APPENDIX B

ZERO SEQUENCES IN p-ADIC COMPACTOIDS

W.H. Schikhof

ABSTRACT. We prove the following theorem, solving Problem 6 of [3]. *Let E, F be Banach spaces over a nonarchimedean valued field K, let A be an absolutely convex closed compactoid in E and $T \in \mathcal{L}(E,F)$. Then, if $y_1, y_2, \ldots$ is a sequence in TA tending to 0 then there is a sequence $x_1, x_2, \ldots$ in some scalar multiple of A tending to 0 such that $Tx_n = y_n$ for each n.* (See Theorem 2.7).

INTRODUCTION. For a proof we decompose T:

$$\begin{tikzcd} A \arrow[rr, "T"] \arrow[dr, "\pi"'] & & TA \\ & A/A \cap \operatorname{Ker} T \arrow[ur, "i"'] & \end{tikzcd}$$

Here, the topology of TA is inherited from the norm on F and on $A/A \cap \operatorname{Ker} T$ we take the quotient topology induced by the quotient map π. Then i (which is the unique map making the diagram commute) is continuous. The object $A/A \cap \operatorname{Ker} T$ is in a natural way a topological module over $\{\lambda \in K : |\lambda| \leq 1\}$ whose topology is induced by the metric $\big(\pi(x), \pi(y)\big) \mapsto \|\pi(x) - \pi(y)\|$ where

$$\|\pi(z)\| = \operatorname{dist}(z, A \cap \operatorname{Ker} T) \qquad (z \in A)$$

This "norm" $\|\ \|$ on $A/A \cap \operatorname{Ker} T$ satisfies the strong triangle inequality and

$$(*) \qquad \|\lambda \pi(z)\| \leq |\lambda|\, \|\pi(z)\| \qquad (z \in A, \lambda \in K, |\lambda| \leq 1)$$

but one does not always have equality in $(*)$. Thus, we shall study a class of tolopogical modules over the valuation ring which properly contains the class of absolutely convex subsets of locally convex spaces. This study involves a careful modification of the theory given in [4] for absolutely convex sets. We then will be able to conclude that although i (see diagram) is not always a homeomorphism it does have the property that $i(t_n) \to 0$ implies $\|\lambda t_n\| \to 0$ for each $\lambda \in K$, $|\lambda| < 1$; the statement announced in the Abstract follows easily.

TERMINOLOGY. (For unexplained terms we refer to [2])

1. Throughout K is a nonarchimedean nontrivially valued field, that is complete under the metric induced by the valuation $|\ |$. We set

$$B_K := \{\lambda \in K : |\lambda| \leq 1\}$$
$$B_K^- := \{\lambda \in K : |\lambda| < 1\}.$$

2. Let A be a module over the ring B_K, let B be a submodule of A, let $\lambda, \mu \in K$. We set

$$\lambda B := \begin{cases} \{\lambda b : b \in B\} & \text{if } |\lambda| \leq 1 \\ \{x \in A : \lambda^{-1} x \in B\} & \text{if } |\lambda| \geq 1 \end{cases}$$

(observe that this causes no ambiguity if $|\lambda| = 1$). We have the following obvious consequences.

(i) λB is a submodule of A, $1 \cdot B = B$.

(ii) If $|\lambda| \leq |\mu|$ then $\lambda B \subset \mu B$. In particular, $|\lambda| = |\mu|$ implies $\lambda B = \mu B$.

(iii) If either $|\lambda|, |\mu| \leq 1$ or $|\lambda|, |\mu| \geq 1$ then $(\lambda\mu)B = \lambda(\mu B) = \mu(\lambda B)$.

We shall say that B *absorbs* a subset X of A if $X \subset \bigcup_{\lambda \in K} \lambda B$. The module generated by $X \subset A$ is denoted $\operatorname{co} X$.

3. A *locally convex topology* on a B_K-module A is a topology τ on A such that

(i) (A, τ) is a topological B_K-module (of course, the topology on B_K is the valuation topology),

(ii) there is a neighbourhood base of 0 consisting of (τ-open) B_K-submodules of A.

Then we call $A = (A, \tau)$ a *locally convex module.*

Clearly, absolutely convex subsets of locally convex spaces over K are examples of locally convex modules. Any submodule of a locally convex module is, with the restriction topology, a locally convex module. If B is a (closed) submodule of a locally convex module A the quotient topology on A/B is (Hausdorff) and locally convex.
Let A be a locally convex module. The closure of a subset X of A is denoted $\overline{X}$. Instead of $\overline{\mathrm{co}X}$ we write $\overline{\mathrm{co}}X$. Let $X_1, X_2, \ldots$ be subsets of A. We shall write $\lim_{n\to\infty} X_n = 0$ if for each neighbourhood U of 0 in A we have $X_n \subset U$ for large n. The following is not hard to see. If $X_1, X_2, \ldots$ are submodules, $\lim_{n\to\infty} X_n = 0$ then

$$\bigcap_{n=1}^{\infty} (\overline{B + X_n}) = \overline{B}$$

for every submodule B of A.

Let $i \mapsto x_i$ be a net in a locally convex module converging to 0. Then, for any net $i \mapsto \lambda_i$ in B_K, the net $i \mapsto \lambda_i x_i$ converges to 0. This is a direct consequence of local convexity.

THROUGHOUT §1 AND §2 WE ASSUME THAT THE VALUATION OF K IS DENSE.

§1. LOCALLY CONVEX B_K-MODULES

In this section A is a locally convex module over B_K.

LEMMA 1.1. *Let B be a closed submodule of A let $a \in A$, let $\lambda \in B_K^-$. If*

$$i \mapsto x_i = b_i + c_i a \qquad (b_i \in B,\ c_i \in B_K)$$

is a net in $B + \mathrm{co}\{a\}$ converging to 0 then $\lim_i c_i = 0$ or $\lambda x_i \in B$ eventually, where the latter case occurs if B absorbs $\{a\}$.

Proof. Set $C := \{\xi \in B_K : \xi a \in B\}$. Then C is absolutely convex; let r be its diameter. Suppose $\overline{\lim_i} |c_i| > r$. Then there exists a $\mu \in K$ and a cofinal $J \subset I$ such that

$$|c_j| \geq |\mu| > r \qquad (j \in J)$$

Then $\mu c_j^{-1} \in B_K$ so that

$$\mu c_j^{-1} b_j + \mu a = \mu c_j^{-1} x_j \to 0$$

We see that $\mu a = -\lim_j \mu c_j^{-1} b_j \in \overline{B} = B$ so that $\mu \in C$ conflicting $|\mu| > r$. Thus, we have proved that $\overline{\lim_i}|c_i| \leq r$. If $r = 0$ then $\lim_i c_i = 0$. If $r > 0$ then, eventually, $|\lambda c_i| < r$, i.e. $\lambda c_i \in C$ i.e. $\lambda c_i a \in B$ i.e. $\lambda x_i = \lambda b_i + \lambda c_i a \in \lambda B + B \subset B$.
To prove the second assertion, let B absorb $\{a\}$ and let $\lim_i c_i = 0$. We have $\mu a \in B$ for some nonzero $\mu \in B_K$. Then, eventually, $|c_i| \leq |\mu|$ yielding $c_i a \in B$ implying $x_i \in B$.

For a submodule B of A we define

$$B^e := \bigcap \{\lambda B : \lambda \in K, |\lambda| > 1\}$$

The following elementary facts are easily verified. *B^e is a submodule, $B \subset B^e$, $B^{ee} = B^e$, $B^e = \{x \in A : \lambda x \in B$ for all $\lambda \in B_K^-\}$. $\overline{B^e} \subset \overline{B}^e$.* Further, we have: *$B^e$ is closed $\iff \overline{B} \subset B^e \iff \lambda \overline{B} \subset B$ for each $\lambda \in B_K^-$.*

LEMMA **1.2.** *Let B be a closed submodule of A, let $a \in A$. Then $(B+\mathrm{co}\{a\})^e$ is closed.*

Proof. Let $i \mapsto x_i = b_i + c_i a$ $(b_i \in B, c_i \in B_K)$ be a net in $B + \mathrm{co}\{a\}$ converging to some $x \in A$. Let $\lambda \in B_K^-$; we shall prove that $\lambda x \in B + \mathrm{co}\{a\}$. In fact, the net $(i,j) \mapsto x_i - x_j$ converges to 0. So by Lemma 1.1 we have either $\lim_{i,j}(c_i - c_j) = 0$ (then $\lim_i c_i = c$ for some $c \in B_K$ and it follows easily that even $x \in B + \mathrm{co}\{a\}$) or $\lambda(x_i - x_j) \in B$ for, say, all $i, j \geq s$. Then $\lambda x_i \in \lambda x_s + B$ $(i \geq s)$ so $\lambda x = \lim_i \lambda x_i \in \overline{\lambda x_s + B} = \lambda x_s + B \in \lambda(B + \mathrm{co}\{a\}) + B \subset B + \mathrm{co}\{a\}$.

LEMMA **1.3.** *Let B be a closed submodule of A, let $a_1, \ldots, a_n \in A$. Then $(B + \mathrm{co}\{a_1, \ldots, a_n\})^e$ is closed.*

Proof. Lemma 1.2 covers the case n=1. Suppose, for some m,
$Z := (B + \mathrm{co}\{a_1, \ldots, a_{m-1}\})^e$ is closed. Then, again by Lemma 1.2,
$\overline{B + \mathrm{co}\{a_1, \ldots, a_m\}} \subset \overline{Z + \mathrm{co}\{a_m\}} \subset (Z + \mathrm{co}\{a_m\})^e = (B + \mathrm{co}\{a_1, \ldots, a_m\})^e$, so $(B + \mathrm{co}\{a_1, \ldots, a_m\})^e$ is closed.

LEMMA 1.4. *Let B be a closed submodule of A, let $a_1, \dots, a_n \in A$, and suppose that B absorbs $\{a_1, \dots, a_n\}$. If $i \mapsto x_i$ is a net in $B + \mathrm{co}\{a_1, \dots, a_n\}$ converging to 0 then $\lambda x_i \in B$ eventually, for each $\lambda \in B_K^-$.*

Proof. Choose $\mu_1, \dots, \mu_n \in B_K^-$ such that $|\prod_{i=1}^{n} \mu_i|^2 \geq |\lambda|$. We have $x_i \in (B+\mathrm{co}\{a_1, \dots, a_{n-1}\}) + \mathrm{co}\{a_n\}$. By Lemma 1.1 we have, eventually,

$$\mu_1 x_i \in \overline{B + \mathrm{co}\{a_1, \dots, a_{n-1}\}} \subset (B + \mathrm{co}\{a_1, \dots, a_{n-1}\})^e \quad \text{(Lemma 1.3)},$$

so that, eventually,

$$\mu_1^2 x_i \in B + \mathrm{co}\{a_1, \dots, a_{n-1}\}$$

Inductively we find

$$\mu_1^2 \cdots \mu_n^2 x_i \in B \qquad \text{eventually}$$

implying $\lambda x_i \in B$ eventually.

§2. COMPACTOID MODULES

DEFINITION 2.1. A locally convex module A is a *compactoid module* if for each $\lambda \in B_K^-$ and each neighbourhood U of 0 in A there exists a finite set $F \subset A$ such that

$$\lambda A \subset U + \mathrm{co}\, F$$

Remark. An absolutely convex subset of a locally convex space over K is a compactoid module if and only if it is a compactoid in the usual sense. So Definition 2.1 generalizes the notion of compactoid to a larger class of objects, and we will see that it suits the purpose of the paper. (Yet we must warn the reader that, in general, compactoid modules are no longer 'compact-like': For any Banach space E the module $\{x \in E : \|x\| \leq 1\}/\{x \in E : \|x\| < 1\}$ has the discrete topology but is a compactoid module.)

We need the following algebraic lemma.

LEMMA 2.2. *Let B, U be submodules of a B_K-module A, let $x_1, \dots, x_n \in A$ and suppose*

$$B \subset U + \mathrm{co}\{x_1, \dots, x_n\}.$$

Then, for each $\lambda \in B_K^-$ there exist $b_1, \ldots, b_n \in B$ such that

$$\lambda B \subset U + \mathrm{co}\{b_1, \ldots, b_n\}.$$

Proof. The proofs of Lemmas 1.1 and 1.2 of [1] apply in this more general situation.

PROPOSITION 2.3. *Every submodule of a compactoid module is compactoid.*

Proof. Let B be a submodule of a compactoid module A, let V be a zero neighbourhood in B, let $\lambda \in B_K^-$. Choose $\mu \in B_K^-$ with $|\lambda| \le |\mu|^2$, choose an open submodule U of A with $U \cap B \subset V$. There is a finite set $F_1 \subset A$ such that

$$\mu B \subset \mu A \subset U + \mathrm{co}\, F_1$$

By Lemma 2.2 there exists a finite set $F_2 \subset B$ such that

$$\mu^2 B \subset U + \mathrm{co}\, F_2$$

It follows that

$$\lambda B \subset \mu^2 B \subset U \cap B + \mathrm{co}\, F_2 \subset V + \mathrm{co}\, F_2$$

which proves that B is a compactoid module.

PROPOSITION 2.4. *Let A be a metrizable compactoid module, let $\lambda \in B_K^-$. Then there exists a sequence $x_1, x_2, \ldots$ with $\lim_{n\to\infty} x_n = 0$ such that*

$$\overline{\lambda A} \subset \overline{\mathrm{co}}\{x_1, x_2, \ldots\} \subset A$$

Proof. Choose $\lambda_1, \lambda_2, \ldots \in B_K^-$ such that $|\Pi\lambda_n| \ge |\lambda|$. By metrizability and local convexity there exist open submodules

$$A = U_1 \supset U_2 \supset \cdots$$

of A forming a neighbourhood base at 0. By compactoidity there is a finite set $F_1 \subset U_1$ such that

$$\lambda_1 U_1 \subset U_2 + \mathrm{co}\, F_1.$$

By Proposition 2.3 U_2 is a compactoid so

$$\lambda_2 U_2 \subset U_3 + \mathrm{co}\, F_2$$

for some finite set $F_2 \subset U_2$, etc. Inductively we find finite sets $F_1, F_2, \ldots$ such that

(i) $F_n \subset U_n$ for each n. Hence $\lim_n F_n = 0$ (see TERMINOLOGY, 3)

(ii) $\lambda_n U_n \subset U_{n+1} + \operatorname{co} F_n$ for each n.

Repeated application of (ii) yields for $n \in \mathbb{N}$

$$\begin{aligned}\lambda A &\subset \lambda_1\lambda_2\ldots\lambda_n A \subset \lambda_2\ldots\lambda_n(\lambda_1 U_1) \subset \lambda_2\ldots\lambda_n(U_2 + \operatorname{co} F_1)\\ &\subset \lambda_2\ldots\lambda_n U_2 + \operatorname{co} F_1 \subset \lambda_3\ldots\lambda_n U_3 + \operatorname{co} F_2 + \operatorname{co} F_1 \subset \cdots\\ &\subset U_{n+1} + \operatorname{co}(F_1 \cup F_2 \cup \ldots \cup F_n).\end{aligned}$$

We see that for each n

$$\overline{\lambda A} \subset U_{n+1} + \operatorname{co}\{F_1 \cup F_2 \cup \ldots\}$$

so that

$$\overline{\lambda A} \subset \overline{\operatorname{co}}\{F_1 \cup F_2 \cup \ldots\}$$

By enumerating $F_1, F_2, \ldots$ successively we obtain a sequence $x_1, x_2, \ldots$ in A tending to 0 (as $\lim F_n = 0$) such that

$$\overline{\lambda A} \subset \overline{\operatorname{co}}\{x_1, x_2, \ldots\}.$$

THEOREM 2.5. *Let E, F be locally convex B_K-modules, let A be a complete metrizable compactoid submodule of E.*

(i) *If F is Hausdorff and $T : E \to F$ is a continuous module homomorphism then $(TA)^e$ is closed.*

(ii) *If $B \subset E$ is a closed submodule of E then $(A + B)^e$ is closed.*

Proof. We show that for a closed submodule B of F the set $(TA + B)^e$ is closed (then (i) follows by taking $B = \{0\}$, (ii) is the case $E = F$, T is the identity). Thus, let $\lambda \in B_K^-$; we prove that $\lambda\overline{(TA + B)} \subset TA + B$.

Let $\lambda_1, \mu_1, \mu_2, \ldots \in B_K^-$ such that $|\lambda_1 \prod \mu_n| \geq |\lambda|$.

By Proposition 2.4 there exist $x_1, x_2, \ldots \in A$ such that

$$(*) \qquad \lim x_n = 0 \quad \text{and} \quad \overline{\lambda_1 A} \subset \overline{\operatorname{co}}\{x_1, x_2, \ldots\} \subset A$$

Write $X_n := \overline{\operatorname{co}}\{x_n\}$ and $A_n := \overline{\sum_{i \geq n} X_i}$. Then $\lim_{n\to\infty} X_n = 0$, $\lim_{n\to\infty} A_n = 0$ and $\lim_{n\to\infty} TA_n = 0$. First observe that by $(*)$

$$\lambda_1\overline{(TA + B)} \subset \overline{TA_1 + B}$$

Also we have by Lemma 1.2

$$\mu_1\overline{(TA_1+B)} = \mu_1\overline{(TX_1+TA_2+B)} \subset TX_1 + \overline{TA_2+B}$$
$$\mu_2\overline{(TA_2+B)} = \mu_2\overline{(TX_2+TA_3+B)} \subset TX_2 + \overline{TA_3+B}, \text{ etc.}$$

Let $x \in \lambda\overline{(TA+B)}$. Then $x \in (\prod \mu_n)\lambda_1\overline{(TA+B)} \subset TX_1 + (\prod_{n\geq 2}\mu_n)\overline{(TA_2+B)}$. There is a $y_1 \in X_1$ such that

$$x - Ty_1 \in \prod_{n\geq 2}\mu_n\overline{(TA_2+B)} \subset TX_2 + (\prod_{n\geq 3}\mu_n)\overline{(TA_3+B)}.$$

So there is a $y_2 \in X_2$ such that

$$x - Ty_1 - Ty_2 \in (\prod_{n\geq 3}\mu_n)\overline{(TA_3+B)}, \text{ etc.}$$

Inductively we find $y_1 \in X_1, y_2 \in X_2, \dots$ such that for each n

$$x = T(y_1 + \cdots + y_n) + z_{n+1}$$

when $z_{n+1} \in \overline{TA_{n+1}+B}$. Now $\lim_{n\to\infty} y_n = 0$, so, by completeness of A, $y := \sum_{n=1}^{\infty} y_n$, exists, and lies in A. Then $z : \lim_{n\to\infty} z_{n+1}$ also must exist and it lies in $\bigcap_n \overline{(TA_n+B)}$ which is B since $\lim_{n\to\infty} TA_n = 0$. We see that $x = Ty + z \in TA + B$.

As a corollary we obtain

THEOREM 2.6. *Let A_1, A_2 be locally convex modules, let A_1 be metrizable, complete and compactoid and let $T : A_1 \to A_2$ be a continuous injective module homomorphism. Let $i \mapsto a_i$ be a net in A_1 such that $Ta_i \to 0$. Then, for each $\lambda \in B_K^-$, $\lambda a_i \to 0$.*

Proof. Let U be an open submodule of A_1, let $\lambda \in B_K^-$. Choose $\mu \in B_K^-$, $|\mu|^3 \geq |\lambda|$. There exists a finite set $F \subset A_1$ such that $\mu A \subset U + \operatorname{co} F$. Hence

$$\mu TA \subset TU + \operatorname{co} TF \subset (TU)^e + \operatorname{co} TF.$$

The net $i \mapsto \mu Ta_i$ lies in $(TU)^e + \operatorname{co} TF$ and converges to 0. Further, $(TU)^e$ is closed (Proposition 2.3 and Theorem 2.5) and it absorbs TF. Thus, by Lemma 1.4 we have, eventually: $\mu^2 Ta_i \in (TU)^e$ which implies $\mu^3 Ta_i \in TU$. As $|\lambda| \leq |\mu|^3$ we have $T(\lambda a_i) \in TU$. By injectivity of T, $\lambda a_i \in U$. If follows that $\lambda a_i \to 0$.

Now we can prove the result announced in the Abstract.

THEOREM 2.7. *Let E, F be K-Banach spaces, let $T \in \mathcal{L}(E,F)$ and let $A \subset E$ be an absolutely convex and closed compactoid. If $y_1, y_2, \dots$ is a sequence in TA tending to 0 then for each $\lambda \in K$, $|\lambda| > 1$ there is a sequence $x_1, x_2, \dots$ in λA tending to 0 such that $Tx_n = y_n$ for each n.*

Proof. Decompose T (see the Introduction):

$$\begin{tikzcd} A \arrow[rr, "T"] \arrow[dr, "\pi"'] & & TA \\ & A/A \cap \operatorname{Ker} T \arrow[ur, "i"'] & \end{tikzcd}$$

Let $z_n := i^{-1}(y_n)$ for each n, let $\lambda \in K$, $|\lambda| > 1$. By Theorem 2.6 $\lim_{n \to \infty} \lambda^{-1} z_n = 0$ in the quotient topology of $A/A \cap \operatorname{Ker} T$. By metrizability there exist $a_1, a_2, \dots \in A$ such that $\lim_{n\to\infty} a_n = 0$ and $\pi(a_n) = \lambda^{-1} z_n$ for each n. Set $x_n := \lambda a_n$ $(n \in \mathbb{N})$. Then $x_n \in \lambda A$, $\lim_{n\to\infty} x_n = 0$ and $Tx_n = y_n$ for each n.

The following is a more general version of Theorem 2.7.

THEOREM 2.8. *Let A_1, A_2 be locally convex modules, let A_1 be metrizable, complete and compactoid and let $T : A_1 \to A_2$ be a continuous surjective module homomorphism. If $U \subset A_1$ is an open submodule then for every $\lambda \in K$, $|\lambda| > 1$ the module $T(\lambda U)$ is open in A_2.*

Proof. We may assume that T is bijective (decompose T in the spirit of above). If 0 is not in the interior of $T(\lambda U)$ then there exists a net $i \mapsto z_i$ in A_2 converging to 0 but $z_i \notin T(\lambda U)$ for each i. Let $x_i := T^{-1}(z_i)$. By Theorem 2.6 we have $\lambda^{-1} x_i \to 0$ so eventually $\lambda^{-1} x_i \in U$ i.e. $x_i \in \lambda U$ i.e. $z_i \in T(\lambda U)$, a contradiction.

§3. THE CASE OF A DISCRETE VALUATION

We extend the definition of a compactoid module to arbitrary ground fields K as follows. For $r \in [0,1]$ and a B_K-module A set

$$rA := \bigcap \{\lambda A : \lambda \in K, |\lambda| \geq r\}$$

A locally convex module A is a *compactoid* if for each $r \in [0,1)$ and each neighbourhood U of 0 in A there exists a finite set $F \subset A$ such that $rA \subset U + \mathrm{co}\, F$. If the valuation of K is dense, this definition is equivalent to 2.1 whereas for discretely valued K one obtains Definition 2.1 but where "$\lambda \in B_K^-$" is replaced by "$\lambda = 1$".

It is not hard to see that all lemmas, propositions and theorems of §1, §2 remain true for discretely valued K even when we take $\lambda := 1$ everywhere and replace Z^e by Z for every occurring module Z.

Problem. Is this last conclusion also true if K is spherically complete?

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KATHOLIEKE UNIVERSITEIT
MATHEMATISCH INSTITUUT
TOERNOOIVELD
6525 ED NIJMEGEN, THE NETHERLANDS