ABSTRACT

Norm Hilbert spaces (NHS) are defined as Banach spaces over valued fields (see 1.4) for which each closed subspace has a norm-orthogonal complement. For fields with a rank 1 valuation, these spaces were characterized already in [10, 5.13, 5.16], where it was proved that infinite-dimensional NHS exist only if the valuation of K is discrete. The first discussion of the case of (Krull) valued fields appeared in [1] and [3]. In this paper we continue and expand this work focusing on the most interesting cases, not covered before. If K is not metrizable then each NHS is finite-dimensional (Corollary 3.2.2), but otherwise there do exist infinite-dimensional NHS; they are completely described in 3.2.5. Our main result is Theorem 3.2.1, where various characterizations of NHS of different nature are presented. Typical results are that NHS are of countable type, that they have orthogonal bases, and that no subspace is linearly homeomorphic to c0.

0. INTRODUCTION

Traditionally, Non-Archimedean Functional Analysis (where the scalar fields \( \mathbb{R} \) and \( \mathbb{C} \) are replaced by a non-Archimedean valued field \( K \)), has been developed for those \( K \) whose valuations have rank 1, i.e., with range in \( \mathbb{R} \). Quite naturally, and also with an eye on building alternative models in Quantum Mechanics, one has been looking within this frame for counterparts of classical Hilbert spaces. Several authors (Kalish, Bayod, Diarra) studied ‘inner products’ on (Banach) spaces over \( K \),

MSC: primary 46S10; secondary 46H35

Key words and phrases: Lipschitz operators, Hilbert spaces, Krull valued fields

E-mails: hochsen@mat.puc.cl (H. Ochsenius), W.Schikhof@math.ru.nl (W.H. Schikhof).

1 Supported by Fondecyt No. 1020710.

2 Supported by Fondecyt No. 7020710.
but it did not bring about spaces that are close to Hilbert space in the sense that, e.g., the Projection Theorem 'each closed subspace has an orthogonal complement' (the property that interests us here) holds. A second approach has been to drop the idea of an inner product, but instead take 'norm orthogonality' \((x \perp y \text{ if } \|x - \lambda y\| \geq \|x\| \text{ for all } \lambda \in K)\) as a starting point, thus defining a Norm Hilbert space (NHS) as a Banach space in which each closed subspace has a norm-orthogonal complement. Such NHS are completely described in [10, 5.13, 5.16], and it is proven that infinite-dimensional NHS do exist but only if the valuation of \(K\) is discrete.

In the meantime, independently, a purely algebraic subject was progressing by studying the so-called 'orthomodular spaces'. These are spaces over a field \(K\), equipped with an inner product such that for each subspace \(D\) we have \(D = D^\perp \perp\) iff \(E = D \oplus D^\perp\) [4]. It was H. Keller [5] who found the first example of an infinite-dimensional orthomodular space, not isomorphic to a classical Hilbert space. His scalar field turned out to have a natural Krull valuation, whose value group is not real, but a more complicated totally ordered group. In [6] this discovery was linked with Functional Analysis. Surprising results concerning bounded and self-adjoint operators appeared in [7] and [8].

In this paper we shall describe the wider class of the NHS over Krull valued fields. A start was made in [1] and [3], but at that moment we restricted ourselves to fields that satisfied a strong countability condition (see [1, 1.4.4]) which among other things implied metrizability. We now develop the theory with far less assumptions, including a discussion of the case of non-metrizable scalar fields. Some ‘known facts’ remain valid in this new setting but under drastically changed proofs, and many new, even stronger results are established. We wish to mention that, with this paper at hand, it is easily seen that the theory of [9] remains valid in this new context.

I. PRELIMINARIES

Below we recall a few notions and facts and add some new ones needed in the paper. For background and examples, see [1].

1.1. Linearly ordered sets

A subset \(S\) of a linearly ordered set \(X\) is bounded or bounded above if there is an \(x \in X\) such that \(s \leq x\) for all \(s \in S\). Similarly we define bounded below. For \(s, t \in X, t \leq s\) we denote \(\{x \in X: t \leq x \leq s\}\) by \([t, s]\). If a set \(S \subset X\) has a smallest upper bound we denote it by \(\sup_X S\). Similarly we define \(\inf_X S\). The (Dedekind) completion of \(X\) is denoted \(X^\#\) [1, 1.1.4].

1.2. Linearly ordered groups

Throughout this paper \(G\) is an abelian multiplicatively written group with unit \(1\), linearly ordered such that \(g, g_1, g_2 \in G, g_1 \leq g_2\) implies \(gg_1 \leq gg_2\). We assume \(G \neq \{1\}\). Then \(G\) is torsion free. A subset \(Z \subset G\) is called convex if \(g_1, g_2 \in Z, g_1 \leq g_2\) implies \([g_1, g_2] \subset Z\). The set of all convex subgroups of \(G\) is linearly ordered by
inclusion. A subgroup $H$ of $G$ is called principal (convex subgroup) if there exists
an $h \in H$ such that $H$ is the smallest convex subgroup of $G$ containing $h$. It is easily
seen that, if $h \geq 1$, then $H = \bigcup_{n \in \mathbb{N}} [h^{-n}, h^n]$. It will be useful to augment $G$ with a
zero element $0_G$ satisfying $0_G < g$, $0_G g = g 0_G = 0_G = 0_G$ for all $g \in G$.

1.3. **$G$-modules**

A linearly ordered set $X$ is called a $G$-module if there is an action $(g, s) \mapsto gs$
($g \in G$, $s \in X$) of $G$ on $X$ that is increasing in both variables and such that for
each $s, t \in X$ there is a $g \in G$ such that $gs < t$. A trivial example is $G$ itself
under multiplication. For $s \in X$ we set $\text{Stab}(s) := \{g \in G : g s = s\}$. It is easy
to see that $\text{Stab}(s)$ is a proper convex subgroup of $G$. For $s, t \in X$ the relation
$s \in Gt$ is an equivalence relation. Let $\sum_X$ (or, shortly, $\sum$) be the collection of
equivalence classes, called the collection of algebraic types of $X$. Let $\sigma : X \to \sum$ be
the canonical map. For $s \in X$ its image $\sigma(s)$ is called the algebraic type of $s$.

In the spirit of 1.2 we adjoin to a $G$-module $X$ a zero element $0_X$ satisfying
$0_X < s$, $0_G s = 0_G 0_X = 0_X$ for all $s \in X$. However, henceforth we omit the subscripts
and write 0 for the zero element augmented to any $G$-module.

Let $I$ be an infinite set, let $i \mapsto s_i$ be a map $I \to X$. We write $\lim_i s_i = 0$ (or,
shortly, $s_i \to 0$) if for each $\varepsilon \in X$ the set $\{i \in I : s_i \geq \varepsilon\}$ is finite.

A sequence $s_1, s_2, \ldots$ in a $G$-module $X$ is said to satisfy the type condition (here,
"type" does not refer to the algebraic types of above) if for all $g_1, g_2, \ldots \in G$, boundedness of $\{g_1 s_1, g_2 s_2, \ldots\}$ implies $g_n s_n \to 0$.

The completion $X^\#$ of a $G$-module $X$ is in a natural way a $G$-module [1, 1.5.4].
In particular, $G^\#$ is a $G$-module.

**Proposition 1.3.1.** (i) Let $H$ be a proper convex subgroup of $G$. Then $s := \sup_{G^\#} H$
and $t := \inf_{G^\#} H$ exist and $\text{Stab}(s) = \text{Stab}(t) = H$.

(ii) The map $g \mapsto g^{-1}$ ($g \in G$) extends uniquely to a decreasing map $\omega : G^\# \to
G^\#$. We have that $\omega \circ \omega$ is the identity, $\omega(gs) = g^{-1} \omega(s)$ for all $g \in G, s \in G^\#$.

(iii) Let $s_1, s_2, \ldots \in G^\#$, $s_n \to 0$. Then there exist $g_1, g_2, \ldots \in G$, $s_n < g_n$ for all
$n$, $g_n \to 0$.

**Proof.** (i) We may assume $H \neq \{1\}$. Let $h \in H$, $h < 1$. Then $hs$ is easily seen to be a
majorant of $H$, so $hs \geq s$, i.e., $hs = s$. Thus $H \subset \text{Stab}(s)$. Conversely if $g \in \text{Stab}(s)$,
g > 1, g \notin H then $g \geq s$, so $1 > s g^{-1} = s$, a contradiction, so also $\text{Stab}(s) \subset H$.

(ii) See [1, 1.3.1].

(iii) There is a strictly decreasing sequence $\varepsilon_1, \varepsilon_2, \ldots$ in $G^\#$ tending to 0 (e.g.,
a suitable subsequence of $n \mapsto \max(s_n, s_{n+1}, \ldots)$). Without loss, assume $\varepsilon_1 >
\max_n s_n$. Then for each $n \in \mathbb{N}$ there is a unique $m(n) \in \mathbb{N}$ such that $\varepsilon_{m(n)+1} \leq s_n <
\varepsilon_{m(n)}$. Now we can choose a $g_n \in G$ such that $s_n < g_n \leq \varepsilon_{m(n)}$. Since $s_n \to 0$ we have
$\lim_{n \to \infty} m(n) = \infty$ so that $g_n \to 0$. □
1.4. Valued fields

A valuation on a field $K$ (with value group $G$) is a surjective map $|: K \to G \cup \{0\}$ such that, for all $\lambda, \mu \in K$, (i) $|\lambda| = 0$ if and only if $\lambda = 0$, (ii) $|\lambda + \mu| \leq \max(|\lambda|, |\mu|)$ and (iii) $|\lambda \mu| = |\lambda||\mu|$. $K = (K, |)$ is called a valued field. Notice that $B_K := \{\lambda \in K : |\lambda| \leq 1\}$ is a subring of $K$. The valuation $|$ induces a Hausdorff field topology on $K$ in the usual way and we have the familiar notions of convergent and Cauchy nets in $K$. $K$ is called complete if each Cauchy net in $K$ converges.

We now characterize metrizability of $K$.

**Proposition 1.4.1.** For a valued field $(K, |)$ with value group $G$ the following are equivalent:

$(\alpha)$ $K$ is (ultra)metrizable.

$(\beta)$ There is a sequence $\lambda_1, \lambda_2, \ldots \in K$ such that $|\lambda_1| > |\lambda_2| > \cdots$ and $\lim_n \lambda_n = 0$ (equivalently, $|\lambda_n| \to 0$).

$(\gamma)$ Either $G$ is principal or $G$ is the union of a strictly increasing sequence of convex subgroups.

$(\delta)$ Each $G$-module has a coinitial (cofinal) sequence.

$(\epsilon)$ There is a $G$-module having a coinitial (cofinal) sequence.

**Proof.** For $(\alpha) \Leftrightarrow (\beta)$ see [1, 1.4.1]. We prove $(\beta) \Rightarrow (\gamma) \Rightarrow (\delta) \Rightarrow (\epsilon) \Rightarrow (\beta)$. Suppose $(\beta)$. Then there are $g_1, g_2, \ldots \in G$, $1 > g_1 > g_2 > \cdots$, $g_n \to 0$. For each $n$, let $H_n$ be the principal convex subgroup generated by $g_n$. Then $H_1 \subset H_2 \subset \cdots$, $\bigcup_n H_n = G$. If the sequence becomes stationary then $G$ is principal. Otherwise there is a strictly increasing subsequence, so we have $(\gamma)$.

To prove $(\gamma) \Rightarrow (\delta)$, let $X$ be a $G$-module, $s \in X$. If $G$ is principal, generated by $g < 1$ then $g^n \to 0$ so $g^n s \to 0$. Otherwise we have a strictly increasing sequence of convex subgroups $H_1 \subset H_2 \subset \cdots$ whose union is $G$. Choose, for each $n$, a $g_n \in H_{n+1} \setminus H_n$, $g_n < 1$. Then $g_1 > g_2 > \cdots$, $g_n \to 0$, hence $g_n s \to 0$.

$(\delta) \Rightarrow (\epsilon)$ is trivial. Now let $X$ be a $G$-module having a coinitial sequence $s_1, s_2, \ldots$. There are $\lambda_1, \lambda_2, \ldots \in K$, $\lambda_n \neq 0$ for all $n$ such that $\lambda_n s_1 < s_n$ for each $n$. Without loss, assume $|\lambda_1| > |\lambda_2| > \cdots$. From $s_n \to 0$ it follows that $\lambda_n s_1 \to 0$, hence $\lambda_n \to 0$ and we have $(\beta)$. \qed

**Corollary 1.4.2.** If $K$ is not metrizable, each sequence in any $G$-module is bounded and bounded below.

We will be concerned mostly with the following class of (metrizable) fields.

**Proposition 1.4.3.** For a valued field $(K, |)$ with value group $G$ the following are equivalent:

$(\alpha)$ $K$ is metrizable, $G$ is not principal.

$(\beta)$ $G$ is the union of a strictly increasing sequence of convex subgroups.
$G$ is the union of a strictly increasing sequence of principal convex subgroups.

There is a $G$-module $X$ and a sequence in $X$ satisfying the type condition.

**Proof.** $(\alpha) \iff (\beta)$ follows from Proposition 1.4.1. To prove $(\beta) \Rightarrow (\gamma)$ let $H_1 \subset H_2 \subset \cdots$ be a strictly increasing sequence of convex subgroups covering $G$. For $n \geq 2$, choose $g_n \in H_n \setminus H_{n-1}$ and let $H_n'$ be the principal convex subgroup generated by $g_n$. Then $H_{n-1} \subseteq H_n' \subset H_n$ for each $n \geq 2$. It follows that $H_2' \subset H_3' \subset \cdots$ is strictly increasing and $\bigcup_n H_n' = G$.

Next we prove $(\gamma) \Rightarrow (\delta)$. Let $H_1 \subset H_2 \subset \cdots$ be a strictly increasing sequence of convex subgroups covering $G$. Let $s_n := \sup_{H_n} H_n$; we show that $s_1, s_2, \ldots$ is a sequence in the $G$-module $G$ satisfying the type condition. Let $g_1, g_2, \ldots \in G$ be such that $\{g_n s_n : n \in \mathbb{N}\}$ is bounded. Without loss, assume $g_n s_n \leq 1$ for all $n$. Then for each $h \in H_n$ we have $h^{-1} \in \text{Stab}(s_n)$ (Proposition 1.3.1(i)) so that $g_n h^{-1} s_n \leq 1$ or $g_n s_n \leq h$. Therefore $g_n s_n \leq \inf_{G} H_n \to 0$.

Finally we prove $(\delta) \Rightarrow (\alpha)$. Let $s_1, s_2, \ldots$ satisfy the type condition in some $G$-module $X$. By 1.4.2 $K$ must be metrizable. Suppose $G$ is principal; we derive a contradiction. Let $g > 1$ be a generator of $G$, let $s_0 \in X$. Then $g s_0, g^2 s_0, \ldots$ is cofinal and $g^{-1} s_0, g^{-2} s_0, \ldots$ is coinitial in $X$. Thus, for each $n$ there is an $m_n \in \mathbb{Z}$ such that $g^{m_n} s_0 \leq s_n \leq g^{m_n+1} s_0$, and the sequence $n \mapsto g^{-m_n} s_n$ is bounded above and below, a contradiction. \[\Box\]

2. BANACH SPACES

From now on in this paper $K = (K, | |)$ is a complete valued field with value group $G$.

For an impression of the immense class of such fields we refer to [2].

Let $E$ be a $K$-vector space. By restricting the scalar multiplication only to $B_K$, $E$ becomes a $B_K$-module. $B_K$-submodules of $E$ are often called absolutely convex sets. For example, the absolutely convex subsets of $K$ are $K, \{\lambda \in K : |\lambda| \leq r\}$ where $r \in G^* \cup \{0\}, \{\lambda \in K : |\lambda| < r\}$ where $r \in G^*$. For a set $S \subset E$ we denote by $[S]$ its $K$-linear span, by $\text{aco} S$ the $B_K$-module generated by $S$. A hyperplane is a linear subspace of codimension 1.

2.1. Normed spaces

Let $X$ be a $G$-module, let $E$ be a $K$-vector space. A $X$-norm on $E$ is a map $\| | : E \to X \cup \{0\}$ such that for all $x, y \in E, \lambda \in K$ we have (i) $\|x\| = 0$ if and only if $x = 0$, (ii) $\|\lambda x\| = |\lambda| \|x\|$ and (iii) $\|x + y\| \leq \max(\|x\|, \|y\|)$. Then $E = (E, \| |)$ is called an $(X)$-normed space. For $a \in E, r \in X$ we define balls as usual:

$$B_E(a, r) := \{x \in E : \|x - a\| \leq r\},$$

$$B_E(a, r^-) := \{x \in E : \|x - a\| < r\}.$$  

They induce a Hausdorff vector topology in the usual way. Convergent and Cauchy nets are defined. $E$ is called an $(X$-normed) Banach space if each Cauchy net
converges. If $K$ is metrizable the induced topology on $E$ is metrizable and $E$ is a Banach space if and only if every Cauchy sequence converges. If $K$ is not metrizable then neither is any nonzero normed $K$-vector space.

The closure of a set $Z \subseteq E$ is denoted by $\overline{Z}$. $E$ is called of countable type if there is a countable set $C \subseteq E$ such that $[C] = E$. A subset $Z \subseteq E$ is called bounded if $Z \subseteq B_E(0, r)$ for some $r \in X$.

Now let $F$ be a $Y$-normed space for some $G$-module $Y$. Then a linear map $E \to F$ is continuous if and only if it maps bounded sets into bounded sets. We write $E \sim F$ to indicate that $E$, $F$ are linearly homeomorphic.

A sequence $x_1, x_2, \ldots \in E \setminus \{0\}$ is said to satisfy the type condition if $\|x_1\|, \|x_2\|, \ldots$ satisfies the type condition. If $T : E \to F$ is a linear homeomorphism then a sequence $x_1, x_2, \ldots \in E \setminus \{0\}$ satisfies the type condition if and only if $T x_1, T x_2, \ldots$ satisfies the type condition.

If $E, F$ are both $X$-normed then a linear map $T : E \to F$ is called a (linear) isometry if $\|T x\| = \|x\|$ for all $x \in E$. We write $E \simeq F$ to indicate that $E, F$ are isometrically isomorphic.

Notice that we did not require surjectivity for the norm function. So, if $E$ is $X$-normed and $X$ is a $G$-submodule of a $G$-module $Y$ then $E$ is automatically $Y$-normed. This enables us to assume without harm that $X$ is complete. If $G$ is a submodule of $X$ (e.g., if $X = G^y$) we sometimes write $B_E := B_E(0, 1)$ and $B_E^\ominus := B_E(0, 1^-)$.

Next, we introduce quotients. Let $E$ be an $X$-normed space, where $X$ is complete, let $D \subseteq E$ be a closed subspace, let $\pi : E \to E/D$ be the canonical map. Then the formula

$$\|\pi(x)\| = \inf_{X \cup \{0\}} \{\|x - d\| : d \in D\}$$

defines an $X$-norm on $E/D$, the so-called quotient norm. If $K$ is metrizable and $E$ is a Banach space then so is $E/D$ [1, 2.5.1].

An $X$-normed space $F$ is called a quotient of $E$ if there exists a continuous linear surjection $T : E \to F$ such that the map $T_1$ in the natural factorization

$$E \xrightarrow{\pi} E/D \xrightarrow{T_1} F$$

is an isometry. Such a $T$ is called a quotient map. It is shown in [1, 2.2.2] that a linear map $T : E \to F$ is a quotient map if and only if $T(B_E(0, r^-)) = B_F(0, r^-)$ for each $r \in X$. Compositions of quotient maps are quotient maps. Notice that, if $E, F$ are $G^y$-normed, a linear map $T : E \to F$ is a quotient map if and only if $T(B_E^-) = B_F^-$. Let $I$ be an infinite set, let $i \mapsto x_i$ be a map $I \to E$ where $E$ is an $X$-normed space. We say that $\lim_{i \in I} x_i = 0$ (or, shortly, $x_i \to 0$) if for each $\varepsilon \in X$ the set $\{i \in I : \|x_i\| \geq \varepsilon\}$ is finite. Similarly, if $x \in E$, we say that $x = \sum_{i \in I} x_i$ if for each $\varepsilon \in X$ there exists a finite set $J_0 \subseteq I$ such that for all finite sets $J, J_0 \subseteq J \subseteq I$, we
have \( \|x - \sum_{i \in J} x_i\| < \varepsilon \). If \( x = \sum_{i \in I} x_i \) then \( x_i \to 0 \). The converse holds if \( E \) is a Banach space. It is not hard to see that if \( I = \mathbb{N} \) the above concepts coincide with the usual ones.

The following examples will play a key role in Section 3. Let \( X \) be a \( G \)-module, let \( s := (s_1, s_2, \ldots) \in X^\mathbb{N} \). Then \( c_0(s) \) is the space of all \( (\lambda_1, \lambda_2, \ldots) \in \mathbb{K}^\mathbb{N} \) for which \( |\lambda_n|s_n \to 0 \). It is a Banach space with respect to the norm

\[
(\lambda_1, \lambda_2, \ldots) \mapsto \max_n |\lambda_n|s_n.
\]

If \( X = G \) and \( s = (1, 1, \ldots) \) we simply write \( c_0 \) instead of \( c_0(s) \). Its norm \( (\lambda_1, \lambda_2, \ldots) \mapsto \max_n |\lambda_n| \) is denoted \( \| \cdot \|_\infty \). Finally, \( c_{00} \) is the subspace of \( c_0 \) consisting of all sequences \( (\lambda_1, \lambda_2, \ldots) \in \mathbb{K}^\mathbb{N} \) for which \( \lambda_n = 0 \) eventually.

### 2.2. Orthogonality

Let \((E, \| \cdot \|)\) be a normed space over \( K \). Two subspaces \( D_1 \) and \( D_2 \) are called (norm) orthogonal, notation \( D_1 \perp D_2 \), if for each \( d_1 \in D_1, d_2 \in D_2 \) we have \( \|d_1 + d_2\| = \max(\|d_1\|, \|d_2\|) \). We sometimes write \( x \perp y \) to indicate that \( Kx \perp Ky \). A subspace \( D \) of \( E \) is said to be orthocomplemented (in \( E \)) if there is a subspace \( S \subseteq D \) such that \( D + S = E \). Such an \( S \) (in general not unique) is called an orthocomplemented or orthocomplement of \( D \). A continuous linear map \( P : E \to E \) is called an orthogonal projection if \( P^2 = P \) and \( \|Px\| \leq \|x\| \) for each \( x \in E \). Then \( PE \) and \( \text{Ker} P \) are orthocomplements of one another. Conversely, a subspace \( D \) of \( E \) is orthocomplemented if and only if there is an orthogonal projection \( P \) with \( PE = D \).

A collection \( \{e_i : i \in I\} \subseteq E \), where \( e_i \neq 0 \) for each \( i \), is called an orthogonal system if, for each \( i \in I, \{e_i\} \perp \{e_j : j \neq i\} \). This is the case if and only if for each finite set \( J \subseteq I \) and \( \{\lambda_j : j \in J\} \subseteq K \)

\[
\left\| \sum_{j \in J} \lambda_j e_j \right\| = \max_{j \in J} \|\lambda_j e_j\|.
\]

A sequence \( e_1, e_2, \ldots \) is called orthogonal if \( \{e_1, e_2, \ldots\} \) is an orthogonal system. If, in addition, \( \|e_1\| > \|e_2\| > \cdots \), we call \( e_1, e_2, \ldots \) a strictly decreasing orthogonal sequence.

We quote two facts.

#### Proposition 2.2.1

Let \( E \) be a normed space, let \( \{e_i : i \in I\} \subseteq E \setminus \{0\} \).

(i) If \( \sigma(\|e_i\|) \neq \sigma(\|e_j\|) \) whenever \( i \neq j \), then \( \{e_i : i \in I\} \) is orthogonal.

(ii) (Perturbation lemma) If \( \{e_i : i \in I\} \) is orthogonal and \( \{f_i : i \in I\} \subseteq E \) is such that \( \|e_i - f_i\| < \|e_i\| \) for each \( i \in I \) then also \( \{f_i : i \in I\} \) is orthogonal.

#### Proof

[1, 3.2.8] and [1, 2.4.8]. \( \square \)
An orthogonal system \{e_i : i \in I\} in a Banach space \(E\) is called an orthogonal base (of \(E\)) if \(\{e_i : i \in I\} = E\). This is the case if and only if each \(x \in E\) has a unique expansion
\[
x = \sum_{i \in I} \lambda_i e_i \quad (\lambda_i \in K, \lambda_i e_i \to 0).
\]

Then, by continuity, \(\|x\| = \max_{i \in I} \|\lambda_i e_i\|\). Conversely, if \(\{\lambda_i : i \in I\} \subset K\) such that \(\sum_{i \in I} \lambda_i e_i \to 0\) then \(\sum_{i \in I} \lambda_i e_i\) exists and represents an element of \(E\).

Let \(X\) be a \(G\)-module and let \(\{E_i : i \in I\}\) be a collection of \(X\)-normed Banach spaces. The orthogonal direct sum \(\bigoplus_{i \in I} E_i\) of \(\{E_i : i \in I\}\) is the subspace of all \(x = (x_i)_{i \in I} \subset \prod_{i \in I} E_i\) for which \(x_i \to 0\), normed by \(x \mapsto \max_i \|x_i\|\). Then \(\bigoplus_{i \in I} E_i\) is a Banach space. In particular we say that a Banach space \(E\) is the orthogonal direct sum of the subspaces \(\{E_i : i \in I\}\) if the map \(\bigoplus_{i \in I} E_i \to E\) given by \(x \mapsto \sum_{i \in I} x_i\) is a surjective isometry. We then also write \(E = \bigoplus_{i \in I} E_i\).

The following proposition contains results, essentially from [1], but modified for our purpose.

**Proposition 2.2.2.** Let \(E\) be an \(X\)-normed Banach space, let \(\{e_i : i \in I\}\) and \(\{f_j : j \in J\}\) be two maximal orthogonal systems in \(E\). For each \(\sigma \in \Sigma = \sum_{i \in I} \lambda_i \|e_i\|\), let \(I_{\sigma} := \{i \in I : \|e_i\| \in \sigma\}\), \(E_{\sigma} := \{e_i : i \in I_{\sigma}\}\), similarly, \(J_{\sigma} := \{j \in J : \|f_j\| \in \sigma\}\), \(F_{\sigma} := \{f_j : j \in J_{\sigma}\}\). Then we have the following:

(i) There are a bijection \(\tau : I \to J\) and \(\lambda_i : i \in I\) \(\subset K\) such that \(\|e_i\| = \|e_i f_{\tau(i)}\|\) for each \(i \in I\).

(ii) There is a linear surjective isometry \([e_i : i \in I] \to [f_j : j \in J]\) mapping \(Ke_i\) onto \(Kf_{\tau(i)}\) for each \(i \in I\).

(iii) If \(\{e_i : i \in I\}\) and \(\{f_j : j \in J\}\) are orthogonal bases of \(E\) then \(E = \bigoplus_{\sigma \in \Sigma} E_{\sigma}\) (and \(F = \bigoplus_{\sigma \in \Sigma} F_{\sigma}\)). For each \(\sigma \in \Sigma\) we have \(E_{\sigma} \cong F_{\sigma}\) and they have orthogonal bases whose members all have the same norm.

**Proof.** Statements (ii) and (iii) are straightforward consequences of (i). To prove (i) observe that by [1, 2.4.13] and [1, 2.4.12] for each \(\sigma \in \Sigma\), the sets \(I_{\sigma}\) and \(J_{\sigma}\) have the same cardinality, so there is a bijection \(\tau_{\sigma} : I_{\sigma} \to J_{\sigma}\), and we can find \(\lambda_i : i \in I_{\sigma}\) \(\subset K\) such that \(\|e_i\| = \|\lambda_i f_{\tau_{\sigma}(i)}\|\) for each \(i \in I_{\sigma}\). Since the \(I_{\sigma}\) (\(J_{\sigma}\)) form a partition of \(I (J)\) we can glue the \(\{\tau_{\sigma} : \sigma \in \Sigma\}\) together to arrive at the desired \(\tau\).

\(\square\)

### 2.3. Line orthogonal spaces

**Definition 2.3.1.** A Banach space over \(K\) is called line orthogonal if every one-dimensional subspace is orthocomplemented.

**Proposition 2.3.2.** (i) Closed subspaces of line orthogonal spaces are line orthogonal.

(ii) Banach spaces with an orthogonal base are line orthogonal.

72
(iii) In a line orthogonal space every finite-dimensional subspace is orthocomplemented.

**Proof.** (i) Let $D_1 \subset D$ be subspaces of a line orthogonal space $E$, where $D$ is closed and $D_1$ is one-dimensional. $D_1$ has an orthocomplement $S$ in $E$. Then $S \cap D$ is an orthocomplement of $D_1$ in $D$.

(ii) Let \( \{e_i : i \in I\} \) be an orthogonal base of a Banach space $E$, let $a \in E \setminus \{0\}$ have the expansion $a = \sum_{i \in I} \lambda_ie_i$. There is a $j \in I$ for which $\|a\| = \|\lambda_je_j\|$. One checks easily that $\{e_i : i \neq j\}$ is an orthocomplement of $Ka$.

(iii) See [1, 3.3.1]. \( \square \)

We do not know whether the converse of (ii) above holds, but it seems doubtful. Yet we do have the following.

**Theorem 2.3.3.** A line orthogonal Banach space of countable type has a countable orthogonal base.

**Proof.** Assume the space $E$ under consideration is infinite-dimensional (for the finite-dimensional case the process below breaks off). There are subspaces $D_1 \subset D_2 \subset \cdots$ with $\dim D_n = n$ for each $n$ whose union is dense in $E$. By 2.3.2 each $D_n$ is orthocomplemented in $D_{n+1}$, so we can choose $e_1 \in D_1$, $e_1 \neq 0$ and, for $n \geq 2$, a nonzero $e_n \in D_n$, $Ke_n \perp D_{n-1}$. Then the sequence $e_1, e_2, \ldots$ is easily seen to be orthogonal and since $[e_1, e_2, \ldots, e_n] = D_n$ for each $n$, the linear hull of $e_1, e_2, \ldots$ is dense in $E$, so it is an orthogonal base. \( \square \)

2.4. Finite-dimensional normed spaces

It was proved in [1, 2.3.4] that all norms on a finite-dimensional space induce the same topology and that it is a Banach space for each norm. These facts, together with the results of Section 2.3 lead to the following corollary.

**Proposition 2.4.1.** For a finite-dimensional normed space over $K$ the following are equivalent.

(\( \alpha \)) $E$ is line orthogonal.

(\( \beta \)) Each subspace of $E$ is orthocomplemented.

(\( \gamma \)) $E$ has an orthogonal base.

(\( \delta \)) Each subspace of $E$ has an orthogonal base.

If $K$ happens to be spherically complete (i.e., for any non-empty collection $C$ of balls in $K$, linearly ordered by inclusion, we have $\bigcap C \neq \emptyset$) every finite-dimensional normed space over $K$ has the properties (\( \alpha \))–(\( \delta \)) of above [1, 3.3.3].

The converse is also true.

**Proposition 2.4.2.** Let $K$ be not spherically complete. Then there exists a two-dimensional $G$-normed space (called $K_2^2$) over $K$ that has no orthogonal base.
Proof. Let \( C \) be a non-empty collection of balls in \( K \), linearly ordered by inclusion, \( \bigcap C = \emptyset \). For \( B_1, B_2 \in C \) write \( B_1 \supseteq B_2 \) if \( B_1 \subseteq B_2 \). Then \( (C, \subseteq) \) is a directed set. For each \( B \in C \), choose a \( b_B \in B \). Now let \( (\lambda, \mu) \in K^2 \). The net \( B \mapsto |\lambda - \mu b_B| \) is easily seen to be eventually constant so we can define

\[
\nu(\lambda, \mu) := \lim_B |\lambda - \mu b_B| \in G \cup \{0\}.
\]

With the same methods as in [10, p. 68], one shows that \( \nu \) is a norm on \( K^2 \) and that the subspace \( \{ (\lambda, 0) : \lambda \in K \} \) has no orthogonal complement. \( \square \)

### 2.5. Spaces of countable type

In this section we will prove the following extension of [1, 3.2.4].

**Theorem 2.5.1.** Let \( E \) be a Banach space over \( K \) of countable type. Suppose

(i) \( K \) is non-metrizable, or
(ii) \( G \) is principal, or
(iii) each \( B_K \)-submodule of \( K \) is countably generated.

Then each closed subspace of \( E \) is of countable type.

We regret to confess that the general case remains open.

**Problem.** Are closed subspaces of Banach spaces of countable type again of countable type?

To attack the problem one has to consider (Proposition 1.4.3) the case where \( G \) is the union of a strictly increasing sequence of convex subgroups and (iii) above does not hold. A major step in solving the problem would be the answer to the question as to whether closed subspaces of \( c_0 \) are of countable type, or, equivalently, have an orthogonal base (Proposition 2.3.2, Theorem 2.3.3). In Section 3 we will study the class of Norm Hilbert spaces and prove that the answer is positive for such spaces (Corollary 3.2.3).

For the general case we have only the following two results.

**Proposition 2.5.2.** Let \( E \) be a Banach space of countable type, let \( D \) be a closed subspace of finite codimension. Then \( D \) is of countable type.

**Proof.** It suffices to consider the case of a hyperplane \( D \). Let \( a \in E \setminus D \). Then each \( x \in E \) has a unique decomposition \( x = \lambda a + d \) where \( \lambda \in K \), \( d \in D \). Then \( x \mapsto d \) is a linear surjection \( E \to D \); if suffices to prove continuity. So let \( i \mapsto x_i = \lambda_i a + d_i \) be a net converging to 0. By applying the quotient map \( \pi : E \to E/D \) we see that \( \pi(x_i) = \lambda_i a \to 0 \). Then also \( d_i \to 0 \) and we are done. \( \square \)
Proposition 2.5.3. Let $E$ be a Banach space of countable type. Let $D$ be a closed subspace with an orthogonal base. Then $D$ is of countable type.

Proof. Let $F$ be a subspace of countable algebraic dimension that is dense in $E$, let $\{e_i: i \in I\}$ be an orthogonal base of $D$. For each $i \in I$, choose a $y_i \in F$ such that $\|e_i - y_i\| < \|e_i\|$. By the Perturbation Lemma 2.2.1(ii) the system $\{y_i: i \in I\}$ is orthogonal, hence linearly independent. Then $I$ is countable. □

We now consider Theorem 2.5.1.

Lemma 2.5.4. Let $E$ be an infinite-dimensional normed space of countable type over a non-metrizable $K$. Then $E \sim c_{00}$.

Proof. There are subspaces $D_1 \subset D_2 \subset \cdots$, with $\dim D_n = n$ for each $n$, and whose union is dense in $E$. Suppose there exists an $x \in E \setminus \bigcup_n D_n$; we derive a contradiction. Let $(x_i)_{i \in I}$ be a net in $\bigcup_n D_n$ converging to $x$. For each $n \in \mathbb{N}$ set $I_n := \{i \in I: x_i \in D_n\}$. Then $I_1 \subset I_2 \subset \cdots \subset I_n = I$. Clearly no $I_n$ is cofinal, so for each $n \in \mathbb{N}$ there is an $i_n \in I$ such that $i < i_n$ for all $i \in I_n$. Then $i_1, i_2, \ldots$ is cofinal in $I$ so the sequence $x_{i_1}, x_{i_2}, \ldots$ converges to $x$, and $n \mapsto \|x - x_{i_n}\|$ is a coinitial sequence in $\|E\|\{0\}$, conflicting Corollary 1.4.2. So $E$ has countable algebraic dimension and there is an algebraic isomorphism $T : E \to c_{00}$. Now all norms on $c_{00}$ induce the same topology [1, 2.4.18] and it follows that $T$ is a homeomorphism. □

Remark. We see that the space $E$ of above is complete, but not a Baire space. Each subspace of $E$ is closed and – since its dimension is at most countable – of countable type.

Lemma 2.5.5. Let $G$ be principal, let $E$ be a Banach space of countable type. Then so is every closed subspace.

Proof. First consider the special case where $E$ is $G^\#$-normed and $G$ has only $\{1\}$ and $G$ as convex subgroups. Then by [11, A. Prop. 1] we may assume $G \subset (0, \infty)$ and so $G^\# \subset (0, \infty)$. Then the valuation and norm are real-valued and the conclusion follows from [10, 3.16]. Now let $E$ be $X$-normed for some $G$-module $X$. Let $G$ be generated as a convex group by $g$. Then the largest convex subgroup $H$ not containing $g$ is a maximal proper convex subgroup. Then the naturally linearly ordered group $G/H$ is isomorphic to a subgroup of $(0, \infty)$. Letting $\pi : G \to G/H$ and $\varphi : X \to X/\sim$ be the canonical maps, where for $s, t \in X$, $s \sim t$ if there are $h_1, h_2 \in H$ such that $h_1 t \leq s \leq h_2 t$, we see that the formula $v(\lambda) = \pi(|\lambda|) (\lambda \neq 0)$, together with $v(0) = 0$ defines a valuation $v$ on $K$ that induces the same topology on $K$ as $|\cdot|$. Also, the formula $\pi(g)\varphi(s) = \varphi(gs) (g \in G, s \in X)$ defines a $G/H$-module structure on $X/\sim$. Then $E$ is an $X/\sim$-normed space over $(K, v)$ and its induced topology is the same as the original one, since $\varphi$ is surjective and increasing. Thus, we have reduced the problem to the case $G \subset (0, \infty)$, $E$ is $Y$-normed for some
G-module Y. Now [1, 2.1.9] shows that we may assume $Y = G^\#$ and we are in the case discussed at the beginning of the proof. □

**Proof of Theorem 2.5.1.** Lemma 2.5.4 and the remark following it take care of (i). Lemma 2.5.5 proves (ii). Finally, (iii) is covered by [1, 3.2.4]. □

### 2.6. Quotients of $c_0$

The results of this subsection will be needed for the next section. Any quotient of $c_0$ is necessarily $G^\#$-normed and of countable type. The converse was proved in [1, 3.2.6] but under the assumption (iii) of our Theorem 2.5.1. We will see in Corollary 2.6.3 that it is no longer true if we drop (iii).

Fortunately, a corollary [1, 4.3.4], needed for the next section, can be ‘saved’ in the general setting by modifying the original proof (Theorem 2.6.5).

**Proposition 2.6.1.** Let $E$ be a $G^\#$-normed Banach space with a (finite or) countable orthogonal base $e_1, e_2, \ldots$. Then $E$ is a quotient of $c_0$ if and only if, for each $n$, $K e_n$ is a quotient of $c_0$.

**Proof.** We only treat the infinite-dimensional case. Let $\pi : c_0 \to E$ be a quotient map, let $P_n : E \to K e_n$ be the canonical projection. Then $P_n \circ \pi : c_0 \to K e_n$ is a quotient map for every $n$.

Conversely, suppose we have quotient maps $\pi_n : c_0 \to K e_n$ for each $n$. By gluing them together we obtain a map

$$\pi : F := \bigoplus_{n \in \mathbb{N}} c_0 \to E$$

via the formula

$$\pi(x) = \sum_{n=1}^{\infty} \pi_n(x_n) \quad (x = (x_n)_{n \in \mathbb{N}} \in F).$$

Then $\pi$ is linear, $\|\pi(x)\| \leq \|x\|$ for all $x \in F$. Now we prove that $\pi$ is a quotient map. (Then we are done since $F$ is easily seen to be $\simeq c_0$.) For that it is enough to show that $\pi(B_F^-) = B_E^-$ (see Section 2.1), in fact only $B_E^- \subset \pi(B_F^-)$ needs proof. So, let $z \in B_E^-$ have expansion $\sum_n \lambda_n e_n$. Then $\|\lambda_n e_n\| < 1$ for all $n$ and $\|\lambda_n e_n\| \to 0$.

By Proposition 1.3.1(iii) there are $g_1, g_2, \ldots \in G$, $g_n < 1$, such that $\|\lambda_n e_n\| < g_n$ for each $n$ and $g_n \to 0$. For each $n$, $\pi_n$ is a quotient map, so there exists an $x_n \in c_0$ such that $\pi_n(x_n) = \lambda_n e_n$ and $\|x_n\|_{\infty} < g_n$. Then $x_n \to 0$ so $x := (x_n)_{n \in \mathbb{N}} \in F$. We have $\pi(x) = \sum_n \pi_n(x_n) = \sum_n \lambda_n e_n = z$ and $\|x\| = \max_n \|x_n\|_{\infty} < \max_n g_n < 1$. □

**Theorem 2.6.2.** Let $E$ be a $G^\#$-normed Banach space with a (finite or) countable orthogonal base $e_1, e_2, \ldots$. Then $E$ is a quotient of $c_0$ if and only if, for each $n$,

$$D_n := \{ \lambda \in K : |\lambda| \leq \omega(\|e_n\|) \}$$

is countably generated as a $B_K$-module.
Proof. Again we only consider the infinite-dimensional case. Suppose $E$ is a quotient of $c_0$. Let $n \in \mathbb{N}$. If $\|e_n\| \in G$ then $\omega(\|e_n\|) = \|e_n\|^{-1} \in G$ so $D_n$ is generated by a single element $\lambda \in K$ for which $|\lambda| = \|e_n\|^{-1}$. Now assume $\|e_n\| \notin G$. By Proposition 2.6.1 there is a quotient map $\pi_n : c_0 \to K e_n$, so that $\pi_n(B_{c_0}^-) = \{\lambda e_n : |\lambda| < \omega(\|e_n\|)\} = D_n e_n$. Now if $x \in c_0$, $\|x\|_\infty = 1$ then $\|\pi_n(x)\| \leq 1$ but also $\|\pi_n(x)\| \notin G$ so that $\|\pi_n(x)\| < 1$. It follows that even $\pi_n(B_{c_0}^-) = D_n e_n$. The $B_K$-module generated by the canonical base of $c_0$ is dense in $B_{c_0}$. Thus, $D_n e_n$ contains a dense countably generated $B_K$-module. But all one-dimensional $B_K$-modules are closed and it follows that $D_n e_n$ itself is countably generated, hence so is $D_n$.

Conversely, suppose $D_n$ is countably generated for each $n$. Let $n \in \mathbb{N}$; by Proposition 2.6.1 it suffices to construct a quotient map $\pi : c_0 \to K e_n$. If $\|e_n\| \in G$ we can take for $\pi$ the map $(\lambda_1, \lambda_2, \ldots) \mapsto \lambda_1 \mu e_n$ where $\mu \in K$, $|\mu| = \|e_n\|^{-1}$; so assume $\|e_n\| \notin G$. Then $D_n = \{\lambda \in K : |\lambda| < \omega(\|e_n\|)\}$ and by assumption there are $\lambda_1, \lambda_2, \ldots \in K$, $|\lambda_1| < |\lambda_2| < \cdots$ such that $\sup_{G^\#}(|\lambda_n| : n \in \mathbb{N}) = \omega(\|e_n\|)$. Now set

$$
\pi(\xi_1, \xi_2, \ldots) = \left( \sum_{i=1}^\infty \xi_i \lambda_i \right) e_n \quad ((\xi_1, \xi_2, \ldots) \in c_0).
$$

Clearly, $\pi$ maps $B_{c_0}^-$ into $D_n e_n = \{x \in K e_n : \|x\| < 1\}$. Conversely, if $\mu \in K$, $\mu \neq 0$, $|\mu e_n| < 1$ then $\mu \in D_n$ so there is an $m$ such that $|\lambda_m| > |\mu|$. Then $\mu e_n = \pi(0, 0, \ldots, \mu \lambda_m^{-1}, 0, \ldots) \in \pi(B_{c_0}^-)$ (here, $\mu \lambda_m^{-1}$ is at the $m$th place) and $\pi$ is a quotient map. □

Now we can see that $[1, 3.2.6]$ is not true for general $K$.

**Corollary 2.6.3.** Suppose there exists an $r \in G^\#$ such that $\{\lambda \in K : |\lambda| \leq r\}$ is not countably generated as a $B_K$-module. Then there exists a one-dimensional $G^\#$-normed space over $K$ that is not a quotient of $c_0$.

**Proof.** Consider the space $K$ with the norm $\lambda \mapsto |\lambda| \omega(r)$ and apply Theorem 2.6.2. □

**Corollary 2.6.4.** Let $K$ be metrizable and let $G$ be not principal. Then there exists a sequence $t := (t_1, t_2, \ldots)$ in $G^\#$ that satisfies the type condition and such that $c_0(t)$ is a quotient of $c_0$.

**Proof.** By Proposition 1.4.3 $G$ is the union of a strictly increasing sequence $H_1 \subset H_2 \subset \cdots$ of principal convex subgroups. Say, $H_n$ is generated by $g_n > 1$. Then $H_n = \bigcup_k [g_n^{-k}, g_n^k]$, so $s_n := \sup_{G^\#} H_n = \sup \{g_n, g_n^2, \ldots\}$. Then, after choosing a $\lambda_n \in K$ with $|\lambda_n| = g_n$, we see that the $B_K$-module $\{\lambda \in K : |\lambda| \leq s_n\}$ is generated by the countable set $\{\lambda_n, \lambda_n^2, \ldots\}$. Now choose $t_n := \omega(s_n) = \inf_{C^\#} H_n$ for each $n$, and apply Theorem 2.6.2 to conclude that $c_0(t)$ is a quotient of $c_0$. A similar reasoning as in the proof of Proposition 1.4.3. $(\gamma) \Rightarrow (\delta)$ shows that $t_1, t_2, \ldots$ satisfies the type condition. □
Now we reach the goal of this section.

**Theorem 2.6.5.** Let $K$ be metrizable and let $G$ be not principal. Then there is a closed subspace $D$ in $c_0$ without closed complement (i.e., there does not exist a closed subspace $S$ such that $S \cap D = \{0\}$, $S + D = c_0$).

**Proof.** Let $c_0(t)$ be as in Corollary 2.6.4. Then we have $c_0(t) \cong c_0/D$ for some closed subspace $D$ of $c_0$. Now suppose that $D$ has a closed complement $S$; we derive a contradiction. Since $K$ is metrizable the Open Mapping Theorem [1, 2.5.3] holds, so we have $c_0 \sim D \oplus S$. Combining this with the orthoprojection $D \oplus S \to S$ we obtain a continuous linear surjection $c_0 \to S$ with kernel $D$. By applying again the Open Mapping Theorem we obtain $c_0(t) \sim S$. Now $c_0(t)$ has a sequence satisfying the type condition, so (see Section 2.1) $S$ must have one. But this is impossible since $S$ is $G$-normed. (If $g_1, g_2, \ldots \in G$ then $g_1^{-1} g_1, g_2^{-1} g_2, \ldots$ is bounded but does not tend to zero.) $\square$

3. NORM HILBERT SPACES

### 3.1. Definition and first properties

**Definition 3.1.1.** A Banach space over $K$ is called a **Norm Hilbert space** (NHS) if each closed subspace has an orthogonal complement.

Clearly, closed subspaces and quotients (which are in fact isometrically isomorphic to closed subspaces) of NHS are NHS. By Proposition 2.4.1 each finite-dimensional space with an orthogonal base is a NHS.

The following proposition has some redundancy for quotational reasons.

**Proposition 3.1.2.** Let $X$ be a $G$-module, let $E$ be an $X$-normed Banach space. Then the following are equivalent.

1. $E$ is Norm Hilbert space.
2. For each closed subspace $D$ there is an orthogonal projection $P : E \to E$ with $P E = D$.
3. Each orthogonal system in $E$ can be extended to an orthogonal base.
4. Each maximal orthogonal system is an orthogonal base.
5. For each closed subspace $D \subset E$ and $x \in E \setminus D$ the set $\{\|x - d\| : d \in D\}$ has a minimum.

**Proof.** $(\alpha) \Leftrightarrow (\beta)$ and $(\gamma) \Leftrightarrow (\delta)$ are immediate. We prove $(\beta) \Rightarrow (\epsilon) \Rightarrow (\delta) \Rightarrow (\beta)$. Suppose $(\beta)$, let $D$ be a closed subspace, let $P$ be an orthoprojection onto $D$, let $x \in E \setminus D$. Then, for all $d \in D$, $\|x - Px\| = \|x - d - P(x - d)\| \leq \max(\|x - d\|, \|P(x - d)\|) \leq \|x - d\|$. We see that $\|x - Px\| = \min(\|x - d\| : d \in D)$ and $(\epsilon)$ follows. To prove $(\epsilon) \Rightarrow (\delta)$, let $\{e_i : i \in I\}$ be a maximal orthogonal system in $E$, let $D$ be its closed linear span. If $D \neq E$, choose $x \in E \setminus D$. By $(\epsilon)$ there exists a $d_0 \in D$ with minimal distance to $x$. But then $K(x - d_0) \perp D$, a contradiction which proves $(\delta)$. Finally, assume $(\delta)$. Let $D$ be a closed subspace of $E$. Let $\{d_i : i \in I\}$ be a maximal
orthogonal system in $D$, let $D_1$ be its closed linear span, let $\{e_j : j \in J\}$ be such that $\{d_i, e_j : i \in I, j \in J\}$ is a maximal orthogonal system in $E$. Then it is an orthogonal base of $E$ and we have $E = D_1 \oplus S$ where $S$ is the closed linear span of $\{e_j : j \in J\}$. There is an orthogonal projection $P$ with $PE = D_1$, $PS = \{0\}$, so $PD = D_1$. If $d \in D_1 \setminus \{0\}$, $Pd = 0$ then $Kd \perp D_1$ conflicting the maximality of $\{d_i : i \in I\}$. Thus $P|_D$ is injective, i.e., $D = D_1$ and we have (β). 

Norm Hilbert spaces with norm and valuation real-valued were extensively characterized in [10, 5.13] and [10, 5.16]. More generally, by using similar methods we presented in [1, 4.1.3] a characterization of NHS over those $K$ whose value group $G$ is principal. We now will consider the remaining cases.

From now on in this paper we assume that the value group $G$ of $K$ is not principal.

Then either $K$ is non-metrizable or $G$ is the union of a strictly increasing sequence of convex subgroups (Proposition 1.4.3).

3.2. The main theorem and its consequences

**Theorem 3.2.1.** Let $E$ be an infinite-dimensional Banach space over $K$. Then the following are equivalent.

1. $E$ is a Norm Hilbert space.
2. $E$ has an orthogonal base and each closed hyperplane is orthocomplemented.
3. Each closed subspace of countable type is orthocomplemented.
4. $E$ is line orthogonal. Each closed subspace of countable type is a Norm Hilbert space.
5. $E$ has an orthogonal base. Each linear isometry $E \rightarrow E$ is surjective.
6. $K$ is metrizable. $E$ is line orthogonal. Each closed subspace of $E$ has a closed complement.
7. $E$ is line orthogonal. No subspace of $E$ is linearly homeomorphic to $c_0$.
8. $K$ is metrizable. $E$ has a countable orthogonal base satisfying the type condition.
9. $E$ has a countable orthogonal base satisfying the type condition.
10. $E$ is line orthogonal. Each bounded orthogonal sequence tends to zero.
11. $E$ is line orthogonal. Each strictly decreasing orthogonal sequence tends to zero.

Before proving the theorem (which is a generalization and an extension of [1, 4.3.7]) we state a few consequences, give some examples and comments. First we see that NHS over non-metrizable $K$ are trivial.

**Corollary 3.2.2.** Norm Hilbert spaces over non-metrizable $K$ are finite-dimensional. In fact, they are precisely the finite-dimensional spaces with an orthogonal base.

**Proof.** Implication $(α) \Rightarrow (η)$ of above and Proposition 2.4.1. □
Corollary 3.2.3. Norm Hilbert spaces are of countable type. Closed subspaces are of countable type (compare Section 2.5). Each orthogonal base of a Norm Hilbert space satisfies the type condition.

Corollary 3.2.4. Let $E$, $F$ be Norm Hilbert spaces over $K$, with norm values in the same $G$-module. Then $E \oplus F$ is a Norm Hilbert space.

Proof. We only need to consider the infinite-dimensional case. By $(\alpha) \Rightarrow (\kappa)$ of above $E$, $F$ have orthogonal bases $e_1, e_2, \ldots, f_1, f_2, \ldots$ respectively both satisfying the type condition. Then $e_1, f_1, e_2, f_2, \ldots$ is an orthogonal base of $E \oplus F$ and it is easily seen to satisfy the type condition. Now apply $(\kappa) \Rightarrow (\alpha)$. \(\Box\)

Corollary 3.2.5. The class of the infinite-dimensional Norm Hilbert spaces consists precisely of those $c_0(s)$ where $s = s_1, s_2, \ldots$ is a sequence in some $G$-module satisfying the type condition.

Proof. This is a restatement of $(\alpha) \Leftrightarrow (\kappa)$ of above. \(\Box\)

If $K$ is metrizable there exist infinite-dimensional Norm Hilbert spaces. For example, let $s_n := \sup_{G_n} H_n$, $t_n := \inf_{G_n} H_n$, where $H_1 \subsetneq H_2 \subsetneq \cdots$ is a sequence of convex subgroups covering $G$. Then $c_0(s)$ and $c_0(t)$, where $s := (s_1, s_2, \ldots)$, $t := (t_1, t_2, \ldots)$, are NHS. Other examples are the Form Hilbert spaces (\(\sqrt{G}\)-normed NHS where there is a symmetric bilinear form $(,)$ such that $\| (x, x) \| = \| x \|^2$ for all $x$, see [1, 4.4] for details).

Remarks. Dropping the metrizability condition in $(\eta)$ above leads to a falsity. Indeed, if $K$ is not metrizable $c_0 = c_{00}$ (Lemma 2.5.4) has an orthogonal base, so it is line orthogonal and (algebraic complements of) subspaces are closed, but $c_0$ is not a NHS (Corollary 3.2.2).

If $K$ is spherically complete every Banach space over $K$ is line orthogonal [1, 2.4.6] so the condition of line orthogonality in $(\eta)$, $(\theta)$, $(\lambda)$, $(\mu)$ of Theorem 3.2.1 is superfluous and can be dropped without harm. However, if $K$ is not spherically complete the condition is needed. In fact, consider the $G^\#$-normed space $E := K^2_v \oplus c_0(s)$ where $K^2_v$ is as in Proposition 2.4.2 and $s := (s_1, s_2, \ldots) \in (G^\#)^N$, $s_n := \sup_{G_n} H_n$ where $\{1\} \neq H_1 \subset H_2 \subset \cdots$ is a strictly increasing sequence of convex subgroups covering $G$. Then $E$ is linearly homeomorphic to $K^2 \oplus c_0(s)$ (where $K^2$ carries the usual max-norm) which is a NHS. Thus, $E$ does not contain subspaces linearly homeomorphic to $c_0$, each closed subspace of $E$ has a closed complement. $E$ is not line orthogonal, $E$ is not a NHS. We conclude that $(\eta)$ and $(\theta)$ fail if we drop line orthogonality. To see that the same happens to $(\lambda)$ and $(\mu)$ consider an orthogonal sequence $x_1 + y_1, x_2 + y_2, \ldots \in E$ where $x_n \in K^2_v$, $y_n \in c_0(s)$. By using the fact that $\| K^2_v \| \setminus \{0\} = G$ and $G \cap (\| c_0(s) \| \setminus \{0\}) = \emptyset$ we find $\| x_n \| \neq \| y_n \|$ for each $n$. Now if $\| x_n \| > \| y_n \|$ for infinitely many $n$ then by the Perturbation Lemma 2.2.1(ii) we would have that $x_{n_1}, x_{n_2}, \ldots$ is orthogonal for some $n_1 < n_2 < \cdots$, conflicting finite-dimensionality. So $\| x_n \| < \| y_n \|$ for $n \geq n_0$ and by
the Perturbation Lemma we have that $y_{n_0}, y_{n_0+1}, \ldots$ is orthogonal, $\|x_n + y_n\| = \|y_n\|$ for $n \geq n_0$. So we see, using the fact that $c_0(s)$ is a NHS, that each bounded (resp. strictly decreasing) orthogonal sequence in $E$ tends to 0, but $E$ is not a NHS. We conclude that $(\lambda)$, $(\mu)$ both fail if we drop line orthogonality.

We do not know whether line orthogonality can be dropped in $(\delta)$ without harm. Neither do we know whether the condition on $E$ having an orthogonal base in $(\beta)$, $(\zeta)$ can be relaxed to, say, line orthogonality.

3.3. Lemmas needed for the proof of Theorem 3.2.1

Lemma 3.3.1. Let $E$ be a Banach space over $K$, let $D_1$ be a closed subspace having an orthocomplement $D_2$. Let $F$ be a closed hyperplane in $D_1$. Suppose $F + D_2$ is orthocomplemented in $E$. Then $F$ is orthocomplemented in $D_1$.

Proof. $F + D_2$ is closed, $F + D_2 \neq E$, so by assumption there is a non-zero $x \in E$ such that $Kx \perp F + D_2$. Write $x = d_1 + d_2$ where $d_1 \in D_1, d_2 \in D_2$. Then, since $x \notin D_2$ we have $d_1 \neq 0$. For any $y \in F$ we have $\|d_1 - y\| = \|x - d_2 - y\| = \max(\|x\|, \|d_2 + y\|) = \max(\|x\|, \|d_2\|, \|y\|) \geq \|y\|$. Then $\|d_1\| \leq \max(\|y\|, \|d_1 - y\|) \leq \|d_1\| - \|y\|$, which proves that $Kd_1 \perp F$. Since $F$ is a hyperplane in $D_1$, $Kd_1$ is an orthocomplement of $F$ in $D_1$.

Corollary 3.3.2. Let $E$ be a Banach space over $K$ having an orthogonal base $\{e_i: i \in I\}$. Suppose that each closed hyperplane in $E$ is orthocomplemented. Then for each $J \subset I$ each closed hyperplane in $D := \overline{\{e_j: j \in J\}}$ is orthocomplemented in $D$.

Lemma 3.3.3. Let $E$ be a Banach space over $K$ with a countable orthogonal base $e_1, e_2, \ldots$ such that $\|e_1\| > \|e_2\| > \cdots$. Suppose that each closed hyperplane in $E$ is orthocomplemented. Then $\|e_n\| \to 0$.

Proof. Suppose $\|e_i\| > s$ for all $i \in \mathbb{N}$ and some non-zero norm value $s$; we derive a contradiction. The formula

$$\phi \left( \sum_{i=1}^{\infty} \xi_i e_i \right) = \sum_{i=1}^{\infty} \xi_i \ (\xi_i \in K, \ \|\xi_i e_i\| \to 0)$$

defines a non-zero $\phi : E \to K$ that is easily seen to be linear and continuous. Then Ker$\phi$ is a closed hyperplane in $E$ and by assumption there is a non-zero $a \in E$ such that $Ka \perp$ Ker$\phi$. Without loss, assume $\phi(a) = 1$. Let $a$ have the expansion $\sum_{i=1}^{\infty} \lambda_i e_i$ where $\lambda_i \in K$, $\|\lambda_i e_i\| \to 0$. From $\|\lambda_i e_i\| \geq |\lambda_i|s$ we see that $\lambda_i \to 0$ and

$$1 = |\phi(a)| = \left| \sum_{i=1}^{\infty} \lambda_i \right| \leq \max_i |\lambda_i|,$$

so that $|\lambda_n| \geq 1$ for some $n$ and $\|a\| = \max_i \|\lambda_i e_i\| \geq \|\lambda_n e_n\| \geq \|e_n\| > \|e_{n+1}\|$.
But, on the other hand, $\phi(a - e_{n+1}) = 0$ so that $a$ and $a - e_{n+1}$ are orthogonal, whence
\[
\|e_{n+1}\| = \|a - (a - e_{n+1})\| = \max(\|a\|, \|a - e_{n+1}\|) \geq \|a\|,
\]
a contradiction. □

3.4. Proof of Theorem 3.2.1

We start with $(\alpha) \Rightarrow (\beta) \Rightarrow (i) \Rightarrow (\kappa) \Rightarrow (\lambda) \Rightarrow (\theta) \Rightarrow (\mu) \Rightarrow (\alpha)$.

$(\alpha) \Rightarrow (\beta)$. Obvious.

$(\beta) \Rightarrow (i)$. First we prove metrizability of $K$. Let $\{e_i: i \in I\}$ be an orthogonal base of $E$. Let $J := \{i_1, i_2, \ldots \} \subset I$ where $i_n \neq i_m$ whenever $n \neq m$ and set $D := \{e_{i_n}: n \in \mathbb{N}\}$. Choose $\lambda_1, \lambda_2, \ldots \in K$ such that $\|\lambda_1 e_{i_1}\| > \|\lambda_2 e_{i_2}\| > \cdots$. By Corollary 3.3.2 each closed hyperplane in $D$ is orthocomplemented and by applying Lemma 3.3.3 to $D$ with its orthogonal base $\lambda_1 e_{i_1}, \lambda_2 e_{i_2}, \ldots$ we conclude that $\|\lambda_i e_{i_n}\| \to 0$. Thus, the $G$-module $\|E\|\{0\}$ has a coinitial sequence and Proposition 1.4.1 $(\varepsilon) \Rightarrow (\alpha)$ tells us that $K$ is metrizable. Now, for each $i \in I$, let $\lambda_i \in K \setminus \{0\}$ be such that $\{\|\lambda_i e_i\|: i \in I\}$ is bounded above by some $M \in \|E\|\{0\}$. By the first part of the proof $\|E\|\{0\}$ has a coinitial sequence $\varepsilon_1 > \varepsilon_2 > \cdots$. We now show that for each $n \in \mathbb{N}$ the set $\{i \in I: \|\lambda_i e_i\| \geq \varepsilon_n\}$ is finite. (Then $I$ is countable and for each enumeration $i_1, i_2, \ldots$ of $I$ the sequence $e_{i_1}, e_{i_2}, \ldots$ satisfies the type condition, and we are done.) Suppose not. Then, for some $n$ and some infinite $J \subset I$ we have
\[
\varepsilon_n \leq \|\lambda_j e_j\| \leq M \quad (j \in J).
\]
Let $j_1, j_2, \ldots$ be pairwise distinct elements of $J$, and choose a nonzero $\mu \in K$ with $|\mu| M < \varepsilon_n$. The principal subgroup of $G$ generated by $|\mu|$ is proper, so there is a $g \in G$ such that $g < |\mu|^l$ for all $j \in \mathbb{N}$. Then
\[
\|\mu \lambda_{j_1} e_{j_1}\| > \|\mu^2 \lambda_{j_2} e_{j_2}\| > \cdots \geq g \varepsilon_n.
\]
Let $D := \{e_{j_k}: k \in \mathbb{N}\}$. Then $k \mapsto \mu^k \lambda_{j_k} e_{j_k}$ is an orthogonal base of $D$ whose norms form a strictly decreasing sequence bounded below by $g \varepsilon_n$. But by Corollary 3.3.2 and Lemma 3.3.3 these norms tend to $0$, a contradiction.

$(i) \Rightarrow (\kappa)$. Trivial.

$(\kappa) \Rightarrow (\lambda)$. Clearly $E$ is line orthogonal (Proposition 2.3.2(ii)). Let $e_1, e_2, \ldots$ be an orthogonal base of $E$ satisfying the type condition and let $f_1, f_2, \ldots$ be a bounded orthogonal system in $E$. We may assume it is maximal. By Proposition 2.2.2(i) there are a permutation $\tau$ of $\mathbb{N}$ and $\{\lambda_i: i \in \mathbb{N}\} \subset K$ such that $\|f_i\| = \|\lambda_i e_{\tau(i)}\|$ for each $i$. Now $e_1, e_2, \ldots$ satisfies the type condition, hence so does $e_{\tau(1)}, e_{\tau(2)}, \ldots$ By boundedness $\|\lambda_i e_{\tau(i)}\| \to 0$, hence $\|f_i\| \to 0$, and $(\lambda)$ is proved.

$(\lambda) \Rightarrow (\theta)$. Suppose we have a linear homeomorphism $T: c_0 \to D$ where $D$ is a closed subspace of $E$. Then $D$ is of countable type and line orthogonal (Proposition 2.3.2(ii)), so it has an orthogonal base (Theorem 2.3.3), say $e_1, e_2, \ldots$. Since $c_0$ is $G$-normed there are $\lambda_1, \lambda_2, \ldots \in K$ such that $\|T^{-1}(\lambda_n e_n)\|_\infty = 1$ for all $n \in \mathbb{N}$. Then $\{\lambda_n e_n: n \in \mathbb{N}\}$ is bounded in $E$ but $\|\lambda_n e_n\| \not\to 0$, a contradiction.
\((\theta) \Rightarrow (\mu)\). Let \(e_1, e_2, \ldots\) be a strictly decreasing orthogonal sequence in \(E\). If not \(\|e_n\| \to 0\) we may assume that there is an \(s \in \|E\|\{0\}\) such that \(\|e_n\| \geq s\) for all \(n\). But then

\[
(\lambda_1, \lambda_2, \ldots) \mapsto \sum_{n=1}^{\infty} \lambda_n e_n
\]
defines a linear homeomorphism of \(c_0\) onto \([e_n; n \in \mathbb{N}]\), a contradiction.

\((\mu) \Rightarrow (\alpha)\). Let \(\{e_i; i \in I\}\) be a maximal orthogonal system in \(E\). We prove (Proposition 3.1.2(\(\delta\))) that \(F := [e_i; i \in I] = E\). First, observe that \(\|E\|\{0\}\) and \(\|F\|\{0\}\) are the same. Indeed, if \(x \in E, x \neq 0\), then by maximality there is a \(y \in F\) such that \(\|x - y\| < \|x\|\) so that \(\|x\| = \|y\|\). Let \(\Sigma\) be the collection of all algebraic types (see Section 1.3) of \(\|E\|\{0\}\), \(\|F\|\{0\}\), let \(\sigma : \|E\|\{0\} \to \Sigma\) be the canonical map. For each \(\sigma \in \Sigma\), let \(F_\sigma := \{e_i; \|e_i\| \in \sigma\}\). Then by Proposition 2.2.2, \(F = \bigoplus_{\sigma \in \Sigma} F_\sigma\). We first prove that each \(F_\sigma\) is finite-dimensional. If not, there would be, by Proposition 2.2.2(iii), an orthogonal sequence \(f_1, f_2, \ldots\) in \(F_\sigma\) with \(\|f_n\| = s\) for all \(n \in \mathbb{N}\) and some \(s \in \|E\|\{0\}\). Choose a \(\lambda \in K, \lambda \neq 0\) such that \(|\lambda|s < s\). Then the sequence \(n \mapsto \lambda^n f_n\) is strictly decreasing and therefore tends to 0 by (\(\mu\)), implying \(\lambda^n \to 0\). This would mean that the principal convex subgroup of \(G\) generated by \(|\lambda|\) is all of \(G\) contradicting our assumptions. Thus, \(F_\sigma\) is finite-dimensional.

Now let \(x \in E\). To prove \(x \in F\) we may assume that \(x\) is not in the algebraic linear span of the \(F_\sigma\). Let \(\sigma_1 := \sigma(\|x\|)\). By finite-dimensionality and Proposition 2.2.2(iii) the space \(F_{\sigma_1}\) is orthocomplemented, so there is a \(y_1 \in F_{\sigma_1}\) such that \(K(x - y_1) \perp F_{\sigma_1}\). By \([1, 2.4.13]\) any orthogonal base of \(F_{\sigma_1}\) is a maximal orthogonal subset of \(\{x \in E; \|x\| \in \sigma_1\}\), so \(\sigma_2 := \sigma(\|x - y_1\|)\) is different from \(\sigma_1\). Next, there is a \(y_2 \in F_{\sigma_1} + F_{\sigma_2}\) such that \(K(x - y_2) \perp F_{\sigma_1} + F_{\sigma_2}\). By the same token \(\sigma_3 := \sigma(\|x - y_2\|)\) is different from \(\sigma_1, \sigma_2\). Continuing this way we obtain distinct \(\sigma_1, \sigma_2, \ldots\) and \(y_n \in F_{\sigma_1} + \cdots + F_{\sigma_n}\) such that \(K(x - y_n) \perp (F_{\sigma_1} + \cdots + F_{\sigma_n})\), \(\sigma_{n+1} = \sigma(\|x - y_n\|)\) for each \(n\). Since \(\|x - y_n\| = \min\{\|x - y\|; y \in F_{\sigma_1} + \cdots + F_{\sigma_n}\}\) we have \(\|x - y_1\| \geq \|x - y_2\| \geq \cdots\) but all their algebraic types are different so \(\|x - y_1\| > \|x - y_2\| > \cdots\) and \(n \mapsto x - y_n\) is orthogonal (Proposition 2.2.1(i)). Then, by (\(\mu\)), \(\|x - y_n\| \to 0\) so that \(x = \lim y_n \in F\).

Next, we establish \((\alpha) \Rightarrow (\gamma) \Rightarrow (\delta) \Rightarrow (\mu)\).

\((\alpha) \Rightarrow (\gamma)\) is trivial.

\((\gamma) \Rightarrow (\delta)\). Clearly \(E\) is line orthogonal. Let \(D\) be a closed subspace of countable type. Then \(D\) is line orthogonal (Proposition 2.3.2 (i)) and therefore has an orthogonal base (Theorem 2.3.3). Let \(D_1\) be a closed hyperplane in \(D\). Then (Proposition 2.5.2) \(D_1\) is of countable type, so by assumption it has an orthocomplement \(S\) in \(E\), hence \(S \cap D\) is an orthocomplement of \(D_1\) in \(D\). Now apply \((\beta) \Rightarrow (\alpha)\) (proved above) to \(D\) and we obtain that \(D\) is a NHS.

\((\delta) \Rightarrow (\mu)\). Let \(e_1, e_2, \ldots\) be a strictly decreasing orthogonal sequence in \(E\). By assumption, \(D := [e_1, e_2, \ldots]\) is a NHS and by applying \((\alpha) \Rightarrow (\mu)\) (proved above) to \(D\) we find \(\|e_n\| \to 0\).

We continue with \((\alpha) \land (i) \Rightarrow (\eta) \Rightarrow (\theta)\).
$(\alpha) \land (i) \Rightarrow (\eta)$ is trivial.

$(\eta) \Rightarrow (\theta)$. Let $D \subset E$ be a closed subspace, and suppose $D \sim c_0$. Then, since $K$ is metrizable and $G$ not principal there is a closed subspace $S$ of $D$ without closed complement in $D$ (Theorem 2.6.5). But by $(\eta)$, $S$ has a closed complement $F$ in $E$. Then $F \cap D$ is a closed complement of $S$ in $D$, a contradiction.

Finally we prove $(\alpha) \land (\lambda) \Rightarrow (\zeta) \Rightarrow (\alpha)$.

$(\alpha) \land (\lambda) \Rightarrow (\zeta)$. Let $T : E \to E$ be a linear isometry, and suppose $TE \neq E$. Then by $(\alpha)$ there is a non-zero $x$ such that $Kx \perp TE$. The sequence $x, Tx, T^2x, \ldots$ is easily seen to be orthogonal ($Kx \perp [Tx, T^2x, \ldots], KTx \perp [T^2x, T^3x, \ldots]$, etc.), but $\|T^nx\| = \|x\|$ for all $n$ so $T^n x \to 0$ conflicting $(\lambda)$. Hence, $TE = E$.

$(\zeta) \Rightarrow (\alpha)$. Let $\{e_i : i \in I\}$ be an orthogonal base of $E$, let $\{f_i : i \in I\}$ be a maximal orthogonal system. According to Proposition 2.2.2(ii) $E$ is linearly isometric to $[f_i : i \in I]$. By $(\zeta)$ this last space must be equal to $E$, so $\{f_i : i \in I\}$ is an orthogonal base of $E$, and by Proposition 3.1.2 $E$ is a NHS. ∎

REFERENCES


(Received March 2005)