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Lipschitz operators on Banach spaces over Krull valued fields

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Introduction

In [5] a start was made with developing a general theory of normed vector spaces over fields with a Krull valuation $|.|: K \rightarrow G \cup \{0\}$. Here, $G$ is a multiplicatively written linearly ordered abelian group augmented with a smallest element 0. (In the case of a rank 1 valuation i.e. where $G$ is a subgroup of the group of positive real numbers such a theory was known already for quite some time and published in the monograph [9].)

The present paper can be viewed as a continuation of [5] and [7]. It contains the basics of the theory of Lipschitz operators i.e. linear maps $T$ between normed spaces for which there exists an element $g \in G$ such that

$$\|Tx\| \leq g\|x\| \quad \text{for all vectors } x.$$  

In Chapter 1 we collect all facts on $G$-modules, the natural home for norm values, that are needed later on. The notion of a continuous $G$-module is introduced and studied; new formulas concerning topological types are proved.

In Chapter 2 we develop a machinery for Lipschitz operators, with as little assumptions on the underlying spaces as possible. Innocent-looking formulas such as $\|Tx\| \leq \|T\|\|x\|$ for a Lipschitz operator $T$ and a vector $x$ are a serious object of study here. Also the notion of a strictly Lipschitz operator is introduced; here the inequality in (1) is replaced by a strict inequality for all nonzero $x$. The corresponding (strict) Lipschitz norms are investigated. Criteria in order that every Lipschitz operator is strict are derived. Similarly, situations are studied in which the invertible Lipschitz operators form an open set in the Banach algebra of all Lipschitz operators from a Banach space into itself, and continuity of inversion is discussed. It is shown that the

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trace function on the ideal of finite rank operators is continuous with respect to the strict Lipschitz norm.

Chapter 3 is the heart of the paper. Here we assume throughout that the underlying Banach space $E$ has a countable orthogonal base. Properties of an operator are translated into properties of the corresponding matrix. This way we can extend the results of [7] to a much wider class of spaces. Especially the ideal of the compact operators i.e. the closure of the ideal of finite rank operators with respect to the Lipschitz norm, and the one of the nuclear operators (replace in the above the Lipschitz norm by the strict Lipschitz norm) are studied.

1 Linearly ordered sets, groups, and modules

For definition and basic facts we refer to [5], Chapter 1, from which we also will use notations freely.

In this Chapter we reconsider and improve the theory of [5], develop some new notions and theory on $G$-modules that we will need later on.

1.1 Linearly ordered sets

Throughout 1.1 $X$ is a linearly ordered set embedded into its completion $X^\#$.

Lemma 1.1.1 Let $V_1, V_2, \ldots, V_n \subset X$ and suppose $\inf V_i$ exists for each $i \in \{1, \ldots, n\}$. Then $\inf \{\max(v_1, \ldots, v_n) : v_i \in V_i \text{ for each } i\} = \max_i (\inf V_i)$.

Proof. Left to the reader.

Lemma 1.1.2 Let $V \subset X$. Then $\inf_X V$ exists if and only if $\inf_{X^\#} V$ exists and lies in $X$. In that case $\inf_X V = \inf_{X^\#} V$.

Proof. Suppose $r = \inf_X V$ exists. Then $V$ is bounded below and non-empty so $s := \inf_{X^\#} V$ exists, and clearly $r \leq s$. If $r < s$ then by [5] 1.1.4 (iv) there is an $r' \in X$ with $r < r' \leq s$. Then $r'$ is a lower bound of $V$, $r' \in X$, $r' > r$, conflicting $r = \inf_X V$.

To complete the proof, let $r \in X$, $r = \inf_{X^\#} V$; we prove that $r = \inf_X V$. Clearly, $r$ is a lower bound of $V$. Let $r' \in X$, $r' > r$; we show that $r'$ is not a lower bound of $V$. In fact, since $r = \inf_{X^\#} V$ there is a $v \in V$ such that $r' > v \geq r$.

Lemma 1.1.3 Let $X \neq \emptyset$ be complete. Let $I, J$ be non-empty sets and let $\phi : I \times J \to X$ be such that $\phi(I \times J)$ is bounded below. Then

$$\inf \phi(I \times J) = \inf_{i \in I} (\inf_{j \in J} \phi(i, j)) = \inf_{j \in J} (\inf_{i \in I} \phi(i, j)).$$

Proof. Left to the reader.
1.2 Linearly ordered groups

From now on in this Chapter \( G \) is a linearly ordered abelian group, written multiplicatively, with unit 1. We assume \( G \neq \{1\} \).

Inspired by the terminology in rank 1 valuation theory we introduce the following.

**Definition 1.2.1** \( G \) is called quasidiscrete if \( \min\{g \in G : g > 1\} \) exists; otherwise \( G \) is called quasidense.

From [5], 1.1.1 it follows that \( G \) is quasidense if and only if \( \inf\{g \in G : g > 1\} = 1 \).

**Remark.** The reason for using the prefix "quasi" lies in the fact that, contrary to the rank 1 case, a quasidiscrete group may have quasidense subgroups! In fact, let \( G_1 \simeq \mathbb{Z} \) (as an ordered group, but written multiplicatively), let \( G_2 = (0, \infty) \) and, for \( n > 2 \), choose for \( G_n \) any linearly ordered group.

Set \[ X := G_1 \oplus G_2 \oplus \ldots \]

With the antilexicographic ordering \( X \) is a linearly ordered group. \( X \) is quasidiscrete as \((a, 1, 1, \ldots)\) is the smallest element of \( X \) that is \( > (1, 1, \ldots) \), where \( a \in G_1, \ a > 1, \ a \) is a generator of \( G_1 \). Now consider the subgroup \[ G := \{1\} \oplus G_2 \oplus \ldots \]

with the inherited ordering. We claim that \( G \) is quasidense. In fact, let \( a = (1, g_2, g_3, \ldots) \in G, \ a > (1, 1, \ldots) \); we construct a \( b \in G \) with \( a > b > (1, 1, \ldots) \).

If \( g_2 > 1 \) choose \( b = (1, \frac{1}{2} g_2, 1, \ldots) \). If \( g_2 = 1 \) choose \( b = (1, 2, 1, \ldots) \).

1.3 (Almost) faithful \( G \)-modules

**Definition 1.3.1** Let \( s \in X \). Set \( \text{Stab}(s) := \{g \in G : gs = s\} \); \( s \) is called faithful if \( \text{Stab}(s) = \{1\} \). If each element of \( X \) is faithful then \( X \) is called faithful.

\( X \) is called almost faithful if there is a proper convex subgroup \( H \) of \( G \) such that \( \text{Stab}(s) \subseteq H \) for each \( s \in X \).

**Proposition 1.3.2** The following are equivalent.

\( (\alpha) \) \( X \) is almost faithful.

\( (\beta) \) There is a \( g \in G \) such that \( s < gs \) for all \( s \in X \).

\( (\gamma) \) \( \bigcup_{s \in X} \text{Stab}(s) \neq G \).

**Proof.** Left to the reader.
1.4 The completion $G^\#$ of $G$

Recall ([5] 1.5.4) that $G^\#$ is in a natural way a $G$-module.

**Proposition 1.4.1** Let $H$ be a proper convex subgroup of $G$. Let $s := \inf_{G^\#} H$, $t := \inf_{G^\#} H$. Then $\text{Stab}(s) = \text{Stab}(t) = H$.

**Proof.** We may assume $H \neq \{1\}$. Then $s, t \notin G$. We prove $\text{Stab}(t) = H$. Firstly, if $h \in H$ then $ht = h \inf_{G^\#} H = \inf_{G^\#} hH = t$, so $H \subseteq \text{Stab}(t)$. Conversely, let $g \in G$, $g \notin H$; we prove that $g \notin \text{Stab}(t)$. We may assume $g > 1$ (otherwise, consider $g^{-1}$). Since $gH \cap H = \varnothing$, by convexity each element of $H$ is strictly smaller than each element of $gH$. For each $h \in H$ we have $h \leq \inf_{G^\#} gH = gt$. But $t \notin G$ so we have $h < gt$. Then certainly $t = \inf_{G^\#} H < gt$ and we are done.

**Corollary 1.4.2** The following are equivalent.

(a) $G^\#$ is almost faithful.

(b) $G$ has a maximal proper convex subgroup.

**Corollary 1.4.3** Let $H, t, s$ be as in Proposition 1.4.1. Assume $H \neq \{1\}$. Then for all $g \in G$ we have

(i) $g < t \iff gt < t \iff gs < s \iff gs \leq t$,

(ii) $g > s \iff gs > s \iff gt > t \iff gt \geq s$.

**Proof.** We only prove (i). Observe that $s, t \notin G$.

(a) $g < t \Rightarrow gs \leq t$: Suppose $gs > t$. Then there is an $h_1 \in H$ with $gs > h_1$, so $s > g^{-1}h_1$, and there is an $h_2 \in H$ with $s > h_2 \geq g^{-1}h_1$. Then $g \geq h_2^{-1}h_1 \in H$ and therefore $g \geq t$, a contradiction.

(b) $gs \leq t \Rightarrow gs < s$: Trivial, since $t < s$.

(c) $gs < s \Rightarrow g < t$: From the assumption it follows that $g < 1$ and $g \notin \text{Stab}(s) = H$. So $g < h$ for all $h \in H$ i.e. $g \leq t$ and, since $t \notin G$, $g < t$.

(d) $g < t \Rightarrow gt < t$: If $gt > t$ then $g > 1$ conflicting $g < t < 1$. If $gt = t$ then $g \in \text{Stab}(t) = H$, conflicting $g < t$. Hence, $gt < t$.

(e) $gt < t \Rightarrow g < t$: From $gt < t$ it follows that $g < 1$ and $g \notin \text{Stab}(t) = H$. Thus $g \leq \inf_{G^\#} H = t$ and, since $g \neq t$, $g < t$.

We now reconsider the 'large' multiplication $*$ and the 'small' multiplication $\cdot$ on $G^\#$ ([5], 1.3).

**Definition 1.4.4** For $s, t \in G^\#$ set

$$s \ast t := \inf_{G^\#} \{ g_1g_2 : g_1, g_2 \in G, g_1 \geq s, g_2 \geq t \}$$

$$s \cdot t := \sup_{G^\#} \{ g_1g_2 : g_1, g_2 \in G, g_1 \leq s, g_2 \leq t \}.$$ 

For $s \in G^\#$, $g \in G$, write $gs := g \cdot s = s \cdot g = g \ast s = s \ast g$.

As an application of Lemma 1.1.3 we have the following auxiliary formulas.

**Proposition 1.4.5** Let $s, t \in G^\#$. Then
\[ s \ast t = \inf_{G^\#}\{gt : g \in G, \ g \geq s\}, \]
\[ s \cdot t = \sup_{G^\#}\{gt : g \in G, \ g \leq s\}. \]

In [5] 1.3 associativity was taken for granted. A closer look, however, shows that it is not completely trivial.

**Proposition 1.4.6** The multiplications \( \ast \) and \( \cdot \) are associative.

**Proof.** We carry out the proof for \( \ast \) (associativity of \( \cdot \) can be proved in the same spirit). Let \( s, t, u \in G^\# \) and put

\[ \Gamma := \{g_1g_2g_3 : g_1, g_2, g_3 \in G, \ g_1 \geq s, \ g_2 \geq t, \ g_3 \geq u\}. \]

We shall prove that \( s \ast (t \ast u) = \inf_{G^\#}\Gamma \). (Then by symmetry and commutativity, \( (s \ast t) \ast u = u \ast (s \ast t) = \inf_{G^\#}\Gamma \) and we are done). To this end we first notice that \( s \ast (t \ast u) \) is a lower bound of \( \Gamma \). Now let \( v \in G^\# \), \( s \ast (t \ast u) < v \); we prove that \( v \) is not a lower bound for \( \Gamma \). There is a \( g \in G \) such that \( s \ast (t \ast u) < g \leq v \), so there are \( g_1, h \in G \) with \( g_1 \geq s, \ h \geq t \ast u \) and \( g_1h < g \). Then \( g_1h \leq h < g_1^{-1}g \), so there are \( g_2, g_3 \in G \) with \( g_2 \geq t, \ g_3 \geq u \) and \( g_2g_3 < g_1^{-1}g \). Then \( g_1g_2g_3 \in \Gamma \), \( g_1g_2g_3 < g \) implying that \( g \) (and, hence, \( v \)) is not a lower bound of \( \Gamma \).

Now the following conclusion is straightforward.

**Proposition 1.4.7** For both multiplications \( G^\# \) is an associative, commutative semigroup with unit \( 1 \). Both multiplications extend the group multiplication in \( G \) and are increasing in both variables.

We quote the following from [5], 1.3.1.

**Theorem 1.4.8** There is precisely one decreasing extension \( \omega : G^\# \to G^\# \) (called the antipode) of the inversion \( g \mapsto g^{-1} \) (\( g \in G \)). It is bijective and equals its inverse.

For \( s \in G^\# \) we have

\[ \omega(s) = \sup_{G^\#}\{g \in G : gs \leq 1\} = \inf_{G^\#}\{g \in G : gs \geq 1\}. \]

To investigate the behaviour of \( \omega \) with respect to the multiplications we first prove the following simple lemma.

**Lemma 1.4.9** Let \( V \subset G^\# \) be bounded above. Then \( \omega(V) \) is bounded below and \( \inf_{G^\#}\omega(V) = \omega(\sup_{G^\#}V) \).

**Proof.** Clearly \( \omega(\sup_{G^\#}V) \) is a lower bound of \( \omega(V) \). Let \( w \in G^\# \), \( \omega(\sup_{G^\#}V) < w \); we show that \( w \) is not a lower bound of \( \omega(V) \). There is a \( g \in G \) with \( \omega(\sup_{G^\#}V) < g \leq w \). Then \( \sup_{G^\#}V > g^{-1} \geq \omega(v) \) and there is a \( v \in V \) with \( v > g^{-1} \geq \omega(w) \), hence \( \omega(v) < g \leq w \). We conclude that \( w \) is not a lower bound of \( \omega(V) \).

**Theorem 1.4.10** Let \( s, t \in G^\# \). Then

(i) \( \omega(s \cdot t) = \omega(s) \ast \omega(t) \),
(ii) \( \omega(s \ast t) = \omega(s) \cdot \omega(t) \).
(iii) \( \omega(s) = s \iff s = 1 \),
(iv) for \( g \in G \), \( \omega(gs) = g^{-1}\omega(s) \).

**Proof.** We only prove (i). We have \( \omega(s) = s \iff \omega(t) = t \)

\[
\inf_{G^*} \{ g_1g_2 : g_1, g_2 \in G, g_1 \geq \omega(s), g_2 \geq \omega(t) \} =
\inf_{G^*} \{ g_1g_2 : g_1, g_2 \in G, g_1^{-1} \leq s, g_2^{-1} \leq t \} =
\inf_{G^*} \{ (g_1g_2)^{-1} : g_1, g_2 \in G, g_1 \leq s, g_2 \leq t \},
\]
which, by Lemma 1.4.9, equals \( \omega(\sup_{G^*} \{ g_1g_2 : g_1, g_2 \in G, g_1 \leq s, g_2 \leq t \}) = \omega(s \cdot t) \).

In the next proposition we compute products of \( \inf H, \sup H \) for a convex subgroup \( H \). Recall that \( G/H \) is in a natural way a linearly ordered group.

**Proposition 1.4.11** Let \( H \subset G \) be a proper convex subgroup, \( H \neq \{1\} \), put \( s := \sup_{G^*} H, t := \inf_{G^*} H \). Then we have

(i) \( s \cdot s = s \cdot t = s; \ t \cdot t = t \cdot t = t; \ \omega(s) = t \).
(ii) If \( G/H \) is quasidense then \( s \cdot s = s, \ t \cdot t = t \).
(iii) If \( G/H \) is quasidiscrete then \( s \cdot s = g_0s > s, \ t \cdot t = g_0^{-1}t < t \) where \( g_0 \in G, g_0 > s \) and where, with \( \pi : G \to G/H \) the canonical map, \( \pi(g_0) = \min \{ u \in G/H : u > 1 \} \).

**Proof.** (i) is straightforward. To prove (ii), let \( g \in G, g > s \); we show that \( s \cdot s < g \). We have \( \pi(g) > 1 \) so by quasidensity of \( G/H \) there exist \( u, v \in G/H, u, v > 1 \) such that \( \pi(g) = uv \). Choose \( g_1, g_2 \in G \) with \( \pi(g_1) = u, \pi(g_2) = v \). Then \( g_1 > s, g_2 > s \) and \( \pi(g_1g_2) = \pi(g) \), so there is an \( h \in H \) with \( g = hg_1g_2 \). Now \( h_{g_1} > hs = s \) by Proposition 1.4.1, so \( s \cdot s < (hg_1)g_2 = g \). We leave the proof of \( t \cdot t = t \) to the reader.

To prove (iii), let \( u_0 \) be the smallest element of \( G/H \) that is \( > 1 \), choose \( g_0 \in G \) with \( \pi(g_0) = u_0 \). Then \( g_0 > s \), so clearly \( s \cdot s \leq g_0s \). To complete the proof, let \( g_1, g_2 \in G, g_1 > s, g_2 > s \); we show that \( g_1g_2 > g_0s \). From \( g_1 > s \) we obtain \( \pi(g_1) > 1 \) so \( \pi(g_1) \geq \pi(g_0) \). Then either \( g_0^{-1}g_1 \in H \) or \( g_0^{-1}g_1 > s \). In either case there is an \( h \in H \) with \( g_0^{-1}g_1 \geq h \). Then \( g_1 \geq g_0h \) and we have \( g_1g_2 \geq g_1s \geq g_0hs = g_0s \). We leave the proof of \( t \cdot t = g_0^{-1}t \) to the reader.

Another interesting conclusion can be drawn.

**Proposition 1.4.12** Let \( H, g_0, s, t \) be as in previous proposition.

(i) If \( G/H \) is quasidense then \( s \notin Gt \).
(ii) If \( G/H \) is quasidiscrete then \( g_0t = s \).

**Proof.** (i) Suppose \( s = gt \) for some \( g \in G \). Then (Proposition 1.4.11 (ii)) \( s = gt = g(t \cdot t) = (gt) \cdot t = s \cdot t = t \), a contradiction.

(ii) We have \( g_0 > s \), so \( g_0t \geq s \) by Corollary 1.4.3 (ii). To prove equality, let \( g \in G, g_0t > g \); it suffices to show that \( g \) is not an upper bound of \( H \). From \( t > g_0^{-1}g \) we obtain (with \( \pi \) as above) \( \pi(g_0^{-1}g) < 1 \), so, by minimality, \( \pi(g_0^{-1}g) \leq \pi(g_0^{-1}) \), so \( \pi(g) \leq 1 \) (by faithfulness of \( \pi(g_0^{-1}) \in G/H \)), implying \( g \in H \) or \( g < 1 \), in neither of which cases \( g \) is an upper bound of \( H \).
Now we consider the behaviour of \( \text{Stab} \) with respect to the multiplications.

**Theorem 1.4.13** Let \( s, t \in G^\# \). Then

(i) \( \text{Stab}(s * t) = \text{Stab}(s \cdot t) = \text{Stab}(s) \cup \text{Stab}(t) \).

(ii) \( \text{Stab}(\omega(s)) = \text{Stab}(s) \).

**Proof.** If \( g \in \text{Stab}(\omega(s)) \) then \( g\omega(s) = \omega(s) \), or \( g^{-1}s = s \), or \( g \in \text{Stab}(s) \). This proves (ii) and (via Theorem 1.4.10) the first equality of (i). It remains to be shown that \( \text{Stab}(s \cdot t) \in \text{Stab}(s) \cup \text{Stab}(t) \) (as the opposite inclusion is trivial). Let \( g \in \text{Stab}(s \cdot t) \), \( g \notin \text{Stab}(s) \); we prove that \( g \in \text{Stab}(t) \). To this end we may assume \( g < 1 \). Then \( gs < s \), so there is a \( \tilde{g} \in G \) with \( \tilde{g}s \leq \tilde{g} \leq s \), which after applying large multiplication by \( t \) becomes \( s \cdot t = g(s \cdot t) = (gs) \cdot t \leq s \cdot t \), implying \( \tilde{g}t = s \cdot t \), hence \( g\tilde{g}t = \tilde{g}t \) i.e. \( g \tilde{t} = t \), as announced.

**Definition 1.4.14** ([5] 1.6.1) Let \( s \in G^\# \). Its **topological type** (with respect to the unit 1) is the set

\[
\tau(s) := \{ g \in G : \tau_1(s) \leq g \leq \tau_u(s) \},
\]

where

\[
\tau_1(s) := \sup_{G^\#} \{ gs : g \in G, gs \leq 1 \},
\]

\[
\tau_u(s) := \inf_{G^\#} \{ gs : g \in G, gs \geq 1 \}.
\]

**Proposition 1.4.15** For all \( s \in G^\# \), \( \tau(s) = \text{Stab}(s) \).

**Proof.** See [8] 3.1.

**Proposition 1.4.16** Let \( s \in G^\# \). Then

(i) \( \tau_u(s) = s * \omega(s) = \sup_{G^\#} \tau(s) \).

(ii) \( \tau_1(s) = s \cdot \omega(s) = \inf_{G^\#} \tau(s) \).

**Proof.** We have \( \omega(s) * s = \inf \{ gs : g \in G, g \geq \omega(s) \} = \inf \{ gs : g \in G, g^{-1} \leq s \} = \inf \{ gs : g \in G, 1 \leq gs \} = \tau_u(s) \). The proof of (ii) runs similarly.

We now introduce a subgroup \( G_0 \) of \( G^\# \)containing \( G \) that will play a role in Theorem 1.7.3.

**Definition 1.4.17** \( G_0 := \{ s \in G^\# : s \text{ is faithful} \} \).

**Proposition 1.4.18** For each \( s \in G_0 \) the map \( t \mapsto t * s = t \cdot s \) is a bijection \( G^\# \to G^\# \), and sends \( G_0 \) onto \( G_0 \) yielding a group structure on \( G_0 \) with inversion \( \omega \).

**Proof.** By Propositions 1.4.15, 1.4.16, and Definition 1.4.17 we have \( s \in G_0 \iff \text{Stab}(s) = \{ 1 \} \iff \tau(s) = \{ 1 \} \iff \tau_u(s) = \tau_1(s) = 1 \iff s * \omega(s) = 1 \iff s \cdot \omega(s) = 1 \).

Now let \( t_1, t_2 \in G^\# \) be such that \( t_1 * s = t_2 * s \); after multiplying by \( \omega(s) \) we obtain \( t_1 = t_2 \) showing injectivity. Surjectivity follows from \( u = u * (\omega(s) * s) = (u * \omega(s)) * s \) for each \( u \in G^\# \). Theorem 1.4.13 shows us that if \( s, t \in G_0 \), then \( s \cdot t, s * t, \omega(s) \) are in \( G_0 \). We complete the proof by showing that for \( s \in G_0, t \in G^\# \) we have \( t \cdot s = t * s \).
Clearly $t \cdot s \leq t \ast s$; we arrive at a contradiction from the assumption $t \cdot s < t \ast s$.
By surjectivity $t \cdot s = u \ast s$ for some $u \in G^\#$, whence $u < t$. There is a $g \in G$ with $u \leq g \leq t$, hence $t \cdot s = u \ast s \leq gs$. By faithfulness of $s$, $t \leq g$, so we have $t = g \in G$, conflicting $t \cdot s < t \ast s$.

1.5 Topological types in $G$-modules

We now study the topological types with respect to different units. The results of 1.5 will be crucial for 3.3 and 3.4. Recall that $X$ is a $G$-module.

Definition 1.5.1 ([5]) For $s, t \in X$ we denote by $\tau(s; t)$ the topological type of $s$ with respect to the unit $t$.

Proposition 1.5.2 Let $s, t \in X$. Then $\tau(s; t)$ is the set of all $h \in G$ satisfying (i) and (ii) below.

(i) If $g \in G$, $t \leq gs$ then $ht \leq gs$.
(ii) If $g \in G$, $t \geq gs$ then $ht \geq gs$.

Alternatively, if $s \in Gt$ then $\tau(s; t) = \text{Stab}(s) = \text{Stab}(t)$, otherwise $\tau(s; t)$ is the largest convex subgroup $H$ of $G$ for which $\text{conv}_X(Ht) \cap Gs = \emptyset$.


Theorem 1.5.3 Let $s, t, u \in X$.

(i) If $s' \in Gs$, $t' \in Gt$ then $\tau(s; t) = \tau(s'; t')$.
(ii) $\text{Stab}(s) \cup \text{Stab}(t) \subset \tau(s; t)$.
(iii) $\tau(s; t) = \tau(t; s)$.
(iv) $\tau(s; u) \subset \tau(s; t) \cup \tau(t; u)$.
(v) If $\tau(s; t) \neq \tau(t; u)$ then $\tau(s; u) = \tau(s; t) \cup \tau(t; u)$.

Proof. (i) and (ii) are clear from the first part of Proposition 1.5.2. To prove (iii) we may assume $s \notin Gt$. Then, by the second part of Proposition 1.5.2 and by symmetry it suffices to prove, for a convex subgroup $H$,

$$\text{conv}_X(Ht) \cap Gs \neq \emptyset \Rightarrow \text{conv}_X(Hs) \cap Gt \neq \emptyset.$$ 

But that is easy: there are $h_1, h_2 \in H$, $g \in G$ such that $h_1t \leq gs \leq h_2t$. Then $h_2^{-1}s \leq g^{-1}t \leq h_1^{-1}s$, so $\text{conv}_X(Hs) \cap Gt \neq \emptyset$. Next we prove (iv); we may assume $s \notin Gu$, $s \notin Gt$, $t \notin Gu$. Let $H := \tau(s; u)$. Assume (iv) does not hold; we derive a contradiction. We have

$$\tau(t; u) \cup \tau(s; t) \subseteq H,$$

so by Proposition 1.5.2 we have $\text{conv}_X(Ht) \cap Gs \neq \emptyset$, $\text{conv}_X(Hu) \cap Gt \neq \emptyset$, (since $\tau(t; u) = \tau(u; t)$). Thus there are $h_1, h_2, h'_1, h'_2 \in H$ and $g, g' \in G$ such that

1. $h_1t \leq gs \leq h_2t$
2. $h'_1u \leq g't \leq h'_2u$.

Multiplying (1) by $g'$ and using (2) yields
\[ h_1 h'_1 u \leq h_1 g't \leq gg's \leq h_2 g't \leq h_2 h'_2 u, \]

and so \( \text{conv}_X(Hu) \cap Gs \neq \emptyset \), a contradiction.

**Theorem 1.5.4** Let \( s, t \in X \). Set

\[
u := \inf_{G'} \{ g \in G : s \leq gt \} \in G#
\]

\[ u'^{-} := \inf_{G'} \{ g \in G : s < gt \} \in G#.
\]

Then \( \text{Stab}(u) = \text{Stab}(u'^{-}) = \tau(t; s) \).

**Proof.** First assume that \( s \in Gt \), say, \( s = g_0 t \) for some \( g_0 \in G \). Then clearly \( u = \inf_{G'} \{ g \in G : g_0 t = gt \} = \inf_{G'} g_0 \text{Stab}(t) = g_0 \inf_{G'} \text{Stab}(t) \). So by Proposition 1.4.1, \( \text{Stab}(u) = \text{Stab}(g_0 u) = \text{Stab}(\inf_{G'} \text{Stab}(t)) = \text{Stab}(t) = \tau(t; s) \).

In the same spirit one proves that \( u'^{-} = \inf_{G'} \{ g \in G : g > g_0 h \text{ for all } h \in \text{Stab}(t) \} = g_0 \inf_{G'} \{ g \in G : g > h \text{ for all } h \in \text{Stab}(t) \} = g_0 \sup_{G'} \text{Stab}(t) \), and that \( \text{Stab}(u'^{-}) = \text{Stab}(t) \).

So, from now on in the proof we may assume \( s \notin Gt \) and, hence, \( u = u'^{-} \) and it suffices to prove that \( \text{Stab}(u) = \tau(t; s) \).

Let \( V := \{ g \in G : s \leq gt \} \).

(a) \( \tau(t; s) \subset \text{Stab}(u) \). Let \( h \in \tau(t; s), g \in V \). Then \( gt \geq s \), so by Proposition 1.5.2 we have \( hs \leq gt \) implying \( h^{-1}V \subset V \). Since \( \tau(t; s) \) is a group we have also \( hV \subset V \) and therefore \( hV = V \). Then \( hu = h \inf_{G'} V = \inf_{G'} hV = u \) and we are done.

(b) \( \text{Stab}(u) \subset \tau(t; s) \). If \( u \in G \) then \( \text{Stab}(u) = \{1\} \) which is trivially contained in \( \tau(t; s) \). So, we may assume \( u \notin G \). Then \( u'^{-} \notin G \).

\[
(*) \quad V = \{ g \in G : g \geq u \} = \{ g \in G : g > u \}.
\]

Now let \( h \in \text{Stab}(u) \). It is enough to prove (Proposition 1.5.2) that for each \( g \in G \)

\[
\begin{align*}
(i) & \quad s \leq gt \text{ implies } hs \leq gt \\
(ii) & \quad s \geq gt \text{ implies } hs \geq gt.
\end{align*}
\]

To prove (i), let \( s \leq gt \). Then \( g \in V \), so \( g \geq u \), hence \( g \geq hu \) or \( h^{-1}g \in V \). Therefore \( s \leq h^{-1}gt \) or \( hs \leq gt \).

Now we prove (ii). Let \( s \geq gt \). By assumption \( s > gt \), so \( g \notin V \), hence \( g < u \), so that \( hg < u \) i.e. \( hg \notin V \) or \( hs > gt \) which proves (ii).

We end with a corollary of the theory in 1.5 for the case \( X = G# \).

**Corollary 1.5.5** Let \( s, t \in G# \). Then

\[ \tau(s; t) = \text{Stab}(s) \cup \text{Stab}(t). \]
Proof. By Theorem 1.5.3 (ii) we have \( \tau(s; t) \supset \text{Stab}(s) \cup \text{Stab}(t) \). On the other hand we have by Theorem 1.5.3 (iv), (iii)

\[
\tau(s; t) \subset \tau(s; 1) \cup \tau(1; t) = \tau(s; 1) \cup \tau(t; 1) = \tau(s) \cup \tau(t)
\]

which equals \( \text{Stab}(s) \cup \text{Stab}(t) \) according to Proposition 1.4.16.

1.6 Continuous \( G \)-modules

In this section \( X \) is a \( G \)-module embedded in its completion \( X^\# \). Let \( r \in X \). We want to give a precise meaning to the intuitive expression '\( \lim_{\gamma \downarrow 1} g r = r \)' as follows.

Definition 1.6.1 We say that \( X \) is right continuous at \( r \in X \) if for every \( W \subset G \) for which \( \inf W \) exists we have

\[
(\inf_G W)r = \inf_X Wr.
\]

Similarly we define left continuity at \( r \).

But we can prove:

Proposition 1.6.2 Left and right continuity are identical.

Proof. It suffices to prove that right continuity implies left continuity. So let \( X \) be right continuous at \( r \in X \), and let \( W \subset G \) be such that \( \sup_G W \) exists. We prove that \( (\sup_G W)r = \sup_X Wr \).

Clearly \((\sup_G W)r\) is an upper bound of \( Wr \). Now let \( t \in X \) be any upper bound of \( Wr \); we prove \( t \geq (\sup_G W)r \). To this end, let \( g \in W^{-1} \). Then \( g^{-1} \in W \), so \( g^{-1}r \leq t \) i.e. \( r \leq gt \). This holds for all \( g \in W^{-1} \) so, by right continuity, \( r \leq \inf_X \{gt : g \in W^{-1}\} = \inf_X W^{-1}t = (\inf_G W^{-1})t \), which equals \((\sup_G W)^{-1}t\) by Lemma 1.4.9.

From now on we will use 'continuity' to express left or right continuity. Also by saying that \( X \) is continuous we mean that \( X \) is continuous at each \( r \in X \).

As first examples we have the following:

Proposition 1.6.3 \( G \) is a continuous \( G \)-module.

Proof. Apply [5], 1.5.3.(i) with \( X := G \).

Proposition 1.6.4 If \( G \) is quasidiscrete, then each \( G \)-module \( X \) is continuous.

Proof. Let \( g_0 := \min\{g \in G : g > 1\} \). Let \( r \in X \), \( W \subset G \) such that \( g_1 := \inf_G W \) exists. Then \( g_1 < g_0 g_1 \), so there is a \( g_2 \in W \) with \( g_1 \leq g_2 < g_0 g_1 \).

Hence \( 1 \leq g_1^{-1} g_2 < g_0 \). By minimality we have \( g_1^{-1} g_2 = 1 \), or \( g_1 \in W \). Thus, \( \inf_G W = \min W \) and then clearly \( (\min W)r = \min(Wr) \).

Proposition 1.6.5 \( G \)-submodules of continuous \( G \)-modules are again continuous.

Proof. Let \( Y \) be a \( G \)-submodule of the continuous \( G \)-module \( X \). Let \( r \in Y \) and \( W \subset G \) be such that \( \inf_G W \) exists. We will show that \((\inf_G W)r = \inf_Y Wr \). First,
notice that \((\inf_G W)r\) is a lower bound of \(Wr\) and lies in \(Y\). Let \(t \in Y\) be any lower bound of \(Wr\). Then \(t \leq \inf_X(\inf_G W)r = (\inf_G W)r\), which shows that \((\inf_G W)r\) is the greatest lower bound of \(Wr\).

The following result is important.

**Theorem 1.6.6** The completion of a continuous \(G\)-module is again continuous.

**Proof.** Let \(X\) be continuous, let \(r \in X\), let \(W \subset G\) be such that \(g : = \inf_G W\) exists. We shall prove that \(gr = s\) where \(s : = \inf_X(Wr)\). Clearly \(gr \leq s\). Suppose \(gr < s\) ; we derive a contradiction. We have \(r < g^{-1}s\), so there is an element \(u \in X\) with \(r < u < g^{-1}s\). By assumption we have \(gu = \inf_X(Wu)\), and via Lemma 1.1.2 we obtain \(\inf_X(Wu) = \inf_X(Wr) = s\) so that \(gu \geq s\) or \(u \geq g^{-1}s\), a contradiction.

**Corollary 1.6.7** \(G^\#\) is a continuous \(G\)-module.

**Proof.** Combine Proposition 1.6.3 and Theorem 1.6.6.

The next proposition shows that to conclude continuity of \(X\) it suffices to check \((\inf_G W)r = \inf_X(\inf_G W)r\) only for the special set \(W = \{g \in G : g > 1\}\). It also links up with the intuitive formula \(\lim_{s \downarrow 1} gr = r\) at the beginning of this section.

**Proposition 1.6.8** Let \(G\) be quasidense. Then \(X\) is continuous if and only if for each \(r \in X\)

\[ (*) \quad \inf_X \{gr : g \in G, g > 1\} = r. \]

**Proof.** We only need to prove the 'if' part. So assume \((*)\) for each \(r \in X\). Let \(W \subset G\) be such that \(w_0 : = \inf_G W\) exists. We will prove that \(w_0r = \inf_X(\inf_G W)r\) \((r \in X)\). Clearly \(w_0r\) is a lower bound of \(Wr\). Now let \(t\) be any lower bound of \(Wr\); we prove \(t \leq w_0r\). Let \(g \in G\), \(g > 1\). Then \(w_0 < gw_0\) so there is a \(w \in W\) with \(w_0 \leq w \leq gw_0\). So \(t \leq w_0r \leq gw_0r\). We then have \(t \leq gw_0r\) for all \(g \in G\), \(g > 1\). Applying \((*)\) (with \(w_0r\) in place of \(r\)), we obtain \(t \leq w_0r\).

An example of a non-continuous \(G\)-module can be found in [5], 1.5.5 (c). Another one is furnished by the group \(X\) defined in Remark following Definition 1.2.1, considered as a \(G\)-module. (Continuity of \(X\) would imply \(\inf_X\{g \in G : g > 1\} = 1\), but this is not true as for each \(g \in G\), \(g > 1\) we have \(g \geq (a, 1, 1, \ldots)\).)

### 1.7 \(G^\#\)-modules

For a \(G\)-module \(X\) we extend the structure map \(G \times X \to X\) to a 'large' multiplication \(G^\# \times X^\# \to X^\#\) in the spirit of Definition 1.4.4 as follows.

**Definition 1.7.1** For \(s \in G^\#, r \in X^\#\) we set

\[ s \ast r = \inf_X\{gu : g \in G, u \in X, g \geq s, u \geq r\}. \]

**Remarks.**
1. This definition generalizes the ‘large’ multiplication defined in Definition 1.4.4.
2. It is possible to define an extension of the ‘small’ multiplication, but we do not need it here.
3. Thanks to Lemma 1.1.3 we have
   \[ s \ast r = \inf_{X^G} \{ gr : g \in G, \ g \geq s \}, \]
   where
   \[ gr = \inf_{X^G} \{ gu : u \in X, \ u \geq r \} \]
   defines the natural G-module structure on \( X^G \).
4. \((s, r) \mapsto s \ast r\) is increasing in both variables.

In view of Remark 3 above, from now on in Section 1.7 we assume that \( X \) is a complete \( G \)-module so that the extended multiplication \( G^G \times X \to X \) is defined by
   \[ s \ast r = \inf_{X^G} \{ gr : g \in G, \ g \geq s \}. \]

The question as to whether this multiplication is associative turns out to be interesting. First a preparatory lemma.

Lemma 1.7.2 Let \( g \in G, \ s, t \in G^G, \ r \in X \). Then
   (i) \((gs) \ast r = s \ast (gr) = g(s \ast r)\),
   (ii) \( s \ast (t \ast r) = t \ast (s \ast r) \geq (s \ast t) \ast r \),
   (iii) if \( s \ast t \notin G \) then we have equality in (ii).

Proof. Let \( V := \{ h \in G : h \geq gs \} \). For \( h \in V \) we have \( g^{-1}h \geq s \) so \( g^{-1}hr \geq s \ast r \), hence \( hr \geq g(s \ast r) \). We find \( (gs) \ast r = \inf_X \{ hr : h \in V \} \geq g(s \ast r) \). Thus
   \[
   (*) \quad (gs) \ast r \geq g(s \ast r) \quad (g \in G).
   \]
   By applying (*) for \( g^{-1} \) in place of \( g \) and \( gs \) in place of \( s \) we obtain \( s \ast r \geq g^{-1}(gs \ast r) \) or \( g(s \ast r) \geq gs \ast r \) and we find
   \[
   (**) \quad (gs) \ast r = g(s \ast r).
   \]
   To prove \( s \ast gr = g(s \ast r) \) we use a similar method. We have \( s \ast (gr) = \inf_X \{ hgr : h \geq s, \ h \in G \} \). If \( h \geq s \) then \( hg \geq gs \) so \( hgr \geq (gs) \ast r \) which by taking inf over \( h \geq s, \ h \in G \), leads to
   \[
   (***) \quad s \ast (gr) \geq (gs) \ast r = g(s \ast r).
   \]
   By taking \( g^{-1}r \) for \( r \) in (***) we get \( s \ast r \geq g(s \ast g^{-1}r) \), which by (***) is \( \geq g^{-1}(s \ast r) \).
   So we find \( g^{-1}(s \ast r) = s \ast (g^{-1}r) \) for all \( g \in G \) which completes the proof of (i).

(ii) Let \( g \in G, \ g \geq s \). Then \( gr \geq s \ast r \), so by using (i) we get \( g(t \ast r) = (gt) \ast r = t \ast gr \geq t \ast (s \ast r) \). By taking the inf over \( g \in G \), \( g \geq s \) we obtain \( s \ast (t \ast r) \geq t \ast (s \ast r) \).
   By symmetry we must have equality. To complete the proof of (ii), let \( g \in G, \ g \geq s \).
   Then \( gt \geq s \ast t \), so by using (i) we get \( g(t \ast r) = (gt) \ast r \geq (s \ast t) \ast r \) and by taking the inf over all \( g \in G \), \( g \geq s \) we arrive at \( s \ast (t \ast r) \geq (s \ast t) \ast r \).

(iii) Let \( V := \{ g \in G : g \geq s \ast t \} \). Then \( (s \ast t) \ast r = \inf_X (V r) \). By assumption \( V = \{ g \in G : g \geq s \ast t \} \). So if \( g \in V \) there are \( g_1, g_2 \in G \) with \( g_1 \geq s, \ g_2 \geq t \) and \( g_1g_2 \leq g \).
   Then \( gr \geq g_1g_2r \geq g_1(t \ast r) \geq s \ast (t \ast r) \). Hence, \( (s \ast t) \ast r = \inf_X (V r) \geq s \ast (t \ast r) \) and we are done.

Theorem 1.7.3 Let \( X \) be a complete \( G \)-module. The following are equivalent.
The multiplication \( G^\# \times X \rightarrow X \) is associative i.e. \( s \ast (t \ast r) = (s \ast t) \ast r \) for all \( s, t \in G^\#, \ r \in X \).

Proof. We first prove \((\beta) \Rightarrow (\alpha)\). Let \( X \) be continuous. From Lemma 1.7.2 (iii) we only have to establish associativity in case \( s \ast t \in G \). Now let \( W = \{g_1 g_2 : g_1 \in G, \ g_2 \in G, \ g_1 \geq s, \ g_2 \geq t \} \). Then \( W \subseteq G \) and \( \inf_{G^\#} W = s \ast t \in G \). Then \( \inf_X W = s \ast t \) (Lemma 1.1.2) and by continuity we have

\[
\ast \ (s \ast t) \ast r = (\inf_X W) r = \inf_X W r.
\]

Now, for \( w \in W, \ w = g_1 g_2, \ g_1, g_2 \in G, \ g_1 \geq s, \ g_2 \geq t \) we have \( w r = g_1 g_2 r \geq s \ast (t \ast r) \), so \( \inf_X W r \geq s \ast (t \ast r) \) which, combined with \((\ast)\), leads to \( (s \ast t) \ast r \geq s \ast (t \ast r) \), proving associativity.

Next, let \( G = G_0 \). To prove associativity we again may restrict ourselves to the case \( s \ast t \in G \). Then \( \text{Stab}(s \ast t) = \{1\} \), so \( \text{Stab}(s) = \text{Stab}(t) = \{1\} \) by Theorem 1.4.13 (i) i.e. \( s, t \in G_0 = G \). But for this case associativity is obvious (e.g Lemma 1.7.2 (i)).

To prove \((\alpha) \Rightarrow (\beta)\), assume \( X \) is not continuous and \( G \neq G_0 \); we show that multiplication is not associative. There is an \( r_1 \in X \) such that \( r_2 := \inf_X \{g r_1 : g \in G, \ g \geq 1\} \) is strictly greater than \( r_1 \). Further, choose \( s \in G_0 \setminus G \). Then \( \omega(s) \) is the inverse of \( s \), so \( s \ast (\omega(s) \ast r_1) = r_1 \). We shall prove, however that \( s \ast (\omega(s) \ast r_1) \geq r_2 \). In fact, suppose \( s \ast (\omega(s) \ast r_1) < r_2 \). Then there is a \( g \in G \) with \( g \geq s \) (hence \( g > s \)) with \( g(\omega(s) \ast r_1) < r_2 \) i.e. \( \omega(s) \ast r_1 < g^{-1} r_2 \). So there is an \( h \in G, \ h \geq \omega(s) \) (hence, \( h > \omega(s) \)) with \( hr_1 < g^{-1} r_2 \), so \( hgr_1 < r_2 \). But \( hg > s \ast \omega(s) = 1 \), so by the definition of \( r_2 \) we have \( hgr_1 \geq r_2 \), a contradiction.

Remarks.

1. It is easy to find a case where \( \ast \) is not associative. In fact, let \( G \) be the multiplicative group of the strictly positive rational numbers, and let \( X := G \cup G^- \) as in [5] 1.5.5 (c). It is not hard to see that \( G^\# \simeq G_0 \simeq (0, \infty) \) and that \( X \) is not continuous. Indeed, one verifies that \((\sqrt{2}) \ast (\sqrt{2}^{-1}) \ast 1^- = 1^- \), but \( \sqrt{2} \ast ((\sqrt{2})^{-1} \ast 1^-) = 1 \).

2. Theorem 1.7.3 together with Corollary 1.6.7 yields an alternative (but round-about) proof of the associativity of the 'large' multiplication in \( G^\# \) (Proposition 1.4.6).

3. Independently of associativity one can prove in general that for \( s \in G^\#, \ r \in X \) we have \( \text{Stab}(s \ast r) = \text{Stab}(s) \cup \text{Stab}(r) \). It suffices to follow the proof of Theorem 1.4.13 (i) where \( t \in G^\# \) is replaced by \( r \in X \).

To complete this section we prove the following.

**Theorem 1.7.4** Let \( X \) be a complete continuous \( G \)-module. Let \( V \subseteq G^\#, \ W \subseteq X \) be bounded below, non-empty. Then

\[
\inf_X (V \ast W) = (\inf_{G^\#} V) \ast (\inf_X W).
\]

Proof. We use three steps.
(1) For each \( r \in X \) we have \( (\inf_{G^\#} V) \ast r = \inf_X (V \ast r) \).

Proof. Obviously "\( \leq \)" holds. Set \( v_0 := \inf_{G^\#} V \). We prove \( \inf_X (V \ast r) \leq v_0 \ast r \). If \( v_0 \) happens to be in \( V \) then this is clear. Now assume \( v_0 \notin G \). Put \( W := \{ g \in G : g \geq v_0 \} = \{ g \in G : g > v_0 \} \). If \( g \in W \) then there is an \( s \in G^\# \) with \( v_0 < s \leq g \) and there is a \( v \in V \) with \( v_0 < v \leq s \). Hence, \( gr \geq v \ast r \geq \inf_X (V \ast r) \), so that \( v_0 \ast r = \inf_X \{ gr : g \in W \} \geq \inf_X (V \ast r) \) and we are done. It remains to consider the case where \( v_0 \in G, v_0 \notin V \). Put \( W := \{ g \in G : g > v_0 \} \). By continuity of \( X \) we have \( \inf_X W \ast r \geq \inf_X (V \ast r) \). For each \( g \in W \) there is a \( s \in G^\# \) with \( v_0 < s < g \) and there is a \( u \in V \) with \( v_0 < u < s \). Hence, \( gr > v \ast r > \inf_X (V \ast r) \), so that \( v_0 \ast r = (\inf_X W) \ast r \geq \inf_X (V \ast r) \) and we are done.

(2) For each \( s \in G^\# \) we have \( \inf_X (s \ast W) = s \ast \inf_X W \).

Proof. Obviously \( \geq \) holds. Let \( w_0 := \inf_X W, q := \inf_X (s \ast W) \). To prove \( s \ast w_0 \geq q \), assume \( s \ast w_0 < q \). Then there is a \( g \in G \) with \( g \geq s \) and \( gw_0 < q \). Also we have \( q < s \ast w < gw \) for each \( w \in W \). Thus we have \( w_0 < g^{-1}q \leq w \) for all \( w \in W \), hence \( w_0 < g^{-1}q \leq w_0 \) a contradiction.

(3) Conclusion of the Proof. Combine (1), (2) and Lemma 1.1.3.

Remark. Notice that for (2) we did not need continuity.

2 Spaces of continuous linear maps

We recall a few notions from [5]. We augment a linearly ordered group \( G \) with an element \( 0 \) and define \( 0 < g, 0 \cdot g = 0 \cdot 0 = 0 \) for all \( g \in G \). A Krull valuation on a field \( K \) with value group \( G \) is a surjective map \( | | : K \rightarrow G \cup \{ 0 \} \) such that, for all \( \lambda, \mu \in K \) (i) \( | \lambda | \geq 0 \) if and only if \( \lambda = 0 \) (ii) \( | \lambda + \mu | \leq \max (| \lambda |, | \mu |) \) (iii) \( | \lambda \mu | = | \lambda | | \mu | \).

From now on in this paper \( K \) is a field with a Krull valuation \( | | \) and value group \( G \neq \{ 1 \} \). We also assume that \( K \) is complete and that satisfies the conditions \( (\alpha) \rightarrow (\delta) \) of [5] Proposition 1.4.4. Notice that these conditions imply that \( K \) is ultrametrizable and that completeness is equivalent to sequential completeness.

Let \( X \) be a \( G \)-module augmented with a smallest element \( 0 \) (see [5] or [6] for details), let \( E \) be a \( K \)-vector space. An \( X \)-norm on \( E \) is a map \( || | : E \rightarrow X \cup \{ 0 \} \) such that for all \( x, y \in E, \lambda \in K \), (i) \( ||x|| = 0 \) if and only if \( x = 0 \) (ii) \( ||\lambda x|| = | \lambda | ||x|| \) (iii) \( ||x + y|| \leq \max (||x||, ||y||) \).

2.1 The spaces

Definition 2.1.1 Let \( X, Y \) be \( G \)-modules. Let \( E \) be an \( X \)-normed space, let \( F \) be an \( Y \)-normed space. The set of all continuous linear operators \( E \rightarrow F \) is denoted by \( L(E, F) \). An element \( A \in L(E, F) \) is said to be of \textbf{finite rank} if \( AE \) is finite-dimensional. The set of all such finite rank operators is denoted by \( FR(E, F) \). We write \( L(E) := L(E, E), FR(E) := FR(E, E) \).

Remark. Under pointwise addition and scalar multiplication the set \( L(E, F) \) is a \( K \)-vector space having \( FR(E, F) \) as a subspace. In addition, the space \( L(E) \) is a
K-algebra under composition as multiplication with the identity map $I$ as a unit. It is easily seen that $FR(E)$ is a two-sided ideal in $L(E)$.

**Definition 2.1.2** Let $X$ be a $G$-module, let $E, F$ be $X$-normed spaces. A linear operator $A : E \rightarrow F$ is called **Lipschitz** (or, in most literature, **bounded**) if there is a $g \in G$ such that $\|Ax\| \leq g\|x\|$ for all $x \in E$. The set of all such Lipschitz operators is denoted by $Lip(E, F)$.

A linear operator $A : E \rightarrow F$ is called **strictly Lipschitz** if there is a $g \in G$ such that $\|Ax\| < g\|x\|$ for all nonzero $x \in E$. The set of all such strictly Lipschitz operators is denoted by $Lip^s(E, F)$. We write $Lip(E) := Lip(E, E)$ and $Lip^s(E) := Lip^s(E, E)$.

**Remark.** Under pointwise addition and scalar multiplication the set $Lip(E, F)$ is a $K$-vector space having $Lip^s(E, F)$ as a subspace. In addition $Lip(E)$ is a subalgebra of $L(E)$ with unit $I$. It is easily seen that $Lip^s(E)$ is a two-sided ideal in $Lip(E)$.

Before defining natural norms on the various spaces of operators we first study some set-theoretic inclusions that will make clear why 'continuous', 'Lipschitz', 'strictly Lipschitz' are not always identical for linear maps between normed spaces.

**Proposition 2.1.3** $FR(E, F) \subset Lip^s(E, F)$.

For the proof we reinvestigate Sec. 2.3 of [5] by proving the following three lemmas.

**Lemma 2.1.4** If $E, F$ are one-dimensional then each linear map $E \rightarrow F$ is strictly Lipschitz.

**Proof.** Such a map has the form $A : \lambda a \mapsto \lambda b (\lambda \in K)$ for some nonzero $a \in E$ and some $b \in F$. By cofinality of $G\|a\|$ there is a $g \in G$ such that $\|b\| < g\|a\|$. Then for each nonzero $\lambda \in K$ we have $\|A(\lambda a)\| = \|\lambda b\| < |\lambda|g\|a\| = g\|\lambda a\|$ proving the assertion.

**Lemma 2.1.5** If $F$ is one-dimensional then each continuous linear map $E \rightarrow F$ is strictly Lipschitz.

**Proof.** We may assume that $X$ is complete. Let $A \in L(E, F)$, $A \neq 0$. Then $KerA$ is closed so the quotient norm $\| \|$ on $E/KerA$ is defined. Let $\pi : E \rightarrow E/KerA$ be the quotient map. Then $\|\pi(x)\| \leq \|x\|$ for all $x \in E$. There is a unique linear map $A_1 : E/KerA \rightarrow F$ such that $A = A_1 \circ \pi$. By the previous lemma $A_1$ is strictly Lipschitz, so there is a $g \in G$ such that $\|Ax\| = \|A_1(\pi(x))\| < g\|\pi(x)\| \leq g\|x\|$ for all $x \in E$ with $\pi(x) \neq 0$. But if $x \neq 0$, $\pi(x) = 0$, then $Ax = 0$ so we find $\|Ax\| < g\|x\|$ for all nonzero $x \in E$.

**Lemma 2.1.6** Two $X$-norms $\| \|$ and $\| \|$ on a finite-dimensional space $E$ are strictly Lipschitz equivalent i.e. there are $g_1, g_2 \in G$ such that $g_1\|x\|_1 < \|x\|_2 < g_2\|x\|_1$ for all nonzero $x \in E$. 

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Proof. Let $e_1, \ldots, e_n$ be a base of $E$, let $\| \|$ be an $X$-norm on $E$; we prove $\| \|$ to be Lipschitz equivalent to $\| \|_\infty : \xi_1 e_1 + \cdots + \xi_n e_n \mapsto \max_i |\xi_i| \|e_i\|$. For each $i \in \{1, \ldots, n\}$ there is a $g_i \in G$ such that $\|e_i\| < g_i \|e_i\|$ (cofinality of $G\|e_i\|$). Then, with $g := \max(g_1, \ldots, g_n)$ we have for $x = \xi_1 e_1 + \cdots + \xi_n e_n \in E$, $x \neq 0$ that $\|x\| \leq \max_i \|\xi_i e_i\| < g \|x\|_\infty$. Conversely, let, for $j \in \{1, \ldots, n\}$, $A_j : E \to K e_j$ be the map $\sum_{i=1}^n \xi_i e_i \mapsto \xi_j e_j$. By [5], 2.3.4 $A_j$ is continuous with respect to $\| \|$, hence strictly Lipschitz by Lemma 2.1.5. So there is an $h_j \in G$ such that $\|A_j x\| < h_j \|x\|$ for all nonzero $x \in E$. Let $g' := \max(h_1, \ldots, h_n)$. Then for nonzero $x \in E$ we have $\|x\|_\infty = \max_i \|A_i x\| < g' \|x\|$.

Proof of Proposition 2.1.3 Let $A \in FR(E, F)$. Then $\text{Ker}A$ is closed, so (again assuming that $X$ is complete) the quotient norm on $E/\text{Ker}A$ is defined. There is a unique linear injection $A_1 : E/\text{Ker}A \to F$ such that $A = A_1 \circ \pi$ where $\pi : E \to E/\text{Ker}A$ is the canonical map. By Lemma 2.1.6 the norm $x \mapsto \|A_1 x\|$ on the finite-dimensional space $E/\text{Ker}A$ is strictly Lipschitz equivalent to the quotient norm. We therefore have a $g \in G$ such that $\|A_1(\pi(x))\| < g \|\pi(x)\|$ for all $x \in E$ with $\pi(x) \neq 0$ implying $\|Ax\| < g \|x\|$ for all nonzero $x \in E$.

We now will have a closer look at the obvious inclusion $\text{Lip}(E, F) \subset L(E, F)$. First we consider equality.

Proposition 2.1.7 Suppose there exists an $s_0 \in X$ such that for each $s \in \|E\| \setminus \{0\}$, $\inf_K \{g_0 : g \in G, \; g_0 \geq s\} = s$. Then $\text{Lip}(E, F) = L(E, F)$.

Proof. Let $A \in L(E, F)$. Then $A$ maps bounded sets into bounded sets so there is a $g \in G$ such that $x \in E$, $\|x\| \leq s_0$ implies $\|Ax\| \leq g s_0$. Now let $x \in E$, $x \neq 0$; we prove that $\|Ax\| \leq g \|x\|$. Choose $\mu \in K$ such that $|\mu| s_0 \geq \|x\|$. Then $\|\mu^{-1} x\| \leq s_0$ so $\|A(\mu^{-1} x)\| \leq g s_0$ or $g^{-1} \|Ax\| \leq |\mu| s_0$. This holds for all $\mu \in K$ with $|\mu| s_0 \geq \|x\|$ so $g^{-1} \|Ax\| \leq \inf_K \{|\mu| s_0 : \mu \in K, \; |\mu| s_0 \geq \|x\|\} = \|x\|$ by assumption.

Corollary 2.1.8 ([8] 3.2) Let $E, F$ be $G\#$-normed spaces. Then $\text{Lip}(E, F) = L(E, F)$.

Proof. Apply the previous proposition to $X := G\#$, $s_0 := 1$.

Proposition 2.1.9 Suppose the norm on $E$ is equivalent to a $G$-norm. Then $\text{Lip}(E, F) = L(E, F)$.

Proof. Let $\| \|$ be the norm on $E$, equivalent to some $G$-norm $\| \|_1$. Let $A \in L(E, F)$ and suppose $A$ is not Lipschitz. Then for each $g \in G$ we can find an $x_g \in E$ such that $\|A x_g\| > g \|x_g\|$. By scalar multiplication we may suppose that $\|x_g\|_1 = 1$ for all $g \in G$. Then $\{x_g : g \in G\}$ is also $\| \|$-bounded so that $\{A x_g : g \in G\}$ is $\| \|$-bounded. On the other hand there is an $s \in X$ such that $\|x_g\| \geq s$ for all $g \in G$ (otherwise 0 is in the $\| \|$-closure, hence $\| \|_1$-closure of $\{x_g : g \in G\}$). Then $\|A x_g\| > g \|x_g\| \geq g s$ for all $g \in G$, showing that $\{A x_g : g \in G\}$ is $\| \|$-unbounded, a contradiction.

Corollary 2.1.10 If $E$ is linearly homeomorphic to $c_0$ then $\text{Lip}(E, F) = L(E, F)$. 16
Theorem 2.1.11 Let \( s_0 \in X \) and let \( \tau(s) := \tau(s; s_0) \) be the topological type of \( s \in X \) with respect to the unit \( s_0 \) (see 1.5). Suppose there is a proper convex subgroup \( H \) of \( G \) for which \( \tau(s) \subset H \) for all \( s \in \|E\| \setminus \{0\} \). Then \( \text{Lip}(E, F) = L(E, F) \).

**Proof.** Let \( A \in L(E, F) \). Then there is a \( g_1 \in G \) such that \( x \in E, \|x\| \leq s_0 \) implies \( \|Ax\| \leq g_1 s_0 \). Choose \( g_0 \in G, g_0 \notin H, g_0 > 1 \). We prove that \( \|Ax\| \leq g_0^1 g_1 \|x\| \) for all \( x \in E, x \neq 0 \). By \([5]\), 1.6.1 \( g_0 s_0 \) is either \( < g_0 \|x\| \leq s_0 \) for some \( g \in G \) (which cannot happen since \( g_0 > 1 \)) or \( g_0 s_0 > g \|x\| \geq s_0 \) for some \( g \in G \). From \( \|x\| \leq g^{-1} g_0 s_0 \) it follows easily that \( \|Ax\| \leq g^{-1} g_0 g_1 s_0 \leq g^{-1} g_0 g_1 \|x\| = g_0 g_1 \|x\| \).

Corollary 2.1.12 If the valuation of \( K \) is of finite rank then \( \text{Lip}(E, F) = L(E, F) \).

**Proof.** Among the proper convex subgroups there is a largest one \( H \). Then clearly \( \tau(s) \subset H \) for all \( s \in X \). Now apply Theorem 2.1.11.

Now we will discuss cases where not every continuous linear operator is Lipschitz. To find such case we must have by Theorem 2.1.11 that \( \bigcup \{\tau(s) : s \in \|E\| \setminus \{0\}\} = G \). By metrizability of \( E \), \( G \) must be the union of a strictly increasing sequence \( \tau(s_1), \tau(s_2), \ldots \). Also recall that \( \text{Stab}(s) \subset \tau(s) \) for all \( s \in X \) (Theorem 1.5.3 (ii)). After these preparations the following theorem will not come as a surprise. (See also \([1]\) and \([8]\) 4.2)

Theorem 2.1.13 Let \( E \) be a Banach space such that each onedimensional subspace has an orthogonal complement. Let \( \tau \) be as in Theorem 2.1.11 and suppose there exist \( s_1, s_2, \ldots \in \|E\| \setminus \{0\} \) such that \( \tau(s_1) \subset \tau(s_2) \subset \ldots \) and \( \bigcap \tau(s_n) = G \), and such that \( \text{Stab}(s_n) \subset \tau(s_n) \) for each \( n \). Then \( \text{Lip}(E) \neq L(E) \).

**Proof.** Choose \( e_1, e_2, \ldots \in E \) such that \( \|e_n\| = s_n \) for each \( n \), let \( P_n \) be an ortho­

projection \( E \to K e_n \). Choose \( \lambda_1, \lambda_2, \ldots \in K \) such that \( |\lambda_n| > 1, |\lambda_n| \in \tau(\|e_n\|) \setminus \text{Stab}(\|e_n\|) \) for all \( n \) and such that \( |\lambda_n| \to \infty \). We shall prove that the formula

\[
Ax = \sum_{n=1}^{\infty} \lambda_n P_n x
\]

defines a continuous linear operator that is not Lipschitz.

To show summability it suffices (since \( E \) is a Banach space and by scalar multiplication) that \( \lim_{n \to \infty} \lambda_n P_n x = 0 \) for all \( x \in E, \|x\| \leq s_0 \). Define \( \xi_1, \xi_2, \ldots \in K \) by \( P_n x = \xi_n e_n \). Then \( \|\xi_n e_n\| = \|P_n x\| \leq \|x\| \leq s_0 \). Thus, for all \( h \in \tau(\|e_n\|) \) we have \( \|\xi_n e_n\| \leq \lambda_0, and, since \( |\lambda_n| \in \tau(\|e_n\|) \), also \( \|\lambda_n P_n x\| \leq \lambda_0 \) for all \( h \in \tau(\|e_n\|) \).

Then, from \( \inf_{G \neq \tau(\|e_n\|)} \|x\| \to 0 \) we obtain \( \lim_{n \to \infty} \lambda_n P_n x = 0 \), so at this point we have shown that \( A \) is a well-defined linear operator \( E \to E \). To establish continuity of \( A \) it suffices to show that the image of \( \{x \in E : \|x\| \leq s_0\} \) is bounded. But that is easy: from the above proof it follows that if \( \|x\| \leq s_0 \) then \( \|\lambda_n P_n x\| \leq \lambda_0 \), hence \( \|Ax\| \leq \lambda_0 \).

Now assume \( A \) is Lipschitz; we derive a contradiction. There is a \( g \in G \) such that \( \|\lambda_n e_n\| = \|A e_n\| \leq g \|e_n\| \) for all \( n \). Since \( |\lambda_n| \to \infty \) there is a \( k \in \mathbb{N} \) such that \( |\lambda_n| > g \) for \( n \geq k \), and it follows that \( \|\lambda_n e_n\| = g \|e_n\| \) for \( n \geq k \). We see that \( g^{-1} |\lambda_n| \in \text{Stab}(\|e_n\|) \) for \( n \geq k \). But from \( g^{-1} |\lambda_n| \to \infty \) it follows that \( \bigcup \text{Stab}(\|e_n\|) = G \),
so \( g \in \text{Stab}(\|e_n\|) \) for infinitely many \( n \). Hence \( |\lambda_n| = gg^{-1}|\lambda_n| \in \text{Stab}(\|e_n\|) \) for infinitely many \( n \), a contradiction.

**Remark.** Unfortunately we have not been able to find necessary and sufficient conditions on \( E \) in order that \( \text{Lip}(E) = L(E) \). With Theorem 2.1.13 in mind, a major step in this direction would be the answer to the following.

**Problem:** Suppose \( \text{Stab}(s) = \tau(s) \) for all \( s \in X \) (or \( \text{Stab}(s) \cup \text{Stab}(t) = \tau(s; t) \) for all \( s, t \in X \)). Does it follow that \( L(E) = \text{Lip}(E) \)?

The answer is not even known for Norm Hilbert Spaces \( E \).

We do have the following obvious conclusions.

**Corollary 2.1.14** If \( E \) is an infinite-dimensional Form Hilbert Space ([5] 4.4) then \( \text{Lip}(E) \neq L(E) \).

**Proof.** \( \|E\| \setminus \{0\} = \sqrt{G} \) (see [5] 4.4), which is a group, so \( \text{Stab}(s) = \{1\} \) for all \( s \in \|E\| \setminus \{0\} \). On the other hand, for any orthogonal base \( e_1, e_2, \ldots \) we have \( \bigcup_n \tau(\|e_n\|) = G \) by the type condition. So \( E \) satisfies the condition of Theorem 2.1.13.

**Corollary 2.1.15** Let \( E \) be a Banach space that does not contain an infinite-dimensional Norm Hilbert Space. Then \( \text{Lip}(E, F) = L(E, F) \).

**Proof.** Suppose \( \text{Lip}(E, F) \neq L(E, F) \). Then by Theorem 2.1.11 there exist \( s_1, s_2, \ldots \in \|E\| \setminus \{0\} \) such that \( n \mapsto \tau(s_n) \) is strictly increasing and \( \bigcup_n \tau(s_n) = G \). Choose \( e_1, e_2, \ldots \in E \) such that \( \|e_n\| = s_n \) for each \( n \). Then \( e_1, e_2, \ldots \) is an orthogonal base of \( D = [e_1, e_2, \ldots] \) and by [5], 1.6.6 the sequence \( \|e_1\|, \|e_2\|, \ldots \) satisfies the type condition. From [5] 4.3.7 \( (\beta) \iff (\gamma) \) it follows that \( D \) is a Norm Hilbert Space, a contradiction.

Now we consider the inclusion \( \text{Lip}^\sim(E, F) \subset \text{Lip}(E, F) \). In this case we do have a characterization for equality.

**Theorem 2.1.16** The following are equivalent.

1. \( \|E\| \setminus \{0\} \) is almost faithful (see 1.3).
2. \( \text{Lip}^\sim(E, F) = \text{Lip}(E, F) \) for all \( X \)-normed spaces \( F \).
3. \( \text{Lip}^\sim(E) = \text{Lip}(E) \).
4. \( I \in \text{Lip}^\sim(E) \).

**Proof.** \( (\alpha) \Rightarrow (\beta) \). Let \( A \in \text{Lip}(E, F) \) and let \( g \in G \) be such that \( \|Ax\| \leq g\|x\| \) for all \( x \in E \). By Proposition 1.3.2 \( (\beta) \) there is a \( g' \in G \) such that \( s < g' \) for all \( s \in \|E\| \setminus \{0\} \). Then \( g\|x\| < g'\|x\| \) for all nonzero \( x \in E \), and \( \|Ax\| < g'\|x\| \).

\( (\beta) \Rightarrow (\gamma) \Rightarrow (\delta) \) are trivial.

\( (\delta) \Rightarrow (\alpha) \). From \( (\delta) \) we obtain a \( g \in G \) such that \( \|x\| < g\|x\| \) for all nonzero \( x \in E \), which implies \( (\alpha) \) by Proposition 1.3.2.

**Example 2.1.17** Let \( E \) be the Norm Hilbert Space of [5], 4.2.2. By construction \( \|E\| \setminus \{0\} \) is not almost faithful, so \( \text{Lip}^\sim(E) \) is properly contained in \( \text{Lip}(E) \).
To conclude this section we consider algebraic properties of the space of finite rank operators. First a lemma.

**Lemma 2.1.18** A linear operator $A : E \to F$ is in $FR(E, F)$ if and only if there exists $a_1, \ldots, a_n \in F$ and $f_1, \ldots, f_n \in E'$ such that $Ax = \sum_{i=1}^{n} f_i(x)a_i$ for all $x \in E$.

**Proof.** (Well known but included for convenience). We only need to prove the 'only if', so let $A \in FR(E, F)$. Choose a base $a_1, \ldots, a_n$ of $AE$. There are functions $f_1, \ldots, f_n : E \to K$ such that $Ax = \sum_{i=1}^{n} f_i(x)a_i$ ($x \in E$). Now $f_i = \delta_i \circ A$ where $\delta_i$ is the $i$th coordinate function $AE \to K$ which is automatically continuous (and linear) by finite dimensionality. Hence, $f_i \in E'$ and we are done.

**Proposition 2.1.19** Let $E' \neq \{0\}$. Then the following are equivalent.

(a) Each nonzero two-sided ideal in $L(E)$ contains $FR(E)$.

(b) Each nonzero two-sided ideal in $FR(E)$ equals $FR(E)$.

(c) $E$ is dual-separating.

**Proof.** $(\gamma) \Rightarrow (\beta)$. Let $J$ be a nonzero two-sided ideal in $FR(E)$. Let $a \in E$, $f \in E'$. By Lemma 2.1.18 it suffices to prove that $B : x \mapsto f(x)a$ is in $J$. Let $A \in J$, $A \neq 0$, choose a $b \in E$ with $Ab \neq 0$. By $(\gamma)$ there is a $g \in E'$ with $g(Ab) = 1$. Define $T_1, T_2 \in FR(E)$ by the formulas

$$T_1x = f(x)b$$

$$T_2x = g(x)a.$$  

An easy computation shows that $T_2AT_1 = B$. Hence $B \in J$.

$(\beta) \Rightarrow (a)$. Let $J$ be a nonzero two-sided ideal in $L(E)$. Then $J \cap FR(E)$ is a two-sided ideal in $FR(E)$. To see that it is nonzero, let $A \in J$, $A \neq 0$; choose a $b \in E$ with $Ab \neq 0$. There is an $a \in E$ and an $f \in E'$ with $f(a) \neq 0$. Now let $T : x \mapsto f(x)b$. Then $AT \in J \cap FR(E)$ and $(AT)(a) = f(a)Ab \neq 0$. From $(\beta)$ it then follows that $J \cap FR(E) = FR(E)$ so $J \supset FR(E)$.

$(\alpha) \Rightarrow (\gamma)$ Suppose $E$ is not dual-separating. Then $E_0 := \bigcap \{\text{Ker} f : f \in E'\}$ is a (closed) subspace of $E$, $\{0\} \neq E_0 \subsetneq E$. Let $J := \{A \in L(E) : AE \subset E_0\}$. We first prove that $J$ is a two-sided ideal in $L(E)$. Let $B \in L(E)$. Then, for $A \in J$, $ABE \subset AE \subset E_0$, so $AB \in J$. To prove that also $BA \in J$, first observe that $BE_0 \subset E_0$ (let $x \in E_0$. For any $f \in E'$ we have $f \circ B \in E'$ so $(f \circ B)(x) = 0$ i.e. $f(Bx) = 0$, so $Bx \in E_0$). Then $BAE \subset BE_0 \subset E_0$. Thus, $J$ is a two-sided ideal in $L(E)$. For any $a \in E_0$, $a \neq 0$ and $f \in E'$, $f \neq 0$ the map $x \mapsto f(x)a$ is in $J$ and nonzero. So by $(\alpha)$ we have $J \supset FR(E)$. But for $b \in E \setminus E_0$, $f \in E'$, $f \neq 0$, the map $x \mapsto f(x)b$ is in $FR(E)$, nonzero, not in $J$, which conflicts $J \supset FR(E)$.

In the same spirit we prove the next result.

**Proposition 2.1.20** Let $E$ be dual-separating, let $A \in L(E)$. Suppose $A$ commutes with every operator in $FR(E)$. Then there is a $\lambda \in K$ such that $A = \lambda I$. 

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Proof. Let $a \in E$, $a \neq 0$. We first prove that $Aa$ is a scalar multiple of $a$. In fact, there is an $f \in E'$ with $f(a) \neq 0$. The formula

$$Bx = f(x)a$$

defines a $B \in FR(E)$. So $AB = BA$. Applied to $a$ this implies $f(a)Aa = f(Aa)a$ and we are done since $f(a) \neq 0$. Thus, there is a function $\lambda : E \setminus \{0\} \to K$ such that $Aa = \lambda(a)a$ for all $a \in E \setminus \{0\}$. To prove that $\lambda$ is constant first observe that $\lambda(a) = \lambda(b)$ if $a, b$ are dependent. If $a, b$ are linearly independent we have $A(a+b) = \lambda(a+b)(a+b)$ and $Aa+Ab = \lambda(a)a+\lambda(b)b$ and we get $\lambda(a+b) = \lambda(a) = \lambda(b)$.

For later use we introduce the trace function on $FR(E)$. The construction below is well-known but we include it here for reference.

For $f \in E'$, $a \in E$ let $A_{f,a}(x) := f(x)a$ ($x \in E$). The map $E' \times E \to FR(E)$ given by $(f,a) \mapsto A_{f,a}$ is bilinear, so by the universal property of the tensor product it induces a linear map $\varphi : E' \otimes E \to FR(E)$.

**Proposition 2.1.21** $\varphi$ is a bijection.

**Proof.** Surjectivity follows from Lemma 2.1.18. To show injectivity, let $z \in E' \otimes E$, $\varphi(z) = 0$. The element $z$ has the form $z = f_1 \otimes x_1 + \ldots + f_n \otimes x_n$ for certain $f_1, \ldots, f_n \in E'$, $x_1, \ldots, x_n \in E$. By elementary tensor product theory we may assume that $x_1, \ldots, x_n$ are linearly independent. From $\varphi(z) = 0$ we obtain $\sum_{i=1}^n f_i(x_i) = 0$ for all $x \in E$, hence all $f_i$ are 0, so $z = 0$.

The bilinear map $E' \times E \to K$ given by $(f,a) \mapsto f(a)$ induces a linear map $\tau : E' \otimes E \to K$ given by $\tau(f \otimes a) = f(a)$ ($f \in E'$, $a \in E$).

**Definition 2.1.22** For $A \in FR(E)$ let $\text{tr}(A) := (\tau \circ \varphi^{-1})(A)$.

We check the usual properties.

**Proposition 2.1.23**

(i) $\text{tr}$ is a linear map $FR(E) \to K$.

(ii) If $A \in FR(E)$ and

$$Ax = \sum_{i=1}^n f_i(x)a_i \quad (x \in E) \quad (f_1, \ldots, f_n \in E', \quad a_1, \ldots, a_n \in E)$$

is any representation of $A$ then

$$\text{tr}(A) = \sum_{i=1}^n f_i(a_i).$$

(iii) For $A \in FR(E)$, $B \in L(E)$

$$\text{tr}(AB) = \text{tr}(BA).$$

**Proof.** (i) is obvious. To prove (ii), from $A = \sum_{i=1}^n A_{f_i,a_i}$ we obtain $\varphi^{-1}(A) = \sum_{i=1}^n f_i \otimes a_i$, so $\tau \varphi^{-1}(A) = \sum_{i=1}^n f_i(a_i)$. Finally we prove (iii). By Lemma 2.1.18 it suffices to prove (iii) for an $A$ of the form $x \mapsto f(x)a$, where $f \in E'$, $a \in E$. Let $x \in E$. For each $B \in L(E)$ we have $AB(x) = f(Bx)a$, so by (ii), $\text{tr}(AB) = f(Ba)$. But also $BA(x) = f(x)Ba$, so $\text{tr}(BA) = f(Ba)$. 

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2.2 Operator norms

In this paper we will not focus on norms on $L(E, F)$; for later use we quote the following from [5] Sec 2.2.

**Theorem 2.2.1** Let $X, Y$ be $G$-modules where $Y$ is complete. Let $E$ be an $X$-normed space, let $F$ be a $Y$-normed space and let $x \in X$. Then the formula

$$\|A\|_s = \sup\{\|Ax\| : x \in E, \|x\| \leq s\}$$

(where the supremum is taken in $Y \cup \{0\}$) defines a $Y$-norm $\| \cdot \|_s$ on $L(E, F)$. For each $s, t \in X$ the norms $\| \cdot \|_s$ and $\| \cdot \|_t$ are Lipschitz equivalent. The induced topology on $L(E, F)$ is the topology of uniform convergence on bounded subsets of $E$. If $F$ is a Banach space then so is $(L(E, F), \| \cdot \|_s)$ for each $s \in X$.

Next we study natural norms on $\text{Lip}(E, F)$ and $\text{Lip}^\sim(E, F)$. From now on in this section $X$ is a $G$-module and $E, F$ are $X$-normed spaces.

**Definition 2.2.2** For $A \in \text{Lip}(E, F)$ put $\Gamma_A := \{g \in G : \|Ax\| \leq g\|x\| \text{ for all } x \in E\}$ and $\|A\| := \inf_{G\setminus\{0\}} \Gamma_A$. Similarly, for $B \in \text{Lip}^\sim(E, F)$ put $\Gamma_B^\sim := \{g \in G : \|Bx\| < g\|x\| \text{ for all nonzero } x \in E\}$ and $\|B\|_\sim := \inf_{G\setminus\{0\}} \Gamma_B^\sim$. We call $\| \cdot \|$ the Lipschitz norm and the $\| \cdot \|_\sim$ the strict Lipschitz norm. (See Proposition 2.2.4.)

**Convention.** To avoid complicated notations we will write henceforth, for a subset $V$ of $G$, $\inf V$ in place of $\inf_{G\setminus\{0\}} V$; similarly for sup.

**Proposition 2.2.3** Let $A \in \text{Lip}(E, F), B \in \text{Lip}^\sim(E, F)$. Then $\|A\|, \|B\|_\sim \in G\setminus\{0\}, \|B\| \leq \|B\|_\sim$ and

(i) $\{g \in G : g > \|A\|\} \subset \Gamma_A \subset \{g \in G : g \geq \|A\|\}$,

(ii) $\{g \in G : g > \|B\|_\sim\} \subset \Gamma_B^\sim \subset \{g \in G : g \geq \|B\|_\sim\}$.

**Proof.** Straightforward.

**Proposition 2.2.4** $\| \cdot \|$ is a norm on $\text{Lip}(E, F)$; $\| \cdot \|_\sim$ is a norm on $\text{Lip}^\sim(E, F)$.

**Proof.** We only prove the triangle inequality for $\| \cdot \|_\sim$ leaving the rest to the reader. Let $A, B \in \text{Lip}^\sim(E, F)$; let $g_1 \in \Gamma_A^\sim, g_2 \in \Gamma_B^\sim$. Then, for $x \in E$, $x \neq 0$, $\|A + B\|_\sim x \leq \max(\|Ax\|, \|Bx\|) < \max(g_1\|x\|, g_2\|x\|) = \max(g_1, g_2)\|x\|$. We see that $\|A + B\|_\sim \leq \inf \{\max(g_1, g_2) : g_1 \in \Gamma_A^\sim, g_2 \in \Gamma_B^\sim\}$, which equals, by Lemma 1.1.1, $\max(\inf \Gamma_A^\sim, \inf \Gamma_B^\sim)$. We see that $\|A + B\|_\sim \leq \inf \{\max(g_1, g_2) : g_1 \in \Gamma_A^\sim, g_2 \in \Gamma_B^\sim\}$, which equals, by Lemma 1.1.1, $\max(\inf \Gamma_A^\sim, \inf \Gamma_B^\sim)$. We see that $\|A + B\|_\sim \leq \inf \{\max(g_1, g_2) : g_1 \in \Gamma_A^\sim, g_2 \in \Gamma_B^\sim\}$, which equals, by Lemma 1.1.1, $\max(\inf \Gamma_A^\sim, \inf \Gamma_B^\sim)$.

**Proposition 2.2.5** Let $F$ be a Banach space. Then $(\text{Lip}(E, F), \| \cdot \|)$ and $(\text{Lip}^\sim(E, F), \| \cdot \|_\sim)$ are Banach spaces.

**Proof.** See [5] 2.3.6 for the Lipschitz case. Now let $A_1, A_2, \ldots$ be a Cauchy sequence in $\text{Lip}^\sim(E, F)$. Since $\| \cdot \| \leq \| \cdot \|_\sim$ the sequence is also Cauchy in $\text{Lip}(E, F)$ so there is
an $A \in \text{Lip}(E, F)$ with $\|A - A_n\| \to 0$. Now let $g \in G$. There is an $n_0$ such that, for $m, n \geq n_0$, $\|A_m - A_n\| \sim < g$, hence

$$\|A_m x - A_n x\| < g \|x\| \quad (x \in E, x \neq 0, n, m \geq n_0)$$

Since ‘open’ balls in $E$ are closed we have

$$\|(A - A_n)x\| = \lim_{m \to \infty} \|A_m x - A_n x\| < g \|x\| \quad (x \in E, x \neq 0, n \geq n_0)$$

showing that $A - A_n$, hence $A$, is in Lip$\sim(E, F)$ and that $\|A - A_n\| \sim \leq g$ for $n \geq n_0$, which proves $\lim_{n \to \infty} \|A - A_n\| \sim = 0$.

As an example we compute the norms of an operator with one-dimensional range. We assume $X$ to be complete.

Example 2.2.6 Let $a \in F, f \in E', f \neq 0$ and let $A \in FR(E, F)$ be the map

$$A : x \mapsto f(x)a.$$ 

Let $b \in E$ be such that $f(b) = 1$. Then

(i) $\|A\| = \inf \{g \in G : \|a\| \leq g \text{ dist}(b, \text{Ker} f)\}$,

(ii) if dist$(b, \text{Ker} f)$ is not attained then $\|A\| \sim = \|A\|$,

(iii) if dist$(b, \text{Ker} f)$ is attained then

$$\|A\| \sim = \inf \{g \in G : \|a\| < g \text{ dist}(b, \text{Ker} f)\}.$$ 

Proof. Set $T := \{g \in G : \|a\| \leq g \text{ dist}(b, \text{Ker} f)\}$ and $

\Gamma := \{g \in G : \|a\| < g \text{ dist}(b, \text{Ker} f)\}.$

We first show that $\Gamma = \Gamma_A$ (which proves (i)). Let $g \in \Gamma_A$. Then $\|f(x) a\| \leq g \|x\|$ for all $x \in E$, so by taking $x = b - y$, $y \in \text{Ker} f$ we get $g^{-1} \|a\| \leq \|b - y\|$. It follows that $g^{-1} \|a\| \leq \text{dist}(b, \text{Ker} f)$, so that $g \in \Gamma$. Conversely, let $g \in \Gamma$. Let $x \in E$. Then $x = \lambda b + y$ for some $\lambda \in K$, $y \in \text{Ker} f$. Then $\|Ax\| = \|f(x) a\| = ||\lambda a|| \leq g ||\lambda|| \text{dist}(b, \text{Ker} f) = g \text{ dist}(\lambda b, \text{Ker} f) = g \text{ dist}(x, \text{Ker} f) \leq g \|x\|$, so $G \in \Gamma_A$.

If dist$(b, \text{Ker} f)$ is not attained the last inequality is strict for $x \neq 0$ so we obtain $\Gamma < \Gamma_A$ i.e. $\Gamma = \Gamma_A$ implying (ii).

Finally, let dist$(b, \text{Ker} f)$ be attained; we prove that $\Gamma_A = \Gamma_A$ (which shows (iii)). Let $g \in \Gamma_A$. Then $\|f(x) a\| < g \|x\|$ for all nonzero $x \in E$ so by taking $x = b \cdot y$ where $y \in \text{Ker} f$, $\|b - y\| = \text{dist}(b, \text{Ker} f)$ we get $\|a\| < g \text{ dist}(b, \text{Ker} f)$ or $g \in \Gamma_A$. Conversely, if $g \in \Gamma_A$, let $x \in E, x \neq 0$. Then $x = \lambda b + y$ for some $\lambda \in K$, $y \in \text{Ker} f$. To show that $g \in \Gamma_A$ we may assume $\lambda \neq 0$. Then $\|Ax\| = \|f(x) a\| = ||\lambda|| \|a\| < |\lambda| g \text{ dist}(b, \text{Ker} f) = g \text{ dist}(\lambda b, \text{Ker} f) = g \text{ dist}(x, \text{Ker} f) \leq g \|x\|$, so $g \in \Gamma_A$ which finishes the proof.

The following observation will be convenient later on, especially in Chapter 3.

Proposition 2.2.7 Let $E$ have an orthogonal base $e_1, e_2, \ldots$. Then for $A \in \text{Lip}(E, F), B \in \text{Lip}(E, F)$ we have

$$\|A\| = \inf \{g \in G : \|A e_n\| \leq g \|e_n\| \text{ for each } n\}$$

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\[ \| B \| \sim = \inf \{ g \in G : \| B e_n \| < g \| e_n \| \text{ for each } n \}. \]

Conversely, let \( g \in G \) and \( y_1, y_2, \ldots \in F \) such that \( \| y_n \| \leq g \| e_n \| \) (resp. \( \| y_n \| < g \| e_n \| \)) for all \( n \). Then \( e_n \mapsto y_n \) \((n \in \mathbb{N})\) extends uniquely to a Lipschitz operator (resp. strictly Lipschitz operator) \( E \to F \).

**Proof.** Straightforward.

To compare the norms \( \| \cdot \| \) and \( \| \cdot \| \sim \) in more detail, we compute the norms of the identity operator \( I \in Lip(E) \). From now on in this Section we write, in case \( G \) is quasidiscrete, \( g_0 := \min \{ g \in G : g > 1 \} \).

**Proposition 2.2.8** Let \( E \neq \{ 0 \} \). Set \( H_1 := \bigcap \{ \text{Stab}(\| x \|) : x \in E, x \neq 0 \} \), \( H_2 := \bigcup \{ \text{Stab}(\| x \|) : x \in E, x \neq 0 \} \). Then we have

(i) \( \| I \| = \inf H_1 \).

(ii) If \( H_2 \neq G \) (i.e. if \( \| E \| \setminus \{ 0 \} \) is almost faithful) then \( \| I \| \sim = \sup H_2 \), except when \( H_2 = \{ 1 \} \) and \( G \) is quasidiscrete in which case \( \| I \| \sim = g_0 \).

**Proof.** (i) It is easily seen that \( \Gamma_I = \{ g \in G : g \geq 1 \) or \( g \in H_1 \} \); hence \( \| I \| = \inf \Gamma_I = \inf H_1 \).

(ii) One verifies that \( \Gamma_I^\sim \) is \( \{ g \in G : g > 1 \) and \( g \notin H_2 \} \). If \( H_2 \neq \{ 1 \} \) we see that \( \| I \| \sim = \inf \Gamma_I^\sim = \sup H_2 \). If \( H_2 = \{ 1 \} \) then \( \Gamma_I^\sim = \{ g \in G : g > 1 \} \) which completes the proof.

From Proposition 2.2.8 it follows that \( \| I \| \leq 1 < \| I \| \sim \), but we may have strict inequalities.

**Example 2.2.9** A space \( E \) for which \( \| I \| < 1 < \| I \| \sim \).

**Construction.** Choose \( G \) such that it admits a convex subgroup \( H, \{ 1 \} \neq H \neq G \). Let \( t := \inf H \). By Proposition 1.4.1 we have \( H = \text{Stab}(t) \). Now let \( E := c_0 \) be equipped with the norm \( x = (\xi_1, \xi_2, \ldots) \mapsto \max_n |\xi_n| t \). We see that for each nonzero \( x \in E \) we have \( \text{Stab}(\| x \|) = H \), so, by the previous Proposition, \( \| I \| = \inf H < 1 < \sup H = \| I \| \sim \).

**Proposition 2.2.10** Let \( \| E \| \setminus \{ 0 \} \) be almost faithful. Then the norms \( \| \cdot \| \) and \( \| \cdot \| \sim \) on \( Lip(E, F) = Lip^\sim(E, F) \) are Lipschitz equivalent. More precisely, let \( g \in G, g > 1, g \notin \bigcup \{ \text{Stab}(\| x \|) : x \in E, x \neq 0 \} \). Then \( \| A \| \leq \| A \| \sim \leq g \| A \| \) for all \( A \in Lip(E, F) \).

**Proof.** Let \( g_1 \in \Gamma_A, x \in E, x \neq 0 \). Then \( \| Ax \| \leq g_1 \| x \| < g_1 \| x \| \). It follows that \( \| A \| \sim \leq g_1 \) or \( g^{-1} \| A \| \sim \leq g_1 \). This holds for each \( g_1 \in \Gamma_A \) so that \( g^{-1} \| A \| \sim \leq \inf \Gamma_A = \| A \| \).

**Proposition 2.2.11** Let \( \| E \| \setminus \{ 0 \} \) be almost faithful. Suppose \( G \) is quasidiscrete. Then the following are equivalent.

(a) \( \| A \| \sim = \| A \| \) for all \( A \in Lip(E, F) \),
Proof. We may assume $E \neq \{0\}$. The implication $(\alpha) \Rightarrow (\beta)$ is trivial. To prove $(\beta) \Rightarrow (\gamma)$ first observe that $||I|| = ||I|| = 1$. If $(\gamma)$ were not true, we could find an $a \in E$, $a \neq 0$ and an $h \in G$, $h > 1$ such that $h||a|| = ||a||$. Let $g \in \Gamma_\gamma$. Then $g > 1$ and $g||a|| > ||a||$ so $g > h$. Then $||I|| = \inf \Gamma_\gamma \geq h > 1$, a contradiction.

Finally we prove $(\gamma) \Rightarrow (\alpha)$. Let $g \in \Gamma_A$, let $g' \in G$, $g' > g$. Then, for $x \in E, x \neq 0$ we have $||Ax|| \leq g\|x\| < g'\|x\|$ by faithfulness, so $g' \in \Gamma_A$. We see that

\[(*) \quad ||A|| = \inf \{g' \in G : g' > g\}\]

and so $||A|| \leq g$ by quasidiscreteness. It follows that $||A|| = \inf \Gamma_A = ||A||$ which proves $(\alpha)$.

Proposition 2.2.12 Let $||E|| \setminus \{0\}$ be almost faithful. Suppose $G$ is quasidiscrete. Then the following are equivalent.

$(\alpha) \quad ||A|| \leq ||A|| = g_0||A||$ for all $A \in Lip(E, F),$

$(\beta) \quad ||I|| = ||I|| \leq g_0||I||,$

$(\gamma) \quad ||E|| \setminus \{0\}$ is faithful.

Proof. Again we may assume $E \neq \{0\}$. The implication $(\alpha) \Rightarrow (\beta)$ is trivial. To prove $(\beta) \Rightarrow (\gamma)$ let $H_2$ be as in Proposition 2.2.8.

If $H_2 \neq \{1\}$ then $g_0 \in H_2$ and $(\beta)$ implies $||I|| \leq ||I||$. Hence $||I|| = ||I|| = 1$ conflicting Proposition 2.2.8 (ii). Thus, $H_2 = \{1\}$ which is $(\gamma)$. Finally we prove $(\gamma) \Rightarrow (\alpha)$. For this follow the proof of $(\gamma) \Rightarrow (\alpha)$ of the previous proposition until formula $(*)$, whose right hand side equals $g_0g$ by quasidiscreteness. So $g_0^{-1}||A|| \leq g$ for all $g_0\Gamma_A$ i.e. $g_0^{-1}||A|| \leq ||A||$.

There is more to it.

Proposition 2.2.13 Let $||E|| \setminus \{0\}$ be faithful and let $G$ be quasidiscrete. Then for each $A \in Lip(E, F)$ we have $||A|| = ||A||$ or $||A|| = g_0||A||$.

Proof. We may assume $A \neq 0$. If Stab($||A||$) $\neq \{1\}$ we have $g_0 \in \text{Stab}(||A||)$, so $||A|| \leq g_0||A|| = ||A||$ and we have $||A|| = ||A|| = g_0||A||$. The same conclusion holds if Stab($||A||$) $\neq \{1\}$. So, we may assume that both $||A||$ and $||A||$ are in $G_0$ (see Definition 1.4.17). Suppose $||A|| < ||A|| < g_0||A||$; we derive a contradiction. We have $1 < t < g_0$ where $t := \omega(||A||) \ast ||A|| \in G_0$ (see Proposition 1.4.8). But $t \notin G$ by assumption so $t = \inf \{g \in G : g > t\}$, which implies the existence of a $g \in G$ with $1 < t < g < g_0$, a contradiction.

Both cases may occur in one and the same space, as the following example shows. Verification is left to the reader.

Example 2.2.14 Let $K = \mathbb{Q}_p$, let $E := K^2$ be normed by $(\xi_1, \xi_2) \mapsto \max (|\xi_1|, \sqrt{p} |\xi_2|)$.

Let
\[ A : (\xi_1, \xi_2) \mapsto (\xi_2, 0) \quad ((\xi_1, \xi_2) \in K^2). \]

Then \[ ||I|| = 1, ||I||^\sim = p, \text{ but } ||A|| = ||A||^\sim = 1. \]

We will now investigate what remains of the classical inequalities \[ ||AB|| \leq ||A|| \cdot ||B|| \]
and \[ ||Ax|| \leq ||A|| \cdot ||x|| \] for operators \( A, B \) and vectors \( x \).

**Lemma 2.2.15**

(i) Let \( A, B \in \text{Lip}(E) \). Then \( \Gamma_A \Gamma_B \subset \Gamma_{AB} \).

(ii) Let \( A \in \text{Lip}^\sim(E), B \in \text{Lip}(E) \). Then \( \Gamma_A \Gamma_B \subset \Gamma_{AB} \cap \Gamma_{BA} \).

**Proof.**

(i) Let \( g_1 \in \Gamma_A, g_2 \in \Gamma_B \). Then for any \( x \in E \), \[ ||ABx|| \leq g_1 ||Bx|| \leq g_1 g_2 ||x||, \]
so \( g_1 g_2 \in \Gamma_{AB} \).

(ii) Let \( g_1 \in \Gamma_A^\sim, g_2 \in \Gamma_B \). Then for any \( x \in E \) with \( Bx \neq 0 \), \[ ||ABx|| < g_1 ||Bx|| \leq g_1 g_2 ||x||, \]
hence \( ||ABx|| < g_1 g_2 ||x|| \), which also holds if \( Bx = 0, x \neq 0 \). So, \( g_1 g_2 \in \Gamma_{AB}^\sim \).

Similarly if \( x \in E, x \neq 0, ||BAx|| \leq g_2 ||Ax|| < g_2 g_1 ||x|| \) so \( g_1 g_2 \in \Gamma_{BA}^\sim \).

Recall (Definition 1.4.4) the ‘large’ multiplication \( * \) in \( G^\# \).

**Proposition 2.2.16** Let \( A, B \in \text{Lip}(E) \). Then

\[ ||AB|| \leq ||A|| \cdot ||B||. \]

Let \( A \in \text{Lip}^\sim(E), B \in \text{Lip}(E) \). Then

\[ \max(||AB||^\sim, ||BA||^\sim) \leq ||A||^\sim \cdot ||B|| \]

**Proof.** By Corollary 1.6.7 \( G^\# \) is a continuous \( G \)-module, so by Theorem 1.7.4 (if \( A, B \) are nonzero), \[ ||A|| \cdot ||B|| = (\inf \Gamma_A)^* (\inf \Gamma_B) = \inf \Gamma_A \Gamma_B \geq \inf \Gamma_{AB} = ||AB||, \]
where for the inequality Lemma 2.2.15 is used. Similarly, \( ||A||^\sim \cdot ||B|| = (\inf \Gamma_A^\sim)^* (\inf \Gamma_B) = \inf \Gamma_A^\sim \Gamma_B \geq \inf \Gamma_{AB}^\sim = ||AB||^\sim. \)

For the second inequality we have to put a continuity condition:

**Proposition 2.2.17** Let \( X \) be continuous and complete. Then, for each \( A \in \text{Lip}(E, F) \) we have \( ||Ax|| \leq ||A|| \cdot ||x|| \) \( (x \in X) \), where \( * \) is as in 1.7.

**Proof.** By Theorem 1.7.4 we have \( ||A|| \cdot ||x|| = (\inf \Gamma_A)^* ||x|| = \inf (\Gamma_A ||x||) \geq ||Ax||. \)

The continuity condition cannot be dropped as we can see from the next Proposition.

**Proposition 2.2.18** Let \( ||E|| \setminus \{0\} \) be noncontinuous. Suppose each one-dimensional subspace of \( E \) has an orthogonal complement. Then there exists an \( A \in FR(E) \) with \( ||A|| = 1 \) but \( ||Aa|| > ||a|| \) for some \( a \in E \).
Proof. Write $X : \|E\| \setminus \{0\}$ By Proposition 1.6.4 $G$ is quasidense and by Proposition 1.6.8 there is an $r \in X$ such that $\inf_{X} \{gr : g \in G, g > 1\} = r$ does not hold, which implies by Lemma 1.1.2 that $t := \inf_{X} \{gr : g \in G, g > 1\} > r$. Then there is an $u \in X$ with $r < u \leq t$. Choose $a, b \in E$ with $\|a\| = r, \|b\| = u$. Let $D$ be an orthocomplement of $Ka$. The formula

$$A(d + \lambda a) = \lambda b \quad (d \in D, \quad \lambda \in K)$$

defines an $A \in FR(E)$. For each $g \in G, g > 1$ we have for $x = d + \lambda a \in E, \|Ax\| = \|\lambda b\| = |\lambda|u \leq |\lambda|gr = g\|\lambda a\| \leq g\|x\|$, so that $\|A\| \leq \inf \{g \in G : g > 1\} = 1$. But $\|Aa\| = \|b\| = u > r = \|a\|$. We see that $\|A\| = 1$ and $\|Aa\| > \|a\|$. 

Remark. We leave it to the reader to make concrete examples of such $E$ and $A$. See the last part of Sec. 1.6.

2.3 The Banach algebra Lip($E$)

Throughout 2.3 $X$ is a $G$-module and $E$ is an $X$-normed Banach space. From Proposition 2.2.5 we infer that $Lip(E)$ is also a Banach space and the inequality $\|AB\| \leq \|A\| \|B\|$ of Proposition 2.2.16 shows that $Lip(E)$ deserves the qualification 'Banach algebra'.

Definition 2.3.1 An operator $A \in Lip(E)$ is called invertible if $A$ is a bijection and $A^{-1} \in Lip(E)$. We denote the set of all invertible operators by Inv($E$).

Let $A \in Lip(E)$ be bijective. By the Open Mapping Theorem [5] 2.5.4, $A^{-1} \in L(E)$. So if $L(E) = Lip(E)$ then automatically $A^{-1} \in Lip(E)$. The next example shows that this conclusion is false in general.

Example 2.3.2 Let $E$ be a Banach space with an orthogonal base $e_1, e_2, \ldots$, such that $\tau(\|e_1\|) \subset \tau(\|e_2\|) \subset \ldots$ and $\bigcup_n \tau(\|e_n\|) = G$ and such that $Stab(\|e_n\|) \neq \tau(\|e_n\|)$ for each $n$. Then there exists a bijective $A \in L(E)$ such that $A^{-1}$ is Lipschitz but $A$ is not.

Proof. Choose $\lambda_1, \lambda_2, \ldots \in K$ such that $|\lambda_n| > 1, |\lambda_n| \in \tau(\|e_n\|) \setminus Stab(\|e_n\|)$ for all $n$ and such that $|\lambda_n| \to \infty$. Define $A$ by the formula

$$A \left( \sum_{n=1}^{\infty} \xi_n e_n \right) = \sum_{n=1}^{\infty} \xi_n \lambda_n e_n$$

In the proof of Theorem 2.1.13 it is shown that $A$ is not Lipschitz. But it is clear that for $x = \sum_{n=1}^{\infty} \xi_n e_n \in E$,

$$A^{-1} \left( \sum_{n=1}^{\infty} \xi_n e_n \right) = \sum_{n=1}^{\infty} \xi_n \lambda_n^{-1} e_n$$

and, since $|\lambda_n^{-1}| \leq 1, \|A^{-1}\| \leq 1$. Hence, $A^{-1}$ is Lipschitz.
Clearly Inv\(E\) is a group under composition. With the inherited topology induced by the Lipschitz norm, composition is continuous. We now investigate when \(\text{Inv}(E)\) is a topological group (i.e. when inversion \(A \mapsto A^{-1}\) is continuous).

**Lemma 2.3.3** Let \(A_1, A_2, \ldots\) be a sequence in Inv\(E\) converging to some \(A \in \text{Lip}(E)\). Then the following are equivalent.

1. (\(\alpha\)) \(A \in \text{Inv}(E)\) and \(\lim_{n \to \infty} \|A^{-1} - A_n^{-1}\| = 0\).
2. (\(\beta\)) \(n \mapsto \|A_n^{-1}\|\) is bounded.

**Proof.** Only (\(\beta\)) \(\Rightarrow\) (\(\alpha\)) needs a proof. For \(m, n \in \mathbb{N}\) we have
\[
A_m^{-1} - A_n^{-1} = A_m^{-1}(A_n - A_m)A_n^{-1}.
\]
Let \(g \in G\) be such that \(\|A_n^{-1}\| \leq g\) for all \(n\). Then
\[
\|A_m^{-1} - A_n^{-1}\| \leq \|A_m^{-1}\| \cdot \|A_n - A_m\| \cdot \|A_n^{-1}\| \leq g^2 \|A_n - A_m\|
\]
so \(n \mapsto A_n^{-1}\) is Cauchy. By completeness there is a \(B \in \text{Lip}(E)\) such that \(\lim_{n \to \infty} \|B - A_n^{-1}\| = 0\); then \(BA - I = \lim_{n \to \infty} (BA - A_n^{-1}A_n) = 0\); similarly we get \(AB = I\).

**Theorem 2.3.4** Let \(\|E\| \setminus \{0\}\) be almost faithful. Then \(A \mapsto A^{-1}\) is a homeomorphism Inv\(E\) \(\to\) Inv\(E\).

**Proof.** Only a continuity proof is required. Let \(A_1, A_2, \ldots\) be a sequence in Inv\(E\) converging to some \(A \in \text{Inv}(E)\). By assumption there is a \(g \in G\) such that \(g\|x\| < \|x\|\) for all nonzero \(x \in E\). Now \(n \mapsto A_n A^{-1}\) converges to \(I\) so there is an \(n_0\) such that \(\|A_n A^{-1} - I\| < g\) for \(n \geq n_0\). Then for each nonzero \(x \in E\), \(\|A_n A^{-1}x - x\| < g\|x\|\) and we see that \(\|A_n A^{-1}x\| = \|x\|\) for all nonzero \(x\) i.e. \(\|A_n x\| = \|Ax\|\) for all \(x \in E\), all \(n \geq n_0\). Now \(A \in \text{Inv}(E)\) so there is an \(h \in G\) such that \(\|Ax\| \geq h\|x\|\) for all \(x \in E\). Then \(\|A_n x\| \geq h\|x\|\) implying \(\|A_n^{-1}x\| \leq h^{-1}\|x\|\) for all \(n \geq n_0, x \in E\). We see that \(n \mapsto \|A_n^{-1}\|\) is bounded. Now apply Lemma 2.3.3 to arrive at \(\lim_{n \to \infty} A_n^{-1} = A^{-1}\).

We have a partial converse.

**Theorem 2.3.5** Suppose each onedimensional subspace of \(E\) has an orthogonal complement. If inversion is continuous on Inv\(E\) then \(\|E\| \setminus \{0\}\) is almost faithful.

**Proof.** Suppose \(\|E\| \setminus \{0\}\) is not almost faithful; we prove that inversion is not continuous. Then exists \(a_1, a_2, \ldots \in E \setminus \{0\}\) such that \(\text{Stab}(\|a_1\|) \subset \text{Stab}(\|a_2\| \subset \ldots\) and \(\bigcup_n \text{Stab}(\|a_n\|) = G\). For each \(n\), choose \(\lambda \in K, 0 < |\lambda_n| < h\) for each \(h \in \text{Stab}(\|a_n\|)\), and \(P_n\) be an orthoprojection onto \(K a_n\). Define
\[
A_n := (\lambda_n - 1)P_n + I.
\]
Then \(A_n \in \text{Inv}(E)\) and \(A_n^{-1} = (\lambda_n^{-1} - 1)P_n + I\). We have \(\|A_n - I\| = \|(\lambda_n - 1)P_n\| = \|P_n\| = \inf \text{Stab}(\|a_n\|) \to 0\).
On the other hand, since $|\lambda_n^{-1} - 1| = |\lambda_n^{-1}|$ we have $\|A_n^{-1} - I\| = |\lambda_n^{-1}|\|P_n\|$. Now $|\lambda_n^{-1}| > \sup \text{Stab} (\|e_n\|)$, so that by Corollary 1.4.3 (ii) $|\lambda_n^{-1}|\|P_n\| \geq \sup \text{Stab} (\|e_n\|) \geq 1$. So, we see that $A_n \to I$, but $A_n^{-1} \not\to I$.

A concrete space $E$ for which inversion is not continuous on $\text{Inv}(E)$ is for example the space of $[5]$ 4.2.2.

Next we consider the question whether $\text{Inv}(E)$ is an open subset of $\text{Lip}(E)$. We regret not to have found a characterization in the spirit of Theorem 2.3.4 and 2.3.5. (However in 3.1.1 we shall present a characterization within the category of Banach spaces with a countable orthogonal base.) But, in this general setting, we have the following two results.

**Proposition 2.3.6** Suppose $G$ has a maximal proper convex subgroup $H$. Then $\text{Inv}(E)$ is an open subset of $\text{Lip}(E)$. More precisely, let $t \in G$, $t < 1$, $t \notin H$. Then, if $A \in \text{Inv}(E)$, $B \in \text{Lip}(E)$ and $\|A - B\| \leq \omega(\|A^{-1}\|)^t^2$, then $B \in \text{Inv}(E)$.

**Proof.** It is easily seen that $\bigcup_{n=1}^{\infty} \{ g \in G : t^n \leq g \leq t^{-n} \}$ is the smallest convex subgroup containing $H$ and $\{t\}$, hence equal to $G$. It follows that $\lim_{n \to \infty} t^n = 0$.

We consider the case $A = I$. Then $\|I - B\| \leq t$. Then $\|(I - B)^n\| \leq \|I - B\| * \ldots * \|I - B\| \leq t^n \to 0$. By completeness $C := \sum_{n=0}^{\infty} (I - B)^n$ is in $\text{Lip}(E)$ and $C = B^{-1}$. To prove the general case, we consider $\|(I - A^{-1}B)\| = \|A^{-1}(A - B)\| \leq \|A^{-1}\| * \|A - B\| \leq \|A^{-1}\| * \omega(\|A^{-1}\|)^t \cdot t^2$. By Proposition 1.4.16 $\|A^{-1}\| * \omega(\|A^{-1}\|)$ is the supremum of some proper convex subgroup, hence $\leq \sup H$. Now $t^{-1} > \sup H$ and we find $\|I - A^{-1}B\| \leq (\sup H)^2 < t^{-1}t^2 = t$. By the first part of the proof we obtain $A^{-1}B \in \text{Inv}(E)$, hence $B \in \text{Inv}(E)$.

**Remark.** The above condition on $G$ implies that $\|E\| \setminus \{0\}$ is almost faithful.

**Proposition 2.3.7** Suppose each onedimensional subspace of $E$ has an orthogonal complement, and suppose that $\text{Inv}(E)$ is open in $\text{Lip}(E)$. Then $\|E\| \setminus \{0\}$ is almost faithful (hence inversion is continuous by Theorem 2.3.4).

**Proof.** Suppose $\|E\| \setminus \{0\}$ is not almost faithful. Then let $a_1, a_2, \ldots \in E$ and $P_1, P_2, \ldots$ be as in the proof of Theorem 2.3.5. From that proof it follows that $\|P_n\| \to 0$ so $I - P_n \to I$. By assumption $I - P_n$ should be invertible for large $n$, which is impossible as $(I - P_n)(a_n) = 0$ for each $n$.

**Remark.** In Lemma 3.1.3 we shall give an example of an $E$ for which inversion is continuous but where $\text{Inv}(E)$ is not open in $\text{Lip}(E)$!

### 2.4 The trace function and compact operators

Recall that in Definition 2.1.22 and Proposition 2.1.23 we introduced and discussed the trace function on $FR(E)$ in an algebraic way. We now consider continuity properties of the trace.

Throughout 2.4 $X$ is a $G$-module and $E$ is an $X$-normed Banach space.
Theorem 2.4.1 Suppose each finite-dimensional subspace of \( E \) has an orthogonal base. Then for each \( A \in \text{FR}(E) \) we have \(|\text{tr}(A)| \leq ||A||^\sim\).

**Proof.** Let \( a_1, \ldots, a_n \) be an orthogonal base of \( AE \). Then \( A \) has the form \( x \mapsto \sum_{i=1}^n f_i(x)a_i \) for some \( f_1, \ldots, f_n \in E' \). Let \( s \in \Gamma_A' \). Then, by orthogonality, \( ||f_i(x)a_i|| < s||x|| \) for all nonzero \( x \in E \) and \( i \in \{1, \ldots, n\} \). Putting \( x = a_i \) we get \( ||f_i(a_i)||a_i|| < s||a_i|| \) for each \( i \). This implies \( |f_i(a_i)| < s \), so \(|\text{tr}(A)| = |\sum_{i=1}^n f_i(a_i)| < s \) which proves the theorem.

Essentially, if the Lipschitz norm is not equivalent to \( || \)\(^\sim|| \), then the trace is not Lipschitz continuous, as can be seen from the next result.

Theorem 2.4.2 Suppose each one-dimensional subspace of \( E \) has an orthogonal complement. Then \( \text{tr} : \text{FR}(E) \to \mathbb{K} \) is Lipschitz continuous if and only if \( ||\text{Id}|| \setminus \{0\} \) is almost faithful (i.e. if and only if \( \text{Lip}^\sim(E) = \text{Lip}(E) \). See Theorem 2.1.6).

**Proof.** Suppose \( ||E|| \setminus \{0\} \) is almost faithful. Then by Theorem 2.1.6 \( \text{Lip}(E) = \text{Lip}^\sim(E) \) and, by Proposition 2.2.10, \( ||\)\ and \( ||||^\sim\) are equivalent. From the assumption it follows that finite-dimensional subspaces of \( E \) have orthogonal bases, so from Theorem 2.4.1 it follows that \( \text{tr} \) is \( ||\)-continuous.

Now let \( ||E|| \setminus \{0\} \) be not almost faithful. We will show Lipschitz discontinuity of \( \text{tr} \) by constructing a sequence \( A_1, A_2, \ldots \) in \( \text{FR}(E) \) such that \( \lim_{n \to \infty} ||A_n|| = 0 \) but \( \text{tr}(A_n) = 1 \) for each \( n \).

There exist \( a_1, a_2, \ldots \in E \setminus \{0\} \) such that, with \( H_n := \text{Stab}(||a_n||), H_1 \subset H_2 \subset \ldots, \bigcup_n H_n = G \). Choose \( f_1, f_2, \ldots \in E' \) with \( f_n(a_n) = 1 \) for all \( n \) and let

\[
A_n(x) := f_n(x)a_n \quad (x \in E).
\]

Then clearly \( A_n \in \text{FR}(E) \) and \( \text{tr}(A_n) = 1 \). But

\[
||A_n|| = \inf \{g \in G : ||f_n(x)a_n|| \leq g||x|| \text{ for all } x \in E\}
\leq \inf \{g \in G : ||a_n|| \leq g||a_n||\}
\leq \inf \{g \in G : ||a_n|| = g||a_n||\} = \inf H_n \to 0
\]

We now introduce compact operators. For a subset \( V \) of \( \text{Lip}(E) \) we denote by \( \overline{V} \) its closure with respect to \( || \)\|. For a subset \( W \) of \( \text{Lip}^\sim(E) \), let \( \overline{W}^\sim \) be its closure with respect to \( ||||^\sim\).

**Definition 2.4.3** Let \( C(E) := \overline{\text{FR}(E)} \). An element of \( C(E) \) is called **compact** (supercompact in [7] 3.3). Similarly, let \( C^\sim(E) := \overline{\text{FR}(E)}^\sim \). Following classical conventions we call an element of \( C^\sim(E) \) **nuclear** or of **trace class**.

Clearly \( C^\sim(E) \subset C(E), C(E) \) is a two-sided ideal in \( \text{Lip}(E) \), \( C^\sim(E) \) is a two-sided ideal in \( \text{Lip}^\sim(E) \). Under the conditions of Theorem 2.4.1 the trace function in \( \text{FR}(E) \) can uniquely be extended to a continuous linear function, again denoted \( \text{tr} \), in \( C^\sim(E) \) and we have \( (\text{tr}(A)) \leq ||A||^\sim \) for all \( A \in C^\sim(E) \).

**Proposition 2.4.4** \( C^\sim(E) \) is a two-sided ideal in \( \text{Lip}(E) \). If \( A \in C^\sim(E), B \in \text{Lip}(E) \) then \( \text{tr}(AB) = \text{tr}(BA) \).
Proof. There are $T_1, T_2, \ldots \in FR(E)$ with $\lim_{n \to \infty} \|A - T_n\| = 0$. Then by Proposition 2.2.16

$$\|BA - BT_n\| = \|B(A - T_n)\| \leq \|B\| \cdot \|A - T_n\| \to 0,$$

hence $BA \in C^\sim(E)$; similarly $AB \in C^\sim(E)$. We have using Theorem 2.4.1 and Proposition 2.1.33: $\text{tr}(BA) = \lim_{n \to \infty} \text{tr}(BT_n) = \lim_{n \to \infty} G(T_nB) = \text{tr}(AB)$.

Proposition 2.4.5 Let $A \in C(E)$, $B \in \text{Lip}^\sim(E)$ Then $AB$ and $BA$ are in $C^\sim(E)$, and $\text{tr}(AB) = \text{tr}(BA)$.

Proof. There are $T_1, T_2, \ldots \in FR(E)$ with $\lim_{n \to \infty} \|A - T_n\| = 0$. Then by Proposition 2.2.16, $\|BA - BT_n\| \leq \|B\| \cdot \|A - T_n\| \to 0$, hence $BA \in C^\sim(E)$. Now proceed as in the previous proof.

3 Lipschitz operators on spaces with an orthogonal base

In this chapter we restrict ourselves to spaces with a countable orthogonal base, a class of spaces containing Norm and Form Hilbert spaces (see [6]) and therefore of main interest. It will turn out that we can say much more than in the more general setting of Chapter 2.

Throughout Chapter 3 $X$ will be a $G$-module and $E$ an $X$-normed Banach space with a countable orthogonal base $e_1, e_2, \ldots$.

3.1 Inversion

We shall characterize those $E$ for which $\text{Inv}(E)$ is open. Recall that from the theory of Chapter 2 we have the following.

(i) If $G$ has a maximal convex proper subgroup then $\text{Inv}(E)$ is open.
(ii) $\text{Inv}(E)$ is open $\Rightarrow \|E\| \setminus \{0\}$ is almost faithful $\iff A \mapsto A^{-1}$ is a homeomorphism of $\text{Inv}(E)$.

(See Propositions 2.3.6, 2.3.7, Theorems 2.3.4, 2.3.5).

Thus, the following theorem will complete the characterization.

**Theorem 3.1.1** Let $G$ have no maximal proper convex subgroups. Then the following are equivalent.

($\alpha$) $\text{Inv}(E)$ is open in $\text{Lip}(E)$.
($\beta$) $\|E\| \setminus \{0\}$ is almost faithful and $E$ is a Norm Hilbert space.

The proof runs in several steps. The proof of ($\beta$) $\Rightarrow$ ($\alpha$) is contained in the next Proposition.
Proposition 3.1.2 Suppose \( (\beta) \). Let \( t \in G \) be such that \( t\|x\| < \|x\| \) for all nonzero \( x \in E \). Let \( A \in \text{Inv}(E), B \in \text{Lip}(E), \|A - B\| < t^2 \omega(\|A^{-1}\|) \). Then \( B \in \text{Inv}(E) \).

Proof. We first treat the case \( A = I \). Then \( \|I - B\| < t \) so for nonzero \( x \in E \) we have \( \|x - Bx\| \leq t\|x\| < \|x\| \) and it follows that \( B \) is an isometry. By [8], 2.5 (ii) \( B \) is surjective, hence \( B \in \text{Inv}(E) \). To prove the general case we consider \( \|I - A^{-1}B\| = \|A^{-1}(A - B)\| \leq \|A^{-1}\| \omega(\|A^{-1}\|) t^2 \). Like in the proof of Proposition 2.3.6 we observe that \( t^{-1} > \|A^{-1}\| \omega(\|A^{-1}\|) \) and so \( \|I - A^{-1}B\| < t \). By the first part of the proof we obtain \( A^{-1}B \in \text{Inv}(E) \), hence \( B \in \text{Inv}(E) \).

For the proof of \( (\alpha) \Rightarrow (\beta) \) we need three Lemmas. Notice that by (ii) above we only have to prove that openness of \( \text{Lip}(E) \) implies that \( E \) is a Norm Hilbert Space. Let us introduce the following Property \((*)\) for \( E \).

\((*)\) There exists a sequence \( A_1, A_2, \ldots \) of nonsurjective operators in \( \text{Lip}(E) \) such that \( \lim_{n \to \infty} \|I - A_n\| = 0 \)

Clearly if \( E \) has \((*)\) then \( \text{Inv}(E) \) is not open.

Lemma 3.1.3 \( c_0 \) has Property \((*)\).

Proof. Let \( h \in G, h < 1 \); we shall construct a nonsurjective \( A \in \text{Lip}(c_0) \) such that \( \|I - A\| \leq h \).

By assumption there is a proper convex subgroup \( H \) with \( h \in H \) ([5] 4.3.1). Let \( S := \sup_{G^\#} H \). Then (Proposition 1.4.1) \( \text{Stab}(s) = H \) so in particular \( s \in G^\# \setminus G \).

(1) Now let \( F \) be the space \( c_0 \) but with the norm

\[ x = (x_1, x_2, \ldots) \mapsto \max_n |x_n| s. \]

Then \( F \) is a \( G^\# \)-normed Banach space of countable type so, by [5] 3.2.6, \( F \) is a quotient of \( c_0 \) i.e. \( c_0 \) has a closed subspace \( D \) such that \( c_0/D \) is isometrically isomorphic to \( F \).

Let \( \pi : c_0 \to c_0/D \) be the quotient map.

(2) We proceed to prove that for each \( x \in c_0 \) with \( \|x\|_\infty = 1 \) there is a \( d \in D \) with \( \|x - d\|_\infty < h \) (here \( \|\cdot\|_\infty \) is the canonical norm on \( c_0 \)). To this end we may suppose \( \pi(x) \neq 0 \). We have \( \|\pi(x)\| \leq 1 \) but also \( \|\pi(x)\| \in Gs \) say, \( \|\pi(x)\| = gs \) for some \( g \in G \). Then \( gs \leq 1 \) but since \( gs \notin G \) we must have \( gs < 1 \). Now \( h^{-1} \in H \) so \( h^{-1} \in \text{Stab}(s) \), hence \( h^{-1} gs < 1 \) or \( gs < h \). We see that \( \|\pi(x)\| < h \) which proves (2).

(3) Now let \( b_1, b_2, \ldots \) be the canonical orthonormal base of \( c_0 \). By (2) there are \( d_1, d_2, \ldots \in D \) such that \( \|b_n - d_n\|_\infty < h \) for all \( n \). By the Perturbation Lemma [5] 2.4.8, the sequence \( d_1, d_2, \ldots \) is orthonormal in \( (D, \|\cdot\|_\infty) \) and the formula \( Ab_n = d_n \) defines a linear isometry \( A \) of \( c_0 \) into \( D \), so \( A \) is not surjective. But on the other hand \( \|(I - A)(b_n)\|_\infty = \|d_n - b_n\|_\infty < h = h\|b_n\|_\infty. \) So by Proposition 2.2.7, \( \|I - A\| \leq h \) and we are done.

Remark. \( c_0 \) is \( G \)-normed so \( \|c_0\|_\infty \setminus \{0\} \) is faithful hence inversion \( A \mapsto A^{-1} \) is continuous on \( \text{Inv}(c_0) \). But \( \text{Inv}(c_0) \) is not open in \( \text{Lip}(E) \).

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Lemma 3.1.4 Let $E$ be linearly homeomorphic to $c_0$. Then $E$ has Property ($\ast$).

Proof. We denote the given $X$-norm on $E$ by $\|\|$ and the canonical norm on $c_0$ by $\|\|_\infty$. We may suppose that $E$ and $c_0$ are identical as vector spaces and that $\|\|$ and $\|\|_\infty$ are equivalent. By Proposition 2.1.9 we have $\text{Lip}(E) = L(E)$, $\text{Lip}(c_0) = L(c_0)$, and by assumption $L(E) = L(c_0)$. By [5], 2.5.5. and the fact that the natural topologies on $L(E)$ and $L(c_0)$ are equal we have that the Lipschitz norms in $\text{Lip}(E)$ and $L(c_0)$ are equivalent. By the previous Lemma there exists a sequence $A_1, A_2, \ldots$ of nonsurjective operators converging to $I$ in the topology of $\text{Lip}(c_0)$, hence in the topology of $\text{Lip}(E)$ and we are done.

Lemma 3.1.5 Let $E$ be not a Norm Hilbert space. Then $E$ has property ($\ast$).

Proof. Clearly $E$ is infinite-dimensional. By assumption and [5] 4.3.7 ($\beta$) $\leftrightarrow$ ($\gamma$) the sequence $e_1, e_2, \ldots$ does not satisfy the type condition so, by suitable scalar multiplication, we may assume that there exists $s_1, s_2 \in X$ and $n_1 < n_2 < \ldots$ in $N$ such that $s_1 \leq \|e_{n_1}\| \leq s_2$ for all $i$. The closed linear span $D$ of $\{e_{n_1}, e_{n_2}, \ldots\}$ is linearly homeomorphic to $c_0$, so by the previous lemma there are nonsurjective $A_1, A_2, \ldots \in \text{Lip}(D)$ such that $A_n \to I$ (identity in $D$) in the Lipschitz norm on $\text{Lip}(D)$. Let $S$ be the closed linear span of $\{e_m : m \in N, m \notin \{n_1, n_2, \ldots\}\}$. Then clearly $S$ is an orthocomplement of $D$ in $E$. The formula

$$B_n x := \begin{cases} A_n x & \text{if } n \in N, x \in D \\ x & \text{if } n \notin N, x \in S \end{cases}$$

defines a sequence $B_1, B_2, \ldots$ of nonsurjective operators in $\text{Lip}(E)$ converging in the Lipschitz norm of $\text{Lip}(E)$ to the identity on $E$.

Proof of Theorem 3.1.1 Combine Proposition 3.1.2 and Lemma 3.1.5.

3.2 Density

We prove two density theorems.

Theorem 3.2.1 $\text{Lip}(E)$ is dense in $\text{Lip}(E)$. $C^\infty(E)$ is dense in $C(E)$.

Proof. Since $\text{Lip}(E)$ is a two-sided ideal in $\text{Lip}(E)$ it is enough to show that $I \in \text{Lip}(E)$. Thus, let $\varepsilon \in G$, $\varepsilon < 1$; we construct a $T \in \text{Lip}(E)$ for which $\|I - T\| \leq \varepsilon$. Write $H_n := \text{Stab}(\|e_n\|)$ and define $Te_n = 0$ if $\varepsilon \in H_n$, $Te_n = e_n$ if $\varepsilon \notin H_n$. If $\varepsilon \notin H_n$ then $\|Te_n\| = \|e_n\| < \varepsilon^{-1}\|e_n\|$ (since $\varepsilon^{-1} \notin H_n$, $\varepsilon^{-1} > 1$), so we have $\|Te_n\| < \varepsilon^{-1}\|e_n\|$ for all $n \in N$. By Proposition 2.2.7 $T$ extends uniquely to a strictly Lipschitz operator, again called $T$. To show that $\|I - T\| \leq \varepsilon$, let $n \in N$. If $\varepsilon \in H_n$ then $\|(I - T)e_n\| = \|e_n\| = \varepsilon\|e_n\|$; if $\varepsilon \notin H_n$ then $\|(I - T)e_n\| = 0$. Thus, $\|(I - T)e_n\| \leq \varepsilon\|e_n\|$ for all $n$ and, by Proposition 2.2.7, $\|I - T\| \leq \varepsilon$. The second statement is an easy consequence of the first and Proposition 2.4.5.

Theorem 3.2.2 $\text{Lip}(E)$ is dense in $L(E)$.
Proof. We may assume that \( X \) is complete. Let \( s_0 \in X \). The topology of \( L(E) \) is defined by the norm

\[
A \mapsto \|A\|_1 = \sup_{x \in \{0\}} \{\|Ax\| : x \in E, \|x\| \leq s_0\}.
\]

Now let \( A \in L(E), \varepsilon \in G \). We construct a \( B \in Lip(E) \) with \( \|A - B\|_1 \leq \varepsilon s_0 \). Choose \( g_0 \in G \) such that \( g_0\|A\|_1 < s_0 \) and put \( \delta := \varepsilon g_0 \). We first define \( Be_n \) for each \( n \) as follows.

Case 1: If there exists a \( \lambda \in K \) with \( \delta s_0 < \|\lambda e_n\| \leq s_0 \) then \( Be_n := A e_n \).

Case 2: If \( \lambda \in K, \|\lambda e_n\| \leq s_0 \) implies \( \|\lambda e_n\| \geq \delta s_0 \) then \( Be_n := 0 \).

We prove that \( \|B e_n\| \leq \varepsilon^{-1} g_0^{-2} \|e_n\| \) for each \( n \). This is clear in case 2, so we may assume the existence of a \( \lambda \in K \) with \( \delta s_0 < \|\lambda e_n\| \leq s_0 \). Then \( \|B(\lambda e_n)\| = \|A(\lambda e_n)\| \leq A11 \), so

\[
\|B e_n\| \leq \|\lambda\|^{-1} \|A\|_1 \leq \|\lambda\|^{-1} g_0^{-1} s_0 \leq \|\lambda\|^{-1} g_0^{-1} \delta^{-1} \|\lambda e_n\|
\]

\[
= g_0^{-1} s_0^{-1} \|e_n\| = \varepsilon^{-1} g_0^{-2} \|e_n\|. \quad \text{By Proposition 2.2.7 \( B \) extends uniquely to a Lipschitz operator, again called \( B \).}
\]

To estimate \( \|A - B\|_1 \), let \( \lambda \in K, n \in N \) be such that \( \|\lambda e_n\| \leq s_0 \). In case 1, \( (A - B)(\lambda e_n) = 0 \). In case 2 we have \( \|\lambda e_n\| \leq \delta s_0 \), so \( \|(A - B)(\lambda e_n)\| = \|A(\lambda e_n)\| \leq \delta \|A\|_1 \leq \delta g_0^{-1} s_0 = \varepsilon s_0 \). Thus, \( \|(A - B)(\lambda e_n)\| \leq \varepsilon s_0 \) as soon as \( n \in N, \|\lambda e_n\| \leq s_0 \). Now take \( x \in E, \|x\| \leq s_0 \). Let \( x = \sum_{n=1}^{\infty} \xi_n e_n \) be the expansion of \( x \). By orthogonality \( \|\xi_n e_n\| \leq s_0 \) for each \( n \) and we just proved that \( \|(A - B)(\xi_n e_n)\| \leq \varepsilon s_0 \). But then \( \|(A - B)(x)\| \leq \max_n \|(A - B)(\xi_n e_n)\| \leq \varepsilon s_0 \). It follows that \( \|A - B\|_1 \leq \varepsilon \).

Question Can one extend Theorem 3.2.1 and 3.2.2 to arbitrary Banach spaces?

3.3 Matrix characterizations

Each \( A \in Lip(E) \) has a matrix

\[
\begin{pmatrix}
  a_{11} & a_{12} & \cdots \\
  a_{21} & \vdots & \\
  \vdots & & \\
\end{pmatrix}
\]

with respect to the given base \( e_1, e_2, \ldots \). Of special interest are the ‘building blocks’ \( P_{mn} \) \( (m, n \in N) \) given by the formula

\[
P_{mn}(e_k) = \delta_{kn} e_m \quad (k \in N).
\]

Clearly \( P_{mn} \in FR(E) \) and its matrix has zero entries except for a one in the \( n \)th column and the \( m \)th row. With this in mind it is natural to compare \( A \) with \( \{a_{mn} P_{mn} : m, n \in N\} \).

Lemma 3.3.1
(i) For each $m, n \in \mathbb{N}$ we have
\[
\|P_{mn}\| = \inf \{g \in G : \|e_m\| \leq g\|e_n\|\}
\]
\[
\|P_{mn}\|^\sim = \inf \{g \in G : \|e_m\| < g\|e_n\|\}.
\]

(ii) $\{P_{mn} : m, n \in \mathbb{N}\}$ is an orthogonal set with respect to $\|\|$ and $\|\|^\sim$.

Proof. (i) By Proposition 2.2.7
\[
\|P_{mn}\| = \inf \{g \in G : \|P_{mn}(e_k)\| \leq g\|e_k\| \text{ for all } k \in \mathbb{N}\}.
\]
Now $P_{mn}(e_k) = 0$ for $k \neq n$ so we get
\[
\|P_{mn}\| = \inf \{g \in G : \|e_m\| \leq g\|e_n\|\}.
\]
The formula for $\|P_{mn}\|^\sim$ is proved in the same fashion.

(ii) Let $A := \sum \lambda_{mn}P_{mn}$ be a finite linear combination of the $P_{mn}$. Let $g \in \Gamma_A$ (resp. $g \in \Gamma_A^\sim$); we show that for $m, n \in \mathbb{N}$, $\|\lambda_{mn}P_{mn}\| \leq g$ (resp. $\|\lambda_{mn}P_{mn}\|^\sim \leq g$). To this end we may assume $\lambda_{mn} \neq 0$. We have $g\|e_n\| \geq (\text{resp. } >) \|Ae_n\| = \|\sum_{i,j} \lambda_{ij}P_{ij}(e_n)\| = \|\sum_{i} \lambda_{in}P_{in}(e_n)\| = \|\sum_{i} \lambda_{in}e_i\| \geq \|\lambda_{mn}e_m\|$. Hence $\|e_m\| \leq (\text{resp. } <) \|\lambda_{mn}^{-1}\|g\|e_n\|$, showing that $|\lambda_{mn}^{-1}|g \geq ||P_{mn}\| \text{ (resp. } ||P_{mn}\|^\sim)$ and we are done.

Corollary 3.3.2
(i) For each $m, n$ we have
\[
\text{Stab}(||P_{mn}||) = \text{Stab}(||P_{mn}||^\sim) = \tau(||e_n||; ||e_m||).
\]
(ii) For each $n$ we have
\[
\|P_{mn}\| = \inf \text{Stab}(||e_n||), \quad ||P_{mn}||^\sim = \sup \text{Stab}(||e_n||).
\]
(iii) Let $A \in \text{Lip}(\mathcal{E})$ have matrix
\[
\begin{pmatrix}
    a_{11} & a_{12} & \cdots \\
    a_{21} & & \\
    & & \\
    & & \\
    & & \\
\end{pmatrix}
\]
with respect to $e_1, e_2, \ldots$ Then $A \in [P_{mn} : m, n \in \mathbb{N}]$
if and only if $\lim_{m+n \to \infty} \|a_{mn}P_{mn}\| = 0; \ A \in [P_{mn}^\sim : m, n \in \mathbb{N}^\sim]\nif and only if $\lim_{m+n \to \infty} \|a_{mn}P_{mn}\|^\sim = 0$.

Proof. Straightforward (for (i) use Theorem 1.5.4).

Lemma 3.3.3 Let $a_{mn} \in K (m, n \in \mathbb{N})$. The following are equivalent.

(a) For each $n$, $\lim_{m \to \infty} a_{mn}e_m = 0$.
(b) For each $n$, $\lim_{m \to \infty} \|a_{mn}P_{mn}\| = 0$.
(c) For each $n$, $\lim_{m \to \infty} \|a_{mn}P_{mn}\|^\sim = 0$.

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Proof. \((a) \Rightarrow (\gamma)\). Let \(n \in \mathbb{N}, \ g \in G\). There is an \(m_0\) such that for \(m \geq m_0\) we have 
\[\|a_{mn}e_m\| < g\|e_n\|, \ \text{i.e.} \quad \|a_{mn}P_{mn}(e_n)\| < g\|e_n\| \]. Thus we have 
\[\|a_{mn}P_{mn}(e_j)\| < g\|e_j\| \] for each \(j\), each \(m \geq m_0\). It follows that 
\[\|a_{mn}P_{mn}\| \leq g \quad \text{for} \quad m \geq m_0; \ \text{in other words we proved} \ (\gamma)\]. The implication \((\gamma) \Rightarrow (\beta)\) is trivial, so we prove \((\beta) \Rightarrow (a)\). Let \(n \in \mathbb{N}, \ \varepsilon \in X\). Choose a \(g \in G\) with \(\|e_n\| < \varepsilon\). There is an \(m_0\) such that for \(m \geq m_0\) we have 
\[\|a_{mn}P_{mn}\| < g\]. Then, by Proposition 2.2.3 (i), 
\[\|a_{mn}P_{mn}(x)\| \leq g\|x\| \] for all \(x \in E\) and \(m \geq m_0\). By taking \(x = e_n\) we find 
\[\|a_{mn}e_m\| \leq g\|e_n\| < \varepsilon \] for \(m \geq m_0\) and we are done.

Theorem 3.3.4 (Characterization of Lipschitz operators by matrices)

(i) Let \(A \in \text{Lip}(E)\) have the matrix
\[
\begin{pmatrix}
  a_{11} & a_{12} & \cdots \\
  a_{21} & & \\
  & & \\
  & & \\
  & & \\
\end{pmatrix}
\]
with respect to \(e_1, e_2, \ldots\). Then, for each \(n\), 
\[\lim_{m \to \infty} \|a_{mn}P_{mn}\| = 0\]
and 
\[\|A\| = \sup\{\|a_{mn}P_{mn}\| : m, n \in \mathbb{N}\}\].

(ii) Conversely, let
\[
\begin{pmatrix}
  a_{11} & a_{12} & \cdots \\
  a_{21} & & \\
  & & \\
  & & \\
  & & \\
\end{pmatrix}
\]
be a matrix with entries in \(K\), such that, for each \(n\), 
\[\lim_{m \to \infty} \|a_{mn}P_{mn}\| = 0\] and such that \((m, n) \mapsto \|a_{mn}P_{mn}\|\) is bounded above. Then the matrix represents a Lipschitz operator.

Proof. (i) Let \(n \in \mathbb{N}\). Then \(Ae_n = \sum_{m=1}^{\infty} a_{mn}e_m\), so \(\lim_{m \to \infty} \|a_{mn}e_m\| = 0\), so by the previous Lemma we have 
\[\lim_{m \to \infty} \|a_{mn}P_{mn}\| = 0\]. Next we prove 
\[\|a_{mn}P_{mn}\| \leq \|A\|\] for each \(m, n \in \mathbb{N}\). Let \(g \in \Gamma_A\). We have 
\[\|a_{mn}e_m\| \leq \|\sum_{j=1}^{\infty} a_{jn}e_j\| = \|Ae_n\| \leq g\|e_n\|\]. Thus (assuming \(a_{mn} \neq 0\)) 
\[\|e_n\| \leq |a_{mn}|^{-1}g\|e_n\|\]; so, by Lemma 3.3.1, 
\[\|P_{mn}\| \leq |a_{mn}|^{-1}g\].
To complete the proof of (i) we suppose that 
\[s := \sup\{\|a_{mn}P_{mn}\| : m, n \in \mathbb{N}\} < \|A\|\]
and derive a contradiction. (The proof looks rather overnice; we would welcome proposals for a more direct proof.) First assume that \(X\) is continuous. (Then \(\Gamma_B = \{g \in G : g \geq \|B\|\} \) for each \(B \in \text{Lip}(E)\).) There is a \(g \in G\) with \(s \leq g < \|A\|\). Thus \(g \notin \Gamma_A\) so there is an \(n\) such that 
\[\|Ae_n\| > g\|e_n\|\] and since \(Ae_n = \sum a_{jn}e_j\) there is an \(m\) such that 
\[\|a_{mn}e_m\| > g\|e_n\|\]. So \(|a_{mn}|^{-1}g \notin \Gamma_{P_{mn}}\) and by assumption 
\[|a_{mn}|^{-1}g < \|P_{mn}\| \quad \text{or} \quad g < |a_{mn}||P_{mn}|\] conflicting \(g \geq s\).
Now suppose that \(X\) is not continuous. Then \(G\) is quasidense (Proposition 1.6.4) which implies the existence of a \(g \in G\) such that 
\(s < g < \|A\|\) (quasidenseness is
used when $s, \|A\| \in G$.) By the same reasoning as above we find an $m, n$ such that $|a_{mn}|^{-1}g \notin \Gamma_{P_{mn}}$, so $g \notin \Gamma_{a_{mn}P_{mn}}$ conflicting $g > s$.

(ii) Let $x \in E$ have expansion $\sum_{n=1}^{\infty} \xi_n e_n$. Set

$$t_{mn} := \xi_n a_{mn} e_m \quad (m, n \in \mathbb{N}).$$

We first show that $\lim_{m+n \to \infty} t_{mn} = 0$. We have 1 and 2 below.

1. For each $n$, $\lim_{m \to \infty} a_{mn} e_m = 0$ (Lemma 3.3.3) so that $\lim_{m \to \infty} t_{mn} = 0$ for each $n$.

2. Let $g \in G$, $g > \|a_{mn} P_{mn}\|$ for each $m, n$. Then $g \in \Gamma_{a_{mn} P_{mn}}$, so

$$\|t_{mn}\| = \|\xi_n a_{mn} P_{mn}(e_n)\| \leq \|\xi_n e_n\| g, \text{ so}$$

$$\lim_{m \to \infty} t_{mn} = 0 \quad \text{uniformly in } n.$$ Together 1 and 2 imply unconditional summability of $t_{mn}$, so the formula

$$Ax = \sum_{n=1}^{\infty} \xi_n \sum_{m=1}^{\infty} a_{mn} e_m$$

defines a map $A : E \to E$. Direct verification tells that $A$ is linear and that its matrix is the required one. To see that $A$ is Lipschitz, let $g \in G$ be as above and $x \in E$. Then

$$\|Ax\| \leq \sup\{\|t_{mn}\| : m, n \in \mathbb{N}\} \leq g \max\{\|\xi_n e_n\| : n \in \mathbb{N}\} = g\|x\|$$

In the same vein we have

**Theorem 3.3.5** (Characterization of strictly Lipschitz operators by matrices)

(i) Let $A \in \text{Lip}^\sim(E)$ have the matrix

$$
\begin{pmatrix}
  a_{11} & a_{12} & \cdots \\
  a_{21} & & \\
  \vdots & & \\
\end{pmatrix}
$$

with respect to $e_1, e_2, \ldots$. Then, for each $n$, $\lim_{m \to \infty} \|a_{mn} P_{mn}\| \sim = 0$ and

$$\|A\| \sim = \sup\{\|a_{mn} P_{mn}\| \sim : m, n \in \mathbb{N}\}.$$ (ii) Conversely, let

$$
\begin{pmatrix}
  a_{11} & a_{12} & \cdots \\
  a_{21} & & \\
  \vdots & & \\
\end{pmatrix}
$$

be a matrix with entries in $K$, such that, for each $n$, $\lim_{m \to \infty} \|a_{mn} P_{mn}\| \sim = 0$ and such that $(m, n) \mapsto \|a_{mn} P_{mn}\| \sim$ is bounded above. Then the matrix represents a strictly Lipschitz operator.
Proof. Straightforward adaptation of the proof of Theorem 3.3.4. We leave the details to the reader.

Now we characterize compact and nuclear operators (see Definition 2.4.3).

**Theorem 3.3.6** Let $A \in \text{Lip}(E)$ have the matrix

$$
\begin{pmatrix}
  a_{11} & a_{12} & \cdots \\
  a_{21} & & \\
  & & \\
  a_{m1} & a_{m2} & \\
  \vdots & & \\
  a_{m+1,1} & \cdots & \\
  \vdots & & \\
  \vdots & & \\
  \vdots & & \\
\end{pmatrix}
$$

with respect to $e_1, e_2, \ldots$. Then

(i) $A \in C(E)$ if and only if $\lim_{m \to \infty} \|a_{mn} P_{mn}\| = 0$ uniformly in $n \in \mathbb{N}$,

(ii) $A \in C^*(E)$ if and only if $\lim_{m \to \infty} \|a_{mn} P_{mn}\| = 0$ uniformly in $n \in \mathbb{N}$.

**Proof.** Suppose $\lim_{m \to \infty} \|a_{mn} P_{mn}\| = 0$ uniformly in $n$. Let $\epsilon \in G$. There is an $m$ such that $\|a_{kn} P_{kn}\| < \epsilon$ for all $k > m$, all $n$. The matrix decomposition

$$
\begin{pmatrix}
  a_{11} & a_{12} & \cdots \\
  a_{21} & & \\
  & & \\
  a_{m1} & a_{m2} & \\
  \vdots & & \\
  a_{m+1,1} & \cdots & \\
  \vdots & & \\
  \vdots & & \\
  \vdots & & \\
\end{pmatrix} =
\begin{pmatrix}
  a_{11} & a_{12} & \cdots \\
  a_{21} & & \\
  & & \\
  a_{m1} & a_{m2} & \\
  \vdots & & \\
  0 & 0 & 0 & \cdots \\
  \vdots & & \\
  \vdots & & \\
  \vdots & & \\
\end{pmatrix} +
\begin{pmatrix}
  0 & 0 & \cdots \\
  0 & & \\
  & & \\
  0 & & \\
  \vdots & & \\
  a_{m+1,1} & a_{m+1,2} & \cdots \\
  \vdots & & \\
  \vdots & & \\
\end{pmatrix}
$$

corresponds to a decomposition $A = A_1 + A_2$; where $A_1, A_2 \in \text{Lip}(E)$. Clearly $A_1 \in FR(E)$ and $\|A_2\| = \sup \{\|a_{kn} P_{kn}\| : k > m, n \in \mathbb{N}\} \leq \epsilon$. We see that $\|A - A_1\| \leq \epsilon$. Thus $A \in C(E)$. A similar proof goes for the 'if' part of (ii).

To prove the 'only if' parts observe that

$$\{A \in \text{Lip}(E) : \lim_{m \to \infty} \|a_{mn} P_{mn}\| = 0 \text{ uniformly in } n \}$$

is a $\|\|$-closed subspace of $\text{Lip}(E)$ and that

$$\{A \in \text{Lip}^*(E) : \lim_{m \to \infty} \|a_{mn} P_{mn}\| = 0 \text{ uniformly in } n \}$$

is a $\|\|_*$-closed subspace of $\text{Lip}^*(E)$. So we are done as soon as the latter set contains $A : x \mapsto f(x)a \ (f \in E', \ a \in E)$ which we shall prove now. There is an $s_0 \in X$ such that $s_0, f(e_n) < \|e_n\|$ for all $n \in \mathbb{N}$ (Lemma 2.1.5). Let $\epsilon \in G$, let $a$ have an expansion $\sum_{i=1}^{\infty} \xi_i e_i$. There is an $m_0$ such that $\|\xi_m e_m\| \leq \epsilon s_0$ for $m \geq m_0$. Then for $m \geq m_0$ and $n \in \mathbb{N}$ we have $\|(a_{mn} P_{mn})(e_n)\| = \|a_{mn} e_m f(e_n)\| \leq \epsilon s_0 \|f(e_n)\| < \epsilon \|e_n\|$. Thus $\|a_{mn} P_{mn}\| \leq \epsilon$ for those $m, n$ and we are done.

We also have the following expected formula for the trace.
Theorem 3.3.7 Let $A \in C^\sim(E)$ have the matrix

\[
\begin{pmatrix}
a_{11} & a_{12} & \cdots \\
a_{21} & & \\
& & \\
& & \\
& & \\
\end{pmatrix}
\]

with respect to $e_1, e_2, \ldots$. Then $\lim_{n \to \infty} a_{nn} = 0$ and $\text{tr}(A) = \sum_{n=1}^\infty a_{nn}$.

Proof. From Theorem 3.3.6 (ii) we get $\lim_{n \to \infty} \|a_{nn} P_{nn}\| = 0$. Now $\|P_{nn}\| \geq 1$ (Corollary 3.3.2 (ii)), so $\lim_{n \to \infty} a_{nn} = 0$, so $\sum_{n=1}^\infty a_{nn}$ exists. Clearly the conclusion of Theorem 3.3.7 holds for operators in $FR(E)$ whose matrices have the form

\[
\begin{pmatrix}
a_{11} & a_{12} & \cdots \\
a_{21} & & \\
& & \\
& & \\
& & \\
0 & 0 & \cdots \\
\end{pmatrix}
\]

(by using Proposition 2.1.23). Those operators form a dense subspace of $C^\sim(E)$ on which the continuous maps $A \mapsto \sum_{n=1}^\infty a_{nn}$ and $A \mapsto G(A)$ coincide, hence they coincide in $C^\sim(E)$.

Remark. Since the choice of the orthogonal base was arbitrary we can conclude that the formula for the trace is 'independent of the choice of orthogonal base' in the sense that, if $b_1, b_2, \ldots$ is a second orthogonal base and $A \in C^\sim(E)$ has matrix $(c_{mn})$ with respect to $b_1, b_2, \ldots$ then $\text{tr}(A) = \sum_n c_{nn}$.

3.4 Matrix properties of subclasses

Theorem 3.4.1 Let $E$ be a Norm Hilbert space, let $A \in L(E)$ have matrix $(a_{mn})$ with respect to $e_1, e_2, \ldots$.

Then the following are equivalent.

(a) $A \in \text{Lip}(E)$.

(b) $(m, n) \mapsto \|a_{mn} P_{mn}\|$ is bounded.

(c) $(m, n) \mapsto \|a_{mn} P_{mn}\|$ is bounded. For each $m \in \mathbb{N}$, $\lim_{n \to \infty} \|a_{mn} P_{mn}\| = 0$. For each $n \in \mathbb{N}$, $\lim_{m \to \infty} \|a_{mn} P_{mn}\| = 0$.

Similarly, the following are equivalent.

(a) $A \in \text{Lip}^\sim(E)$.

(b) $(m, n) \mapsto \|a_{mn} P_{mn}\|^\sim$ is bounded.
(γ) \( (m,n) \mapsto \|a_{mn}P_{mn}\| \) is bounded. For each \( m \in \mathbb{N} \), \( \lim_{n \to \infty} \|a_{mn}P_{mn}\| = 0 \).
For each \( n \in \mathbb{N} \), \( \lim_{m \to \infty} \|a_{mn}P_{mn}\| = 0 \).

**Proof.** We only prove the equivalence of \((α) \Rightarrow (γ)\) leaving the other case to the reader. The implications \((α) \Rightarrow (β)\) and \((γ) \Rightarrow (α)\) follow from Theorem 3.3.4, so we prove \((β) \Rightarrow (γ)\). By Corollary 3.3.2 (i) we have \( \text{Stab}(\|P_{mn}\|) = \tau(\|e_n\|; \|e_m\|) \).
Since \( E \) is a Norm Hilbert space we have \( \lim_{n \to \infty} \tau(\|e_n\|; \|e_m\|) = \infty \) in the sense of [5] 1.6.4 (ii), for each \( m \). Thus \( n \mapsto \|P_{mn}\| \) and (since \( \text{Stab}(\|P_{mn}\|) = \text{Stab}(\|P_{nm}\|) \)) \( m \mapsto \|P_{mn}\| \) satisfy the type condition, so \((β)\) implies \( \lim_{n \to \infty} \|a_{mn}P_{mn}\| = 0, \lim_{m \to \infty} \|a_{mn}P_{mn}\| = 0 \).

**Corollary 3.4.2** Let \( E \) be a Norm Hilbert space, let \( A \in \text{Lip}(E) \) have matrix \((a_{mn})\) with respect to \( e_1, e_2, \ldots \). Then \( A \in C(E) \) if and only if \( \lim_{m+n \to \infty} \|a_{mn}P_{mn}\| = 0 \); and the \( P_{mn} \) form an orthogonal base of \( C(E) \). Similarly, \( A \in C^*(E) \) if and only if \( \lim_{m+n \to \infty} \|a_{mn}P_{mn}\| = 0 \); and the \( P_{mn} \) form an orthogonal base of \( C^*(E) \).

**Proof.** Combine Theorem 3.3.6, 3.4.1, Lemma 3.3.1 (ii) and Corollary 3.3.2 (ii).

The properties of Theorem 3.4.1 and Corollary 3.4.2 also characterize Norm Hilbert spaces, as is shown by the following

**Theorem 3.4.3** Let \( E \) be not a Norm Hilbert space. Then there exists an \( A \in \text{FR}(E) \) with matrix \((a_{mn})\) such that not \( \lim_{n \to \infty} \|a_{1n}P_{1n}\| = 0 \).

**Proof.** There are a subsequence \( e_{n_1}, e_{n_2}, \ldots \) of \( e_1, e_2, \ldots \), \( \lambda_1, \lambda_2, \ldots \in K \) and \( c_1, c_2 \in X \) such that
\[
c_1 \leq |\lambda_i| \|e_{n_i}\| \leq c_2 \quad (i \in \mathbb{N}).
\]
For all \( x \in E \) with expansion \( \sum_{i=1}^{\infty} \xi_i e_i \) set
\[
f(x) = \sum_{i=1}^{\infty} \lambda_i^{-1} \xi_{n_i},
\]
\((f \text{ is easily seen to be in } E')\) and put
\[
Ax := f(x)e_1.
\]
Then \( A \in \text{FR}(E) \) and \( a_{1n} = f(e_n) \) for all \( n \). We will show that the sequence \( i \mapsto \|a_{1n}P_{1n}\| \) does not tend to 0. We have for each \( i \)
\[
\|a_{1n_i}P_{1n_i}\| = \|\lambda_i^{-1}P_{1n_i}\| = \inf V_i
\]
where
\[
V_i = \{ g \in G : \|e_i\| \leq g|\lambda_{n_i}| \|e_{n_i}\| \}.
\]
Now let \( g \in V_i \). Then \( \|e_i\| \leq gc_2 \). Choose \( g_1 \in G \) such that \( \|e_i\| > g_1 c_2 \). Then \( g > g_1 \), so \( g_1 \) is a lower bound of \( V_i \) for each \( i \) and we have \( \|a_{1n_i}P_{1n_i}\| \geq g_1 \) for each \( i \).

To a classical Functional analyst the following feature will appear surrealistic.

**Theorem 3.4.4** Let \( E \) be infinite-dimensional. Then the following are equivalent.
(α) $C(E) = \text{Lip}(E)$.
(β) $C^\sim(E) = \text{Lip}^\sim(E)$.
(γ) $\lim_{n \to \infty} \text{Stab}(\|e_n\|) = \infty$ (i.e. for every proper convex subgroup $H$ of $G$ we have $\text{Stab}(\|e_n\|) \supseteq H$ for large $n$).

**Proof.** (α) $\Rightarrow$ (β). Let $A \in \text{Lip}^\sim(E)$, $\varepsilon \in G$. Choose $\delta \in G$ such that $\delta \|A\|^\sim < \varepsilon$. By assumption there is a $B \in \text{FR}(E)$ with $\|I - B\| < \delta$. Then (Proposition 2.2.16)

$$\|A - BA\|^\sim \leq \|I - B\| \ast \|A\|^\sim \leq \delta \|A\|^\sim < \varepsilon.$$  

(β) $\Rightarrow$ (α). Let $A \in \text{Lip}(E)$, $\varepsilon \in G$. By Theorem 3.2.1 there is a $B \in \text{Lip}^\sim(E)$ with $\|A - B\| < \varepsilon$. By assumption there is a $C \in \text{FR}(E)$ with $\|B - C\|^\sim < \varepsilon$, hence $\|B - C\| < \varepsilon$. Then $\|A - C\| \leq \max(\|A - B\|, \|B - C\|) < \varepsilon$.

(α) $\Rightarrow$ (γ). We have that $I \in C(E)$. For its matrix entries we have $a_{mn} = \delta_{mn}$, so by Theorem 3.3.6 (i), $\lim_{n \to \infty} \|P_{nn}\| = 0$. But $\|P_{nn}\| = \inf \text{Stab}(\|e_n\|) \to 0$, so (γ) follows.

(γ) $\Rightarrow$ (α). From (γ) we obtain $\|P_{nn}\| \to 0$ then $I = \sum_{n=1}^{\infty} P_{nn} \in C(E)$.

**Remarks.** Condition (γ) implies that $E$ is a Norm Hilbert space. For a concrete example of a space $E$ satisfying (α) – (γ), see [5], 4.2.2. In fact, any $C\#$-normed Norm Hilbert space satisfies (α) – (γ) of above ($\lim_{n \to \infty} \tau(\|e_n\|) = \infty$, and $\tau(\|e_n\|) = \text{Stab}(\|e_n\|)$ (Proposition 1.4.15)).

We would like to conclude this paper by describing a class of Norm Hilbert spaces thereby generalizing the results of [6] considerably.

Let us call momentarily $E$ **type-separating** if there exists an $s_0 \in X$ such that $n \neq m$ implies

$$\tau(\|e_n\|; s_0) \neq \tau(\|e_m\|; s_0)$$

Examples of such spaces can be found in [2], [3], [4].

Type-separating spaces are Norm Hilbert spaces ([3] and [6]).

For an $A \in \text{Lip}(E)$ with matrix $(a_{mn})$ the matrix decomposition

\[
\begin{pmatrix}
a_{11} & a_{12} & \cdots \\
a_{21} & a_{22} & \\
\vdots & \\
\end{pmatrix}
= \begin{pmatrix}
a_{11} & 0 & 0 & \cdots \\
0 & a_{22} & \\
\vdots & \\
\end{pmatrix}
+ \begin{pmatrix}
0 & a_{12} & \cdots \\
0 & a_{21} & \\
\vdots & \\
\end{pmatrix}
\]

represents a decomposition $A = D + S$ (which we will call henceforth the **standard decomposition**), where $D, S \in \text{Lip}(E)$, $D$ has diagonal matrix, $S$ has zero diagonal.
Proposition 3.4.5 Let $E$ be type-separating. Let $D+S$ be the standard decomposition of an $A \in \text{Lip}(E)$. Then $S$ is nuclear.

Proof. From Corollary 3.3.2 and Theorem 1.5.3 (v), we infer $\text{Stab}(\|P_{mn}\|) = \tau(\|e_m\|; \|e_n\|) = \tau(\|e_n\|; s_o) \cup \tau(\|e_m\|; s_o)$ whenever $m \neq n$. We see that $\{\|P_{mn}\| : m \neq n\}$ satisfy the type condition and therefore

$$\lim_{m+n \to \infty} (m \neq n) \|a_{mn}P_{mn}\| = 0$$

showing (Theorem 3.3.6) that $S$ is compact. But, since the algebraic types of $\|e_m\|$ and $\|e_n\|$ must differ whenever $m \neq n$ we have $\|P_{nn}\| = \|P_{nn}\|_\sim$ according to Lemma 3.3.1 (i).

The following Corollary obtains (Compare [7], 3.8 and 4.3)

Theorem 3.4.6 Let $E$ be type separating.

(i) If $A, B \in \text{Lip}(E)$ then $AB - BA \in C^{-}(E)$ and $\text{tr}(AB - BA) = 0$.

(ii) Let $A \in \text{Lip}(E)$. Then $A \in C(E)$ if and only if $\lim_{n \to \infty} \|a_{nn}P_{nn}\| = 0$, and $A \in C^{-}(E)$ if and only if $\lim_{n \to \infty} \|a_{nn}P_{nn}\|_\sim = 0$.

(iii) The Calkin algebra $\text{Lip}(E)/C(E)$ is commutative.

(iv) If $A \in \text{Lip}(E)$, $n \in \mathbb{N}$, $A^n \in C(E)$ then $A \in C(E)$.

Proof. By considering the standard decomposition of $A$ and $B$ one easily verifies that the diagonal of the matrix of $AB - BA$ is 0. Now (i) follows from Proposition 3.4.5 and Theorem 3.3.7. From Proposition 3.4.5 it follows that an $A \in \text{Lip}(E)$ is compact (nuclear) if its ‘diagonal part’ is compact (nuclear). This yields (ii). (iii) follows directly from (i). To prove (iv) it suffices to consider the case $n = 2$. So let $A \in \text{Lip}(E)$ have standard decomposition $D+S$ and suppose $A^2 \in C(E)$. Then, since $DS, SD, S^2$ are in $C(E)$, we have $D^2 \in C(E)$ which means $\lim_{n \to \infty} |a_{nn}^2| \|P_{nn}\| = 0$. It is not hard to see, using boundedness of $n \mapsto \|P_{nn}\|$, that also $\lim_{n \to \infty} |a_{nn}| \|P_{nn}\| = 0$ i.e. $D \in C(E)$. Then $A = D + S \in C(E)$.

Question. Does (iv) hold with $C^{-}(E)$ in place of $C(E)$?

References