A NOTE ON p-ADIC REFLEXIVITY

by

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ABSTRACT. For a nonarchimedean nontrivially valued complete field $K$ consider the following statements (A), (B), (C) (for terminology, see §1).

(A) If $D_1$ and $D_2$ are closed subspaces of a $K$-Banach space $E$ such that $D_1 + D_2 = E$ and $D_1 \cap D_2 = \{0\}$ and if $D_1$ and $D_2$ are (pseudo)reflexive then so is $E$.

(B) If $D$ is a finite-dimensional subspace of a $K$-Banach space $E$ and if $E/D$ is pseudoreflexive then so is $E$.

(C) If $D$ is a finite-dimensional subspace of a $K$-Banach space $E$ and if $E/D$ is reflexive then so is $E$.

The purpose of this note is to show that (A) is false and that (B) implies (C) rendering a solution of one Problem and a reduction of two other Problems of [1], §8.
§1. Preliminaries.

In this note $K$ is a nonarchimedean valued field whose valuation $| |$ is complete and non-trivial. Norms on $K$-vector spaces are always assumed to be nonarchimedean i.e. to satisfy the strong triangle inequality.

**PROPOSITION 1.1** Let $(E, || ||)$ be a normed space over $K$, let $D$ be a subspace of $E$, let $q$ be a norm on $D$ satisfying

$$\frac{1}{2}||d|| \leq q(d) \leq ||d|| \quad (d \in D).$$

Then $q$ can be extended to a norm $|| ||_1$ on $E$ for which

$$\frac{1}{2}||x|| \leq ||x||_1 \leq ||x|| \quad (x \in E).$$

**Proof.** Set

$$||x||_1 := \inf_{d \in D} \max(||x - d||, q(d)).$$

Immediate verification shows that $|| ||_1$ satisfies the requirements.

A $K$-Banach space $E$ is called pseudoreflexive if the canonical map $j_E : E \to E''$ is an isometry, reflexive if $E$ is pseudoreflexive and $j_E$ is surjective. If $K$ is spherically complete each $K$-Banach space is pseudoreflexive and the reflexive spaces are precisely the finite-dimensional ones. For such $K$ the statements (A), (B), (C) of above are trivially true. Therefore from now on we assume that $K$ is NOT SPHERICALLY COMPLETE implying that the valuation of $K$ is dense. Then it is easy to see that a $K$-Banach space $E$ is pseudoreflexive if and only if the norm is polar (recall that a seminorm $p$ is polar if $p = \sup\{|f| : f \in E^*, |f| \leq p\}$ where $E^*$ is the algebraic dual of $E$). Also, each closed subspace of a pseudoreflexive $K$-Banach space is pseudoreflexive.

A subspace $D$ of a $K$-Banach space $E$ is said to have the Weak Extension Property (WEP) if every $f \in D'$ can be extended to an element of $E'$, in other words, if the adjoint $i' : E' \to D'$ of the inclusion map $i : D \to E$ is surjective.

A subspace $D$ of a $K$-Banach space $E$ is said to have the Extension Property (EP) if for each $\varepsilon > 0$ and $f \in D'$ there is an extension $\widehat{f} \in E'$ of $f$ such that $||\widehat{f}|| \leq (1 + \varepsilon)||f||$, in other words, if the adjoint $i' : E' \to D'$ of the inclusion map $i : D \to E$ is a quotient map.

**PROPOSITION 1.2.** Let $D$ be a closed subspace of a $K$-Banach space $E$, let $f \in D'$, $\varepsilon > 0$, $x \in E\setminus D$. Then $f$ can be extended to an $\widehat{f} \in (Kx + D)'$ such that $||\widehat{f}|| \leq (1+\varepsilon)||f||$. 

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Proof. We may suppose that dist \((x,D) \geq t\|x\|\) where \(t := (1 + \varepsilon)^{-1}\). Then \(\|\lambda x + d\| \geq t \max(\|\lambda x\|, \|d\|)\) for all \(\lambda \in K\) and \(d \in D\). The formula \(\overline{f}(\lambda x + d) = f(d)\) defines an extension \(\overline{f} \in (Kx + D)^*\) of \(f\). For each \(\lambda \in K\), \(d \in D\) we have \(|\overline{f}(\lambda x + d)| = |f(d)| \leq \|f\| \max(\|\lambda x\|, \|d\|) \leq \|f\| t^{-1} \|\lambda x + d\|\) and we see that \(\|\overline{f}\| \leq (1 + \varepsilon)\|f\|\).

Corollary 1.3. Any finite codimensional subspace of a \(K\)-Banach space has the EP.

Proposition 1.4. Let \(F,G,H\) be \(K\)-Banach spaces, let

\[ F \xrightarrow{T_1} G \xrightarrow{T_2} H \]

be continuous linear maps such that \(\text{Im } T_1 = \text{Ker } T_2\). Suppose that \(T_2 G\) is a closed subspace of \(H\) with the WEP. Then for the adjoints

\[ H^1 \xrightarrow{T_2'} G' \xrightarrow{T_1'} F' \]

we have \(\text{Im } T_2' = \text{Ker } T_1'\).

Proof. Obviously \(T_1' \circ T_2' = (T_2 \circ T_1)' = 0\) whence \(\text{Im } T_2' \subset \text{Ker } T_1'\). Conversely, suppose \(f \in \text{Ker } T_1'\) i.e. \(f \circ T_1 = 0:\)

\[
\begin{array}{ccc}
F & \xrightarrow{T_2} & G & \xrightarrow{T_3} & H \\
\downarrow f \\
K
\end{array}
\]

Then \(f = 0\) on \(\text{Ker } T_2\) so there exist unique linear maps \(f_1 : G/\text{Ker } T_2 \to K\) and \(i : G/\text{Ker } T_2 \to H\) making

\[
\begin{array}{ccc}
G & \xrightarrow{T_2} & H \\
\downarrow f \\
K & \xleftarrow{f_1} & G/\text{Ker } T_2
\end{array}
\]

commute (here \(\pi\) is, of course, the quotient map). The maps \(f_1\) and \(i\) are continuous and \(i\) is even a homeomorphism onto \(T_2 G\) by the Banach Open Mapping Theorem. The map \(z \mapsto f_1(i^{-1}(z))\) \((z \in T_2 G)\) extends to an \(f_2 \in H'\) which obviously makes
It is known that the spaces $c_0$ and $\ell^\infty$ (whose standard norms will be denoted $\| \|$) are reflexive and each others dual by means of the pairing

$$(x, y) \mapsto \sum_{i=1}^{\infty} \xi_i \eta_i$$

$(x = (\xi_1, \xi_2, \ldots) \in c_0, y = (\eta_1, \eta_2, \ldots) \in \ell^\infty)$. Then, if $y = (\eta_1, \eta_2, \ldots) \in \ell^\infty$ then

$$\lim_{n \to \infty} y_n = y$$

weakly, where $y_n := (\eta_1, \eta_2, \ldots, \eta_n, 0, 0, \ldots)$. commute. It follows that $f = f_2 \circ T_2 = T_2(f_2) \in \text{Im } T_2'$. 
§2. (A) is false.

2.1. We construct an equivalent nonpolar norm $p$ on $\ell^\infty$.

Set
\[ p(x) := \max\left(\frac{1}{2}||x||, \text{dist}(x, c_0)\right). \]

We have $\frac{1}{2}||x|| \leq p(x) \leq ||x||$ for all $x \in \ell^\infty$ and, since $p(1, 1, 1, \ldots) = ||(1, 1, \ldots)|| = 1$, $p \neq \frac{1}{2}||x||$. To arrive at the non-polarness of $p$ we prove that $f \in (\ell^\infty)^*$, $|f| \leq p$ implies $|f| \leq \frac{1}{2}||x||$. Let $x = (\xi_1, \xi_2, \ldots) \in \ell^\infty$. Then $x = \lim_{n \to \infty} x_n$ weakly where $x_n := (\xi_1, \xi_2, \ldots, \xi_n, 0, 0, \ldots)$. Now $f$ is in $(\ell^\infty)^*$ and $x_n \in c_0$ and therefore $|f(x)| = \lim_{n \to \infty} |f(x_n)| \leq \sup p(x_n) = \sup_n \frac{1}{2}||x_n|| \leq \frac{1}{2} \sup_n |\xi_n| = \frac{1}{2}||x||$.

This way we have obtained a Banach space $E$ with two norms $||.||_1$, $||.||_2$, each one defining the topology while $(E, ||.||_1)$ is reflexive and $(E, ||.||_2)$ is not.

2.2. Let us denote the product norm on $\ell^\infty \times \ell^\infty$ again by $||.||$. We construct a second norm $||.||_1$ on $\ell^\infty \times \ell^\infty$ as follows. First define a norm $q$ on the diagonal $\Delta := \{(x, x) : x \in \ell^\infty\}$ via
\[ q(x, x) := p(x) \]

where $p$ is as in 2.1. By Proposition 1.1 the formula
\[ ||(x, y)||_1 := \inf_{t \in \ell^\infty} \left(\max\left(||(x, y) - (t, t)||, q(t, t)\right)\right) \]
defines a norm $||.||_1$ on $\ell^\infty \times \ell^\infty$. This norm is not polar since its restriction to $\Delta$ is not polar, but satisfies $\frac{1}{2}||x|| \leq ||x|| \leq ||x||$ for all $x \in \ell^\infty \times \ell^\infty$.

2.3. Now we show that (A) is false. Let $E := (\ell^\infty \times \ell^\infty, ||.||_1)$, and set $D_1 := \{(x, 0) : x \in \ell^\infty\}$, $D_2 := \{(0, x) : x \in \ell^\infty\}$. Then $D_1 + D_2 = E$, $D_1 \cap D_2 = \{0\}$. For each $x \in \ell^\infty$ we have $||(x, 0)|| \geq ||(x, 0)||_1 = \inf_{t \in \ell^\infty} \max\left(||(x - t, -t)||, q(t)\right) \geq \inf_{t \in \ell^\infty} ||(x - t, -t)||$
\[ = \inf_{t \in \ell^\infty} \max(||x - t||, ||t||) \geq ||x|| = ||(x, 0)||. \]

It follows that $D_1$ is isometrically isomorphic to $\ell^\infty$, hence reflexive. By the same token $D_2$ is reflexive. But in 2.2 we have seen that $E$ is not even pseudoreflexive.
§3. (B) implies (C).

In the next two lemmas $D$ is a finite-dimensional subspace of a $K$-Banach space $E$ with inclusion map $i : D \rightarrow E$ and quotient map $\pi : E \rightarrow E/D$. We consider the commutative diagram

$$
\begin{array}{ccc}
D & \xrightarrow{i} & E \\
\downarrow{j_D} & & \downarrow{j_E} \\
D'' & \xrightarrow{i''} & E'' \\
\end{array} \quad \begin{array}{c}
\xrightarrow{\pi} \\
\downarrow{j_{E/D}} \\
(E/D)' \\
\end{array}
$$

**Lemma 3.1.** $\text{Im } i'' = \text{Ker } \pi''$ and $\pi''$ is a quotient map.

*Proof.* Since $\pi$ is surjective we have by Proposition 1.4 in $(E/D)' \xrightarrow{\pi'} E' \xrightarrow{i'} D'$ that $\text{Im } \pi' = \text{Ker } i'$. Now $D'$ is finite-dimensional so $\text{Im } i'$ is closed and has the WEP. Hence, again by Proposition 1.4, $\text{Im } i'' = \text{Ker } \pi''$. To prove the second statement observe that $\pi'$ is an isometry whose image has finite codimension and therefore has the EP by Corollary 1.3, so $\pi''$ is a quotient map.

**Lemma 3.2.** $j_E$ is surjective if and only if $j_{E/D}$ is surjective.

*Proof.* If $j_E$ is surjective then so is $\pi'' \circ j_E = j_{E/D} \circ \pi$. Then $j_{E/D}$ must be surjective. Conversely, if $j_{E/D}$ is surjective, let $\Theta \in E''$. Then there is an $x \in E$ with $\pi''(\Theta) = j_{E/D}(\pi(x)) = \pi''(j_E(x))$. We see that $\Theta - j_E(x) \in \text{Ker } \pi'' = \text{Im } i''$ so by surjectivity of $j_D$ we can find a $d \in D$ such that

$$
\Theta - j_E(x) = i'' j_D(d) = j_E i(d)
$$

Then $\Theta = j_E(x + i(d)) \in j_E(E)$ and the surjectivity of $j_E$ is proved.

**Remark.** In the same vein one can prove $j_E$ injective $\Rightarrow j_{E/D}$ injective; a counterexample to the converse is given in [1], §8.

Also one has $j_E$ is isometrical $\Rightarrow j_{E/D}$ is isometrical. Its converse is just statement (B) the truth of which is an open problem.

If we assume (B) and if $E/D$ is reflexive we have that $j_E$ is isometrical by (B) and surjective by Lemma 3.2. We may conclude that (B) implies (C).