A NOTE ON p-ADIC REFLEXIVITY

by

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ABSTRACT. For a nonarchimedean nontrivially valued complete field K consider the following statements (A), (B), (C) (for terminology, see §1).

(A) If D_1 and D_2 are closed subspaces of a K-Banach space E such that $D_1 + D_2 = E$ and $D_1 \cap D_2 = \{0\}$ and if D_1 and D_2 are (pseudo)reflexive then so is E.

(B) If D is a finite-dimensional subspace of a K-Banach space E and if E/D is pseudoreflexive then so is E.

(C) If D is a finite-dimensional subspace of a K-Banach space E and if E/D is reflexive then so is E.

The purpose of this note is to show that (A) is false and that (B) implies (C) rendering a solution of one Problem and a reduction of two other Problems of [1], §8.

§1. Preliminaries.

In this note K is a nonarchimedean valued field whose valuation $|\ |$ is complete and non-trivial. Norms on K-vector spaces are always assumed to be nonarchimedean i.e. to satisfy the strong triangle inequality.

PROPOSITION 1.1 *Let $(E, \|\ \|)$ be a normed space over K, let D be a subspace of E, let q be a norm on D satisfying*

$$\tfrac{1}{2}\|d\| \leq q(d) \leq \|d\| \quad (d \in D).$$

Then q can be extended to a norm $\|\ \|_1$ on E for which

$$\tfrac{1}{2}\|x\| \leq \|x\|_1 \leq \|x\| \quad (x \in E).$$

Proof. Set

$$\|x\|_1 := \inf_{d \in D} \max(\|x - d\|, q(d)).$$

Immediate verification shows that $\|\ \|_1$ satisfies the requirements.

A K-Banach space E is called *pseudoreflexive* if the canonical map $j_E : E \to E''$ is an isometry, *reflexive* if E is pseudoreflexive and j_E is surjective. If K is spherically complete each K-Banach space is pseudoreflexive and the reflexive spaces are precisely the finite-dimensional ones. For such K the statements (A), (B), (C) of above are trivially true. Therefore from now on we assume that K is NOT SPHERICALLY COMPLETE implying that the valuation of K is dense. Then it is easy to see that a K-Banach space E is pseudoreflexive if and only if the norm is polar (recall that a seminorm p is *polar* if $p = \sup\{|f| : f \in E^*, |f| \leq p\}$ where E^* is the algebraic dual of E). Also, each closed subspace of a pseudoreflexive K-Banach space is pseudoreflexive.

A subspace D of a K-Banach space E is said to have the *Weak Extension Property* (**WEP**) if every $f \in D'$ can be extended to an element of E', in other words, if the adjoint $i' : E' \to D'$ of the inclusion map $i : D \to E$ is surjective.
A subspace D of a K-Banach space E is said to have the *Extension Property* (**EP**) if for each $\varepsilon > 0$ and $f \in D'$ there is an extension $\overline{f} \in E'$ of f such that $\|\overline{f}\| \leq (1+\varepsilon)\|f\|$, in other words, if the adjoint $i' : E' \to D'$ of the inclusion map $i : D \to E$ is a quotient map.

PROPOSITION 1.2. *Let D be a closed subspace of a K-Banach space E, let $f \in D'$, $\varepsilon > 0$, $x \in E \backslash D$. Then f can be extended to an $\overline{f} \in (Kx+D)'$ such that $\|\overline{f}\| \leq (1+\varepsilon)\|f\|$.*

Proof. We may suppose that dist $(x, D) \geq t\|x\|$ where $t := (1+\varepsilon)^{-1}$. Then $\|\lambda x + d\| \geq t \max(\|\lambda x\|, \|d\|)$ for all $\lambda \in K$ and $d \in D$. The formula $\overline{f}(\lambda x + d) = f(d)$ defines an extension $\overline{f} \in (Kx + D)^*$ of f. For each $\lambda \in K$, $d \in D$ we have $|\overline{f}(\lambda x + d)| = |f(d)| \leq \|f\|\,\|d\| \leq \|f\| \max(\|\lambda x\|, \|d\|) \leq \|f\| t^{-1} \|\lambda x + d\|$ and we see that $\|\overline{f}\| \leq (1+\varepsilon)\|f\|$.

COROLLARY 1.3. *Any finite codimensional subspace of a K-Banach space has the* EP.

PROPOSITION 1.4. *Let F, G, H be K-Banach spaces, let*

$$F \xrightarrow{T_1} G \xrightarrow{T_2} H$$

be continuous linear maps such that $\mathrm{Im}\ T_1 = \mathrm{Ker}\ T_2$. *Suppose that T_2G is a closed subspace of H with the WEP. Then for the adjoints*

$$H^1 \xrightarrow{T_2'} G' \xrightarrow{T_1'} F'$$

we have $\mathrm{Im}\ T_2' = \mathrm{Ker}\ T_1'$.

Proof. Obviously $T_1' \circ T_2' = (T_2 \circ T_1)' = 0$ whence $\mathrm{Im}\ T_2' \subset \mathrm{Ker}\ T_1'$. Conversely, suppose $f \in \mathrm{Ker}\ T_1'$ i.e. $f \circ T_1 = 0$:

$$\begin{array}{ccccc} F & \xrightarrow{T_2} & G & \xrightarrow{T_2} & H \\ & & \downarrow{\scriptstyle f} & & \\ & & K & & \end{array}$$

Then $f = 0$ on $\mathrm{Ker}\ T_2$ so there exist unique linear maps $f_1 : G/\ \mathrm{Ker}\ T_2 \to K$ and $i : G/\ \mathrm{Ker}\ T_2 \to H$ making

$$\begin{array}{ccc} G & \xrightarrow{T_2} & H \\ f\downarrow & \searrow^{\pi} & \uparrow i \\ K & \xleftarrow[f_1]{} & G/\mathrm{Ker}\ T_2 \end{array}$$

commute (here π is, of course, the quotient map). The maps f_1 and i are continuous and i is even a homeomorphism onto T_2G by the Banach Open Mapping Theorem. The map $z \mapsto f_1(i^{-1}(z)) \quad (z \in T_2G)$ extends to an $f_2 \in H'$ which obviously makes

$$\begin{array}{ccc} G & \xrightarrow{T_2} & H \\ f\downarrow & \swarrow f_2 & \\ K & & \end{array}$$

commute. It follows that $f = f_2 \circ T_2 = T_2'(f_2) \in \operatorname{Im} T_2'$.

It is known that the spaces c_o and ℓ^∞ (whose standard norms will be denoted $\| \ \|$) are reflexive and each others dual by means of the pairing

$$(x, y) \mapsto \sum_{i=1}^{\infty} \xi_i \eta_i$$

($x = (\xi_1, \xi_2, \ldots) \in c_o$, $y = (\eta_1, \eta_2, \ldots) \in \ell^\infty$). Then, if $y = (\eta_1, \eta_2, \ldots) \in \ell^\infty$ then $\lim_{n \to \infty} y_n = y$ weakly, where $y_n := (\eta_1, \eta_2, \ldots, \eta_n, 0, 0, \ldots)$.

§2. (A) is false.

2.1. We construct an equivalent nonpolar norm p on ℓ^∞.
Set

$$p(x) := \max\left(\tfrac{1}{2}\|x\|, \operatorname{dist}(x, c_o)\right).$$

We have $\frac{1}{2}\|x\| \leq p(x) \leq \|x\|$ for all $x \in \ell^\infty$ and, since $p(1,1,1,\ldots) = \|(1,1,\ldots)\| = 1$, $p \neq \frac{1}{2}\|\ \|$. To arrive at the non-polarness of p we prove that $f \in (\ell^\infty)^*$, $|f| \leq p$ implies $|f| \leq \frac{1}{2}\|\ \|$. Let $x = (\xi_1, \xi_2, \ldots) \in \ell^\infty$. Then $x = \lim_{n\to\infty} x_n$ weakly where $x_n := (\xi_1, \xi_2, \ldots, \xi_n, 0, 0, \ldots)$. Now f is in $(\ell^\infty)'$ and $x_n \in c_o$ and therefore $|f(x)| = \lim_{n\to\infty} |f(x_n)| \leq \sup_n p(x_n) = \sup_n \frac{1}{2}\|x_n\| \leq \frac{1}{2}\sup_n |\xi_n| = \frac{1}{2}\|x\|$.

This way we have obtained a Banach space E with two norms $\|\ \|_1$, $\|\ \|_2$, each one defining the topology while $(E, \|\ \|_1)$ is reflexive and $(E, \|\ \|_2)$ is not.

2.2. Let us denote the product norm on $\ell^\infty \times \ell^\infty$ again by $\|\ \|$. We construct a second norm $\|\ \|_1$ on $\ell^\infty \times \ell^\infty$ as follows. First define a norm q on the diagonal $\Delta := \{(x,x) : x \in \ell^\infty\}$ via

$$q(x,x) := p(x)$$

where p is as in 2.1. By Proposition 1.1 the formula

$$\|(x,y)\|_1 := \inf_{t\in\ell^\infty} \left(\max \|(x,y) - (t,t)\|, q(t,t)\right)$$

defines a norm $\|\ \|_1$ on $\ell^\infty \times \ell^\infty$. This norm is not polar since its restriction to Δ is not polar, but satisfies $\frac{1}{2}\|z\| \leq \|z\|_1 \leq \|z\|$ for all $z \in \ell^\infty \times \ell^\infty$.

2.3. Now we show that (A) is false. Let $E := (\ell^\infty \times \ell^\infty, \|\ \|_1)$, and set $D_1 := \{(x,0) : x \in \ell^\infty\}$, $D_2 := \{(0,x) : x \in \ell^\infty\}$. Then $D_1 + D_2 = E$, $D_1 \cap D_2 = \{0\}$. For each $x \in \ell^\infty$ we have $\|(x,0)\| \geq \|(x,0)\|_1 = \inf_{t\in\ell^\infty} \max(\|(x-t,-t)\|, q(t)) \geq \inf_{t\in\ell^\infty} \|(x-t,-t)\|$

$$= \inf_{t\in\ell^\infty} \max(\|x-t\|, \|t\|) \geq \|x\| = \|(x,0)\|.$$

It follows that D_1 is isometrically isomorphic to ℓ^∞, hence reflexive. By the same token D_2 is reflexive. But in 2.2 we have seen that E is not even pseudoreflexive.

§3. (B) implies (C).

In the next two lemmas D is a finite-dimensional subspace of a K-Banach space E with inclusion map $i : D \to E$ and quotient map $\pi : E \to E/D$. We consider the commutative diagram

$$\begin{array}{ccccc} D & \xrightarrow{i} & E & \xrightarrow{\pi} & E/D \\ \downarrow{\scriptstyle j_D} & & \downarrow{\scriptstyle j_E} & & \downarrow{\scriptstyle j_{E/D}} \\ D'' & \xrightarrow[i'']{} & E'' & \xrightarrow[\pi'']{} & (E/D)'' \end{array}$$

LEMMA 3.1. Im i'' = Ker π'' *and* π'' *is a quotient map.*

Proof. Since π is surjective we have by Proposition 1.4 in $(E/D)' \xrightarrow{\pi'} E' \xrightarrow{i'} D'$ that Im π' = Ker i'. Now D' is finite-dimensional so Im i' is closed and has the WEP. Hence, again by Proposition 1.4, Im i'' = Ker π''. To prove the second statement observe that π' is an isometry whose image has finite codimension and therefore has the EP by Corollary 1.3, so π'' is a quotient map.

LEMMA 3.2. j_E *is surjective if and only if* $j_{E/D}$ *is surjective.*

Proof. If j_E is surjective then so is $\pi'' \circ j_E = j_{E/D} \circ \pi$. Then $j_{E/D}$ must be surjective. Conversely, if $j_{E/D}$ is surjective, let $\Theta \in E''$.
Then there is an $x \in E$ with $\pi''(\Theta) = j_{E/D}(\pi(x)) = \pi''(j_E(x))$. We see that $\Theta - j_E(x) \in$ Ker π'' = Im i'' so by surjectivety of j_D we can find a $d \in D$ such that

$$\Theta - j_E(x) = i'' j_D(d) = j_E i(d)$$

Then $\Theta = j_E(x + i(d)) \in j_E(E)$ and the surjectivity of j_E is proved.

REMARK. In the same vein one can prove j_E injective $\Rightarrow j_{E/D}$ injective; a counterexample to the converse is given in [1], §8.
Also one has j_E is isometrical $\Rightarrow j_{E/D}$ is isometrical. Its converse is just statement (B) the truth of which is an open problem.

If we assume (B) and if E/D is reflexive we have that j_E is isometrical by (B) and surjective by Lemma 3.2. We may conclude that (B) implies (C).

REFERENCE

[1] A.C.M. van Rooij and W.H. Schikhof: *Open problems.* In *p-adic Functional Analysis*, edited by J.M. Bayod, N. De Grande - De Kimpe and J. Martinez - Maurica, Marcel Dekker New York (1991), 209-219.