n}\). We consider the corresponding elements \( a^k_{l_m,j_m} \) of \( A^k \). There are several possibilities.

a) The subsequence \((a^k_{l_m,j_m})_{m,n}\) contains an infinite number of elements of some row of \( A^k \).

If this row is between the rows 1, \ldots, \( k \), take \( h = k + 1 \). Then the sequence of the quotients \((a^h_{l_m,j_m} / a^k_{l_m,j_m})_{m,n}\) is unbounded.

If this row is the \((k + r)\)-th row for some \( r \geq 1 \), then take \( h = k + r \).

b) The subsequence \((a^k_{l_m,j_m})_{m,n}\) consists of finitely many elements of an infinite number of rows. Consider then a subsequence with one element in an infinite number of rows below the \( k \)-th row. Such a subsequence looks like

\[ (k + l_1)^k, (k + l_2)^k, (k + l_3)^k, \ldots \]

with \((l_n)\) increasing to infinity. Take now \( h = k + 1 \).
Finally we investigate what the situation exactly is.

**Definition 4.2.** A locally convex space $X$ is said to be *quasinormable* if for every zero-neighbourhood $U$ in $X$ there exists a zero-neighbourhood $V$ in $X$, $V \subset U$, such that on $U'$ the topology $\beta(X', X)$ coincides with norm topology of $X'_{V'}$.

**Definition 4.3.** Let $X$ be a locally convex space. A sequence $(a_n) \subset X'$ is said to be *locally convergent to zero* if there exists a zero-neighbourhood $U$ in $X$ such that $(a_n) \subset X'_{U'}$ and $\lim_{n} \|a_n\|_{U'} = 0$.

**Theorem 4.4.** For an FM-space $E$ the following properties are equivalent.

(i) $E$ is nuclear.
(ii) $E$ is quasinormable.
(iii) Every $\beta(E', E)$-convergent sequence in $E'$ is locally convergent.

**Proof.** The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) follow by [2, Proposition 14 and [5], 5.2 respectively.

(iii) $\Rightarrow$ (i) Since $E$ is of countable type (Theorem 3.1) its topology can be described by the $\sigma(E', E)$-null sequences on $E'$ ([4], Theorem 3.2). By Theorem 3.3 (i) $\Rightarrow$ (iv) these sequences are null-sequences in $\beta(E', E)$ and by (iii) they are locally convergent to zero. The conclusion then follows from [5], 4.6.i). □

**Corollary 4.5.** The Köthe space in 4.1 is also an example of an FM-space which is not quasinormable.

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