Banach spaces over fields with an infinite rank valuation

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Abstract. For a field $K$, complete with respect to a valuation $|\cdot|$ of infinite rank, a basic theory of normed and Banach spaces is being developed. A crucial part is played by the $G$-modules introduced in 1.5. The results are applied to the class of the Norm Hilbert Spaces (NHS) i.e. Banach spaces for which closed subspaces admit projections of norm $\leq 1$. We characterize NHS in several ways (Theorem 4.3.7). Bounded orthogonal sequences tend to 0 implying that every ball is a compactoid, a property that in rank 1 theory is shared only by the finite-dimensional spaces. Finally we describe in Section 5 those NHS for which there exists a Hermitean form $(\cdot, \cdot)$, satisfying $|(x,x)| = |x|^2$ for all $x$. The first such so-called Form Hilbert Space (FHS) was discovered by Keller in 1980 [5] and its class was studied in several papers ([1], [3], [11], [18]).

INTRODUCTION

In real and complex Functional Analysis the basic and most elegant examples of Banach spaces are Hilbert spaces; therefore one might wonder why they are so absent in p-adic theory. In fact, attempts in the past, in Banach spaces over a complete field $K$ with a rank 1 valuation $|\cdot|$ (i.e. $|\cdot|$ is real-valued), to introduce 'inner products' $(\cdot, \cdot)$ that were compatible with the norm never resulted, for $K$ not $\mathbb{R}$ or $\mathbb{C}$, into Hilbert-like spaces. It was a consequence of a beautiful theorem of M.P. Soler [18] (see below) that enabled one to understand that this feature was not accidental. In fact, we will show in the present paper (by independent means) that infinite-dimensional Banach spaces over finite rank valued fields are never 'form Hilbert' i.e. there is no inner product such that every closed subspace

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has an orthogonal complement, see Corollary 4.4.6 for a precise formulation. We may summarize roughly by the slogan

‘There are no $p$-adic Hilbert Spaces’.

One can generalize the perspective by removing in the above topology and norms obtaining a purely algebraic setting as follows. Let $K$ be a field with an involution $a \mapsto a^*$ (that is allowed to be the identity). A vector space $E$ over $K$ with a Hermitean form $(\cdot, \cdot): E \times E \to K$ (linearity in the first variable and $(x, y) = (y, x)^*$ for all $x$ and $y$) is called orthomodular if the projection theorem

$$X = X^\perp \perp \to E = X \bigoplus X^\perp$$

holds (where, as usual, for a subset $X$ of $E$, $X^\perp := \{y \in E : (x, y) = 0 \text{ for all } x \in X\}$).

**QUESTION** Do there exist infinite-dimensional orthomodular spaces, different from the Hilbert spaces over $\mathbb{R}$ and $\mathbb{C}$?

It has been open for quite some years until Keller [5] in 1980 gave an affirmative answer by constructing an example. The base field he employed turned out to have a valuation $| \cdot |$ of infinite rank and $x \mapsto \sqrt{|(x, y)|}$ behaves like a norm. In Keller’s space every topologically closed subspace $S$ is also orthogonally closed (i.e. $S^{\perp \perp} = S$) and it has a countable orthogonal base, but surprisingly enough, no base is ever orthonormal. In other examples that were found later one meets the same state of affairs. A systematic study of orthomodular spaces was made in [3], and in [10] one finds connections with non-archimedean analysis. A breakthrough came in 1995 with the following result.

**THEOREM (Soler) [18]** If $E$ is an orthomodular space over $K$ admitting an orthonormal sequence then $K = \mathbb{R}$ or $\mathbb{C}$ and $E$ is (linearly homeomorphic to) a Hilbert space.

Thus, we have a contrasting catch-phrase.

‘There do exist Hilbert spaces over valued fields with infinite rank’.

However one is tempted to add that such Hilbert spaces have peculiar properties. This is confirmed not only by Soler’s Theorem, but by study of operators on such spaces, see [6], [7] where it is shown that self-adjoint operators behave in a strange way. From 4.3.7 of the present paper it even follows that every operator is compact!

We feel that these interesting phenomena more than justify our paper, which aims at setting up a theory of Banach spaces over fields with an infinite rank valuation, to form a new branch of Non-Archimedean Functional Analysis, encompassing Hilbert spaces as a special class.

Basics can be found in Sections 1-3. As one might expect, several results of rank 1 theory can easily be carried over. However there is one crucial point of difference making this part interesting; it lies in the choice of the range of the norm function. In fact, to include Hilbert space we must admit, by Soler’s Theorem, the range $X$ of the norm function to be strictly bigger than the set of values $|K|$ (otherwise,
orthogonal bases could be transformed into orthonormal ones by suitable scalar multiplication). In addition, to be able to define the norm of operators, norms on quotient spaces, etc. one wants to take infima and suprema of bounded sets in $X$, so it is convenient to require the ordered set $X$ to be Dedekind complete. All these considerations have lead to the introduction of so-called $G$-modules (see 1.5) as a natural 'home' for norm values. The notions of (algebraic and topological) types and the type condition (see 1.6) have no counterparts in rank 1 theory and play a crucial role in Sections 4-5 in which we apply the general theory to so-called norm Hilbert spaces i.e. spaces for which every closed subspace admits a projection of norm $\leq 1$. We characterize them in various ways and describe the subclass of the 'form Hilbert spaces' of above.

Needless to say that this work needs continuation. The whole world of operators in Norm and Form Hilbert Spaces is still unexplored.

1 THE RANGE OF THE NORM FUNCTION

Section 1 deals mainly with the range sets of arbitrary rank valuations on a field $K$ (linearly ordered groups $G$, 1.3 and 1.4) and of norms on $K$-vector spaces (the so-called $G$-modules, 1.5). Ultrametrics are being generalized to so-called scales (1.2) that have values in linearly ordered sets (1.1). These concepts are fundamental for the theory of normed and Hilbert-like spaces in Sections 2, 3 and 4.

A number of basic facts of Section 1 can be found in standard books such as [2], [13], [19]. The concepts of topological type and type condition of [3], Def. 21, 31 have been generalized in 1.6 to arbitrary $G$-modules so as to make them useful for general normed spaces.

We haven't found in the literature the $G$-modules, the scaled spaces, and the antipode in $G^\#$ (1.3.1).

1.2 Linearly ordered sets

Let $X$ be an ordered set. A subset $A$ of $X$ is called cofinal (coinitial) in $X$ if for every $x \in X$ there exists an $a \in A$ such that $a \geq x$ ($a \leq x$). In the same spirit we define cofinal (coinitial) sequences and, more generally, nets.

Let $f : X \rightarrow Y$ where $X, Y$ are linearly ordered sets. We say that $f$ is increasing (strictly increasing) if $x, y \in X, x < y$ implies $f(x) \leq f(y)$ ($f(x) < f(y)$). In the same spirit we define decreasing (strictly decreasing) maps. Let $A \subseteq B \subseteq X$.

We say that $s = \sup_B A$ if $s \in B, s \geq a$ for all $a \in A$, and if $t \in B, t \geq a$ for all $a \in A$ then $t \geq s$. In the same spirit we define $\inf_B A$. If $\sup_B A$ and $\sup_X A$ both exist then $\sup_B A \geq \sup_X A$, but we do not always have equality. If it is clear with respect to which set the supremum (infimum) is taken we sometimes omit the subscript $X$ in $\sup_X A$ ($\inf_X A$).

**Lemma 1.1.1** Let $X$ be a linearly ordered set, let $a \in X$. Then either $\min\{x \in X : x > a\}$ exists or $\inf\{x \in X : x > a\} = a$.

Similarly, either $\max\{x \in X : x < a\}$ exists or $\sup\{x \in X : x < a\} = a$. 


Proof. It suffices to prove the first assertion. Let \( V := \{ x \in X : x > a \} \) and suppose \( \min V \) does not exist. Clearly \( a \) is a lower bound of \( V \). If \( b \in X, b > a \), then \( b \in V \) and by assumption there is a \( v \in V, v < b \). Hence, \( b \) is no lower bound of \( V \) i.e. \( a \) is the greatest lower bound (Remark. If \( V = \emptyset \) then \( a \) is the largest element of \( X \) and \( a = \inf V \)).

1.1.2. Continuity at 0

In the sequel it will be useful to extend a given linearly ordered set by adjoining one element called 0, for which \( 0 < x \) for all \( x \in X \). (See, e.g. 1.2, 1.4, 2.1) Then the extended set \( X \cup \{ 0 \} \) is again linearly ordered, and has 0 as smallest element. Let \( Y \) be a second linearly ordered set, let \( f : X \rightarrow Y \). The (natural) extension \( f_0 : X \cup \{ 0 \} \rightarrow Y \cup \{ 0 \} \) extends \( f \) and maps 0 into 0. We will say that \( f \) (or \( f_0 \)) is continuous at 0 if for each \( \varepsilon \in Y \) there is a \( \delta \in X \) such that \( x < \delta \) implies \( f(x) < \varepsilon \). It is called bicontinuous at 0 if in addition to each \( \delta \in X \) there is an \( \varepsilon \in Y \) such that \( f(x) < \varepsilon \) implies \( x < \delta \).

For a net \( i \mapsto x_i \) in \( X \) we say that \( \lim x_i = 0 \) if for each \( \varepsilon \in Y \) there exists an \( i_0 \in I \) such that \( x_i < \varepsilon \) for all \( i > i_0 \). Then \( f : X \rightarrow Y \) (or its extension \( f_0 \)) is continuous at 0 if and only if, for each net \( i \mapsto x_i \) in \( X \), \( \lim x_i = 0 \) implies \( \lim f(x_i) = 0 \). It is bicontinuous at 0 if and only if, for each net \( i \mapsto x_i \) in \( X \), \( \lim x_i = 0 \iff \lim f(x_i) = 0 \).

For our purpose it is not necessary to enrich \( X \) with a largest element \( \infty \). We only define the following. For a net \( i \mapsto x_i \) in \( X \) \( \cup \{ 0 \} \) we say that \( \lim x_i = \infty \) if for each \( s \in X \) there is an \( i_0 \) such that \( x_i > s \) for \( i \geq i_0 \).

LEMMA 1.1.3 Let \( X,Y \) be linearly ordered sets without a smallest element and let \( f : X \rightarrow Y \) be increasing. If \( f(X) \) is coinitial in \( Y \) then \( f \) is bicontinuous at 0.

Proof. Let \( i \mapsto x_i \) be a net in \( X \) and suppose that not \( \lim f(x_i) = 0 \). Then there is an \( \varepsilon \in Y \) such that \( J := \{ i \in I : f(x_i) \geq \varepsilon \} \) is cofinal. Since \( \varepsilon \) is not the smallest element of \( Y \), and \( f(X) \) is coinitial there is a \( \delta \in X \) such that \( f(x_i) > f(\delta) \) for all \( i \in J \). Then \( x_i > \delta \) for all \( i \in J \). Hence not \( \lim x_i = 0 \).

Now suppose \( \lim f(x_i) = 0 \). To show that \( \lim x_i = 0 \), let \( \epsilon \in X \). Then \( f(\varepsilon) \in Y \) so there is an \( i_0 \in I \) such that \( f(x_i) < f(\varepsilon) \) for \( i \geq i_0 \). Then \( x_i < \varepsilon \) for \( i \geq i_0 \) and we are done.

A linearly ordered set \( X \) is called (Dedekind) complete if each nonempty subset of \( X \) that is bounded above has a supremum. Then also each nonempty subset of \( X \) that is bounded below has an infimum. (Proof. Let \( V \neq \emptyset \) be bounded below. Then the set \( W \) consisting of all lower bounds of \( V \) is nonempty, and bounded above since \( V \neq \emptyset \), so \( s = \sup W \) exists; one verifies easily that \( s \in W \).)

We now describe the construction of the completion of a linearly ordered set \( X \). A subset \( S \) of \( X \) is called a cut if

1. \( S \neq \emptyset, S \) is bounded above,
2. if \( x \in S, y < x \) then \( y \in S \),
3. if \( \sup X S \) exists then \( \sup_X S \in S \).
Let $X^#$ be the collection of all cuts of $X$. With the ordering by inclusion $X^#$ becomes a linearly ordered set. To prove that $X^#$ is complete, let $A \subseteq X^#$ be nonempty and bounded above. There is a cut $T$ such that $S \subseteq T$ for all $S \in A$. Then $V := \bigcup_{S \in A} S$ is nonempty and bounded above by $T$, and by adding $\sup_X V$ (if it exists) to $V$ we obtain a cut that is easily seen to be $\sup_X A$. We have the natural embedding $\varphi : X \rightarrow X^#$ given by

$$\varphi(a) = \{x \in X : x \leq a\}.$$  

$\varphi$ is strictly increasing. Often we shall identify $X$ and $\varphi(X)$, in other words we shall view $\varphi$ as an inclusion. $X^#$ is called the completion of $X$.

For reasons of quoting the next Proposition contains some redundancy.

**PROPOSITION 1.1.4** Let $X$ be a linearly ordered set with completion $X^#$. Then we have the following.

(i) If $X$ is complete then $X = X^#$.  
(ii) $X$ is cofinal and coinitial in $X^#$.  
(iii) For every $s \in X^#$, $\{x \in X : x \leq s\}$ is a cut in $X$; every cut in $X$ has this form.  
(iv) If $s, t \in X^#$, $s < t$ then there exist $x, y \in X$ with $s \leq x < t$, $s < y \leq t$.  
(v) For each $s \in X^#$,

$$s = \sup_{X^#} \{x \in X : x \leq s\} = \inf_{X^#} \{x \in X : x \geq s\}.$$  

(vi) If $A \subseteq X$, $s = \sup_X A$ then $s = \sup_{X^#} A$. If $t = \inf_X A$ then $t = \inf_{X^#} A$.

**Proof.** (i) Suppose $X$ is complete, let $S$ be a cut. Then, letting $s := \sup S$, we have $S = \{x \in X : x \leq s\}$, so $S = \varphi(s)$ where $\varphi$ is as above, i.e. $X^# = \varphi(X)$ or $X^# = X$ by identification.

(ii) Let $s \in X^#$. Then $s$ is a cut, hence bounded above in $X$, so there is an $x \in X$ with $v \leq x$ for all $v \in s$. Hence $s \leq x$. It follows that $X$ is cofinal in $X^#$. Coinitiality follows from the fact that cuts are nonempty.

(iii) Obvious.

(iv) Let $C_1 := \{x \in X : x \leq s\}$, $C_2 := \{x \in X : x \leq t\}$. Then $C_1 \subseteq C_2$, but $C_1 \neq C_2$ so there exists a $y \in C_2 \setminus C_1$ i.e. $s < y \leq t$. To find an $x \in X$ with $s \leq x < t$, take $x := s$ if $s \in X$ and $x := y$ if $t \not\in X$. If $s \not\in X$ and $t \in X$ and there is no $z \in X$ with $s \leq z < t$ then $t = \sup_X C_1$, so $t \in C_1$ and $C_1 = C_2$, a contradiction.

(v) Let $V = \{x \in X : x \leq s\}$. Then clearly $s$ is an upper bound of $V$ in $X^#$. If $v \in X^#$, $v < s$ then by (iv) there is a $y \in X$ with $v < y \leq s$, so $v$ is no upper bound of $V$. Hence $s = \sup_{X^#} V$. Let $W = \{x \in X : x \geq s\}$. Then $s$ is a lower bound of $W$ in $X^#$. If $t \in X^#$, $t > s$ then by (iv) there is an $x \in X$ with $s \leq x < t$, so $t$ is no lower bound of $W$. Hence $s = \inf_{X^#} W$.

(vi) $A$ is nonempty and bounded above in $X^#$ so, by completeness, $t := \sup_{X^#} A$ exists and $t \leq s$. If $t < s$ we would have a $y \in X$ with $t < y \leq s$ by (iv), so $t$ is not an upper bound of $A$, contradiction. We leave the 'inf' part of the proof to the reader.
PROPOSITION 1.1.5 Let $X$ be a linearly ordered set, let $Y \subset X$ be a subset that is complete as a linearly ordered set. If $Y$ is both cofinal and coinitial in $X$ then there exists an increasing projection $P : X \to Y$ (i.e. $Py = y$ for each $y \in Y$).

Proof. Set

$$Px := \sup_Y \{y \in Y : y \leq x\} \quad (x \in X).$$

(By cofinality the set $\{y \in Y : y \leq x\}$ is bounded above in $Y$, by coinitiality it is not empty.) One checks easily that $P$ satisfies the requirements.

REMARK In the above proof the map $x \mapsto \inf_Y \{y \in Y : y \geq x\}$ would also have solved the problem.

1.2 Scaled spaces

Let $M$ be a set, let $X$ be a linearly ordered set enriched with a smallest element, called $0$. An \textit{(X-valued) scale on $M$} is a map $d : M \times M \to X \cup \{0\}$ satisfying

(i) $d(x,y) = 0 \iff x = y$.

(ii) $d(x,y) = d(y,x)$.

(iii) $d(x,z) \leq \max(d(x,y), d(y,z))$

for all $x, y, z \in M$. The set $M = (M, X, d)$ is called a scaled space (ultrametric space if $X = (0,\infty)$). For a nonempty subset $S$ of $M$ for which $\{d(x,y) : x,y \in S\}$ is bounded above in $X \cup \{0\}$ we define its diameter as $\text{diam} S = \sup_{x \in S, y \in S} \{d(x,y) : x, y \in S\}$. For $a \in M, \varepsilon \in X$, we define, as usual, $B_M(a, \varepsilon) := \{x \in M : d(x,a) \leq \varepsilon\}$ (the 'closed' ball) and $B_M(a, \varepsilon^-) := \{x \in M : d(x,a) < \varepsilon\}$ (the 'open' ball). A subset $U$ of $M$ is called open if for each $a \in U$ there exists an $\varepsilon \in X$, such that $B(a, \varepsilon^-) \subset U$. The collection of those open sets form a topology, the topology induced by $d$. Each ball is clopen (= closed and open), two balls are either disjoint or ordered by inclusion, every point of a ball is a center. The induced topology is zerodimensional.

A nest of balls in a scaled space is a nonempty collection of balls that is linearly ordered by inclusion. If $\mathcal{C}$ is a nonempty collection of balls such that any two members have a nonempty intersection then $\mathcal{C}$ is a nest.

DEFINITION 1.2.1 A scaled space is spherically complete if each nest of balls has a nonempty intersection.

If $B_1 \subset B_2$ are balls in a scaled space and $B_1 \neq B_2$ then there exist an 'open' ball $S$ and a 'closed' ball $T$ such that $B_1 \subset S \subset B_2, B_1 \subset T \subset B_2$. Therefore, a scaled space is spherically complete if and only if each nest of 'open' ('closed') balls has a nonempty intersection. We use this fact to show that spherical completeness 'does not depend on the range space of $d$' in the following sense.

PROPOSITION 1.2.2 Let $(M, X, d)$ be a scaled space, let $Y := \{d(x,y) : x, y \in M, x \neq y\}$. Then $(M, X, d)$ is spherically complete if and only if $(M, Y, d)$ is spherically complete.
Proof. It suffices to prove that spherical completeness of \((M,Y,d)\) implies that of \((M,X,d)\). Thus, let \(C\) be a nest of 'open' balls in \((M,X,d)\). To show that 
\[ \bigcap C \neq \emptyset \] we may assume that \(C\) has no smallest element. Let \(B \in C\). Then there exists a \(B' \in C, B' \subset B, B' \neq B\). Thus, let \(B' = \{x \in M : d(x,a) < r'\}\), \(B = \{x \in M : d(x,a) < r\}\) for some \(r,r' \in X\) with \(r' < r\) and some \(a \in B'\). If 
\[ \{s \in X : r' < s < r\} \cap Y = \emptyset \] then it would follow that \(B = B'\), a contradiction. Thus, there is an \(s \in Y, r' < s < r\) is a ball in \((M,Y,d)\), between \(B'\) and \(B\). It is easily seen that \(D := \{\tau(B) : B \in C\}\) is a nest of balls in \((M,Y,d)\), so \(\emptyset \neq \bigcap D \subset \bigcap C\).

**PROPOSITION 1.2.3** Let \((M,X,d)\) be a scaled space, let \(V \subset M\) be a spherically complete subspace. Then each \(x \in M\) has a best approximation in \(V\) i.e. \(\min\{d(x,v) : v \in V\}\) exists.

Proof. The collection \(\{B(x,r) \cap V : r \in X, B(x,r) \cap V \neq \emptyset\}\) is a nest of balls in \((V,Y,d)\). By spherical completeness of \(V\) (thanks to Proposition 1.2.2 we do not have to specify the range space of \(d\)) it has a nonempty intersection, so there exists \(v \in V\) such that \(d(x,v) \leq d(x,w)\) for all \(w \in V\) and we are done.

**PROPOSITION 1.2.4** Let \((M,X,d)\) be a scaled space. The following are equivalent.

(a) \(M\) is ultrametrizable.

(\(\beta\)) \(M\) is discrete or there exist \(s_1 > s_2 > \ldots \) in \(X\) such that \(\lim_n s_n = 0\).

Proof. To prove (\(\beta\)) \(\Rightarrow\) (a), suppose we have \(s_1 > s_2 > \ldots \) in \(X\) with \(\lim_n s_n = 0\). For each \(r \in X\), let \(n_r := \min\{m \in \mathbb{N} : s_m \leq r\}\) and set \(\phi(r) := 2^{-n_r}\). By adding the requirement \(\phi(0) := 0\) we obtain an increasing map \(\phi : X \cup \{0\} \to \{0, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}\) and it is easily checked that \(\phi \circ d\) is an ultrametric on \(M\) yielding the same topology as \(d\).

Conversely suppose (a) and let \(M\) be not discrete. Then it has a non-isolated point \(a\). Let \(U_1 \supset U_2 \supset \ldots\) be a neighbourhood base at \(a\). There exist \(u_n \in U_n (n \in \mathbb{N})\) with \(u_n \neq a\) for each \(n\). As \(\lim_{n \to \infty} u_n = a\) we have \(\lim_n d(a,u_n) = 0\), so (\(\beta\)) is proved, for \(n \mapsto s_n\), a suitably chosen subsequence of \(n \mapsto \min\{d(a,u_j) : 1 \leq j \leq n\}\).

### 1.3 Linearly ordered groups

Throughout this paper \(G\) is an abelian multiplicatively written group with unit element \(1\). If \(G\) is linearly ordered such that \(x,y,z \in G, x \leq y\) implies \(xz \leq yz\) we call \(G = (G, \leq)\) a linearly ordered group. Then \(x,y,z \in G, x < y\) implies \(xz < yz\) (if \(x < y\) then \(x = xzz^{-1} \geq yzz^{-1} = y\), a contradiction). It follows easily that \(G\) is torsion free and that, if \(G \neq \{1\}\), \(G\) has no smallest or largest element.

A subset \(H\) of a linearly ordered group \(G\) is convex if \(x,y,z \in H, z \leq y\) implies \(xz \leq yz\) implies \(z \in H\). Each proper convex subgroup is bounded from below and from above. The set of convex subgroups is linearly ordered by inclusion. A convex subgroup \(H\) is called principal if there is an \(a \in G\) such that \(H\) is the smallest convex subgroup of \(G\) containing \(a\). The order type of the set of all principal subgroups \(\neq \{1\}\) is called the rank of \(G\). \(G\) has rank 1 if and only if it is, as an ordered group, isomorphic to a subgroup of \((0, \infty)\). For a proof, see [13].
group $\bigoplus_{i \in \mathbb{N}} \mathbb{Z}_i$, where $\mathbb{Z}_i = \mathbb{Z}$ for each $i$, with the antilexicographic ordering, is an example of a group with infinite rank.

If $H$ is a convex subgroup then $G/H$ is in a natural way a linearly ordered group and the canonical quotient map $G \to G/H$ is an increasing homomorphism. If rank $G > 1$ then $G$ is not complete. In fact, let $H$ be a proper convex subgroup, $H \neq \{1\}$. If $s := \text{sup}_G H$ would exist then $s^2 = s$, so $s = 1$, a contradiction.

Let $G$ be a linearly ordered group. We extend the multiplication to its completion (see 1.1) $G^\#$ as follows. For $s, t \in G^\#$ set

$$st = \text{sup}\{g_1g_2 : g_1, g_2 \in G, \ g_1 \leq s, \ g_2 \leq t\}.$$ 

Clearly this multiplication extends the one of $G$, is associative, commutative and has a unit $1$. But the semigroup $G^\#$ is in general not a group. In fact, if rank $G > 1$ then we have $s^2 = s$ if $s = \text{sup} H$ when $H$ is a convex subgroup, $H \neq \{1\}$, $H \neq G$, but $s \neq 1$, so $G^\#$ is not a group. The order on $G^\#$ satisfies the following. If $s, t, s', t' \in G^\#$, $s \leq t$, $s' \leq t'$ then $ss' \leq tt'$.

**REMARK** The extension of the multiplication from $G$ to $G^\#$ is, in general, not unique. In fact, the formula $s \ast t = \inf\{g_1g_2 : g_1, g_2 \in G, \ g_1 \leq s, \ g_2 \leq t\}$ defines an extension $\ast$ to $G^\#$ of the multiplication of $G$ that is also associative, commutative, for which $1$ is a unit and such that $s, t, s', t' \in G^\#$, $s \leq t$, $s' \leq t'$ implies $s \ast s' \leq t \ast t'$. We have $s \ast t \geq st$ for all $s, t \in G^\#$. To see that $\ast$ differs from the multiplication of above, let $H \neq \{1\}$ be a proper convex subgroup, $s := \text{sup} H$, $t := \text{inf} H$. Then one proves easily that $s \ast t = s$ but $st = t$. (However, we will see in 1.5.4 that for any two extensions $\ast$ and $\cdot$ of the multiplication that are increasing in both variables we have $s \ast g = s \cdot g$ for each $s \in G^\#$, $g \in G$.) In contrast to this we will show now that the inversion map $g \mapsto g^{-1}$ extends uniquely to a decreasing map $G^\# \to G^\#$.

**PROPOSITION 1.3.1** Let $G$ be a linearly ordered group. Then there is a unique decreasing map $\omega : G^\# \to G^\#$ extending $g \mapsto g^{-1}$ ($g \in G$). It has the following properties.

(i) $\omega(s) = \text{sup}\{g \in G : sg \leq 1\} = \inf\{g \in G : 1 \leq sg\}$ \ ($s \in G^\#$).
(ii) $\omega(gs) = g^{-1}\omega(s)$ \ ($s \in G^\#, \ g \in G$).
(iii) $\omega(st) \geq \omega(s)\omega(t)$ \ ($s, t \in G^\#$).
(iv) $\omega$ is the identity.
(v) For each net $i \mapsto s_i$ in $G^\#$, $\lim_i s_i = 0 \iff \lim_i \omega(s_i) = \infty$.

Proof. Let $\omega$ be any decreasing extension of $g \mapsto g^{-1}$. Then if $s \in G^\#$ and $g \in G$ is such that $1 \leq sg$, then $g^{-1} \leq s$ so $g \geq \omega(s)$. We see that $\omega(s) \leq \inf\{g \in G : 1 \leq sg\}$. In the same vein we have $\omega(s) \geq \text{sup}\{g \in G : sg \leq 1\}$. So, to prove existence, uniqueness and (i) it suffices to prove that $\sup A = \text{inf} B$ where $A = \{g \in G : sg \leq 1\}$, $B = \{g \in G : sg \geq 1\}$. This is clear if $s \in G$ (then $\sup A = \text{inf} B = s^{-1}$), so assume that $s \notin G$. Because $G$ is coinitial and cofinal in $G^\#$ the sets $A, B$ are nonempty. We have $A \cap B = \emptyset$, $A \cup B = G$, and $A$ is a cut. (In fact, suppose $h := \text{sup}_G A$ exists, but $h \notin A$. Then $sh > 1$ so $h^{-1} < s$. By 1.1.4 (iv) there exists a $b \in G$ with $h^{-1} < b < s$. We have $b \neq s$ so $b < s$ i.e. $1 < b^{-1}s$. But $b^{-1} < h$ so $b^{-1} \notin A$, a contradiction.) So $t := \text{sup}_B A$ exists and $t \notin B$, so $B = \{x \in G : x \geq t\}$. By 1.1.4 (v), $t = \text{inf}_{G^\#} B$. To prove (iii),
let \( V := \{ g \in G : sg \leq 1 \} \), \( W := \{ g \in G : tg \leq 1 \} \), \( X := \{ g \in G : stg \leq 1 \} \). Then \( VW \) \( V \subset X \), so \( \omega(s)\omega(t) = \sup VW \leq \sup X = \omega(st) \). From (iii) it follows that \( g^{-1}\omega(s) = \omega(g)\omega(s) \leq \omega(gs) \). But also \( \omega(gs) = \omega(g^{-1})\omega(gs) \leq \omega(s) \), so \( \omega(gs) \leq g^{-1}\omega(s) \) and (ii) is proved. To prove (iv), let \( s \in G^\# \). If \( g_1, g_2 \in G \), then \( g_1 \leq g_2 \) if \( g_1 \leq \omega^2(s) \leq g_2 \). It follows that \( s \leq \omega^2(s) \leq s \) i.e. \( \omega^2(s) = s \). Finally (v) follows from bijectivity and \( x < \omega(\epsilon) \iff \omega(x) > \epsilon \) for each \( \epsilon \in G^\# \).

We will call the map \( \omega : G^\# \to G^\# \) of 1.3.1 the antipode.

**Remark** It is easy to prove that, for a proper convex subgroup \( H \), \( \omega(\text{sup} H) = \text{inf} H \).

### 1.3.2 Example. The completion of \( \bigoplus \mathbb{Z} \)

Let \( G = \bigoplus_{i \in \mathbb{N}} G_i \) be the direct sum of the groups \( G_i \), \( i \in \mathbb{N} \), where \( G_i \) is the infinite cyclic group generated by \( g_i \). Hence if \( g \in G \), then \( g = (g_i^n)_{i \in \mathbb{N}} \), \( n_i \in \mathbb{Z} \), \( n_i \neq 0 \) only for a finite number of indexes \( i \). We order each \( G_i \) by \( g_i^1 < g_i^m \) if and only if \( n < m \); with the antilexicographical order, \( G \) becomes a linearly ordered group.

Below we shall describe \( G^\# \), the completion of \( G \) (as in 1.1.4 and its preamble), as the set obtained by adjoining to \( G \) all the symbols of the form \( (g_i^n)_{i \in \mathbb{N}} \), such that for a given \( m \in \mathbb{N}, m \geq 2, n_i = \infty \) for all \( i, m_i \neq 0 \) only for a finite number of indexes \( i \). Such a symbol will denote the supremum of the set \( \{(a_1, a_2, \ldots, a_{m-1}, g_{m_i}^m, \ldots, g_{r_i-1}^0) : a_i \in G_i \text{ for } i < m \} \). \( G^\# \) is ordered antilexicographically, taking into account that for all \( i, g_i^\infty > g_i^0 \) for any \( r \in \mathbb{Z} \).

The infimum of the set \( \{(a_1, a_2, \ldots, a_{m-1}, g_{m_i}^m, \ldots, g_{r_i-1}^0) : a_i \in G_i \text{ for } i < m \} \) corresponds to the symbol \( (g_i^\infty, g_i^0, \ldots, g_i^{m-1}, g_i^{m-1+n}, g_i^{m+1}, \ldots, g_i^{r_i}, 1, \ldots) \).

We will prove that the set of all proper convex subgroups of \( G \) is the set of all \( H_n, n \geq 0 \), where \( H_0 = \{1\} \), and \( H_k = \{(g_i^n)_{i \in \mathbb{N}} : n_i = 0 \text{ if } i > k \} \). We will denote by \( s_k \) the supremum of \( H_k \), and by \( t_k \) its infimum. Therefore \( s_k = (g_i^\infty, g_i^0, \ldots, g_i^{k-1}, 1, \ldots) \) and \( t_k = (g_i^\infty, g_i^0, \ldots, g_i^{k-1}, 1, \ldots) \). We will extend the multiplication of \( G \) to its completion, according to the formulas in 1.3. Then we have that for two elements \( g = (g_i^n)_{i \in \mathbb{N}} \), \( h = (h_i^n)_{i \in \mathbb{N}} \), where \( n_i, t_i \in \mathbb{Z} \cup \{\infty\} \), \( gh = (g_i^{n_i+t_i})_{i \in \mathbb{N}} \), with \( \infty + m = m + \infty = \infty + \infty = \infty \) for all \( m \in \mathbb{Z} \). This shows that any element \( w \) in the completion of \( G \) can be written as \( w = s_n w' \) for some \( n \in \mathbb{N} \) and some \( w' \in G \); the number \( n \) is completely determined by \( w \), but there are many possible choices for \( w' \). As to the second extension * of the multiplication in \( G \), we shall prove that \( g * h = gh \), \( h * g \), \( g * h' \), \( h * g' \). In that case \( g * h = (1, \ldots, 1, g_n+1, \ldots)gh \).

The construction of \( G^\# \). Let \( n_i \in \mathbb{Z} \) for \( i = m, m + 1, \ldots, r \), define \( A = \{(a_1, a_2, \ldots, a_{m-1}, g_{m_i}^m, \ldots, g_{r_i}^0, 1, \ldots) : a_i \in G_i \text{ for } i < m \} \) and \( C_A = \{g \in G : g \geq x \text{ for some } x \in A \} \). We will prove first that \( \text{sup} C_A \) does not exist in \( G \). In fact, if \( (g_i^n)_{i \in \mathbb{N}} \) was the supremum of that set, then we would have the following inequalities. \( (1, \ldots, 1, g_{m-1}^1, g_{m}^m, \ldots, g_{r_i}^0, 1, \ldots) \leq (g_i^n)_{i \in \mathbb{N}} \leq (1, \ldots, 1, g_{m-1}^1, g_{m}^{m+n}, g_{m+1}^{m+n}, \ldots, g_{r_i}^0, 1, \ldots) \), so \( t_i = n_i \) for \( i \geq m + 1 \). Therefore \( t_m = n_m \) or \( t_m = 1 + n_m \), but in the first case \((1, \ldots, 1, g_{m-1}^1, g_{m}^{m+n}, g_{m+1}^{m+n}, \ldots, g_{r_i}^0, 1, \ldots) \) is an element of \( C_A \) bigger than \( \text{sup} C_A \), and in the second case \((1, \ldots, 1, g_{m-1}^1, g_{m}^{m+n}, g_{m+1}^{m+n}, \ldots, g_{r_i}^0, 1, \ldots) \) is an upper bound of \( C_A \) that is smaller than \( \text{sup} C_A \). Therefore \( \text{sup} C_A \) does not exist.
in G. We shall denote the cut CA by the symbol \((g_{1}^{\infty}, g_{2}^{\infty}, \ldots, g_{m-1}^{\infty}, g_{m}^{n_{m}}, \ldots, g_{n_{r}}^{n_{r}}, 1, \ldots)\). Now let \(F\) be a cut of G such that \(\text{sup } F\) does not exist in G, we will show that \(F\) is equal to a cut \(C_{A}\) for some set \(A\) as described before. Let \(t\) be an upper bound of \(F\), pick some \(f \in F\) and let \(F' = \{x \in F : f \leq x < t\}\), it is clear that \(F = \{g \in G : g \leq x \text{ for some } x \in F'\}\). By the definition of \(F'\) there is an \(m \in N\) such that for all elements \(x = (g_{i}^{x_{i}})_{i \in N} \in F'\) we have that \(x_{i} = 0\) if \(i > m\), while for some \(x \in F'\), \(x_{m} \neq 0\). We examine the \(m\)'th coordinate of the elements of \(F'\), if that set has no maximum, then define \(A := \{(a_{1}, a_{2}, \ldots, a_{m-1}, a_{m}, 1, \ldots) : a_{i} \in G_{i} \text{ for } i < m + 1\}\) and by a direct argument we see that \(F\) is the cut \(C_{A}\), therefore \(F = (g_{1}^{\infty}, g_{2}^{\infty}, \ldots, g_{m}^{\infty}, 1, 1, \ldots)\). If the set has a maximum, say \(\xi_{m}\), we look at the set of the \((m - 1)\)th coordinates of the elements of \(F'\) whose \(m\)th coordinate is \(\xi_{m}\). If that set has no maximum, then \(F = (g_{1}^{\infty}, \ldots, g_{m-1}^{\infty}, \xi_{1}, \ldots, \xi_{m}, 1, 1, \ldots)\), but if \(\xi_{m-1}\) is a maximum, then we continue with the elements of \(F'\) of the form \((*, *, \ldots, *_{m}, \xi_{m}, \xi_{m-1}, 1, \ldots)\). It is not possible to find in such a way elements \(\xi_{1}, \xi_{2}, \ldots, \xi_{m}\) as indicated above, because in that case \((\xi_{1}, \xi_{2}, \ldots, \xi_{m}, 1, \ldots) \in G\) would be the supremum of \(F\), contrary to our hypothesis. Therefore there has to be an \(r \in N, r > 1\), such that \(\xi_{r}\) exists, but \(\xi_{r-1}\) does not exist. Then \(F = (g_{1}^{\infty}, \ldots, g_{r-1}^{\infty}, \xi_{r}, \ldots, \xi_{m}, 1, \ldots)\). Then \(G^{\#}\), the collection of all cuts of \(G\), is the union of the set of all cuts designed by the symbols of the form \((g_{m}^{n_{m}})_{i \in N}\), such that for a given \(m \in N\), \(m \geq 2\), \(n_{i} = \infty\) for all \(i < m\), \(n_{i} \in Z\) if \(i \geq m\), \(n_{i} \neq 0\) only for a finite number of indexes \(i\), and the set of all the cuts that have a supremum in \(G\). As in the preamble to 1.1.4 we identify those cuts with the elements of \(G\). The order in \(G^{\#}\) given by inclusion corresponds to the antilexicographic ordering, requiring that for all \(i\), \(g_{i}^{\infty} > g_{i}^{r}\) for any \(r \in Z\). By 1.1.4 (v), every element \(s \in G^{\#}\) is the supremum of \(\{x \in G : x \leq s\}\). Hence if \(A := \{(a_{1}, a_{2}, \ldots, a_{m-1}, g_{m}^{n_{m}}, \ldots, g_{n_{r}}^{n_{r}}, 1, \ldots) : a_{i} \in G_{i} \text{ for } i < m\}\) for some particular choice of \(n_{m}, n_{m+1}, \ldots, n_{r}\), then \(\sup A = (g_{1}^{\infty}, g_{2}^{\infty}, \ldots, g_{m-1}^{\infty}, g_{m}^{n_{m}}, \ldots, g_{n_{r}}^{n_{r}}, 1, \ldots)\).

**Lemma 1.3.3** Let \(A\) be as in the previous paragraph. Then the infimum of \(A\) is the element \(g = (g_{1}^{\infty}, g_{2}^{\infty}, \ldots, g_{m-1}^{\infty}, g_{m}^{n_{m}+1}, g_{m+1}^{n_{m}+1}, \ldots, g_{n_{r}}^{n_{r}}, 1, \ldots)\).

Proof. Clearly \(g \leq x\) for all \(x \in A\). Suppose there exists a \(b = (g_{i}^{b_{i}})_{i \in N}, b_{i} \in Z\cup\{\infty\}\) such that \(g < b\), we shall prove that there is an \(a \in A\) such that \(g < a \leq b\). Without loss of generality we can assume that \(g < b < (1, 1, 1, g_{m}^{n_{m}}, \ldots, g_{n_{r}}^{n_{r}}, 1, \ldots)\), so \(b_{i} = 0\) if \(i > r\), \(b_{i} = n_{i}\) for \(m < i \leq r\). Any element of the form \((*, *, \ldots, *, g_{m}^{n_{m}+1}, g_{m+1}^{n_{m}+1}, \ldots, g_{n_{r}}^{n_{r}}, 1, \ldots)\) is smaller than or equal to \(g\), therefore we must have \(b_{m} = n_{m}\). Then the second inequality implies that \(b_{m-1} \leq 0\). Then \(a = (1, 1, 1, g_{m-1}^{n_{m}+1}, g_{m}^{n_{m}}, \ldots, g_{n_{r}}^{n_{r}}, 1, \ldots)\) is an element in \(A\) that satisfies \(g < a \leq b\). Hence \(g = \inf A\).

**Convex subgroups of \(G\)** It is readily seen that the set of all proper convex subgroups of \(G\) is the set of all \(H_{n}, n \geq 0\), where \(H_{0} = \{1\}\) and \(H_{k} = \{(g_{i}^{n_{i}})_{i \in N} : n_{i} = 0\text{ if } i > k\}\). In what follows we denote by \(s_{n}\) the supremum of \(H_{n}\), and by \(t_{n}\) its infimum. It is clear now that \(s_{0} = t_{0} = 1\), and for \(k > 0\),

\[
s_{k} = (g_{1}^{\infty}, g_{2}^{\infty}, \ldots, g_{k}^{\infty}, 1, \ldots)\\
t_{k} = (g_{1}^{\infty}, g_{2}^{\infty}, \ldots, g_{k}^{\infty}, g_{k+1}^{1}, 1, \ldots)
\]

**Multiplication in \(G^{\#}\)** For \(s, t \in G^{\#}\) we have defined in 1.3 two products, both
of them extending the multiplication in $G$, in fact,

$$st = \sup\{xy : x, y \in G, x \leq s, y \leq t\} \quad \text{and} \quad s \ast t = \inf\{xy : x, y \in G, x \geq s, y \geq t\}.$$

We look now for a simple formula for these products in $G^\#$, the crucial point is the following fact.

**Lemma 1.3.4** Let $g \in G^\# \setminus G$ be the element $g = (g_i^n)_{i \in \mathbb{N}}$ with $n_i = \infty$ if and only if $i \leq r$. Then $g$ can be written as $g = s_r t = s_r \ast t$ for $t = (g_i^k)_{i \in \mathbb{N}}$ with $t_i = 0$ if $i \leq r$, $t_i = n_i$ if $i > r$.

**Proof.** Since $t \in G$ we have that $s_r t = \sup\{\langle s, t \rangle : s, t \in \mathbb{Z}, s \leq s_r, t \leq t\}$ and $s \ast t = \inf\{\langle s, t \rangle : s, t \in \mathbb{Z}, s \geq s_r, t \geq t\}$.

Then $s_r t = \sup\{(g_i^k)_{i \in \mathbb{N}} : k_i = n_i \text{ if } i > r\} = (g_1^\infty, g_2^\infty, \ldots, g_r^\infty, g_{r+1}^\infty, \ldots) = g$. On the other hand $s_r \ast t = \inf\{\langle s, t \rangle : s, t \in \mathbb{Z}, s \geq s_r, t \geq t\}$, but as $s_r = (g_1^\infty, g_2^\infty, \ldots, g_r^\infty, 1, \ldots)$, by Lemma 1.3.3 we see that $s_r = \inf B$ with $B = \{(b_1, \ldots, b_r, g_{r+1}, 1, \ldots) : b_i \in G_i \text{ for } i \leq r \}$. Hence

$$s_r t = \sup\{\langle s, t \rangle : s, t \in \mathbb{Z}, s \leq s_r, t \leq t\} = \sup\{(g_i^k)_{i \in \mathbb{N}} : k_i = 1 + n_{r+1}, k_i = n_i \text{ if } i > r + 1\} \quad (**)$$

**Lemma 1.3.5** If $t \in H_r$, then $s_r t = s_r \ast t = s_r$.

**Proof.** By (*) we have $s_r t = \sup\{\langle s, t \rangle : s, t \in \mathbb{Z}, s \leq s_r, t \leq t\} = \sup H_r$, and by (**) $s_r \ast t = \inf\{\langle s, t \rangle : s, t \in \mathbb{Z}, s \geq s_r, t \geq t\} = \inf\{(g_i^k)_{i \in \mathbb{N}} : k_i = 1, k_i = 0 \text{ if } i > r + 1\} = s_r$.

**Lemma 1.3.6** Let $n, r \in \mathbb{N}$. If $r < n$ then $s_r s_n = s_r \ast s_n = s_n$. If $r = n$ then $s_r s_n = s_n$ but $s_n \ast s_n = \delta_{n+1} s_n$ with $\delta_{n+1} = (1, \ldots, 1, g_{n+1}, 1, \ldots)$.

**Proof.** i) $s_r s_n = \sup\{\langle x, y \rangle : x, y \in G, x \leq s_r, y \leq s_n\} = \sup\{\langle x, y \rangle : x \in H_r, y \in H_n\}$. If $r \leq n$ then $H_r \subseteq H_n$ and $xy \in H_n$. Therefore the set above is contained in $H_n$, and since it clearly contains $H_n$, we have that $s_r s_n = s_n s_n$. If $r < n$ then $s_r s_n = \inf\{\langle x, y \rangle : x, y \in G, x \geq s_r, y \geq s_n\}$. As in (**) in the proof of Lemma 1.3.4, $s_r \ast s_n = \inf\{\langle x, y \rangle : x, y \in G, x \leq s_r, y \leq s_n\}$, where $B_r = \{(b_1, \ldots, b_r, g_{r+1}, 1, \ldots) : b_i \in G_i \text{ for } i \leq r \}$ and $B_n = \{c_1, \ldots, c_i, g_{n+1}, 1, \ldots : c_i \in G_i \text{ for } i \leq n\}$. If $r < n$ then $s_r \ast s_n = \inf\{\langle d_1, \ldots, d_n, g_{n+1}, 1, \ldots \rangle : d_i \in G_i \text{ for } i \leq n\} = s_n$, but if $r = n$, then $s_r \ast s_n = \inf\{\langle d_1, \ldots, d_n, g_{n+1}, 1, \ldots \rangle : d_i \in G_i \text{ for } i \leq n\} = (g_1^\infty, g_2^\infty, \ldots, g_n^\infty, g_{n+1}, 1, \ldots) = \delta_{n+1} s_n$.

**Corollary 1.3.7** Let $s, t \in G^\#$. Then $s t = s \ast t$ for all cases, except when $s = s_n g$ and $t = s_n h$ for some $g, h \in G$. In that case $s \ast t = \delta_{n+1} s t$ with $\delta_{n+1} = (1, \ldots, 1, g_{n+1}, 1, \ldots)$. 

Finally, from lemmas 1.3.4, 1.3.5, 1.3.6 we obtain

**COROLLARY 1.3.8** Let \( g, h \in G^\# \), \( g = (g_n)_{n \in \mathbb{N}}, h = (g_n^k)_{n \in \mathbb{N}}, n, k \in \mathbb{Z} \cup \{\infty\} \).

Then \( gh = (g_n^{n+k})_{n \in \mathbb{N}} \) with \( \infty + m = \infty = \infty = \infty \) for all \( m \in \mathbb{Z} \).

We will leave the description of the antipode in \( G^\# \) (see 1.3.1) to the reader.

### 1.4 Valued fields

Let \( G \) be a linearly ordered group. Like in 1.1.2 we add an element 0 to \( G \), extend the ordering and the multiplication by declaring that \( 0 < g \) and \( 0.g = 0 = 0 \) for all \( g \in G \). A **valuation** on a field \( K \) (with value group \( G \)) is a surjective map \( | | : K \to G \cup \{0\} \) such that for all \( x, y \in K \)

(i) \( |x| = 0 \) if and only if \( x = 0 \)

(ii) \( |x + y| \leq \max(|x|, |y|) \)

(iii) \( |xy| = |x||y| \).

**REMARK** In this paper we prefer the multiplicative notation over the more commonly used additive one, to link up with the conventions in classical Functional Analysis.

The **rank** of the valued field \( K = (K, | |) \) is the rank of \( G \). We shall exclude the trivial valuation i.e. we assume \( G \neq \{1\} \). The map \( (x, y) \mapsto |x - y| \) \((x, y \in K)\) is a scale in the sense of 1.2 and its topology is a non-discrete field topology. The **valuation ring** \( B_K := \{\lambda \in K : |\lambda| \leq 1\} \) has a unique maximal ideal \( B_k^* := \{\lambda \in K : |\lambda| < 1\} \). The residue class field of \( K \) is \( k := K/B_k^* \). The following theorem concerns metrizability of \( K \).

**THEOREM 1.4.1** Let \( (K, | |) \) be a valued field. The following are equivalent.

(\( \alpha \)) \((K, | |) \) is (ultra) metrizable.

(\( \beta \)) \( G \) has a coinitial (cofinal) sequence.

(\( \gamma \)) \( K^\times := K \setminus \{0\} \) contains a countable set \( C \) for which \( 0 \in \overline{C} \).

(\( \delta \)) \( K \) contains a countable subset that is not closed.

Proof. \( (\alpha) \iff (\beta) \) follows from 1.2.4. The implications \( (\beta) \Rightarrow (\gamma) \Rightarrow (\delta) \) are trivial. To prove \( (\delta) \Rightarrow (\beta) \), let \( \{\alpha_1, \alpha_2, \ldots\} \subset K \) be not closed, let \( \alpha \) be in the closure, \( \alpha \neq \alpha_n \) for each \( n \). Then \( n \mapsto \min(\{\alpha - \alpha_1, |\alpha - \alpha_2|, \ldots, |\alpha - \alpha_n|\}) \) is a sequence in \( G \) tending to 0 i.e. is coinitial.

**DEFINITION 1.4.2** Let \( E \) be a vector space over a valued field \( K \). A subset \( A \) of \( E \) is **absolutely convex** if it is a \( B_k^* \)-submodule of \( E \), in other words, if \( 0 \in A \) and \( x, y \in A \), \( \lambda, \mu \in B_k^* \) implies \( \lambda x + \mu y \in A \). A subset \( S \) of \( E \) is called **convex** if \( x, y, z \in S \), \( \lambda, \mu, \nu \in B_k^* \), \( \lambda + \mu + \nu = 1 \) implies \( \lambda x + \mu y + \nu z \in S \).

It is easy to see that a nonempty \( S \subset E \) is convex if and only if it is an additive coset of an absolutely convex set. The following Proposition describes the absolutely convex subsets of \( K \).
PROPOSITION 1.4.3 ([19], 20.6, (5)) Let $K$ be a valued field with value group $G$. The sets $\{0\}, K, B(0,r^-) := \{\lambda \in K : |\lambda| < r\}, B(0,r) := \{\lambda \in K : |\lambda| \leq r\}$ $(r \in G^\#)$ are absolutely convex. Each absolutely convex subset of $K$ is of one of these forms.

Proof. We only prove the second statement. Let $A \subset K$ be absolutely convex. If $A$ is unbounded (i.e. if $\{\mu : \mu \in A\}$ is not bounded above), let $\lambda \in K$. Then there is a $\mu \in A$, $|\mu| > |\lambda|$. Then $\lambda = (\lambda \cdot \mu^{-1}) \mu \in A$ and it follows that $A = K$. Now suppose that $A$ is bounded above and contains at least one nonzero element. Then $r := \sup_G \{|\lambda| : \lambda \in A\}$ exists. Clearly, if $\lambda \in A$, $|\lambda| < r$ then there is a $\mu \in A$, $|\mu| > |\lambda|$, so $\lambda = (\lambda \mu^{-1}) \mu \in A$. It follows that $B(0,r^-) \subset A \subset B(0,r)$. If the first inclusion is strict there is a $\mu \in A$, $|\mu| = r$. If $\lambda \in K$, $|\lambda| \leq r$ then $\lambda = (\lambda \mu^{-1}) \mu \in A$, so $A = B(0,r)$.

In the main part of this paper (Sections 3 and 4) we shall have to put a restriction upon $K$ namely that each absolutely convex subset of $K$ is countably generated as a $B_K$-module. In the following Proposition we describe the situation. For a linearly ordered set $X$ the interval topology is defined to be the topology generated by the sets $\{x \in X : x > s\}$ and $\{x \in X : x < t\}$ $(s,t \in X)$.

PROPOSITION 1.4.4 Let $K$ be a valued field with value group $G$. The following are equivalent.

(a) Each absolutely convex subset of $K$ is countably generated as a $B_K$-module.
(b) $G$ has a cofinal sequence. For each $s \in G^\#$ there are $g_1, g_2, \ldots \in G$, $g_n < s$ for all $n$, such that $\sup_G \{t \in G^\# : t < s\} = \sup_G \{g_1, g_2, \ldots\}$.
(c) $G$ has a coinitial sequence. For each $s \in G^\#$ there exist $g_1, g_2, \ldots \in G$, $g_n > s$ for all $n$ such that $\inf_G \{t \in G^\# : t > s\} = \inf_G \{g_1, g_2, \ldots\}$.
(d) The interval topology on $G^\#$ satisfies the first axiom of countability. $G^\#$ has a cofinal sequence.

Proof. $(\alpha) \Rightarrow (\beta)$. Let $K$ be generated as a $B_K$-module by $\alpha_1, \alpha_2, \ldots$ which we may suppose to be non-zero. We claim that $|\alpha_1|, |\alpha_2|, \ldots$ is cofinal in $G$. In fact, let $\lambda \in K$. Then there are $n \in \mathbb{N}$, $\xi_1, \ldots, \xi_n \in B_K$ such that $\lambda = \sum_{i=1}^n \xi_i \alpha_i$. Then $|\lambda| \leq \max_{1 \leq i \leq n} |\xi_i \alpha_i| \leq \max_{1 \leq i \leq n} |\alpha_i|$. Now let $s \in G^\#$. By Lemma 1.1.1 either $s_0 := \max\{t \in G^\# : t < s\}$ exists (then by Proposition 1.1.4 (iv) $s_0, s \in G$ and we can choose $g_n = s_0$ for each $n$), or $\sup\{t \in G^\# : t < s\} = s$. Let $\alpha_1, \alpha_2, \ldots \in K^\times$ generate $B(0,s^-)$. Then $|\alpha_n| < s$ for each $n$. To prove that $\sup_n |\alpha_n| = s$, let $t \in G^\#$, $t < s$. By 1.1.4 (iv) there is a $\lambda \in K$, $s > |\lambda| > t$. There are $n \in \mathbb{N}$ and $\xi_1, \ldots, \xi_n \in B_K$ such that $\lambda = \sum_{i=1}^n \xi_i \alpha_i$. Then $|\lambda| \leq \max_{1 \leq i \leq n} |\alpha_i|$. Hence, $\sup_n |\alpha_n| > 0$ for each $t \in G^\#$, $t < s$, so $\sup_n |\alpha_n| = s$. The implications $(\beta) \iff (\gamma)$ can be proved by applying the antipode $\omega$ of Proposition 1.3.1. We now prove $(\beta) \& (\gamma) \Rightarrow (\delta)$. Let $s \in G^\#$. If $s_0 = \max\{t \in G^\# : t < s\}$ exists then $s_0, s \in G$ and $s_1 = \min\{t \in G^\# : t > s\}$ exists and $\{s\}$ is an open set, so trivially there exists a countable neighbourhood base at $s$. Now suppose $\sup\{t \in G^\# : t < s\} = s = \inf\{t \in G^\# : t > s\}$. Let $g_1, g_2, \ldots \in G$, $g_n < s$ for each $n$, $\sup\{g_1, g_2, \ldots\} = \sup\{t \in G^\#, t < s\}$. We may suppose $g_1 < g_2 < \ldots$. Let $h_1, h_2, \ldots \in G$, $h_n > s$ for each $n$, $h_1 > h_2 > \ldots$, $\inf\{h_1, h_2, \ldots\} = \inf\{t \in G^\# : t > s\}$. Then for each $n \in \mathbb{N}$, $U_n := \{x \in G^\# : g_n < x < h_n\}$ is an open neighbourhood
of $s$ in the interval topology; we prove the $U_n$ to be a neighbourhood base. So let $U$
be open in the interval topology, $s \in U$. As the sets $(a,b) := \{x \in G^\# : a < x < b\}$
$(a,b \in G^\#)$ from a base for the interval topology of $G^\#$, we may suppose $U = (a,b)$.
Then $a < s < b$. There is an $n$ such that $a < g_n$ and $b > h_n$. We see that $U_n \subset (a,b)$ and we are done. Finally we prove $(\delta) \Rightarrow (\alpha)$. Let $s_1, s_2, \ldots$ be a cofinal sequence in $G^\#$. By cofinality of $G$ in $G^\#$ we may suppose that $s_n \in G$
for each $n$. Choose $\lambda_n \in K$ such that $|\lambda_n| = s_n$ ($n \in \mathbb{N}$). It is easy to see that $K$ is generated by \{$\lambda_1, \lambda_2, \ldots$\} as a $B_K$-module. Obviously any set of the form $B(0, r)$ where $r \in G$ is generated by a single element, so to finish the proof we show that $B(0, s^-)$ is countably generated where $s \in G^\#$, $\sup\{t \in G : t < s\} = s$. Let $U_1 \supset U_2 \supset \cdots$ be a countable base of the interval topology at $s$, we may suppose that $U_n = \{t \in G^\# : a_n < t < b_n\}$ for some $a_n, b_n \in G^\#$. By assumption and 1.1.4 (iv) we may assume $a_n, b_n \in G$. Choose, for each $n$, a $\lambda_n \in K$ with $|\lambda_n| = a_n$. One proves easily that $B(0, s^-)$ is generated by \{$\lambda_1, \lambda_2, \ldots$\}.

REMARKS (i) If $K$ is separable or, more generally, if $G$ is countable we obviously have $(\alpha) - (\delta)$ of above. By 1.4.1 $(\alpha) - (\delta)$ imply ultrametrizability of $K$. Statement $(\delta)$ implies the first axiom of countability for $G$, and hence, since $G$ is a group, metrizability of $G$ (See [8], Problem O, p. 210). But we do not know if $(\delta)$ implies that $G^\#$ is metrizable.

(ii) It is not hard to see that $(\alpha) - (\delta)$ are equivalent to: each subset $A$ of $G^\#$ that is bounded above has a countable subset $S$ such that $\sup_{G^\#} A = \sup_{G^\#} S$. This property is known in Riesz space theory as 'super Dedekind completeness'.

1.5 $G$-modules

The $G$-modules we introduce below will serve as a natural range set for norms on $K$-vector spaces, see 2.1 and 2.2.

DEFINITION 1.5.1 Let $G$ be a linearly ordered group. A linearly ordered set $X$
is called a $G$-module if there exists a map $G \times X \rightarrow X$, written $(g,x) \mapsto gx$, called
multiplication, such that for all $g, g_1, g_2, \in G$ and all $x, x_1, x_2 \in X$ we have

(i) $g_1(g_2x) = (g_1g_2)x$

(ii) $1x = x$

(iii) $g_1 \geq g_2 \Rightarrow g_1x \geq g_2x$

(iv) $x_1 \geq x_2 \Rightarrow gx_1 \geq gx_2$

(v) $Gx$ is cofinal in $X$

(vi) $X$ has no smallest element.

Thus, the requirements (i) - (iv) mean that $G$ acts on $X$ and that this action
preserves the ordering $\leq$ in $G$ and $X$. The pair $(v) \& (vi)$ is equivalent to "for each $e \in X$ there is a $g \in G$ such that $gx < e$". It follows that modules over the group
$\{1\}$ do not exist. If $X$ is a $G$-module we have for all $x_1, x_2 \in X$, $g \in G$

(iv)' $x_1 > x_2 \Rightarrow gx_1 > gx_2$

(otherwise $gx_1 \leq gx_2$ hence $x_1 = g^{-1}gx_1 \leq g^{-1}gx_2 = x_2$ by (iv), a contradiction),
but the formula $g_1 > g_2 \Rightarrow g_1x > g_2x$ does not hold in general. In fact, the semigroup $G^\#$ is, a fortiori, a $G$-module; if $H$ is a proper convex subgroup $\neq \{1\}$ then $h \sup H = \sup H$ for all $h \in H$. Let $X$ be a $G$-module. Then for each $x \in X$ the set $Gx$ is cofinal in $X$. In fact, let $x, y \in X$. We just saw that there is a $g \in G$ with $gx < y$. Then $x < g^{-1}y$ by (iv)'. This proof also shows that $X$ has no largest element.

For an element $s$ of a $G$-module $X$, let $\text{Stab}(s) := \{g \in G : gs = s\}$. It is a proper convex subgroup of $G$. If $\text{Stab}(s) = \{1\}$ the element $s$ is called 
faithful.

Let $X, Y$ be $G$-modules. A map $\phi : X \to Y$ is called a 
$G$-module map if $\phi$ is increasing and if $\phi(gs) = g\phi(s)$ for all $g \in G, s \in X$. Its extended map $X \cup \{0\} \to Y \cup \{0\}$ is called an extended $G$-module map.

**Proposition 1.5.2** Let $G$ be a linearly ordered group.
(i) (Extended) $G$-module maps are bicontinuous at $0$.
(ii) Let $X$ be a $G$-module. Then for a net $t \to g_t$ in $G$ we have
$$\lim_t g_t s = 0 \iff \lim_t g_t = 0 \quad (s \in X),$$
and for a net $t \to s_t$ in $X$ we have
$$\lim_t g s_t = 0 \iff \lim_t s_t = 0 \quad (g \in G).$$

Proof. (i) Follows from Lemma 1.1.3 and 1.5.1 (v). Statement (ii) follows from (i) and the fact that $g \mapsto gs$ and $s \mapsto gs$ are $G$-module maps $G \to X, X \to X$ respectively.

**Proposition 1.5.3** Let $G$ be a linearly ordered group, let $X$ be a $G$-module.
(i) Let $V \subset X, g \in G$. If $\sup V$ exists then $g\sup V = \sup gV$. If $\inf V$ exists then $g\inf V = \inf gV$. If $V$ is not bounded above (below) then neither is $gV$.
(ii) Let $W \subset G, s \in X$. If $\sup_G W$ and $\sup_X Ws$ exist then $\sup Ws \leq (\sup W)s$.
If $\inf_G W$ and $\inf_X Ws$ exist then $\inf Ws \geq (\inf W)s$. If $W$ is not bounded above (below) then neither is $Ws$, and conversely.

Proof. (i) Let $s := \sup V$. Then $gs$ is an upper bound of $gV$. If $t \in X, t < gs$ then $g^{-1}t < s$, so there is a $v \in V$ with $g^{-1}t < v$ i.e. $t < gv$. We see that $t$ is not an upper bound of $gV$. The proof of the second statement is similar. If $s$ were an upper (lower) bound of $gV$ then $g^{-1}s$ would be an upper (lower) bound of $V$, which finishes the proof of (i). (ii) The first two statements are obvious. Let $W$ be not bounded above. Let $t \in X$. Since $Gs$ is cofinal in $X$ there is a $g \in G$ with $gs > t$. By unboundedness there is a $w \in W$ with $w > g$. Then $ws \geq gs > t$. Conversely, suppose $Ws$ is not bounded above. Let $g \in G$. There is a $w \in W$ such that $ws > gs$. Then $w > g$, so $W$ is not bounded above. The proof for the ‘inf’ case runs similarly.
REMARGE (1) To express the fact that some subset $V$ of a $G$-module $X$ is not bounded below we sometimes write $\inf V = 0$ (this can be interpreted as the infimum taken in the linearly ordered set $X \cup \{0\}$).

(2) For an example in which the inequalities in (ii) above are strict, see 1.5.5 (c).

We now turn to the completion of $G$-modules and show that, unlike the (semi)group structure on $G$ (see 1.3 Remark) the $G$-module structure on a set $X$ (in particular, on $G$) can uniquely be extended to its completion.

**THEOREM 1.5.4** Let $G$ be a linearly ordered group, let $X$ be a $G$-module. Then the multiplication $G \times X \to X$ can uniquely be extended to a multiplication $G \times X^* \to X^*$ making $X^*$ into a $G$-module. (In particular, $X^*$ has no smallest or largest elements.)

Proof. Let $g \in G$, $s \in X^* \setminus X$. Then $A := \{gx : x \in X, x \leq s\}$ is a cut and $B := \{gy : y \in X : y \geq s\}$ is its complement in $X$. Clearly $\sup X^* A \not\subset X$, hence $\sup X^* A = \inf X^* B$ and we are forced to define

$$gs = \sup X^* A.$$ 

Straightforward verification shows that with respect to this extended multiplication $X^*$ is a $G$-module.

**EXAMPLES 1.5.5**

(a) For every subgroup $G$ of $X := (0, \infty) \subset \mathbb{R}$, $X$ is in a natural way a (complete) $G$-module. Every element of $X$ is faithful.

(b) If $G$ has rank $> 1$ there are always $G$-modules having non-faithful elements. In fact, let $H$ be a convex subgroup $\neq \{1\}, \neq G$. Then $G/H$ is in a natural way a $G$-module and $h \cdot 1 = 1$ for each $h \in H$ so 1 (the unit element of $G/H$) is not faithful. To construct such $G$, let for each $n \in \mathbb{N}$, $A_n$ be a subgroup of the multiplicative group $(0, \infty), \neq \{1\}$ and take $G := \bigoplus_{n \in \mathbb{N}} A_n$, with the antilexicographic ordering. For each $n$, $A_1 \oplus \cdots \oplus A_n$ is a convex subgroup.

(c) We now construct a $G$-module $X$ for which $\inf W s > (\inf W) s$ for some $W \subset G$, $s \in X$ (see Proposition 1.5.3 (ii)). Let $G$ be such that $1 = \inf \{g \in G : g > 1\}$, e.g. $G = (0, \infty)$. Let $G^- := \{g^- : g \in G\}$ be a copy of $G$, let $X := G \cup G^-$ be ordered by stating that

$$t < s^- < s$$

for all $s, t \in G$, $t < s$. (Thus, every $s \in G$ is given an immediate predecessor). $X$ becomes a $G$-module by extending the multiplication by

$$gs^- := (gs)^- \quad (g \in G, s \in G).$$

Now take $W := \{g \in G : g > 1\}, s := 1^-$. Then $\inf W = 1$, so $(\inf W) s = 1^-$. However $\inf W s = \inf \{g^- : g > 1\} = 1$.

(d) In the sequel we will encounter the following situation. $G = \{|x| : x \in K, x \neq 0\}$, where $K$ is some valued field, $\Gamma$ is a linearly ordered group containing $G$ as a cofinal (hence coinitial) ordered subgroup, $X := \Gamma^#$. We will consider $X$ sometimes as a $G$-module, sometimes as a $\Gamma$-module. Although often $G = \Gamma$, 

...
there are some cases where \( \Gamma \) contains \( G \) properly. A natural example in rank 1 case is given by \( G := \{ |x| : x \in \mathbb{Q}_p \} \), \( \Gamma := \{ |x| : x \in \mathbb{C}_p \} \). (Then \( \Gamma^\# = (0, \infty) \).) In 1.6.3, 1.6.8, 4.4 we will meet the following example having infinite rank. For each \( n \), let \( A_n \) be the free cyclic group generated by \( a_n \) (then \( A_n \cong \mathbb{Z} \) for each \( n \)) with the usual ordering, let \( \sqrt{A_n} \) be the free group generated by, say, \( b_n \), where \( b_n^2 = a_n \). Let \( G := \bigoplus_{n \in \mathbb{N}} A_n \), \( \Gamma := \bigoplus_{n \in \mathbb{N}} \sqrt{A_n} \) with the antilexicographic ordering.

(e) A \( G \)-module is called cyclic if it has the form \( Gs \) for some element \( s \). An arbitrary \( G \)-module is the disjoint union of its cyclic submodules. Conversely, if we are given a collection \( \{Gs_i : i \in I \} \) of cyclic \( G \)-modules, one can form the (disjoint) union \( X := \bigcup_{i \in I} Gs_i \). We can extend the ordering on the subsets \( Gs_i \) to a linear ordering on \( X \) such that \( X \) becomes a \( G \)-module, for example, by putting a linear ordering on \( I \) and by declaring that \( g s_i > g' s_j \) if either \( g > g' \) or \( g = g' \) and \( i > j \).

(f) Let \( X \) be a \( G \)-module, let \( H \subset G \) be a proper convex subgroup, let \( \pi : G \to G/H \) be the natural map. For \( s, t \in X \) define

\[
  s \sim t \quad \text{if} \quad s \in \text{conv}_X(\pi H t)
\]

where, for \( Z \subset X \), \( \text{conv}_X(Z) \) the \( X \)-convex hull of \( Z \), is the set

\[
  \{ x \in X : \text{there are } z_1, z_2 \in Z \text{ with } z_1 \leq x \leq z_2 \}.
\]

Then \( \sim \) is an equivalence relation on \( X \). Let \( \rho : X \to X/\sim \) be the natural map. The requirement

\[ 'v \leq w \text{ if there exist } s, t \in X \text{ with } s \leq t, \rho(s) = v, \rho(t) = w' \]

defines a linear ordering on \( X/\sim \) for which \( \rho \) is increasing and the formula

\[
  \pi(g)\rho(s) = \rho(g s) \quad (g \in G, s \in X)
\]

defines a multiplication \( G/H \times X/\sim \to X/\sim \) making \( X/\sim \) into a \( G/H \)-module.

The proof consists of straightforward verification.

The following observation will be needed in 2.1.9.

**THEOREM 1.5.6** Let \( G \) be a linearly ordered group, let \( X \) be a \( G \)-module. Then there exists a \( G \)-module map \( X \to G^\# \).

**Proof.** Choose any \( s_0 \in X \) and set

\[
  \phi(s) = \inf_{G^\#} \{ g \in G : g s_0 \geq s \} \quad (s \in X).
\]

(The definition makes sense as \( G s_0 \) is coinitial and cofinal in \( X \) so \( \{ g \in G : g s_0 \geq s \} \) is bounded below and non-empty). Obviously \( \phi \) is increasing. By 1.5.3(i) we have for \( g \in G \) that \( g^{-1} \phi(g s) = g^{-1} \inf \{ h \in G : h s_0 \geq g s \} = g^{-1} \inf \{ h \in G : g^{-1} h s_0 \geq s \} = \inf \{ g^{-1} h \in G : g^{-1} h s_0 \geq s \} = \phi(s) \).

**REMARK** The formula \( \phi(s) = \sup_{G^\#} \{ g \in G : g s_0 \leq s \} \) would also have proved our theorem, likewise would \( \phi(s) = \inf_{G^\#} \{ g \in G : g s_0 > s \} \) and \( \phi(s) = \sup_{G^\#} \{ g \in G : g s_0 < s \} \).
1.6 The type condition

Let \( G \) be a linearly ordered group, let \( X \) be a \( G \)-module. The \textit{algebraic type} of an element \( s \in X \) is the set \( Gs \), or, equivalently, the element \( \pi(s) \), where \( \pi : X \to X/\sim \) is the canonical surjection and the equivalence relation \( \sim \) is defined by \( x \sim y \) if \( x \in Gy \). Now choose \( s_0 \in X \). (We may view \( s_0 \) as some sort of unit. If \( G \subseteq X \) it is natural to put \( s_0 := 1 \).) The following constructions depend on the choice of \( s_0 \). For each \( s \in X \) the set \( Gs \) is cofinal and coinitial so there are elements in \( Gs \) that are smaller than \( s_0 \) but also ones that are greater than \( s_0 \) and hence the definitions

\[
\tau_\ell(s) = \sup_{x \in \#} \{ x \in Gs : x \leq s_0 \}
\]

\[
\tau_u(s) = \inf_{x \in \#} \{ x \in Gs : x \geq s_0 \}
\]

make sense. It follows directly that \( \tau_\ell(s) \leq \tau_u(s) \) and that \( \tau_\ell(s) \) and \( \tau_u(s) \) depend only on the algebraic type of \( s \). It may happen that \( \tau_\ell(s) < \tau_u(s) \). (In fact, let \( H \) be a proper convex subgroup of \( G \), \( H \neq \{1\} \), \( X := G^\#, s_0 := 1 \), \( s := \sup_X H \), \( t := \inf X H \). Then \( \tau_\ell(s) = t, \tau_u(s) = s \).) If \( s_0 \in Gs \) then \( \tau_\ell(s) = \tau_u(s) = s_0 \).

**DEFINITION 1.6.1** Let \( G, X, s_0 \) be as above. The \textit{topological type} of an element \( s \in X \) is the set \( \tau(s) := \{ h \in G : \tau_\ell(s) \leq hs_0 \leq \tau_u(s) \} \).

The following theorem shows that this definition ties in with the one given in [3], see Example 1.6.3. For the definition of \( \text{conv}_X \) see 1.5.5 (f).

**THEOREM 1.6.2** The topological type \( \tau(s) \) of an element \( s \) of a \( G \)-module \( X \) is a proper convex subgroup of \( G \). If \( s_0 \in Gs \) then \( \tau(s) = \{ h \in G : hs_0 = s_0 \} \), if \( s_0 \not\in Gs \) then \( \tau(s) \) is the largest among the convex subgroups \( H \) of \( G \) for which \( \text{conv}_X (Hs_0) \cap Gs = \emptyset \).

**Proof.** To show the first statement we may suppose \( s_0 \not\in Gs \). The convexity is clear, as is properness, so it remains to prove that \( \tau(s) \) is a group. Clearly \( 1 \in \tau(s) \). Now let \( h_1, h_2 \in \tau(s) \). Let \( g \in G \) be such that \( gs \geq s_0 \). From \( h_2 \in \tau(s) \) it follows that \( gs \geq h_2s_0 \) i.e. \( h_2^{-1}gs \geq s_0 \). This, combined with \( h_1 \in \tau(s) \) yields \( h_1^{-1}gs \geq h_1s_0 \) i.e. \( gs \geq h_1h_2s_0 \). This result holds for all \( g \in G \) for which \( gs \geq s_0 \) i.e. \( h_1h_2s_0 \leq \tau_u(s) \). Similarly one proves \( h_1h_2s_0 \geq \tau_\ell(s) \) and it follows that \( h_1h_2 \in \tau(s) \). Now let \( h \in \tau(s) \). To prove \( h^{-1}s_0 \leq \tau_u(s) \), let again \( g \in G \) be such that \( gs \geq s_0 \). If \( h^{-1}s_0 > gs \) then \( s_0 > hgs \) and, since \( h \in \tau(s) \), \( h \in Gs \) or \( s_0 \geq gs \) i.e. \( s_0 \not\in Gs \), contradiction. Hence, \( h^{-1}s_0 \leq gs \) for all \( g \in G \) with \( gs \geq s_0 \) i.e. \( h^{-1}s_0 \leq \tau_u(s) \). Similarly one proves that \( h^{-1}s_0 \geq \tau_\ell(s) \) and we are done. To prove the second statement we may assume \( s_0 \not\in Gs \). Let \( H := \tau(s) \). Clearly \( Hs_0 \cap Gs = \emptyset \), so let \( t \in \text{conv}_X (Hs_0) \setminus Hs_0 \). There exist \( h_1, h_2 \in H \) with \( h_1s_0 < t < h_2s_0 \), so \( \tau_\ell(s) < t < \tau_u(s) \) implying \( t \not\in Gs \) and we have proved \( \text{conv}_X (Hs_0) \cap Gs = \emptyset \). Conversely, let \( H \) be a convex subgroup of \( G \) such that \( \text{conv}_X (Hs_0) \cap Gs = \emptyset \); we must prove \( H \subseteq \tau(s) \). Now \( \text{conv}_X (Hs_0) \) contains \( s_0 \) and does not meet \( Gs \). Hence by convexity it is contained in \( \{ t \in X : \tau_\ell(t) \leq t \leq \tau_u(t) \} \), implying \( H \subseteq \tau(s) \).
EXAMPLE 1.6.3 Like in 1.5.5(d), let $G := \bigoplus_{n \in \mathbb{N}} A_n$, $\Gamma := \bigoplus_{n \in \mathbb{N}} \sqrt{A_n}$. Choose $X := \Gamma$, $s_0 := 1$. The definition of the topological type of an element $s \in \Gamma \backslash G$, given in [3], Def. 31 is the largest convex subgroup of $\Gamma$ that does not meet $G$s. According to 1.6.2, however, the topological type $\tau(s)$ is the largest convex subgroup $H$ of $G$ for which $\text{conv}_\Gamma H$ does not meet $G$s. The difference between these definitions is quite immaterial as there exists a 1-1 correspondence between the convex subgroups $H$ of $G$ and the convex subgroups $S$ of $\Gamma$ given by

$$H \leftrightarrow \text{conv}_\Gamma H$$

$$S \cap G \leftrightarrow S.$$  

The verification is immediate.

Now we shall define the type condition for $G$-modules, thereby extending the definition given in [3], Def. 21, and prove a connection with the notion of type of 1.6.1. We will need all this in Section 4.

DEFINITION 1.6.4 Let $G$ be a linearly ordered group, let $X$ be a $G$-module and let $s_1, s_2, \ldots$ be a sequence in $X$.

(i) We say that $s_1, s_2, \ldots$ satisfies the type condition if, for any sequence $g_1, g_2, \ldots$ in $G$, boundedness above of $\{g_1 s_1, g_2 s_2, \ldots\}$ implies $\lim_n g_n s_n = 0$.

(ii) Let $s_0, \tau$ be as in 1.6.1. We say that $\lim_n \tau(s_n) = \infty$ if for each proper convex subgroup $H$ of $G$ we have $\tau(s_n) \not < H$ for large $n$.

LEMMA 1.6.5 Let $G, X, s_0, \tau$ be as above. Let $s_1, s_2, \ldots$ be a sequence in $X$ satisfying the type condition. Then

(i) each subsequence of $s_1, s_2, \ldots$ satisfies the type condition;

(ii) if $g_1, g_2, \ldots \in G$ are such that $\{g_1 s_1, g_2 s_2, \ldots\}$ is bounded below then $\lim_n g_n s_n = 0$.

Proof. (i) Let $s_{n_1}, s_{n_2}, \ldots$ be a subsequence of $s_1, s_2, \ldots$. Let $g_n, g_{n_1}, \ldots \in G$ be such that $\{g_n s_{n_1}, g_n s_{n_2}, \ldots\}$ is bounded above, say by $t \in X$. For $j \in \mathbb{N}$, $j \not \in \{n_1, n_2, \ldots\}$ we can choose, by coinitiality of $G$, a $g_j \in G$ such that $g_j s_j < t$. Then $\{g_1 s_1, g_2 s_2, \ldots\}$ is bounded above and by assumption $\lim_n g_n s_n = 0$, so certainly $\lim_n g_{n_j} s_{n_j} = 0$. (ii) Let $\varepsilon \in X$ be such that $g_n s_n \geq \varepsilon$ for each $n$. Suppose not $\lim_n g_n s_n = \infty$. Then there are $n_1 < n_2 < \cdots$ in $\mathbb{N}$ such that $\{g_n s_{n_1}, g_n s_{n_2}, \ldots\}$ is bounded above. But then, by (i), $\lim_j g_{n_j} s_{n_j} = 0$, conflicting $g_n s_n \geq \varepsilon$ for each $n$.

THEOREM 1.6.6 Let $G$ be a linearly ordered group, let $X$ be a $G$-module, let $s_0 \in X$ and let $\tau(s)$ be the corresponding topological type of $X$, defined in 1.6.1. Then, for a sequence $s_1, s_2, \ldots$ in $X$ the following are equivalent.

(a) $s_1, s_2, \ldots$ satisfies the type condition.

(b) $\lim_n \tau(s_n) = \infty$.

Proof. $(a) \Rightarrow (b)$. We first show that $(a)$ implies that $G$ does not have a maximal proper convex subgroup. Suppose it does, say $H \subset G$ is a maximal proper convex subgroup. Let $\varepsilon \in G \backslash H$, $\varepsilon < 1$. Then $1, \varepsilon, \varepsilon^2, \ldots$ is decreasing. If we had a $\delta \in G$,
\( \delta \leq \varepsilon^n \) for all \( n \in \mathbb{N} \) then \( H_1 := \{ g \in G : \delta \leq g^n \leq \delta^{-1} \text{ for all } n \in \mathbb{N} \} \) is a convex subgroup containing \( H \) (since \( \varepsilon < h < \varepsilon^{-1} \) for all \( h \in H \) and \( \varepsilon \), but not \( \delta \) conflicting the maximality of \( H \). Thus, \( \lim_n \varepsilon^n = 0 \) and \( \{ \varepsilon^n s_0 : n \in \mathbb{N} \} \) is cofinal and coinitial. So, for each \( m \in \mathbb{N} \) there is an \( n_m \in \mathbb{Z} \) such that \( \varepsilon^{n_m+1} s_0 \leq s_m \leq \varepsilon^{n_m} s_0 \) i.e. \( \varepsilon s_0 \leq \varepsilon^{-m} s_m \leq s_0 \). By (\( \alpha \)) we would have \( \lim_n \varepsilon^{-m} s_m = 0 \), a contradiction. Now we come to the proof of (\( \alpha \)) \( \Rightarrow \) (\( \beta \)) proper. Suppose not \( \lim_n \tau(s_n) = \infty \). Then there is a proper convex subgroup \( H \) of \( G \) and a subsequence \( t_1, t_2, \ldots \) of \( s_1, s_2, \ldots \) such that \( \tau(t_n) \in H \) for all \( n \). By the first result of this proof there is a proper convex subgroup \( H' \supset H \), \( H' \neq H \). By 1.6.2 the intersection \( \text{conv}_X(H)^s_0 \cap G t_n \) is non-empty for each \( n \), so there exist \( g_1, g_2, \ldots \in G \) with \( g_n t_n \in \text{conv}_X(H^s_0) \). Now \( H' \) is bounded above and below hence so are \( \text{conv}_X(H^s_0) \) and \( \{g_1 t_1, g_2 t_2, \ldots\} \). By 1.6.5 (i), \( \lim_n g_n t_n = 0 \), a contradiction. (\( \beta \)) \( \Rightarrow \) (\( \alpha \)). Like in the previous part, we first observe that (\( \beta \)) implies that \( G \) does not have a maximal proper convex subgroup (this follows directly from Definition 1.6.4(ii)). Put \( H_0 := \{ g \in G : g s_0 = s_0 \} \). By (\( \beta \)), \( \tau(s_n) \) properly contains \( H_0 \) for large \( n \), so without loss, to prove (\( \alpha \)), we may assume that \( \tau(s_n) \neq H_0 \), for each \( n \), implying \( s_n \notin G s_0 \) for each \( n \) (see 1.6.2). To prove (\( \alpha \)), let \( g_1, g_2, \ldots \in G \) be such that \( \{g_n s_n : n \in \mathbb{N}\} \) is bounded above, say, by \( t \in X \). If not \( \lim_n g_n s_n = 0 \) we could find a \( u \in X \) and \( n_1 < n_2 < \cdots \) in \( \mathbb{N} \) such that, with \( m_n := g_n s_n, t_n := s_n \) we had \( u \leq h_n t_n \leq t \) for all \( n \). By cofinality and coinitiality of \( G s_0 \) there exists a \( v \in G \) such that

\[
v^{-1} s_0 \leq h_n t_n \leq v s_0 \tag{(*)}
\]

for all \( n \). Now let \( H \) be the smallest convex subgroup of \( G \) containing \( v \). Then \( H \neq G \) (If \( H = G \), \( V := \{ h \in G : v^{-1} \leq h \leq v \} \) is a convex subgroup not containing \( v \). If \( H_1 \) is a proper convex subgroup of \( H \) then it cannot contain \( v \) so \( v^{-1} \leq h \leq v \) for all \( h \in H_1 \), so \( H_1 \subset V \) and \( V \) is maximal, a contradiction.) By (\( \beta \)), \( \tau(t_n) \nsubseteq H \) for large \( n \), i.e. \( G t_n \) does not meet \( \text{conv}_X(H s_0) \) for large \( n \). But (\( \ast \)) yields \( h_n t_n \in \text{conv}_X(H s_0) \) for all \( n \), a contradiction.

**Remark** We see that, although \( \tau \) depends on the choice of \( s_0 \) (see the beginning of 1.6), property (\( \beta \)) does not.

**Theorem 1.6.7** Let \( \Gamma \) be a linearly ordered abelian group containing \( G \) as a cofinal subgroup. Suppose in the \( G \)-module \( \Gamma^# \) the sequence \( s_1, s_2, \ldots \) satisfies the type condition. Then so does \( \omega(s_1), \omega(s_2), \ldots \), where \( \omega : \Gamma^# \rightarrow \Gamma^# \) is the antipode (see Proposition 1.3.1).

Proof. Let \( g_1, g_2, \ldots \in G \) and \( t \in \Gamma^# \) be such that \( g_n \omega(s_n) \leq t \) for all \( n \). Then \( \omega(t) \leq \omega(g_n \omega(s_n)) = g_n^{-1} \omega^2(s_n) = g_n^{-1} s_n \) for all \( n \). By Lemma 1.6.5(ii), \( \lim_n g_n^{-1} s_n = \infty \). By 1.3.1(\( v \)), \( \lim_n g_n \omega(s_n) = \lim_n \omega(g_n^{-1} s_n) = 0 \).

**Example 1.6.8** (Continuation of 1.3.2) The topological type of an element \( h \in \sqrt{G} \). Given \( G = \bigoplus_{i \in \mathbb{N}} G_i \), as in 1.3.2, we describe \( \sqrt{G} \) as the direct sum \( \sqrt{G} = \bigoplus_{i \in \mathbb{N}} K_i \), where the cyclic group \( K_i \) is generated by an element \( k_i \) such that \( k_i^2 = g_i \). We order \( \sqrt{G} \) antilexicographically, the order of \( K_i \) being the natural one. With the componentwise product \( \sqrt{G} \) is a group, and identifying \( g = (g_{\alpha n})_{i \in \mathbb{N}} \) with \( (k_{\alpha n})_{i \in \mathbb{N}} \) we can consider \( G \) as a subgroup of \( \sqrt{G} \). Therefore \( \sqrt{G} \) is a \( G \)-module. Taking
Danach spaces over fields with an infinite rank valuation

s₀ := 1 in the definition of topological type τ(k) of an element k ∈ √G, we have by
Theorem 1.6.2 that if 1 ∈ Gk, then τ(k) = H₀ = {1}, but if 1 ∉ Gk then τ(k) is
the largest among the convex subgroups Hₙ of G for which conv√G(Hₙ) ∩ Gk = ∅.

**LEMMA 1.6.9** Let k ∈ √G, k = (kᵢ)ᵢ∈ℕ₁.

i) If tᵢ ∈ 2ℤ for all i, then τ(k) = H₀.

ii) If that is not the case, let j = max{i ∈ ℕ : tᵢ ∉ 2ℤ}. Then τ(k) = Hᵢ₋₁.

Proof. i) If for all i there exist nᵢ ∈ ℤ such that tᵢ = 2nᵢ, then k ∈ √G. Therefore
1 ∈ Gk. ii) Since H₀ = {(gᵢ)ᵢ∈ℕ₁ : nᵢ = 0 if i > n}, any element of conv√G(H₀) =
{x ∈ √G : there are z, w ∈ H₀ with z ≤ x ≤ w} must be of the form (kᵢ)ᵢ∈ℕ₁ with
sᵢ = 0 for i > n. Now let g = (kᵢ)ᵢ∈ℕ₁ ∈ G, then for gk = (kᵢ)ᵢ∈ℕ₁ we have that
2nᵢ₊₁ + 1, and gk does not belong to conv√G(H₀). Hence conv√G(Hᵢ₋₁) ∩
Gk = ∅. But there exists g ∈ G such that x = (kᵢ, ..., kᵢ₋₁, kᵢ, 1, ...) = gk; since
1 ≤ x ≤ (1, ..., 1, kᵢ₋₁, kᵢ₋₁, 1, ...) we have conv√G(Hᵢ) ∩ Gk ≠ ∅ for all n > j.

2 NORMED SPACES

In this Section we establish some basic theory of normed spaces; it is mainly the
material we need for the main subject of this paper to be treated in Sections Three
and Four. Many notations, statements and proofs will run similarly to the rank 1
case. However there are a few sharp deviations (2.1.5, 2.1.8, 2.4.10, 2.4.11, 2.4.18)
which will of course get special attention.

**Throughout K will be a valued field with a surjective valuation | · | : K → G ∪ {0_G},
where G is a linearly ordered group and 0_G is a zero element adjoined to G having
the properties 0_G < g, 0_G · g = 0_G · 0_G = 0_G for all g ∈ G. More generally, to each
G-module X we adjoin a zero element 0_X for which 0_X < x, 0_G · x = 0_G · 0_X
= 0_X for each x ∈ X. However, from now on we will omit the subscripts and write
0 for the zero element of any G-module.**

2.1 Seminorms

**DEFINITION 2.1.1** Let E be a K-vector space, let X be a G-module. An X-
seminorm on E is a map p : E → X ∪ {0} such that for all x, y ∈ E, λ ∈ K
(i) p(0) = 0
(ii) p(λx) = |λ|p(x)
(iii) p(x + y) ≤ max{p(x), p(y)}. If, in addition, p(x) = 0 implies x = 0, then p is
called an X-norm; in that case we often prefer the notation ||x|| rather than
p(x). When there is no danger of confusion we often omit the prefix "X-" and
just write "seminorm" ("norm").

**REMARK** In contrast to the requirements for valuations (see 1.4) we are not
asking seminorms to be surjective. If p is an X-seminorm and Y is a G-module
containing X, then p is, in a natural way, also a Y-seminorm. In particular this
holds if $Y = X^\#$. It will turn out that at many instances it is useful to assume $X$ to be complete; in general this will not restrict the problem we are dealing with.

Let $X, Y$ be $G$-modules, let $p$ be an $X$-seminorm, $q$ a $Y$-seminorm on a $K$-vector space $E$. We say that $p$ is weaker than $q$ (or $q$ is stronger than $p$) if, for each net $x_i \to x$, $p(x_i) = 0$ implies $q(x_i) = 0$, or, equivalently, if for each $\varepsilon \in X$ there exists a $\delta \in Y$ such that $x \in E$, $q(x) < \delta$ implies $p(x) < \varepsilon$. Otherwise stated, $p$ is weaker than $q$ if the topology induced by $p$ in the usual way, is weaker than the topology induced by $q$. To express this notion in yet another way we introduce the concept of boundedness.

**DEFINITION 2.1.2** Let $p$ be an $X$-seminorm on a $K$-vector space $E$. A subset $S$ of $E$ is called $p$-bounded if $\{p(x) : x \in S\}$ is bounded above in $X \cup \{0\}$.

**PROPOSITION 2.1.3** Let $p$ and $q$ be seminorms on a $K$-vector space $E$. Then $p$ is weaker than $q$ if and only if each $q$-bounded set in $E$ is $p$-bounded.

Proof. Let $p$ be an $X$-seminorm, let $q$ be a $Y$-seminorm, where $X$ and $Y$ are $G$-modules. Suppose $p$ is weaker than $q$ and let $S$ be a $q$-bounded set, say $q(x) \leq t \in Y$ for all $x \in S$. Choose $\varepsilon \in X$. There is a $\delta \in Y$ such that $q(x) < \delta$ implies $p(x) < \varepsilon$. There is a $g \in G$ with $gt < \delta$. Choose $\lambda \in K$ for which $|\lambda| = g$, and let $x \in S$. Then $q(\lambda x) = |\lambda| q(x) \leq |\lambda|t = gt < \delta$, so $p(\lambda x) < \varepsilon$. Then $p(x) < g^{-1}\varepsilon$, so $S$ is $p$-bounded. Conversely, let each $q$-bounded set be $p$-bounded. Let $\varepsilon \in X$, choose $t \in Y$. There is an $s \in X$ such that $q(x) \leq t$ implies $p(x) \leq s$. Let $\lambda \in K^X$ be such that $|\lambda|s < \varepsilon$. Then we see that for all $x \in E$, $q(x) \leq |\lambda|t$ implies $p(x) < \varepsilon$ i.e. $p$ is weaker than $q$.

In the theory over fields with rank 1 valuation (i.e. $G$ is a subgroup of $(0, \infty)$, see 1.3), a seminorm $p$ is weaker than a seminorm $q$ if and only if $p \leq Cq$ for some real constant (that can be taken in $G$). In our theory such a statement does not make sense if $p$ is an $X$-seminorm, $q$ is a $Y$-seminorm and $X$ is not a subset of $Y$. If $X \subset Y$ we may $p$ also consider as a $Y$-seminorm, so we may assume $X = Y$. Thus, we define the following.

**DEFINITION 2.1.4** Let $p, q$ be $X$-seminorms on a $K$-vector space $E$, where $X$ is a $G$-module. We say that $p$ is $q$-Lipschitz if there is a $g \in G$ such that $p(x) \leq g q(x)$ for all $x \in E$.

Clearly, if $p$ is $q$-Lipschitz then $p$ is weaker than $q$. The converse does not hold, see [1], [9], 3.7. But if $p$ is weaker than $q$ we do have an increasing function $\phi$ for which $p(x) \leq \phi(q(x))$ for all $x \in E$, as is shown in the next Proposition.

**PROPOSITION 2.1.5** Let $p \neq 0$ be an $X$-seminorm, let $q$ be a $Y$-seminorm on a $K$-vector space $E$, where $X, Y$ are complete $G$-modules. Suppose $p$ is weaker than $q$. Then there exist increasing functions $\xi : Y \cup \{0\} \to X \cup \{0\}$ such that $p \leq \xi \circ q$. Among them there is a smallest one, $\phi$, given by the formula

$$\phi(t) = \sup\{p(x) : x \in E, q(x) \leq t\} \quad (t \in Y \cup \{0\}).$$  (*)
Moreover, $\phi$ is an extended $G$-module map $Y \cup \{0\} \to X \cup \{0\}$ (and therefore bicontinuous at 0, see 1.5.2(i)).

Proof. By 2.1.3 the set $\{p(x) : q(x) \leq t\}$ is bounded above for each $t \in Y \cup \{0\}$, so (*) defines a map $Y \cup \{0\} \to X \cup \{0\}$, which is obviously increasing. For each $x \in E$ we have $\phi(q(x)) = \sup \{p(z) : z \in E, q(z) \leq q(x)\} \geq p(x)$, so $p \leq \phi \circ q$. If $\xi : Y \cup \{0\} \to X \cup \{0\}$ is increasing and $p \leq \xi \circ q$ then we have for $t \in Y \cup \{0\}$ that $\phi(t) = \sup \{p(x) : q(x) \leq t\} \leq \sup \{\xi(q(x)) : q(x) \leq t\} = \xi(t)$. It remains to be shown that $\phi$ is an extended $G$-module map. Let $t \in Y \cup \{0\}$, $g \in G$. Choose $\lambda \in K$ such that $|\lambda| = g$. Then $g^{-1} \phi(gt) = |\lambda|^{-1} \phi(|\lambda| t) = |\lambda|^{-1} \sup \{p(x) : q(x) \leq |\lambda| t\}$. Now by 1.5.3 (i) this equals $\sup \{|\lambda|^{-1} p(x) : |\lambda|^{-1} q(x) \leq t\} = \sup \{p(y) : q(y) \leq t\} = \phi(t)$. Hence $\phi(gt) = g \phi(t)$ for all $g \in G$, $t \in Y \cup \{0\}$. To finish the proof we show $t = 0 \iff \phi(t) = 0$ for all $t \in Y \cup \{0\}$. By the above, if $t = 0$ then from $g \phi(0) = \phi(g \cdot 0) = \phi(0)$ and cofinality of $G$s if $s \neq 0$ we conclude $\phi(t) = \phi(0) = 0$. Conversely, if $t \neq 0$ then on $\{x \in E : q(x) \leq t\}$, $p$ must be not identically 0 (otherwise, $p$ is zero on the whole of $E$ against the assumption), so, by (*), $\phi(t) \neq 0$.

In the same vein one can prove the following. We leave the proof to the reader.

**Proposition 2.1.6** Let $E, X, Y, p, q$ be as in 2.1.5. Then there exist increasing functions $\xi : X \cup \{0\} \to Y \cup \{0\}$ for which $\xi \circ p \leq q$. Among them there is a largest one, $\psi$, given by the formula

$$\psi(s) = \inf \{q(x) : x \in E, p(x) \geq s\} \quad (s \in X \cup \{0\}).$$

Moreover, $\psi$ is an extended $G$-module map $X \cup \{0\} \to Y \cup \{0\}$ (and therefore bicontinuous at 0).

The following definition will not come as a suprise.

**Definition 2.1.7**

(i) Two seminorms $p$ and $q$ on a $K$-vector space are called equivalent if $p$ is weaker than $q$ and $q$ is weaker than $p$.

(ii) Let $X$ be a $G$-module. Two $X$-seminorms $p$ and $q$ on a $K$-vector space are called Lipschitz-equivalent if $p$ is $q$-Lipschitz and $q$ is $p$-Lipschitz.

**Remark** If $p$ is an $X$-seminorm on a $K$-vector space $E$ and $\phi : X \cup \{0\} \to Y \cup \{0\}$ is an extended $G$-module map then $\phi \circ p$ is an $Y$-seminorm on $E$ that is equivalent to $p$. This follows from 1.5.2(i).

The previous Propositions 2.1.5 and 2.1.6 yield the following corollary. Observe that in (i) below $\phi_1 \circ q$ and $\phi_2 \circ q$ are equivalent to $q$.

**Corollary 2.1.8** Let $X, Y$ be complete $G$-modules, let $p \neq 0$ be an $X$-seminorm, $q$ a $Y$-seminorm on a $K$-vector space $E$. Then we have the following.

(i) $p$ and $q$ are equivalent if and only if there exist extended $G$-module maps $\phi_1, \phi_2 : Y \cup \{0\} \to X \cup \{0\}$ such that

$$\phi_1(q(x)) \leq p(x) \leq \phi_2(q(x)) \quad (x \in E).$$
(ii) Let $X = Y$. Then $p$ and $q$ are Lipschitz-equivalent if and only if there exist $g_1, g_2 \in G$ such that
\[ g_1 q(x) \leq p(x) \leq g_2 q(x) \quad (x \in E). \]

The following theorem shows that if one is interested in locally convex topologies rather than (geo)metrical properties it suffices to consider seminorms with values in $G^\#$.

**THEOREM 2.1.9** Each seminorm is equivalent to a $G^\#$-seminorm.

**Proof.** Let $p$ be an $X$-seminorm for some $G$-module $X$. Extend the map $\phi$ of 1.5.6 by $\phi(0) = 0$ so as to obtain an extended $G$-module map $\phi : X \cup \{0\} \to G^\# \cup \{0\}$. Then $\phi \circ p$ is a $G^\#$-seminorm equivalent to $p$.

### 2.2 Normed spaces

A normed space, more precisely, an $X$-normed space, is a pair $(E, \| \cdot \|)$ where $E$ is a $K$-vector space and where $\| \cdot \|$ is an $X$-norm for some $G$-module $X$. The map $(x, y) \mapsto \|x - y\|$ is a scale (see 1.2) on $E$, the induced topology is a Hausdorff vector topology i.e. addition and scalar multiplication are continuous (use 1.5.2(ii)). Often we will write $E$ (rather than $(E, \| \cdot \|)$).

Of course one can define easy generalizations of well-known spaces from rank 1 theory in order to obtain examples of $X$-normed spaces (e.g. see 2.4.15). Typically infinite rank examples will appear in Sections 3 and 4. At the present stage it might be useful to consider the following example that is non-trivial, also in rank 1 case.

**EXAMPLE 2.2.1** (Compare also [15]) Let $X$ be a complete $G$-module. Let $X^\sim$ be a copy of $X$ and define the $G$-module $X \times X^\sim$ in the spirit of 1.5.5(c) by requiring $y < x^\sim < x$ for all $x, y \in X$, $y < x$, and $gx^\sim = (gx)^\sim$. Let $S$ be a topological space, let $E$ be an $X$-normed space. The space $BC(S \to E)$, consisting of all continuous functions $S \to E$ that are bounded (above) is an $X$-normed space under
\[ f \mapsto \|f\|_\infty = \sup_{x \in S} \{|f(x)| : x \in S\}. \]

But it is also an $X \times X^\sim$-normed space with respect to
\[ f \mapsto \|f\|'_\infty = \sup_{x \in X \times X^\sim} \{|f(x)| : x \in S\} \]
i.e.
\[ \|f\|'_\infty = \begin{cases} \|f\|_\infty & \text{if } \max\{|f(x)| : x \in S\} \text{ exists} \\ \|f\|_\infty^\sim & \text{otherwise.} \end{cases} \]

Let $X, Y$ be $G$-modules, let $E$ be an $X$-normed space, $F$ a $Y$-normed space. We consider two types of maps $E \to F$.

1. The collection $C(E, F)$ of all continuous linear maps $T : E \to F$ is a $K$-vector space. The seminorm $x \mapsto \|Tx\|$ is weaker than $\| \cdot \|$; so by 2.1.3 for each $s \in X$ the set $\{|\|Tx\| : \|x\| \leq s\}$ is bounded above in $Y$ and the formula
\[ \|T\|_s := \sup\{|\|Tx\| : \|x\| \leq s\} \]
defines a $Y^\#$-norm on $\mathcal{L}(E, F)$. It is easily seen that, for $t \in X$, the norms $|| \cdot ||_s$ and $|| \cdot ||_t$ are Lipschitz-equivalent. The induced topology on $\mathcal{L}(E, F)$ is the topology of uniform convergence on bounded sets. We call the norms $|| \cdot ||_s$ the uniform norms. If $E = F$ as normed spaces we write $\mathcal{L}(E)$ rather than $\mathcal{L}(E, E)$. The dual space $E'$ of $E$ is $\mathcal{L}(E, K)$ where $K$ is assumed to be normed by the valuation.

2. If $X = Y$, the collection Lip$(E, F)$ of all linear Lipschitz maps $T : E \to F$ (i.e. there is a $g \in G$ such that $||Tx|| \leq g||x||$ for all $x \in E$; in most literature such $T$ are called bounded maps) forms a $K$-linear subspace of $\mathcal{L}(E, F)$. The formula

$$||T|| = \inf\{g \in G : ||Tx|| \leq g||x|| \text{ for all } x \in E\}$$

defines a $G^\#$-norm on Lip$(E, F)$, called the Lipschitz norm. Clearly, for each $s \in X$, $|| \cdot ||_s$ is weaker than $|| \cdot ||$. See [1] for an example of an element of $\mathcal{L}(E, F) \setminus \text{Lip}(E, F)$. The terms ‘linear homeomorphism’, ‘isometrical isomorphism’ between normed spaces will need no explanation.

We now look into the forming of quotients in some detail because we will need this precise information later on. Let $E$ be an $X$-normed space, where $X$ is some $G$-module. Let $D \subseteq E$ be a closed subspace, let $\pi : E \to E/D$ be the canonical map. Like in the classical case one proves that the formula

$$||\pi(a)|| = \inf\{||x|| : x \in E, \pi(x) = \pi(a)\}$$

$$= \inf\{||a - d|| : d \in D\} \quad (a \in E)$$

defines an $X^\#$-norm on $E/D$, the so-called quotient norm. We have $||\pi(a)|| \leq ||a||$ for each $a \in E$, so $\pi$ is Lipschitz. The norm topology on $E/D$ is the quotient topology induced by $\pi$. If $F$ is a second normed space and $T \in \mathcal{L}(E, F)$ then the map $T_1$ in the factorization

$$\begin{array}{ccc}
E & \overset{T}{\longrightarrow} & F \\
\downarrow \pi & & \nearrow T_1 \\
E/\text{Ker}T
\end{array}$$

(where $E/\text{Ker} T$ is equipped with the quotient norm and $\pi$ is the canonical map) is in $\mathcal{L}(E/\text{Ker} T, F)$. For each $s, t \in X^\#$ with $s < t$ we have $||T||_s \leq ||T_1||_s \leq ||T||_t$. If $T$ is Lipschitz then so is $T_1$ and $||T|| = ||T_1||$.

$F$ is called a quotient of $E$ if there is a $T \in \mathcal{L}(E, F)$ such that the map $T_1$ in the above diagram is an isometrical isomorphism; such a $T$ is called quotient map. Obviously, the canonical map $\pi : E \to E/D$ of above is a quotient map. A surjective $T \in \mathcal{L}(E, F)$ is a quotient map if and only if for each $y \in F$ we have $||y|| = \inf\{||x|| : Tx = y\}$. A quotient map $T \in \mathcal{L}(E, F)$ is called strict quotient map (and $F$ is called a strict quotient of $E$) if for all $y \in F$ we have $||y|| = \min\{||x|| : Tx = y\}$.

In the following lemma we characterize the (strict) quotient maps.
LEMMA 2.2.2  Let $X$ be a $G$-module, let $E, F$ be $X$-normed spaces, let $\pi : E \to F$ be a linear map. Then $\pi$ is a quotient map if and only if, for each $s \in X$, $\pi(B_E(0,s^-)) = B_F(0,s^-)$ while $\pi$ is a strict quotient map if and only if, for each $s \in X$, $\pi(B_E(0,s)) = B_F(0,s)$.

Proof. Suppose $\pi$ is a quotient map. Let $s \in X$. Obviously $\pi(B_E(0,s^-)) \subset B_F(0,s^-)$. Conversely, if $y \in F$, $||y|| < s$ there is by definition an $x \in E$ with $||x|| < s$ and $\pi(x) = y$. Now suppose $\pi(B_E(0,s^-)) = B_F(0,s^-)$ for each $s \in X$. Then clearly $\pi$ is surjective. Let $y \in F$, $||y|| = t \in X$. By Lemma 1.1.1 there are two cases.

1. $s_1 := \min\{s \in X : s > t\}$ exists. Then $||y|| < s_1$ so there is an $x \in B_E(0,s_1^-)$ with $\pi(x) = y$. Then $||x|| \leq t$. If $||x||$ were $t$ then $||\pi(x)|| = t$, a contradiction. Hence $||x|| = t$, so $||y|| = \min\{||x|| : \pi(x) = y\}$.

2. $\inf\{s \in X : s > t\} = t$. For each $s \in X$, $s > t$, $y$ is in $B_F(0,s^-)$ so there is an $x \in B_E(0,s^-)$ with $\pi(x) = y$. If $||x||$ were $t$ then $||\pi(x)|| < t$, a contradiction. Hence $||x|| > t$, so $||y|| = \inf\{s : s > ||y||\} = \inf\{||x|| : \pi(x) = y\}$.

Now suppose $\pi$ is a strict quotient map. Let $s \in X$. Obviously, $\pi(B_E(0,s)) \subset B_F(0,s)$. Conversely, let $y \in B_F(0,s)$. There exists an $x \in E$ with $\pi(x) = y$ and $||x|| = ||y||$. Hence $B_F(0,s) \subset \pi(B_E(0,s))$, and we have equality. Conversely, let $\pi(B_E(0,s)) = B_F(0,s)$ for each $s \in X$. Then $\pi$ is surjective. If $y \in F$, $||y|| = s \in X$ then there is an $x \in B_E(0,s)$ with $\pi(x) = y$. If $||x||$ were $s$ then $\pi(x) \in B_F(0,||x||)$, so $||\pi(x)|| \leq ||x|| < s$. Hence, $||x|| = s$ and the Lemma is proved.

The following corollaries obtain.

COROLLARY 2.2.3 Let $X$ be a $G$-module, let $E, F$ be $X$-normed spaces, let $\pi : E \to F$ be a quotient map. If $B$ is an 'open' ball in $F$ with radius $s \in X$ then $\pi(B_E(a,s^-)) = B$ for each $a \in \pi^{-1}(B)$. If moreover, $\pi$ is a strict quotient map and $B$ is a 'closed' ball in $F$ with radius $s \in X$ then $\pi(B_E(a,s)) = B$ for each $a \in \pi^{-1}(B)$.

COROLLARY 2.2.4 Let $X$ be a $G$-module, let $E, F$ be $X$-normed spaces, let $\pi : E \to F$ be a quotient map. Then, if $B_1 \supset B_2 \supset \cdots$ are 'open' balls in $F$ there are 'open' balls $C_1 \supset C_2 \supset \cdots$ in $E$ such that $\pi(C_n) = B_n$ for each $n$. If, in addition, $\pi$ is a strict quotient map then, for any nest $C$ of balls in $F$ there is a nest $C'$ in $E$ such that $B \mapsto \pi(B)$ is a bijection $C' \to C$.

Proof (of 2.2.4). The first assertion can be proved by induction and 2.2.3. To prove the second one we use Zorn’s Lemma. We may assume that $C$ is a maximal nest (by adding all balls in $F$ that contain some element of $C$). Let $V$ be the collection of all non empty sets $D$ of balls in $E$ with the properties: 1. If $B \in D$ and $B'$ is a ball, $B' \supset B$ then $B' \in D$. 2. $\pi(B) \in C$ for all $B \in D$. Order $V$ by declaring that $D_1 \preceq D_2$ if $D_1 \subset D_2$. $V$ is not-empty: choose any ball $B$ in $E$ with $\pi(B) \in C$, the collection of all balls $B'$ in $E$ with $B' \supset B$ belongs to $V$. Clearly each chain in $V$ has an upper bound, so by Zorn’s Lemma, $V$ has a maximal element $D'$. It suffices
to prove that $\mathcal{D}'$ is a maximal nest. If $\bigcap \mathcal{D}' = \emptyset$ this is true, so suppose $\bigcap \mathcal{D}' \neq \emptyset$.

If $\mathcal{D}'$ has a smallest element $B_0$ then, since $\mathcal{C}$ has no smallest element, there is a $C \in \mathcal{C}$ such that $C \subseteq \pi(B_0)$ strictly and there is a ball $B' \subset B_0$ with $\pi(B') = C$ so $\{B : B \text{ ball in } \mathcal{E}, B \supset B'\}$ is in $\mathcal{V}$ and strictly larger than $\mathcal{D}'$, a contradiction.

If $\mathcal{D}'$ has no smallest element then $B_0 := \bigcap \mathcal{D}'$ is a ball of the form $B_E(a, s)$ for some $s \in X$, $B_0$ not in $\mathcal{D}'$. But then $\bigcap_{B \in \mathcal{D}'} \pi(B)$ is a ‘closed’ ball of radius $s$ and contains $\pi(B_0)$, hence $\pi(B_0) = \bigcap_{B \in \mathcal{D}'} \pi(B)$ and $\mathcal{D}' \cup B_0$ is in $\mathcal{V}$ and strictly larger than $\mathcal{D}'$, a contradiction.

**COROLLARY 2.2.5** Strict quotients of spherically complete spaces are spherically complete. If $K$ satisfies the countability conditions of 1.4.4 then all quotients of spherically complete spaces are spherically complete.

Proof. The first assertion follows from the second assertion of 2.2.4. Now let $F$ be a quotient of an $X$-normed spherically complete space $\mathcal{E}$, let $\mathcal{C}$ be a nest of ‘open’ balls in $\mathcal{F}$. To prove $\bigcap \mathcal{C} \neq \emptyset$ we may assume that $\mathcal{C}$ has no smallest element. Let $r := \inf\{\text{diam } B : B \in \mathcal{C}\}$. By assumption there are $r_1 > r_2 > \cdots$ in $X^*$ such that $r_n = r$. For each $n$, choose a $B_n \in \mathcal{C}$ with diameter $\leq r_n$. Then $\bigcap_n B_n = \bigcap \mathcal{C}$. Now the result follows after applying the first assertion of 2.2.4 and using the spherical completeness of $\mathcal{E}$.

### 2.3 Linear operators with finite rank. Banach spaces

In 2.3 we extend results that were already observed in [11] for special normed spaces.

**LEMMA 2.3.1** Let $E, F$ be onedimensional spaces, both equipped with an $X$-norm for some $G$-module $X$. Then every linear map $f : E \to F$ is Lipschitz.

Proof. Such a map has the form $f : \lambda a \mapsto \lambda b$ ($\lambda \in K$) for some non-zero $a \in E$ and some $b \in F$. Let $g \in G$ be such that $\|b\| \leq g\|a\|$. Then for each $\lambda \in K$ we have $\|f(\lambda a)\| = \|\lambda b\| \leq |\lambda|g\|a\| = g\|\lambda a\|$. 

**LEMMA 2.3.2** Let $E, F$ be normed spaces, let $\dim F = 1$, let $f : E \to F$ be linear. Then $f$ is continuous if and only if Ker $f$ is closed. If $E, F$ are both $X$-normed for some (complete) $G$-module $X$ we even have that $f$ is Lipschitz if and only if Ker $f$ is closed.

Proof. By 2.1.9 we can choose on $E$ and $F$ equivalent $G^*$-norms, so it suffices to prove the second statement. Suppose Ker $f$ is closed. From the text following 2.2.1 it follows that in the canonical factorization

\[
\begin{array}{ccc}
E & \overset{f}{\longrightarrow} & F \\
\pi \downarrow & & \uparrow f_1 \\
E/\text{Ker } f
\end{array}
\]

$f_1$ is continuous, hence Lipschitz by 2.3.1. Then so is $f_1 \circ \pi = f$. 

DEFINITION 2.3.3 Let $K$ be complete. A normed space over $K$ is called complete (or a Banach space) if each Cauchy net converges.

We now prove the non-surprising theorem on finite-dimensional spaces. It extends 1.3 of [9].

THEOREM 2.3.4 Let $K$ be complete, let $E$ be a finite-dimensional space over $K$. Then all norms are equivalent, $E$ is a Banach space with respect to each norm. For a $G$-module $X$ all $X$-norms on $E$ are Lipschitz equivalent.

Proof. We prove by induction on dim $E$ that all $X$-norms are Lipschitz equivalent (then we are done since, for each $n$, $K^n$ is complete under the norm $(\xi_1, \xi_2, \ldots, \xi_n) \mapsto \max_i |\xi_i|$, and by 2.1.9). If dim $E = 1$ we have 2.3.1. Suppose the statement holds for spaces with dimension $\leq n - 1$ and let $E$ be an $n$-dimensional space, let $\| \|$ be an $X$-norm on $E$. Let $e_1, \ldots, e_n$ be a base of $E$, we prove $\| \|$ to be Lipschitz equivalent to $\| \|_\infty : \xi_1 e_1 + \cdots + \xi_n e_n \mapsto \max_i |\xi_i| \|e_i\|$. Obviously $\| \| \leq \| \|_\infty$. To prove that $\| \|_\infty$ is $\| \|$-Lipschitz, let, for each $j \in \{1, \ldots, n\}$, $f_j : E \to K e_j$ be the map $\sum_{i=1}^n \xi_i e_i \mapsto \xi_j e_j$. Then dim $\operatorname{Ker} f_j = n - 1$; from the induction hypothesis it follows that $\operatorname{Ker} f_j$ is complete, hence closed in $(E, \| \|_{\infty})$ and by Lemma 2.3.2 the $f_j$ are Lipschitz. So there is a $g \in G$ such that $\| f_j (x) \|_\infty \leq g \| x \|$ $(x \in E, j \in \{1, \ldots, n\})$ and then for $x \in E$ we have $\| x \|_\infty = \| \sum_{i=1}^n f_i (x) \|_\infty \leq \max_i \| f_i (x) \| \leq g \| x \|$ and we are done.

COROLLARY 2.3.5 (Continuous linear operators of finite rank are Lipschitz). Let $K$ be complete, let $X$ be a (complete) $G$-module. Then every continuous linear map of an $X$-normed space $E$ into a finite-dimensional $X$-normed space $F$ is Lipschitz.

Proof. Let $T : E \to F$ be such a map; we may assume that $T$ is surjective. In the canonical decomposition

$$
\begin{array}{ccc}
E & \xrightarrow{T} & F \\
\pi \searrow & & \nearrow T_1 \\
& E/\operatorname{Ker} T & \\
\end{array}
$$

we have that $T_1$ is Lipschitz (by 2.3.4 the norm $x \mapsto \| T_1 x \|$ on $E/\operatorname{Ker} T$ is Lipschitz equivalent to the quotient norm), hence so is $T_1 \circ \pi = T$.

PROPOSITION 2.3.6 Let $E, F$ be normed spaces over $K$. If $F$ is a Banach space then so is $\mathcal{L}(E, F)$. If, in addition, $E, F$ are both $X$-normed for some (complete) $G$-module $X$ then $\operatorname{Lip}(E, F)$ is a Banach space.

Proof. Let $i \mapsto T_i$ $(i \in I)$ be a Cauchy net in $\mathcal{L}(E, F)$. Let $s \in X$. From $\lim_{i,j} \| T_i - T_j \|_s = 0$ and completeness of $F$ it follows that $T := \lim_i T_i$ exists pointwise. From $\| T x - T_j x \| \leq \max \{ \| T x - T_j x \|, \| T_j - T_i \|_s \}$ for all $x \in B_E(0, s)$ and $i, j \in I$ it follows easily that $T \in \mathcal{L}(E, F)$ and $\lim_i \| T - T_i \|_s = 0$. Now let $i \mapsto T_i$ be Cauchy in $\operatorname{Lip}(E, F)$. By the first part there is a $T \in \mathcal{L}(E, F)$ such that $\lim_i T_i = T$ in the topology of $\mathcal{L}(E, F)$. Now let $e \in G$. There is an $i_0$ such that for
which after taking $\lim_i$ becomes

$$
\|(T_{i} - T_{j})x\| \leq \varepsilon \|x\| \quad (x \in E)
$$

implying $\lim_j \|T - T_j\| = 0$.

**COROLLARY 2.3.7** Let $K$ be complete, let $E, F$ be normed spaces over $K$. Then, if $F$ is finite-dimensional, $\mathcal{L}(E, F)$, in particular $E'$, is a Banach space.

### 2.4 The Hahn Banach Theorem. Orthogonality

In 2.4 we extend results in rank 1 theory [14] to $X$-normed spaces. This section contains no surprises, apart from the fact that the proof of 2.4.12 is somewhat more complicated than its rank 1 counterpart [14], Th. 5.4, and apart from Example 2.4.18 for non-metrizable $K$.

**THEOREM 2.4.1** (Hahn Banach). Let $E$ be an $X$-normed space, let $F$ be a $Y$-normed space over $K$ where $X, Y$ are $G$-modules. Suppose $F$ is spherically complete with respect to the induced scale, let $D$ be a subspace of $E$ and let $T \in \mathcal{L}(D, F)$ be such that $\|T x\| \leq \phi(\|x\|)$ $(x \in D)$ where $\phi : X \cup \{0\} \rightarrow Y \cup \{0\}$ is an extended $G$-module map. Then $T$ can be extended to a $\bar{T} \in \mathcal{L}(E, F)$ for which $\|\bar{T} x\| \leq \phi(\|x\|)$ $(x \in E)$.

**Proof.** (Basically classical) By a simple application of Zorn's Lemma we may assume $E = D + Ka$ where $a \in E \setminus D$. It suffices to find $\bar{T} a \in F$ such that

$$
\|\lambda \bar{T} a - T d\| \leq \phi(\|\lambda a - d\|) \quad (\lambda \in K^\times, d \in D).
$$

Now for each $d \in D$, $\lambda \in K$ we have $\phi(\|\lambda a - \lambda d\|) = \phi(|\lambda| \|a - d\|) = |\lambda| \phi(\|a - d\|)$, so it is enough to find $\bar{T} a$ for which

$$
\|\bar{T} a - T d\| \leq \phi(\|a - d\|) \quad (d \in D),
$$
in other words, we have to show that

$$
\bigcap_{d \in D} B_F(Td, \phi(\|a - d\|)) \neq \emptyset. \quad (*)
$$

Let $d_1, d_2 \in D$. Then, by increasingness of $\phi$, $\|Td_1 - Td_2\| \leq \phi(\|d_1 - d_2\|) \leq \phi(\max \|d_1 - a\|, \|a - d_2\|) = \max(\phi(\|a - d_1\|), \phi(\|a - d_2\|))$, so $B_F(Td_1, \phi(\|a - d_1\|)) \cap B_F(Td_2, \phi(\|a - d_2\|)) \neq \emptyset$ showing that the balls in (*) form a nest. By spherical completeness of $F$ the intersection (*) is not empty which is finishing the proof.
DEFINITION 2.4.2 Let \((E, \| \cdot \|)\) be a normed space over \(K\). Two subspaces \(D_1\) and \(D_2\) of \(E\) are called \text{(norm) orthogonal} (notation \(D_1 \perp D_2\)) if for each \(d_1 \in D_1\), \(d_2 \in D_2\)

\[\|d_1 + d_2\| = \max(\|d_1\|, \|d_2\|).\]

A subspace \(D\) is called \text{(norm)orthocomplemented in \(E\)} if there exists a subspace \(S \perp D\) (called an \text{orthocomplement} of \(D\)) such that \(D + S = E\). An operator \(P \in \mathcal{L}(E)\) is called a \text{projection} if \(P^2 = P\). If, in addition, \(\|Px\| \leq \|x\|\) \((x \in E)\), \(P\) is called a \text{(norm)orthogonal projection}.

LEMMA 2.4.3 For an orthogonal projection \(P\), \(\text{Im } P\) and \(\text{Ker } P\) are orthocomplements of each other. A subspace \(D\) of \(E\) is orthocomplemented in \(E\) if and only if there is an orthogonal projection \(P \in \mathcal{L}(E)\) with \(PE = D\).

Proof. Left to the reader.

THEOREM 2.4.4 A spherically complete subspace of a normed space is orthocomplemented.

Proof. Let \(D\) be a spherically complete subspace of a normed space \(E\). By 2.4.1 the identity \(D \to D\) can be extended to a map \(P \in \mathcal{L}(E, D)\) with \(\|Px\| \leq \|x\|\) \((x \in E)\). Then \(P\), viewed as an element of \(\mathcal{L}(E)\) is an orthogonal projection, \(PE = D\). Now apply 2.4.3.

LEMMA 2.4.5 If \(K\) is spherically complete then so is every onedimensional normed space over \(K\).

Proof. Let \(E = Ke\) be a onedimensional normed space, let \(C\) be a nest of balls in \(Ke\). For each \(B \in C\) the set

\[C_B := \{\lambda \in K : \lambda e \in B\}\]

is convex, \(\neq \emptyset\). By spherical completeness of \(K\), 1.4.3 and its preamble, \(\bigcap C_B \neq \emptyset\) so \(\bigcap C \neq \emptyset\).

COROLLARY 2.4.6 Let \(K\) be spherically complete. Then any onedimensional subspace of a normed space over \(K\) is orthocomplemented.

Proof. Combine 2.4.4 and 2.4.5.

DEFINITION 2.4.7 A collection \(\{e_i : i \in I\}\) of vectors of a normed space is called \text{(norm)orthogonal} if for each finite set \(J \subset I\) and \(\lambda_j \in K\)

\[\| \sum_{j \in J} \lambda_j e_j \| = \max_{j} \|\lambda_j e_j\|.\]

Clearly, \(\{e_i : i \in I\}\) is orthogonal if and only if \([e_i, e_j] \uparrow e_j : j \neq i\) for each \(i \in I\).

LEMMA 2.4.8 (Perturbation Lemma) Let \(\{e_i : i \in I\}\) be an orthogonal set in a normed space \(E\), let \(\{f_i : i \in I\} \subset E\) be such that \(\|f_i - e_i\| < \|e_i\|\) for each \(i\). Then \(\{f_i : i \in I\}\) is orthogonal.
Proof. (Compare [14], 5.B) Let \( J \subset I \) be finite, let \( \lambda_j \in K \) for each \( j \in J \). To prove \( \| \sum_{j \in J} \lambda_j f_j \| = \max_{j \in J} \| \lambda_j f_j \| \) we may assume \( \lambda_j \neq 0 \) for all \( j \in J \). From \( \| f_i - e_i \| < \| e_i \| \) it follows that \( \| f_i \| = \| e_i \| \) for each \( i \in I \). For each \( j \in J \) we have \( \| \lambda_j (f_j - e_j) \| < \| \lambda_j e_j \| \) so that \( \| \sum_{j \in J} \lambda_j (f_j - e_j) \| = \max_{j \in J} \| \lambda_j e_j \| = \| \sum_{j \in J} \lambda_j f_j \| = \max_{j \in J} \| \sum_{j \in J} \lambda_j e_j \| = \| \sum_{j \in J} \lambda_j f_j \| . \)

We will show in 2.4.14 that all maximal orthogonal sets in a normed space have the same cardinality. Because the situation is somewhat more complicated than in the rank 1 case (compare [14], 5.4) we shall develop some machinery.

Let \( H \) be a convex subgroup of \( G \). Consider

\[
D_H := \{ \lambda \in K : |\lambda| \leq \sup H \} \\
D_H^- := \{ \lambda \in K : |\lambda| < \inf H \}
\]

(observe that, unless \( H = \{1\} \), \( \sup H \) and \( \inf H \) belong to \( G^\# \setminus G \)). We have

\[
D_H = \{ \lambda \in K : |\lambda| \leq 1 \text{ or } |\lambda| \in H \} \\
D_H^- = \{ \lambda \in K : |\lambda| < 1 \text{ and } |\lambda| \notin H \} \\
|D_H| = |D_H^- \cup H|, |D_H^- \cap H| = \emptyset.
\]

\( D_H \) is a subring of \( K \), \( D_H^- \) is a (unique) maximal ideal so we can define the field \( k_H \) by \( k_H := D_H/D_H^- \).

**DEFINITION 2.4.9** For each convex subgroup \( H \) we call the field \( k_H \) of above the **residue class field with respect to** \( H \). The canonical map \( D_H \to k_H \) is denoted \( \lambda \mapsto \lambda^- \).

**REMARK** \( k_H \) is the residue class field (in the traditional sense) of the field \( K \) with respect to the valuation \( \lambda \mapsto |\lambda| \text{ mod } H \) (with value group \( G/H \)), but for our purpose we prefer the above point of view.

**PROPOSITION 2.4.10** Let \( E \) be an \( X \)-normed space where \( X \) is a \( G \)-module. For each \( s \in X \) the balls \( B_E(0, s) \) and \( B_E(0, s^-) \) are modules over \( D_H \), when \( H_s := \{ g \in G : gs = s \} \). The quotient \( \overline{E}_s := B_E(0, s)/B_E(0, s^-) \) is in a natural way a vector space over \( k_H \).

Proof. \( B_E(0, s) \) and \( B_E(0, s^-) \) are absolutely convex. If \( x \in B_E(0, s) \), \( |\lambda| \in H_s \) then \( \| \lambda x \| = |\lambda| \| x \| \leq |\lambda| |s = s| \), so \( B_E(0, s) \) is a \( D_{H'} \)-module. If \( x \in B_E(0, s^-) \), \( |\lambda| \in H_s \) and \( \| \lambda x \| \) were \( \geq s \) then \( \| x \| = |\lambda|^{-1} \| \lambda x \| \geq |\lambda|^{-1} s = s \) (as \( |\lambda|^{-1} \in H_s \)), a contradiction, so \( B_E(0, s^-) \) is a \( D_{H'} \)-module. If \( \lambda \in D_{H'} \) then \( |\lambda| < 1 \) and \( |\lambda| \notin H_s \) so \( |\lambda| s \leq s \) but not \( |\lambda| s = s \) i.e. \( |\lambda| s < s \). This implies \( D_{H'}, B_E(0, s) \subset B_E(0, s^-) \) showing that \( \overline{E}_s \) is a \( k_{H'} \)-vector space under

\[
\bar{\lambda} \cdot \bar{x} = \overline{\lambda x} \quad (\lambda \in D_{H'}, \ x \in B_E(0, s))
\]

where the canonical map \( D_{H'} \to D_{H'}, D_{H'}^- = k_{H'} \) is denoted \( x \mapsto \bar{x} \).
LEMMA 2.4.11 Let $E$ be an $X$-normed space for some $G$-module $X$, let $s \in X$ and let $\{e_i : i \in I\} \subset E$ be such that $||e_i|| = s$ for all $i$. Then the following are equivalent.

(a) $\{e_i : i \in I\}$ is orthogonal.

(b) $\{\overline{e}_i : i \in I\}$ is linearly independent in $\overline{E}_s$.

Proof. (a) $\Rightarrow$ (b). Let $J \subset I$ be finite, let $\lambda_j \in D_{H_s}$ for each $j \in J$ and suppose that $\sum_{j \in J} \lambda_j \overline{e}_j = 0$. Then $\sum_{j \in J} \lambda_j e_j = 0$ so $||\sum_{j \in J} \lambda_j e_j|| < s$. By orthogonality, $||\lambda_j e_j|| < s$ for each $j$. Then $|\lambda_j| < 1$ and $|\lambda_j| \not\in H_s$, so $|\lambda_j| \in D_{H_s}$, i.e. $\lambda_j = 0$. (b) $\Rightarrow$ (a). Let $J \subset I$ be finite, let $\lambda_j \in K$ for each $j \in J$; we show that $||\sum_{j \in J} \lambda_j e_j|| = (\max |\lambda_j|) \cdot s$. To this end we may suppose that $\max |\lambda_j| = 1$. If $||\sum_{j \in J} \lambda_j e_j||$ were $< s$ then $\sum_{j \in J} \lambda_j e_j \in B_E(0, s^-)$ so $0 = \sum_{j \in J} \lambda_j e_j = \sum_{j \in J} \lambda_j \overline{e}_j$. By (b) $\lambda_j = 0$ for each $j$ i.e. $\lambda_j \in D_{H_s}$. But then $|\lambda_j| s < s$ for each $j$ conflicting $\max |\lambda_j| = 1$.

COROLLARY 2.4.12 For each $s \in X$ all maximal orthogonal sets in $\{x \in E : ||x|| \in G s\}$ have the same cardinality.

Proof. Each such orthogonal set can, via suitable scalar multiplications, be transformed into an orthogonal set of which each vector has length $s$. Now use 2.4.11 and the fact that maximal linear independent sets in vector spaces have the same cardinality.

PROPOSITION 2.4.13 Let $E$ be an $X$-normed space, where $X$ is a $G$-module, let $\{e_i : i \in I\}$ be a maximal orthogonal set of nonzero vectors in $E$. Then, for each $s \in X$, $\{e_i : ||e_i|| \in G s\}$ is a maximal orthogonal subset of $\{x \in E : ||x|| \in G s\}$.

Proof. Suppose for some $s \in X$ we do not have maximality. Then there is an $f \in E$ with $||f|| \in G s$ such that $\{f\} \cup \{e_i : ||e_i|| \in G s\}$ is orthogonal. We claim that $\{f\} \cup \{e_i : i \in I\}$ is orthogonal (which leads to a contradiction proving the Proposition). In fact, let $J \subset I$ be finite, $\lambda_j \in K$ for each $j \in J$, $\lambda_0 \in K$. Let $J_1 = \{j \in J : ||e_j|| \in G s\}$, $J_2 = \{j \in J : ||e_j|| \not\in G s\}$. Then $||\lambda_0 f + \sum_{j \in J_1} \lambda_j e_j|| = \max(||\lambda_0 f||, \max_{j \in J_1} ||\lambda_j e_j||) \in G s$ while $||\sum_{j \in J_2} \lambda_j e_j|| = \max_{j \in J_2} ||\lambda_j e_j|| \not\in G s$. We see that $||\lambda_0 f + \sum_{j \in J} \lambda_j e_j|| = \max(||\lambda_0 f||, \max_{j \in J} ||\lambda_j e_j||)$ and we are done.

COROLLARY 2.4.14 In a normed space each two maximal orthogonal subsets of nonzero vectors have the same cardinality.

We now introduce (norm)orthogonal bases. For the sequel we only need the concept of a countable orthogonal base.

DEFINITION 2.4.15 Let $X$ be a $G$-module, let $s : N \to X$. Then $c_0(N, s)$ is the space of all sequences $(\lambda_1, \lambda_2, \ldots) \in K^N$ for which $\lim_n |\lambda_n| s(n) = 0$ with coordinatewise operations and with $X$-norm $(\lambda_1, \lambda_2, \ldots) \mapsto \max_n |\lambda_n| s(n)$. If $X = G$, $s(n) = 1$ for all $n$ we write $c_0$ rather than $c_0(N, s)$. By $c_{00}$ we denote the space of all $(\lambda_1, \lambda_2, \ldots) \in K^N$ for which $\lambda_n = 0$ for large $n$. 
If $K$ is complete then the space $c_0(\mathbb{N}, s)$ is complete. The proof is standard.

**DEFINITION 2.4.16** A sequence $e_1, e_2, \ldots$ in a normed space $E$ is called Schauder base of $E$ if for each $x \in E$ there are unique $\lambda_1, \lambda_2, \ldots \in K$ such that $x = \sum_{n=1}^{\infty} \lambda_n e_n$. An orthogonal Schauder base is simply called orthogonal base. Then, with $x$ as above, $||x|| = \max_n ||\lambda_n e_n||$.

**PROPOSITION 2.4.17** Let $E$ be an infinite-dimensional $K$-Banach space. For an orthogonal sequence $e_1, e_2, \ldots$ the following are equivalent.

(a) $e_1, e_2, \ldots$ is an orthogonal base.

(β) $e_n \neq 0$ for each $n$. The linear span of $e_1, e_2, \ldots$ is dense in $E$.

Proof. $(\alpha) \Rightarrow (\beta)$. Obvious. To prove $(\beta) \Rightarrow (\alpha)$ we define a linear map $T : c_0(\mathbb{N}, s) \to E$ as follows

$$T : (\lambda_1, \lambda_2, \ldots) \mapsto \sum_{n=1}^{\infty} \lambda_n e_n,$$

where $s(n) := ||e_n||$ for each $n$. (Since $||\lambda_n e_n|| \to 0$ and $E$ is complete $\sum_{n=1}^{\infty} \lambda_n e_n$ exists, so $T$ is well-defined.) Clearly $T$ is a linear isometry. By $(\beta)$, $\text{Im } T$ is dense, but also complete by completeness of $c_0(\mathbb{N}, s)$. Then, $\text{Im } T = E$ and the result follows.

**EXAMPLE 2.4.18** (Weird spaces if $K$ is nonmetrizable). Let $K$ be complete and non-metrizable, let $E = c_0(\mathbb{N}, s)$ be as in 2.4.15.

1. Every sequence in $X$ is bounded below (and above). In fact, let $s_1 > s_2 > \ldots$ be a strictly decreasing sequence in $X$. By coinitiality of $G_{s_n}$ we can find $\lambda_1, \lambda_2, \ldots \in K^\times$ such that $\lambda_n s_1 < s_n$ for each $n$. If $\lim_n s_n = 0$ then $\lim_n \lambda_n s_1 = 0$, so $\lim_n \lambda_n = 0$ (1.5.2), conflicting 1.4.1.

2. For each $(\lambda_1, \lambda_2, \ldots) \in E$, $\lambda_n = 0$ for large $n$. This follows from 1. We see that $E = c_{00}$.

3. $E$ is complete but no Baire space. Clearly $E = \bigcup_n D_n$ where

$$D_n = \{(\lambda_1, \lambda_2, \ldots, \lambda_n, 0, 0 \ldots) : n \in \mathbb{N}, \lambda_i \in K \text{ for } i \in \{1, \ldots, n\}\}$$

Each $D_n$ is a finite-dimensional subspace hence complete (2.3.4) hence closed in $E$. However the interior of $D_n$ is empty.

4. All norms on $c_{00}$ are equivalent! Let $|| \cdot ||_1$ and $|| \cdot ||_2$ be $X$-norms on $E$. By 2.3.4 they are Lipschitz equivalent on $D_n$ for each $n$ so there are $g_1, g_2, \ldots$ and $h_1, h_2, \ldots \in G$ such that

$$h_n ||x||_1 \leq ||x||_2 \leq g_n ||x||_1 \quad (x \in D_n).$$

By nonmetrizability (1.4.1) there are $h, g \in G$ such that $h_n \geq h, g_n \leq g$ for all $n$ and we have

$$h ||x||_1 \leq ||x||_2 \leq g ||x||_1 \quad (x \in E).$$
2.5 Metrizable normed spaces

Throughout 2.5 we will assume that $K$ is complete and metrizable. Recall (1.4.1) that this implies that there is a sequence $\lambda_1, \lambda_2, \ldots$ in $K$ such that $|\lambda_1| > |\lambda_2| > \cdots$, and $\lim_n \lambda_n = 0$. Then each $G$-module $X$ has a coinitial (cofinal) sequence (let $s \in X$ and $\lambda_n$ be as above. Then $|\lambda_1|s, |\lambda_2|s, \ldots$ is coinitial). If $E$ is an $X$-normed space the balls $B_E(0, |\lambda_n|s)$ form a neighbourhood base at 0 for the norm topology. Hence the norm topology of any normed space over $K$ is (ultra)metrizable; it is a Banach space if and only if each Cauchy sequence converges. Observe that a sequence $x_1, x_2, \ldots$ is Cauchy if and only if $\lim_n(x_{n+1} - x_n) = 0$, so that the question as to whether a normed space is Banach depends on the topology, not on the particular norm. Each Banach space is a Baire space (compare Example 2.4.18). The proofs in this section are basically classical.

**PROPOSITION 2.5.1** Let $E$ be a Banach space, $D$ a closed subspace. Then $E/D$ is a Banach space.

Proof. Let the norm have values in a complete $G$-module $X$, let $\varepsilon_1 > \varepsilon_2 > \cdots$ be a coinitial sequence in $X$. Let $z_1, z_2, \ldots$ be a Cauchy sequence in $E/D$. It has a subsequence $y_1, y_2, \ldots$ for which $\|y_{n+1} - y_n\| < \varepsilon_{n+1}$ for each $n$. There are $v_0, v_1, \ldots \in E$ such that, with $\pi : E \to E/D$ the natural map, $\pi(v_0) = y_1$, $\pi(v_n) = y_{n+1} - y_n, \|v_n\| < \varepsilon_n$ for each $n \in \mathbb{N}$. Then $m \mapsto \sum_{n=0}^m v_n$ is Cauchy, hence convergent. Set $x := \sum_{n=0}^\infty v_n$. Then $\pi(x) = \lim_m \pi(\sum_{n=0}^m v_n) = \lim_m y_m$. Thus, $y_1, y_2, \ldots$ converges and therefore so does $z_1, z_2, \ldots$.

We now prove the Open Mapping Theorem 2.5.4, generalizing the results of [11]. We use the easily proved fact that an additive subgroup (in particular, an absolutely convex subset) of $E$ with a non-empty interior is open.

**PROPOSITION 2.5.2** Let $E$ be a normed space, let $F$ be a Banach space, let $T \in \mathcal{L}(E,F)$ be surjective. Then, for each ball $B$ about 0 in $E$, $\overline{TB}$ is a zero neighbourhood in $F$.

Proof. Suppose $E$ is $X$-normed for some $G$-module $X$. Let $s_1 < s_2 < \cdots$ be a cofinal sequence in $X$. Set $B_n := \{x \in E : \|x\| \leq s_n\}$ ($n \in \mathbb{N}$). Then $\bigcup_n \overline{TB_n} = F$, so by the Baire Category Theorem and absolute convexity, $\overline{TB_n}$ is open for some $n$. Then $\overline{TB_n}$ is open for all $n$ and $\overline{TB}$ is open.

**PROPOSITION 2.5.3** Let $E$ be a Banach space, let $F$ be a normed space, let $T \in \mathcal{L}(E,F)$. If $B \subset E$ is a ball about 0 and $\overline{TB}$ is a zero neighbourhood in $F$ then $\overline{TB} = TB$. In particular $\overline{TB}$ is clopen, $T$ is surjective and open.

Proof. Suppose $S := \{z \in F : \|z\| < s\} \subset \overline{TB}$. It suffices to prove $S \subset TB$. Let $z \in S$, let $\lambda_1, \lambda_2, \ldots \in K$ be such that $1 > |\lambda_1| > |\lambda_2| > \cdots$, $\lim_n \lambda_n = 0$. Set $\mu_1 := 1, \mu_n := \prod_{i=1}^{n-1} \lambda_i$ ($n \geq 2$). Inductively we can select $b_1, b_2, \ldots \in B$ and $z_1, z_2, \ldots \in S$ such that for all $n \in \mathbb{N}$

$$z = \sum_{i=1}^n \mu_i T b_i + \mu_{n+1} z_n, \quad (*)$$
Now $\|\mu_n b_n\| \leq |\lambda_n - 1| \|b_n\| \to 0$ by boundedness of $B$ so, by completeness of $E$, $a := \sum_{i=1}^{\infty} \mu_i b_i$ exists. We also have $\|\mu_{n+1} z_n\| \leq |\mu_{n+1}| s \to 0$, so from (*) we obtain $z = Ta$.

The following corollary is obtained from 2.5.2 and 2.5.3 by standard classical arguments.

**COROLLARY 2.5.4** Let $E, F$ be Banach spaces.

(i) (Open Mapping Theorem). Let $T \in \mathcal{L}(E, F)$ be continuous and surjective. Then $T$ is open.

(ii) (Closed Graph Theorem). Let $T : E \to F$ be linear. If the graph of $T$ is closed in $E \times F$ then $T$ is continuous.

As an application we obtain the following.

**THEOREM 2.5.5** Let $X$ be a complete $G$-module, let $E, F$ be $X$-normed Banach spaces and suppose $\mathcal{L}(E, F) = \text{Lip}(E, F)$ (e.g. if $\dim F < \infty$, see 2.3.5). Then the uniform norms and the Lipschitz norm are equivalent.

Proof. By 2.3.6 both $\mathcal{L}(E, F)$ and $\text{Lip}(E, F)$ are Banach spaces. As the uniform norms are weaker than the Lipschitz norm, the identity $\text{Lip}(E, F) \to \mathcal{L}(E, F)$ is continuous. Now apply 2.5.4 (i).

**REMARK** We failed to prove 2.5.5 directly (i.e. without using the Open Mapping Theorem).

**THEOREM 2.5.6** (Uniform Boundedness Principle). Let $E$ be an $X$-normed Banach space, let $F$ be a $Y$-normed space, where $X, Y$ are $G$-modules. If $\{T_i : i \in I\} \subset \mathcal{L}(E, F)$ is pointwise bounded then it is bounded in the uniform topology of $\mathcal{L}(E, F)$.

Proof. Let $t_1, t_2, \ldots$ be a cofinal sequence in $Y$. For each $n$ let $A_n := \{x \in E : ||T_i x|| \leq t_n$ for all $i \in I\}$. Then each $A_n$ is closed and absolutely convex. By assumption $\bigcup A_n = E$, by the Baire Category Theorem $A_n$ is open for some $n$, and hence it contains a ball $\{x \in E : ||x|| \leq s\}$ for some $s \in X$. We see that $||T_i||_s \leq t_n$ for each $i$.

**COROLLARY 2.5.7** (Banach Steinhaus). Let $E, F$ be Banach spaces, let $T_1, T_2, \ldots$ be in $\mathcal{L}(E, F)$ such that, for each $x \in E$, $T_1 x, T_2 x, \ldots$ is Cauchy in $F$. Then $T : x := \lim_n T_n x$ exists for each $x \in E$ and $T \in \mathcal{L}(E, F)$. For each $s \in ||E|| \setminus \{0\}$ we have $||T||_s \leq \sup_m \inf_{n \geq m} ||T_n||_s$.

Proof. The set $\{T_1, T_2, \ldots\}$ is pointwise bounded, hence, by 2.5.6, $n \mapsto ||T_n||_s$ is bounded by, say, $t$. Then, for each $x \in B_E(0, s)$ we have either $Tx = 0$ or $||Tx|| = ||T_n x||$ for large $n$, so $||T||_s \leq t$ i.e. $T \in \mathcal{L}(E, F)$. Obviously, for each $x \in B_E(0, s)$ there is an $m \in \mathbb{N}$ with $||T x|| = \inf_{n \geq m} ||T_n x|| \leq \inf_{n \geq m} ||T_n||_s$, hence $||T x|| \leq \sup_m \inf_{n \geq m} ||T_n||_s$ and we are done.
REMARK For a counterexample to the ‘Lipschitz version’ of 2.5.6 or 2.5.7 see [10].

3 SPACES OF COUNTABLE TYPE

From now on we assume that $K$ is complete and satisfies the conditions $(\alpha) - (\delta)$ of Proposition 1.4.4 i.e. we assume that each absolutely convex subset of $K$ is countably generated as a $B_K$-module. Then $K$ is metrizable (Remark following 1.4.4).

3.1 Countably generated $B_K$-modules

As an algebraic introduction we prove that $B_K$-submodules of countably generated $B_K$-modules are themselves countably generated (Theorem 3.1.4).

Clearly, if $B$ is a $B_K$-submodule of a countably generated $B_K$-module $A$, then $A/B$ is countably generated. We also have the following.

LEMMA 3.1.1 Let $B$ be a submodule of a $B_K$-module $A$. If $B$ and $A/B$ are countably generated then so is $A$.

Proof. Let $\pi : A \to A/B$ be the canonical homomorphism. Let $a_1, a_2, \ldots \in A$ be such that \{\(\pi(a_1), \pi(a_2), \ldots\)\} generates $A/B$, let $b_1, b_2, \ldots \in B$ be such that \{\(b_1, b_2, \ldots\)\} generates $B$. We prove that \{\(a_1, a_2, \ldots\)\} $\cup$ \{\(b_1, b_2, \ldots\)\} generates $A$. In fact, let $x \in A$. Then there exist an $m \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_m \in B_K$ such that $\pi(x) = \sum_{i=1}^{m} \lambda_i \pi(a_i) = \pi(\sum_{i=1}^{m} \lambda_i a_i)$. So $x - \sum_{i=1}^{m} \lambda_i a_i \in B$ and there exists an $n \in \mathbb{N}$ and $\mu_1, \mu_2, \ldots, \mu_n \in B_K$ such that $x - \sum_{i=1}^{m} \lambda_i a_i = \sum_{i=1}^{n} \mu_i b_i$, and we are done.

LEMMA 3.1.2 Absolutely convex subsets of finite-dimensional vector spaces over $K$ are countably generated.

Proof. For one-dimensional vector spaces this is just our assumption made at the beginning of this Section. Suppose the conclusion of the Lemma holds for absolutely convex subsets of vector spaces of dimension $\leq n-1$. Let $A$ be an absolutely convex subset of an $n$-dimensional space $E$. To prove that $A$ is countably generated we may suppose $A \neq \{0\}$, so let $a \in A$, $a \neq 0$ and $Ka := \{\lambda a : \lambda \in K\}$. We have the sequences

$$Ka \rightarrow E \rightarrow E/Ka$$

and

$$(Ka) \cap A \rightarrow A \rightarrow A/(Ka) \cap A$$

(the arrows indicating the natural maps) and the inclusions $(Ka) \cap A \rightarrow Ka$ and $A \rightarrow E$. There is a unique $B_K$-module homomorphism $\varphi : A/(Ka) \cap A \rightarrow E/Ka$
making the diagram
\[
\begin{array}{ccc}
K\alpha & \rightarrow & E \\
\uparrow & & \uparrow \\
(K\alpha) \cap A & \rightarrow & A \\
& & \uparrow \varphi \\
& & A/(\varphi)^{-1}(A)
\end{array}
\]
commute. This \( \varphi \) is injective. Now \( \dim E/K\alpha \leq n - 1 \), so by the induction hypothesis \( A/(\varphi)^{-1}(A) \cap A \) is countably generated and so is \( (K\alpha) \cap A \). Now apply 3.1.1 to conclude that \( A \) is countably generated.

Let us denote the direct sum \( \{(\lambda_1, \lambda_2, \ldots) \in B_K^N : \lambda_n = 0 \text{ for large } n \} \) by \( B_K^{(N)} \).

**Lemma 3.1.3** Every \( B_K \)-submodule of \( B_K^{(N)} \) is countably generated.

**Proof.** For each \( n \in \mathbb{N} \), let \( D_n := \{(\lambda_1, \lambda_2, \ldots) \in K^N : \lambda_m = 0 \text{ for } m > n \} \). If \( A \) is a submodule of \( B_K^{(N)} \) then \( A = \bigcup_n D_n \cap A \). By 3.1.2 each \( D_n \cap A \) is countably generated, hence so is \( A \).

**Theorem 3.1.4** Any submodule of a countably generated \( B_K \)-module is countably generated.

**Proof.** Let \( \{e_1, e_2, \ldots\} \subset A \) generate \( A \), let \( B \) be a submodule of \( A \). The formula
\[
\pi((\lambda_1, \lambda_2, \ldots)) = \sum_i \lambda_i e_i
\]
defines a surjective homomorphism \( \pi : B_K^{(N)} \rightarrow A \). By Lemma 3.1.3 \( \pi^{-1}(B) \) is countably generated, hence so is \( \pi(\pi^{-1}(B)) = B \).

### 3.2 Spaces of countable type, their subspaces and quotients

**Definition 3.2.1** (see [14] p. 66). A normed space over \( K \) is called of countable type if there is a countable set whose linear hull is dense.

If \( K \) is separable then 'of countable type' is identical to 'separable'. Spaces with a Schauder base (2.4.16) are of countable type, but we will see in 3.2.13 that the converse is not true (like in the complex case). Quotients (by closed subspaces) of spaces of countable type are of countable type. That subspaces of spaces of countable type are again of countable type is more difficult to prove. (If \( K \) is separable there is no problem: it is simply the fact that a subset of a separable space is separable. If the valuation is of rank 1 we have [14], 3.16 but that proof uses the existence of a Schauder base, which is no longer true in infinite rank case. We shall give a proof in 3.2.4 based upon ideas used in [12] to prove a similar theorem for locally convex \( B_K \)-modules. Observe that 3.2.4 only works for base fields satisfying 1.4.4. It is an intriguing open problem whether subspaces of spaces of countable type are of countable type in case the base field does not satisfy 1.4.4). To this end we introduce the following. We will say that an absolutely convex subset \( A \) of a normed space \( E \) is a \( B_K \)-module of countable type if there is a countable set \( S \subset A \)
such that \(co S := \{\lambda_1 s_1 + \cdots + \lambda_n s_n : n \in \mathbb{N}, s_1, \ldots, s_n \in A, \lambda_1, \ldots, \lambda_n \in B_K\}\) is dense in \(A\). For normed spaces this notion coincides with 3.2.1:

**PROPOSITION 3.2.2** A normed space is of countable type if and only if it is a \(B_K\)-module of countable type.

Proof. We need to prove the 'only if' part. Suppose \(E\) is an \(X\)-normed space of countable type where \(X\) is a \(G\)-module, let \(\{e_1, e_2, \ldots\} \subset E\) be such that its linear hull is dense in \(E\). By our assumption made at the beginning of this Section there are \(\lambda_1, \lambda_2, \ldots \in K\) such that \(0 < |\lambda_1| < |\lambda_2| < \cdots\) is cofinal in \(G\). Set \(S := \{\lambda_i e_j : i, j \in \mathbb{N}\}\). Then \(S\) is countable. To show that \(co S\) is dense in \(E\), let \(x \in E, \varepsilon \in X\). There are \(m \in \mathbb{N}, \mu_1, \ldots, \mu_m \in K\) such that \(\|x - \sum_{i=1}^{m} \mu_i e_i\| < \varepsilon\). By cofinality there is an \(n \in \mathbb{N}\) such that \(|\lambda_n| > \max\{\|\mu_i\| : 1 \leq i \leq m\}\). Then \(\lambda_n e_i \in S, \mu_i \lambda_n^{-1} \in B_K\) for each \(i\) and \(\|x - \sum_{i=1}^{m} \mu_i \lambda_n^{-1}(\lambda_n e_i)\| < \varepsilon\).

**PROPOSITION 3.2.3** Let \(A\) be an absolutely convex subset of an \(X\)-normed space \(E\). Then \(A\) is a \(B_K\)-module of countable type if and only if for each \(\varepsilon \in X\) the module \(A/A \cap B_E(0, \varepsilon)\) is countably generated.

Proof. The 'only if' is obvious, so suppose \(A/A \cap B_E(0, \varepsilon)\) is countably generated for each \(\varepsilon \in X\). Let \(e_1 > e_2 > \cdots\) be a coinitial sequence in \(X\), let \(S_1, S_2, \ldots\) be countable subsets of \(A\) such that, for each \(n\), \(\pi_n(S_n)\) generates \(A/A \cap B_E(0, \varepsilon_n)\) (where \(\pi_n : A \to A/A \cap B_E(0, \varepsilon_n)\) is the canonical map). Then \(S := \bigcup_n S_n\) is countable. We show that \(co S\) is dense in \(A\). In fact, let \(a \in A, \varepsilon \in X\), choose \(n\) such that \(\varepsilon_n < \varepsilon\). There are \(m \in \mathbb{N}, \lambda_1, \ldots, \lambda_m \in B_K, a_1, \ldots, a_m \in S_n\) such that \(\pi_n(a) = \sum_{i=1}^{m} \lambda_i a_i\).

Hence, \(\pi_n(a - \sum_{i=1}^{m} \lambda_i a_i) = 0\) \(i.e.\) \(\|a - \sum_{i=1}^{m} \lambda_i a_i\| \leq \varepsilon_n < \varepsilon\).

**THEOREM 3.2.4** Let \(E\) be a normed space of countable type. Then each subspace is of countable type. More generally, each absolutely convex subset of \(E\) is a \(B_K\)-module of countable type.

Proof. Suppose \(E\) is \(X\)-normed for some \(G\)-module \(X\). By 3.2.2 and 3.2.3, for each \(\varepsilon \in X\) the \(B_K\)-module \(E/B_E(0, \varepsilon)\) is countably generated. Now let \(A \subset E\) be absolutely convex. There is a natural injective homomorphism \(A/A \cap B_E(0, \varepsilon) \to E/B_E(0, \varepsilon)\), so by 3.1.4 the \(B_K\)-module \(A/A \cap B_E(0, \varepsilon)\) is countably generated for each \(\varepsilon \in X\). Again by 3.2.3 we conclude that \(A\) is of countable type.

The following Proposition shows that being of countable type is a so-called 3-space property.

**PROPOSITION 3.2.5** Let \(E\) be a normed space, let \(D\) be a closed subspace. If \(D\) and \(E/D\) are of countable type then so is \(E\).

Proof. We may assume that \(E, D, E/D\) are \(X\)-normed spaces for some complete \(G\)-module \(X\). Let \(S \subset D, T \subset E\) be countable sets such that the linear hulls of \(S\)
and \( \pi(T) \) are dense in \( D, E/D \) respectively. (Here, \( \pi : E \rightarrow E/D \) is the canonical map.) We claim that the linear hull of \( S \cup T \) is dense in \( E \). In fact, let \( a \in E \), let \( \varepsilon \in X \). There is an element \( x \) in the linear hull of \( T \) such that \( ||\pi(a) - \pi(x)|| < \varepsilon \). By 2.2.2 there is \( y \in E, ||y|| < \varepsilon \) with \( \pi(a-x) = \pi(y) \), i.e. \( a-x-y \in D \). There is an element \( z \) in the linear hull of \( S \) such that \( ||a-x-y-z|| < \varepsilon \). Then \( ||a-x-z|| < \varepsilon \) and we are done.

In [17] we described the strict quotients of \( c_0 \). In this paper we need the following characterization of all quotients of \( c_0 \). Clearly, if \( E \) is such a quotient it is a Banach space (2.5.1) of countable type and it is \( G^\# \)-normed. Surprisingly this turns out to be sufficient, as the following theorem shows.

**THEOREM 3.2.6** Let \( E \) be a \( G^\# \)-normed Banach space of countable type. Then \( E \) is a quotient of \( c_0 \).

**Proof.** Let \( B := \{x \in c_0 : ||x|| < 1\} \), \( S := \{z \in E : ||z|| < 1\} \). By 3.2.4, \( S \) is of countable type as a \( B_K \)-module, so let \( z_1, z_2, \ldots \in S \setminus \{0\} \) be such that \( \operatorname{co}\{z_1, z_2, \ldots\} \) is dense in \( S \). By 1.1.4 (iv) we can choose for each \( n \) a \( \lambda_n \in K \) such that \( ||z_n|| \leq ||\lambda_n|| < 1 \). Let \( e_1, e_2, \ldots \) be the canonical base of \( c_0 \). The formula

\[
\pi(\sum_{n=1}^{\infty} \xi_n e_n) = \sum_{n=1}^{\infty} \xi_n \lambda_n^{-1} z_n \quad (\xi_n \in K, ||\xi_n|| \rightarrow 0)
\]

defines a continuous linear map \( \pi : c_0 \rightarrow E \). Obviously \( \pi(B) \subset S \). For each \( n \), \( z_n = \pi(\lambda_n e_n) \in \pi(B) \), so \( \operatorname{co}\{z_1, z_2, \ldots\} \subset \pi(B) \); it follows that \( \pi(B) \) is dense in \( S \).

Proposition 2.5.3 tells us now that \( \pi(B) = S \). Via scalar multiplication we arrive at \( \pi(B_{c_0}(0,r^-)) = B_E(0,r^-) \) for all \( r \in G \). If \( r \in G^\# \setminus G \) observe that

\[
\pi(B_{c_0}(0,r^-)) = \pi(\bigcup_{g \in G} B_{c_0}(0,g^-)) = \bigcup_{g \in G} B_E(0,g^-) = B_E(0,r^-).
\]

Now apply 2.2.2 to conclude that \( \pi \) is a quotient map.

**COROLLARY 3.2.7** For each Banach space \( E \) of countable type there exists a linear continuous open surjection \( c_0 \rightarrow E \).

**Proof.** By 2.1.9 there is a \( G^\# \)-norm \( || \cdot ||' \) on \( E \), equivalent to the initial norm \( || \cdot || \). By 3.2.6 there is a quotient map \( \pi : c_0 \rightarrow (E, || \cdot ||') \). Then \( \rho \circ \pi \), where \( \rho : (E, || \cdot ||') \rightarrow (E, || \cdot ||) \) is the identity map, is the required surjection.

To prove the related result 3.2.11 we need some preparatory observations. As usual, \( [X] \) is the linear span of \( X \subset E \), \( \overline{[X]} \) its closure. Further, \( ||E|| := \{||x|| : x \in E\} \).

**PROPOSITION 3.2.8** Let \( E \) be a normed space, \( x_1, \ldots, x_n \in E \setminus \{0\} \). If \( ||x_i|| \notin G||x_j|| \) whenever \( i \neq j \) then \( x_1, \ldots, x_n \) are orthogonal.

**Proof.** Let \( \lambda_1, \ldots, \lambda_n \in K \). If \( i, j \in \{1, \ldots, n\} \), then either \( ||\lambda_i x_i|| = ||\lambda_j x_j|| = 0 \) or \( ||\lambda_i x_i|| \neq ||\lambda_j x_j|| \). So, if not all \( \lambda_i \) are 0 there is a unique \( j \) for which \( \max_i ||\lambda_i x_i|| = ||\lambda_j x_j|| \). It follows that \( ||\sum_{i \neq j} \lambda_i x_i|| < ||\lambda_j x_j|| \) so \( ||\sum_{i=1}^{n} \lambda_i x_i|| = ||\lambda_j x_j|| = \max_i ||\lambda_i x_i|| \).
PROPOSITION 3.2.9 In a space of countable type each orthogonal subset of vectors is (at most) countable.

Proof. Suppose $E$ is a space of countable type, let $\{e_i : i \in I\}$ be an orthogonal set in $E$, where $I$ is uncountable. Set $D := \{ e_i : i \in I \}$. Then $D$ is of countable type (3.2.4), let $x_1, x_2, \ldots \in D$ be such that its linear hull is dense in $D$. Clearly for each $n$ there is a countable set $I_n \subset I$ such that $x_n \in \{ e_i : i \in I_n \}$. It follows that $D = \{ e_i : i \in J \}$ where $J = \bigcup_n I_n$, a countable set, which is a contradiction.

COROLLARY 3.2.10 Let $E$ be an $X$-normed space of countable type, where $X$ is some $G$-module. Then there is a countable set $S \subset X$ such that $\|E\| \{0\} = GS$.

Proof. If the conclusion were false we could find an uncountable set $\{x_i : i \in I\} \subset E \backslash \{0\}$ for which $\|x_i\| \notin G\|x_j\|$ whenever $i \neq j$. By 3.2.8 the set $\{x_i : i \in I\}$ is orthogonal, so $I$ is at most countable because of 3.2.9. This is a contradiction.

THEOREM 3.2.11 Each Banach space $E$ of countable type is a quotient of a Banach space with a (countable) orthogonal base.

Proof. Let $S = \{r_1, r_2, \ldots \} \subset X$ be such that $\|E\| \{0\} = GS$ (3.2.10). Without loss, assume that $X = GS$. For each $n \in \mathbb{N}$ let $z_{n1}, z_{n2}, \ldots \in B_E(0, r_n^-)$ be nonzero vectors such that $co\{z_{n1}, z_{n2}, \ldots \}$ is dense in $B_E(0, r_n^-)$. Let $F$ be the space of all $(\xi_{nm}) \in K^{\mathbb{N} \times \mathbb{N}}$ for which $\lim_{m,n} |\xi_{nm}| = 0$, normed by $(\xi_{nm}) \mapsto \max_{n,m} |\xi_{nm}|$. Let $(e_{nm})$ be the natural orthogonal base of $F$. The formula

$$\pi((\xi_{nm})) = \pi(\sum_{n,m=1}^{\infty} \xi_{nm} e_{nm}) = \sum_{n,m=1}^{\infty} \xi_{nm} z_{nm}$$

defines a continuous linear map $\pi : F \to E$. Obviously, $\|\pi(x)\| \leq \|x\|$ for each $x \in F$ so $\pi(B_F(0, r_n^-)) \subset B_E(0, r_n^-)$ for each $n$. For each $n, m$ we have $z_{nm} = \pi(e_{nm}) \in \pi(B_F(0, r_n^-))$, so $co\{z_{n1}, z_{n2}, \ldots \} \subset \pi(B_F(0, r_n^-))$, it follows that $\pi(B_F(0, r_n^-))$ is dense in $B_E(0, r_n^-)$. From 2.5.3 we obtain $\pi(B_F(0, r_n^-)) = B_E(0, r_n^-)$ for each $n$. Scalar multiplication shows that $\pi(B_F(0, r^-)) = B_E(0, r^-)$ for all $r \in GS = \|E\| \{0\}$.

To conclude this section we will present an example of a Banach space of countable type (in fact, a separable Banach space) without a Schauder base. To this end we first prove the following Proposition whose proof is basically classical.

PROPOSITION 3.2.12 Let $E$ be a Banach space with a Schauder base $e_1, e_2, \ldots$. Then $E$ is linearly homeomorphic to a Banach space with an orthogonal base.

Proof. Let $T : c_0(\mathbb{N}, s) \to E$ be the linear map given by

$$(\lambda_1, \lambda_2, \ldots) \mapsto \sum_{n=1}^{\infty} \lambda_n e_n$$

where $s(n) = \|e_n\|$ for each $n \in \mathbb{N}$. $T$ is well-defined since $\|\lambda_n e_n\| \to 0$. For $x = (\lambda_1, \lambda_2, \ldots) \in c_0(\mathbb{N}, s)$ we have $\|Tx\| = \| \sum_n \lambda_n e_n \| \leq \max_n \|\lambda_n e_n\| = \|x\|$, so
$T$ is continuous. Bijectivity follows from the fact that $e_1, e_2, \ldots$ is a Schauder base. By the Open Mapping Theorem 2.5.4, $T$ is a homeomorphism.

**EXAMPLE 3.2.13** (A separable Banach space $E$ without Schauder base). In classical analysis over $\mathbb{R}$ or $\mathbb{C}$ one has Enflo's famous example of a separable Banach space without a Schauder base. In contrast to this, in non-archimedean rank 1 theory any Banach space of countable type has a Schauder base [14]. The same conclusion holds in arbitrary rank case when the base field is spherically complete (3.4.2). Surprisingly we can construct a separable Banach space $E$ without Schauder base over an infinite rank valued, non-spherically complete base field as follows. By [17] 2.2 there is such a (separable) field for which $C_0$ admits a closed subspace $S$ and a $g \in S'$ that cannot be extended to an element of $c_0$. Let $E := c_0/\text{Ker } g$, let $\pi : c_0 \to c_0/\text{Ker } g$ be the quotient map. Let $a \in S$ be such that $g(a) = 1$. If $\varphi \in (c_0/\text{Ker } g)'$ and $\varphi(\pi(a)) = 1$ then $\varphi \circ \pi = g$ on $S = K + \text{Ker } g$ conflicting the non-extendability of $g$. Thus, the elements of $E'$ do not separate the points of $E$. If $E$ had a Schauder base $e_1, e_2, \ldots$ then the coordinate functions would be continuous by 3.2.12 so if $f(x) = 0$ for all $f \in E'$ and some $x = \sum_{n=1}^{\infty} \lambda_n e_n \in E$ then all $\lambda_n = 0$ i.e. $x = 0$, a contradiction. It follows that $E$ has no Schauder base.

### 3.3 Finite-dimensional spaces with an orthogonal base

This section is a stepping stone for 3.4. The results do not differ from the ones in [14]. We start with a general lemma.

**LEMMA 3.3.1** Let $E$ be a normed space for which every onedimensional subspace is orthocomplemented. Then so is every finite-dimensional subspace.

Proof. (After [14] 4.35 (iii)). It suffices to prove the following. If $D_1 \subset D$ are subspaces, $\dim D / D_1 = 1$, $D_1$ has an orthogonal complement, then so has $D$. To prove this, let $S_1$ be an orthogonal complement of $D_1$. Then $D \cap S_1$ is an orthogonal complement of $D_1$ in $D$, so $\dim D / S_1 = 1$. By assumption $D \cap S_1$ has an orthogonal complement $S_2$. One verifies directly that $S_1 \cap S_2$ is an orthogonal complement of $D$. (Clearly $S_1 \cap S_2$ is orthogonal to $D_1$ from $E = S_2 + D \cap S_1$ it follows that $S_1 = S_1 \cap S_2 + D \cap S_1$, so $E = D_1 + S_1 = D_1 + S_1 \cap S_2 + D \cap S_1 = D + S_1 \cap S_2$.)

**THEOREM 3.3.2** For a finite-dimensional normed space $E$ the following are equivalent.

1. $E$ has an orthogonal base.
2. Every subspace has an orthogonal complement.
3. Every subspace has an orthogonal base.
4. Every orthogonal set of nonzero vectors can be extended to an orthogonal base of $E$.

Proof. $(\alpha) \Rightarrow (\beta)$. By Lemma 3.3.1 it suffices to prove that each onedimensional subspace has an orthogonal complement. To this end, let $e_1, \ldots, e_n$ be an orthogonal base of $E$ and let $a = \sum_{i=1}^{n} \lambda_i e_i$ ($\lambda_i \in K$) be a non-zero vector. There is an $m \in \{1, \ldots, n\}$ for which $\|a\| = \|\lambda_m e_m\|$; we prove that $K a \perp S := \{ e_i : i \neq m \}$;
it suffices to show that \( \|a - s\| \geq \|a\| \) for all \( s \in S \). So let \( s = \sum_{i \neq m} \mu_i e_i \in S \). Then \( \|a - s\| = \|\lambda_m e_m + \sum_{i \neq m} (\lambda_i - \mu_i) e_i\| \geq \|\lambda_m e_m\| = \|a\| \). To prove \((\beta) \Rightarrow (\delta)\), let \( e_1, \ldots, e_m \) be a maximal orthogonal set of nonzero vectors in \( E \), let \( D = \{ e_1, \ldots, e_m \} \). By \((\beta)\), \( D \) has an orthogonal complement; by maximality this complement must be \([0] \). Hence \( E = \{ e_1, \ldots, e_m \} \) and we are done. Obviously \((\delta) \Rightarrow (\alpha)\), so at this stage we have proved the equivalence of \((\alpha), (\beta), (\delta)\). Now if \((\beta)\) holds for \( E \) it holds for every subspace of \( E \). But then also \((\alpha)\) holds for subspaces i.e. we have \((\alpha) \Rightarrow (\gamma)\). As trivially \((\gamma) \Rightarrow (\alpha)\) this completes the proof.

**COROLLARY 3.3.3** If \( K \) is spherically complete each finite-dimensional normed space has an orthogonal base and is spherically complete.

Proof. Combining 2.4.6, 3.3.1 and 3.3.2 we conclude that a finite-dimensional normed space \( E \) has an orthogonal base, say \( e_1, \ldots, e_n \). Spherical completeness can be proved inductively, using 2.4.5 and the easily proved fact that if \( D_1 \) and \( D_2 \) are orthogonal subspaces both spherically complete then \( D_1 + D_2 \) is spherically complete.

**REMARK** If \( K \) is not spherically complete there exist two-dimensional normed spaces without an orthogonal base (14, p. 69, 17, Lemma 1.4).

### 3.4 Spaces of countable type with an orthogonal base

**THEOREM 3.4.1** Let \( E \) be a Banach space of countable type. Then the following are equivalent.

1. \( E \) has an orthogonal base.
2. Each closed subspace has an orthogonal base.
3. Each finite-dimensional subspace has an orthogonal base.
4. Each one-dimensional subspace has an orthogonal complement.
5. Each finite-dimensional subspace has an orthogonal complement.
6. For each finite-dimensional subspace \( D \) and \( a \in E \) the set \( \{\|a - d\| : d \in D\} \) has a minimum.

Proof. \((\alpha) \Rightarrow (\delta)\) (similar to \((\alpha) \Rightarrow (\beta)\) of 3.3.2). Let \( e_1, e_2, \ldots \) be an orthogonal base of \( E \), let \( a = \sum_{i=1}^{\infty} \lambda_i e_i \) \( (\lambda_i \in K, \lambda_i e_i \rightarrow 0) \) be a nonzero vector. There is an \( m \in \mathbb{N} \) such that \( \|a\| = \|\lambda_m e_m\| \). Then \( [e_i, i \neq m] \) is an orthogonal complement of \( Ka \). \((\delta) \Rightarrow (\varepsilon)\) is Lemma 3.3.1. Now we prove \( (\varepsilon) \Rightarrow (\zeta)\). Let \( D \) have an orthogonal complement \( S \), write \( a = d_1 + s \) when \( d_1 \in D, s \in S \). Then for each \( d \in D \) we have \( \|a - d\| = \|s + d_1 - d\| = \max(\|s\|, \|d_1 - d\|) \geq \|s\| \), so \( \min(\|a - d\| : d \in D) = \|a - d_1\| = s \). To prove \((\zeta) \Rightarrow (\beta)\), let \( D \) be a closed subspace of \( E \). By 3.2.4 \( D \) is of countable type, let \( x_1, x_2, \ldots \) be linearly independent elements of \( D \) such that \( D = [x_1, x_2, \ldots] \). We construct inductively an orthogonal sequence \( e_1, e_2, \ldots \) in \( D \) such that \( [e_1, \ldots, e_n] = D_n := [x_1, \ldots, x_n] \) for each \( n \). (Then we will be done by 2.4.17.) Set \( e_1 := x_1 \). Suppose we have constructed \( e_1, \ldots, e_{m-1} \). By assumption there is a \( d_1 \in D_{m-1} \) such that \( \|x_m - d_1\| \geq \|x_m - d_1\| \) for all \( d \in D_{m-1} \). Set \( e_m := x_m - d_1 \). Let \( \lambda_1, \ldots, \lambda_{m-1} \in K \). Then \( \|e_m - \sum_{i=1}^{m-1} \lambda_i e_i\| \geq \inf(\|x_m - d\| : d \in D_{m-1}) \geq \|x_m - d_1\| = \|e_m\| \), which proves orthogonality. As
(β) ⇒ (α) is trivial we now have established the equivalence of (α), (β), (δ), (ε), (ζ). Obviously (β) ⇒ (γ). We complete the proof by showing (γ) ⇒ (ζ). From (γ) we obtain that \( D + Ka \) has an orthogonal base so, by 3.3.2, \( D \) has an orthogonal complement, say, \( Kb \) in \( D + Ka \). Then \( a = \lambda b + d_1 \) where \( \lambda \in K \), \( d_1 \in D \) and clearly \( \|a - d\| : d \in D \} \subseteq \{\|\lambda b + d_1 - d\| : d \in D \} \subseteq \{\|\lambda b - d\| : d \in D \} \) has a minimum viz. \( \|\lambda b\| \).

COROLLARY 3.4.2 Each Banach space of countable type over a spherically complete field has an orthogonal base.

Proof. 3.3.3 and 3.4.1 (γ) ⇒ (α).

REMARK Spaces of countable type with the property that every closed subspace has an orthogonal complement will be treated in Section 4.

We now will define and prove a canonical orthogonal decomposition of a Banach space with an orthogonal base.

DEFINITION 3.4.3 Let \( E_1, E_2, \ldots \) be \( X \)-normed Banach spaces, where \( X \) is some \( G \)-module. The orthogonal direct sum \( \bigoplus_n E_n \) of \( E_1, E_2, \ldots \) is the subspace of \( \prod_n E_n \) consisting of all \( x = (x_1, x_2, \ldots) \) for which \( \lim_n \|x_n\| = 0 \), normed by \( x \mapsto \max_n \|x_n\| \). (which makes \( \bigoplus_n E_n \) into a Banach space.) In particular we say that a Banach space \( E \) is the orthogonal direct sum of the subspaces \( E_1, E_2, \ldots \) if the map \( \bigoplus_n E_n \to E \) given by \( (x_1, x_2, \ldots) \mapsto \sum_{n=1}^{\infty} x_n \) is a bijective isometry.

Let \( E \) be an \( X \)-normed Banach space of countable type, where \( X \) is a \( G \)-module. Then \( Y := \{\|x\| : x \in E, x \neq 0\} \) is a \( G \)-submodule of \( X \). From 3.2.10 we know that there is a countable set \( S \) such that \( Y = GS \). Let \( \Sigma := Y/\sim \), where \( s \sim t \) if and only if \( s \in Gt \), be the collection of algebraic types of \( Y \) (see 1.6); \( \Sigma \) is countable.

DEFINITION 3.4.4 Let, as above, \( \Sigma \) be the collection of algebraic types of the \( G \)-module \( \|E\|\{0\} = \{\|x\| : x \in E, x \neq 0\} \), where \( E \) is a Banach space of countable type. A canonical (orthogonal) decomposition of \( E \) is a decomposition into an orthogonal direct sum

\[
E = \bigoplus_{\sigma \in \Sigma} E_{\sigma}
\]

where each \( E_{\sigma} \) is a closed subspace and \( \|E_{\sigma}\|\{0\} = \sigma \).
THEOREM 3.4.5 Each Banach space $E$ of countable type with an orthogonal base has a canonical decomposition. It is unique in the following sense. If $E = \bigoplus_{\sigma \in \Sigma} E_{\sigma} = \bigoplus_{\sigma \in \Sigma} F_{\sigma}$ are two canonical decompositions then, for each $\sigma$, $E_{\sigma}$ and $F_{\sigma}$ are isometrically isomorphic.

Proof. Let $\{d_{i} : i \in V\}$ be an orthogonal base of $E$ where either $V = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$ or $V = \mathbb{N}$. For each $\sigma \in \Sigma$ set $V_{\sigma} := \{i \in V : ||d_{i}|| \in \sigma\}$ and set $E_{\sigma} := \{d_{i} : i \in V_{\sigma}\}$. Since the $V_{\sigma}$ ($\sigma \in \Sigma$) form a partition of $V$ we clearly have that $E = \bigoplus_{\sigma \in \Sigma} E_{\sigma}$ is a canonical decomposition which proves existence. To prove uniqueness, let $E = \bigoplus_{\sigma \in \Sigma} E_{\sigma} = \bigoplus_{\sigma \in \Sigma} F_{\sigma}$ be two canonical decompositions, let $\sigma \in \Sigma$, let $\{e_{i} : i \in W\}$ be an orthogonal base of $E_{\sigma}$, where $W$ is $\{1, \ldots, m\}$ for some $m \in \mathbb{N}$ or $W = \mathbb{N}$. Decompose each $e_{i}$ as follows

$$e_{i} = \sum_{\mu \in \Sigma} f_{\mu}^{i}$$

where $f_{\mu}^{i} \in F_{\mu}$ for each $\mu \in \Sigma$. Then $||e_{i}|| = \max_{\mu \in \Sigma} ||f_{\mu}^{i}||$. But $||e_{i}|| \in \sigma$, $||f_{\mu}^{i}|| \in \mu$, so $||f_{\mu}^{i}|| \neq ||e_{i}||$ i.e. $||f_{\mu}^{i}|| < ||e_{i}||$ if $\mu \neq \sigma$ and therefore we have $||e_{i}|| = ||f_{\sigma}^{i}||$ and even $||e_{i} - f_{\sigma}^{i}|| < ||e_{i}||$. By the Perturbation Lemma 2.4.8 the system $\{f_{\sigma}^{i} : i \in W\}$ is orthogonal in $F_{\sigma}$. By 2.4.14 each maximal orthogonal set in $F_{\sigma}$ has cardinality $\geq \#W$. By symmetry we have equality. It follows that orthogonal bases of $E_{\sigma}$ and $F_{\sigma}$ have the same cardinality. As $||E_{\sigma}|| = ||F_{\sigma}|| = \sigma \cup \{0\}$ one can construct a bijective isometry $E_{\sigma} \to F_{\sigma}$.

REMARK It is not difficult to see that each $E_{\sigma}$ occurring in the canonical decomposition is either finite-dimensional or linearly homeomorphic to $c_{0}$.

3.5 Compactoids

In rank 1 theory the notion of compactoidity plays a fundamental role in Functional Analysis ([14], 133-146). We recall the definition.

DEFINITION 3.5.1 A subset $A$ of a normed space $E$ is called (a) compactoid in $E$ if for each neighbourhood $U$ of zero in $E$ there exists a finite set $\{x_{1}, \ldots, x_{n}\} \subset E$ such that $A \subset co\{x_{1}, \ldots, x_{n}\} + U$. Here, $co\{x_{1}, \ldots, x_{n}\}$ is the absolutely convex hull of $\{x_{1}, \ldots, x_{n}\}$ i.e. $\{\lambda_{1}x_{1} + \cdots + \lambda_{n}x_{n} : \lambda_{1}, \ldots, \lambda_{n} \in B\mathbb{K}\}$.

PROPOSITION 3.5.2 In (i)-(viii), $E, F$ are normed spaces.

(i) Subsets of a compactoid are compactoid.
(ii) The absolutely convex hull of a compactoid is a compactoid.
(iii) The closure of a compactoid is a compactoid.
(iv) The sum of two compactoids is compactoid.
(v) If $A$ is a compactoid in $E$, $T \in L(E, F)$ then $TA$ is compactoid in $F$.
(vi) A bounded finite-dimensional subset of $E$ is compactoid in $E$.
(vii) If $Z \subset E$ is precompact then $\overline{co} Z$ is compactoid in $E$. In particular, if $x_{1}, x_{2}, \ldots \in E$, $\lim_{n \to \infty} x_{n} = 0$ then $\overline{co}\{x_{1}, x_{2}, \ldots\}$ is compactoid in $E$.
(viii) Compactoids are bounded.
Proof. Only (vi) may need a proof. Let \( A \subset K^n \) be bounded with respect to the norm \( (\xi_1, \xi_2, \ldots, \xi_n) \mapsto \max_i |\xi_i| \). There is a \( \lambda \in K \) such that \( |\lambda| > \|a\| \) for each \( a \in A \). Then \( A \subset \text{co}\{\lambda e_1, \lambda e_2, \ldots, \lambda e_n\} \) where \( e_1, \ldots, e_n \) is the canonical base of \( K^n \). It follows that \( A \) is a compactoid in \( K^n \). Now let \( A \) be a bounded subset of an arbitrary normed finite-dimensional space \( F \). By 2.3.4 there is a linear homeomorphism \( T : F \rightarrow K^n \). By the above, \( TA \) is bounded in \( K^n \), hence a compactoid. Then so is \( A = T^{-1}TA \).

**Proposition 3.5.3** Let \( A \) be a compactoid in a normed space \( E \). Then there exists a space \( H \) of countable type with \( A \subset H \subset E \) such that \( A \) is compactoid in \( H \).

Proof. Let \( E \) be \( X \)-normed for some \( G \)-module \( X \). Choose \( \delta_1 > \delta_2 > \cdots \) in \( X \), \( \inf \delta_n = 0 \). There is a finite set \( F_1 \subset E \) such that \( A \subset \text{co}\ F_1 + B_E(0, \delta_1) \). Then \( A \subset \text{co}\ F_1 + A_1 \) where \( A_1 = (A + \text{co}\ F_1) \cap B_E(0, \delta_1) \), which is a compactoid by 3.5.2. There is a finite set \( F_2 \subset E \) such that \( A_1 \subset \text{co}\ F_2 + B_E(0, \delta_2) \), etc. Inductively we arrive at finite sets \( F_1, F_2, \ldots \subset E \) such that for each \( n \)

\[
A \subset \text{co}\ (F_1 \cup \cdots \cup F_n) + B(0, \delta_n).
\]

It follows that \( A \subset \overline{[F]} \) where \( F = \bigcup F_n \). We see that \( H := [F] \) is of countable type and that \( A \) is compactoid in \( H \).

**Corollary 3.5.4** The (closed) linear hull of a compactoid is of countable type.

Proof. 3.5.3 and 3.2.4.

**Remark** In rank 1 theory we always have that if \( A \) is a compactoid in a normed space \( E \) then it is a compactoid in \([A]\), or even in \([A]\). (See [4].) In our case we don’t know this is in general, even if \( K \) is spherically complete. But we will prove 4.3.7 \( (\beta) \iff (\varepsilon) \iff (\zeta) \).

**Theorem 3.5.5** In a compactoid each orthogonal sequence tends to 0.

Proof. Suppose not. Then we could find an orthogonal sequence \( e_1, e_2, \ldots \) in some compactoid \( A \) and a \( \delta \in \|E\| \setminus \{0\} \) such that \( \|e_n\| \geq \delta \) for all \( n \). Choose \( \delta' \in \|E\| \setminus \{0\} \), \( \delta' < \delta \). By compactoidity there is a finite-dimensional subspace \( D \subset E \) such that \( A \subset D + B_E(0, \delta') \). For each \( n \), write \( e_n = d_n + \delta_n \) when \( d_n \in D, \|\delta_n\| < \delta' \). Then \( \|e_n - d_n\| < \|e_n\| \) for each \( n \), so by the Perturbation Lemma 2.4.8, \( d_1, d_2, \ldots \) are orthogonal, and non-zero hence linearly independent which conflicts the finite-dimensionality of \( D \).

**Theorem 3.5.6** Let \( A \) be a compactoid in a Banach space \( E \) with an orthogonal base \( e_1, e_2, \ldots \). Then there are absolutely convex subsets \( C_1, C_2, \ldots \) in \( K \) such that \( \text{diam} \ C_n e_n \rightarrow 0 \) and such that \( A \subset \overline{C_1 e_1 + C_2 e_2 + \cdots} \).

Proof. Let \( E \) be \( X \)-normed for some \( G \)-module \( X \). For each \( n \in \mathbb{N} \), let \( P_n \) be the canonical projection \( E \rightarrow Ke_n \). We prove that \( \sum P_n A \) is a compactoid and
that diam $P_n A \to 0$ (which will finish the proof since we may assume that $A$ is absolutely convex, so $P_n A$ has the form $C_n e_n$ for some absolutely convex set $C_n$ in $K$, and since $A \subseteq \bigcap_n P_n A$). Let $\varepsilon \in X$. There exists a finite set $F \subseteq E$ such that $A \subseteq \operatorname{co} F + B_E(0, \varepsilon)$. There is an $m \in \mathbb{N}$ such that $F \subseteq [e_1, \ldots, e_m] + B_E(0, \varepsilon)$, so we may assume that $F \subseteq [e_1, \ldots, e_m]$. For each $n$ we have, since $\|P_n x\| \leq \|x\|$ for all $x$, $P_n A \subseteq \operatorname{co} P_n F + B_E(0, \varepsilon)$. Adding up and observing that $P_n F = \{0\}$ for $n > m$ we arrive at $\sum_n P_n A \subseteq \operatorname{co} \bigcup_{1 \leq n \leq m} P_n F + B_E(0, \varepsilon)$, proving that $\sum_n P_n A$ is a compactoid. We also see that $P_n A \subseteq B_E(0, \varepsilon)$ for $n > m$ implying diam $P_n A \to 0$.

**REMARK** In rank 1 theory one can even prove that there are $\lambda_1, \lambda_2, \ldots \in K$ with $\lim_n \lambda_n e_n = 0$ and $A \subseteq \overline{\operatorname{co}}\{\lambda_1 e_1, \lambda_2 e_2, \ldots\}$, see [4]. However, we will show that this result no longer holds in our theory.

**EXAMPLE 3.5.7** (A compactoid, not contained in the closed absolutely convex hull of a sequence tending to 0.) Let $G$ be the union of a strictly increasing sequence of convex subgroups

\[
\{1\} \not\subseteq H_1 \not\subseteq H_2 \not\subseteq \ldots
\]

For each $n$, let $t_n := \inf_{G^*} H_n$, $s_n := \sup_{G^*} H_n$. Let (2.4.15) $E := c_0(\mathbb{N}, s)$ where $s(n) = t_n \ (n \in \mathbb{N})$ i.e. the space of all sequences $(\lambda_1, \lambda_2, \ldots)$ in $K^\mathbb{N}$ for which $\lim_n |\lambda_n| t_n = 0$. For the canonical base $e_1, e_2, \ldots$ of $E$ we have $\|e_n\| = t_n$ for each $n$. From 1.5.4 and its proof we have (in the $G$-module $G^\#$) for each $h \in H_n$ that $h t_n = \inf \{ h g : g \in G : g > t_n \} = \inf H_n = t_n$, so, if $C_n := \{ \lambda \in K : |\lambda| < s_n \}$ then diam $C_n e_n = t_n \to 0$. So $A := \sum_n C_n e_n$ is a compactoid. We now prove that, if $\lambda_n \in K$, $\lambda_n \not\in C_n$ then diam $C_n |\lambda_n| e_n \to 0$. In fact, by using 1.5.4 we have $\|\lambda_n e_n\| = |\lambda_n| t_n = \sup \{|\lambda_n| g : g \in G, g < t_n\} \geq |\lambda_n| |\lambda_n|^{-1} = 1$. We see that $A$ is contained in $\overline{\operatorname{co}}\{\lambda_n e_n : n \in \mathbb{N}\}$, for no $\lambda_n$ for which $\lim_n \lambda_n e_n = 0$. We finish the proof by applying the next lemma.

**LEMMA 3.5.8** Let $E$ be a Banach space with an orthogonal base $e_1, e_2, \ldots$. Then, for every sequence $x_1, x_2, \ldots$ in $E$ tending to 0, there are $\lambda_1, \lambda_2, \ldots \in K$ such that $\overline{\operatorname{co}}\{x_1, x_2, \ldots\} \subseteq \overline{\operatorname{co}}\{\lambda_1 e_1, \lambda_2 e_2, \ldots\}$ and $\|\lambda_n e_n\| \to 0$.

**Proof.** For each $n$, let

\[
x_n = \sum_{i=1}^n \xi_i^n e_i
\]

be the expansion of $x_n$. Then from $\|\xi_i^n e_i\| \leq \|x_n\|$ for each $i$ and $n$, and from $\lim_{n \to \infty} x_n = 0$ we obtain $\lim_{n \to \infty} \xi_i^n = 0$ (1.5.2), so there is a $\lambda_i \in K$ with $|\lambda_i| = \max_n |\xi_i^n|$. It is easily seen that $\lim_i |\lambda_i| |e_i| = 0$ and that $x_n = \sum_i \xi_i^n e_i \in \overline{\operatorname{co}}\{\lambda_i e_i : i \in \mathbb{N}\}$ for each $n$.

The *weak topology* on a normed space $E$ is the weakest topology for which all $f \in E'$ are continuous. In rank 1 theory it is shown that the weak and norm topology coincide on compactoids under the assumption that $E$ is a so-called polar space (see [16], 5.12). The proof does not carry over to the arbitrary rank case. We have only the following corollary of 3.5.6.

**COROLLARY 3.5.9** In a Banach space with an orthogonal base the weak and norm topology coincide on compactoids.
Proof. By 3.5.6 we may assume that $A = \sum_{n} C_{n}e_{n}$ where $e_{1}, e_{2}, \ldots$ is an orthogonal base of $E$ and $C_{1}, C_{2}, \ldots$ are absolutely convex subsets in $K$ such that $s_{n} := \text{diam } C_{n}e_{n} := \sup_{X} \{ ||\lambda e_{n}|| : \lambda \in C_{n} \} \to 0$, where $E$ is $X$-normed for some complete $G$-module $X$. Then let $i \mapsto a_{i} \ (i \in I)$ be a net in $A$ converging weakly to 0. Let, for each $i$,

$$a_{i} = \sum_{n=1}^{\infty} \lambda_{n}^{i}e_{n} \quad (\lambda_{n}^{i} \in C_{n})$$

be the expansion of $a_{i}$. The coordinate maps are continuous so $\lim_{i} \lambda_{n}^{i} = 0$ for each $n$, hence $\lim_{i} \lambda_{n}^{i}e_{n} = 0$ for each $n$. Let $\varepsilon \in X$. There is an $m \in \mathbb{N}$ such that $s_{n} < \varepsilon$ for $n > m$. There is an $i_{0} \in I$ such that $||\lambda_{n}^{i}e_{n}|| < \varepsilon$ for $n \in \{1, \ldots, m\}, i \geq i_{0}$. It follows that $||a_{i}|| < \varepsilon$ for $i \geq i_{0}$ proving that $\lim_{i} ||a_{i}|| = 0$.

4 HILBERT-LIKE SPACES

Recall that we assume throughout that $K$ is complete and satisfies the conditions $(\alpha) - (\delta)$ of Proposition 1.4.4. In this Section we study the class of norm-Hilbert spaces over $K$ (see Definition 4.1.1 below) and the -what will turn out to be- subclass of the form-Hilbert spaces introduced by H. Gross, H. Keller and U.M. Künzi in [5], [3], see 4.4.

4.1 Norm-Hilbert spaces

DEFINITION 4.1.1 A Banach space $E$ of countable type will be called norm-Hilbert space if for each (norm-) closed subspace $D$ of $E$ there exists a linear surjective projection $P : E \to D$ for which $||Px|| \leq ||x|| \ (x \in E)$ (Compare the form-Hilbert spaces of 4.4.3).

We leave the proof of the following Proposition to the reader.

PROPOSITION 4.1.2 Let $E$ be a Banach space of countable type. Then the following are equivalent.

$(\alpha)$ $E$ is a norm-Hilbert space.

$(\beta)$ Each norm-closed subspace of $E$ has a normorthogonal complement.

$(\gamma)$ Every orthogonal system of nonzero vectors in $E$ can be extended to an orthogonal base of $E$.

$(\delta)$ For every closed subspace $D$ of $E$ and every $a \in E$ the set $\{||a - d|| : d \in D\}$ has a minimum.

Theorem 4.1.3 below characterizes the norm-Hilbert spaces. It is an extension of [14], 5.16 which treats the rank 1 case. Let us say, by abuse of language, that a sequence $x_{1}, x_{2}, \ldots$ in a normed space $E$ is decreasing if $||x_{1}|| \geq ||x_{2}|| \geq \cdots$, strictly decreasing if $||x_{1}|| > ||x_{2}|| > \cdots$. In the same spirit we define (strictly) increasing sequences in $E$. 

THEOREM 4.1.3 For an infinite-dimensional Banach space of countable type $E$ the following statements $(\alpha)$ and $(\beta)$ are equivalent.

$(\alpha)$ $E$ is a norm-Hilbert space.

$(\beta)$

(i) $E$ has an orthogonal base.

(ii) Every strictly decreasing orthogonal sequence in $E$ tends to 0.

(iii) If $G$ has a maximal proper convex subgroup $H$ then $G/H \simeq \mathbb{Z}$.

The proof runs in several steps. First two lemmas.

LEMMA 4.1.4 If there exists an infinite-dimensional norm-Hilbert space over a field $K$ whose valuation has rank 1 then the valuation of $K$ is discrete.

Proof. Let $E$ be such a space, let $E$ be $X$-normed for some $G$-module $X$. We may assume that $G \subset (0,\infty)$ (see [13]). Then $G^\# \subset (0,\infty)$. Now let $\phi : X \to G^\#$ be as in 1.5.6. Like in the proof of 2.1.9 it follows that its extension $\phi : X \cup \{0\} \to [0,\infty)$ is an extended $G$-module map and that $x \mapsto \phi(||x||)$ is equivalent to $|| \cdot ||$. If $P$ is a linear projection onto a closed subspace and $||Px|| \leq ||x||$ then $\phi(||Px||) \leq \phi(||x||)$, so $E$ is a norm-Hilbert space with respect to $\phi(|| \cdot ||)$. Now apply [14], 5.16 to conclude that $K$ has discrete valuation.

LEMMA 4.1.5 Let $E$ be a Banach space.

(i) If $D_1$ and $D_2$ are spherically complete subspaces and $D_1$ and $D_2$ are normorthogonal then $D_1 + D_2$ is spherically complete.

(ii) If every strictly decreasing sequence in $E$ tends to 0 then $E$ is spherically complete.

Proof. The proof of (i) is straightforward. To prove (ii) observe that to prove spherical completeness it suffices to show that any sequence of 'open' balls $B(a_1,r_1^-) \supset B(a_2,r_2^-) \supset \cdots$ has a nonempty intersection (see our assumption at the beginning of this Section). We may assume $B(a_n,r_n^-) \neq B(a_{n+1},r_{n+1}^-)$ for all $n$. Choosing $b_n \in B(a_n,r_n^-) \setminus B(a_{n+1},r_{n+1}^-)$ we obtain $||b_1 - b_2|| > ||b_2 - b_3|| > \cdots$, so by assumption, $b_{n+1} - b_n \to 0$. By completeness $b := \lim_{n \to \infty} b_n$ exists and it follows easily that $b \in \bigcap_n B(a_n,r_n^-)$.

Proof of Theorem 4.1.3. $(\alpha) \Rightarrow (\beta)$. Clearly we have (i) (every maximal orthogonal family in $E \setminus \{0\}$ is an orthogonal base). We proceed to prove (ii) (compare the proof of $(\alpha) \Rightarrow (\iota)$ of [14], 5.16). Let $e_1,e_2,\ldots$ be a strictly decreasing orthogonal sequence in $E$. Suppose $||e_n|| > s$ for each $n \in \mathbb{N}$ and some nonzero norm value $s$. Let $D := [e_1,e_2,\ldots]$. The formula

$$\phi\left(\sum_{i=1}^{\infty} \xi_i e_i\right) = \sum_{i=1}^{\infty} \xi_i, \quad (\xi_i \in K, ||\xi_i e_i|| \to 0)$$

defines an element $\phi \in D'$. In fact, $||\xi_i e_i|| \to 0$ is equivalent to $\xi_i \to 0$, so $\phi$ is a well-defined linear map $D \to K$. To prove continuity, let $n \mapsto x_n = \sum_{i=1}^{\infty} \xi_i^n e_i$ be a sequence in $D$ tending to 0. Then since $||x_n|| \geq (\max_i ||\xi_i^n||)s$ we have $|\sum_i \xi_i^n| \leq \max_i ||\xi_i^n|| \to 0$. Now $D$ is a norm-Hilbert space so Ker $\phi$ has a normorthogonal complement. So there is an $a \in D$ such that $K a$ is normorthogonal to Ker $\phi$,
\[
\phi(a) = 1, \quad a = \sum_{n=1}^{\infty} \lambda_n e_n \quad (\lambda_n \in K, \lambda_n \to 0). \]

From
\[
1 = |\phi(a)| = |\sum_{n=1}^{\infty} \lambda_n| \leq \max |\lambda_n|
\]
it follows that \(|\lambda_i| \geq 1\) for some \(i\) and
\[
|a| = \max |\lambda_n e_n| \geq |\lambda_i e_i| > |\lambda_{i+1}|.
\]

On the other hand, \(\phi(a - e_{i+1}) = 0\) so \(a\) and \(a - e_{i+1}\) are normorthogonal whence
\[
|e_{i+1}| = |a - (a - e_{i+1})| \geq |a|,
\]
a contradiction which proves (ii). To prove (iii), let \(X\) be the \(G\)-module \(|E|\{0\}\), and let \(X/\sim\) be the canonical \(G/H\)-module constructed in 1.5.5 (f), let \(\pi : G \to G/H, \rho : X \to X/\sim\) be the natural maps. Let \(N(x) := \rho(||x||) (x \in E, x \neq 0), N(0) := 0, \) let \(\nu(\lambda) = \pi(|\lambda|) (\lambda \in K, \lambda \neq 0), \nu(0) := 0. \) Then \(\nu\) is a valuation on \(K\) of rank 1, equivalent to \(|\cdot|\), and \(N\) is a norm on \(E\). For any sequence \(x_1, x_2, \ldots\) in \(E\) we have \(\|x_n\| \to 0\) if and only if \(N(x_n) \to 0. \) If \(P\) is a linear projection onto a closed subspace \(D\) and \(\|P x\| \leq \|x\| (x \in E)\) then \(N(P x) \leq N(x) (x \in E). \) It follows that \((E, N)\) is a norm-Hilbert space over \((K, \nu)\). By 4.1.4 \(\nu\) is a discrete valuation i.e. \(G/H \simeq \mathbb{Z}. \)

Proof of Theorem 4.1.3. \((\beta) \Rightarrow (\alpha). \) Let \(e_1, e_2, \ldots\) be a maximal orthogonal sequence of nonzero vectors in \(E\), let \(F := [e_1, e_2, \ldots]; \) we prove that \(F = E. \) To this end, let \(F = \bigoplus_{\sigma \in \Sigma} F_{\sigma}\) be the canonical orthogonal decomposition of \(F\) in the sense of 3.4.4. Let \(\sigma \in \Sigma, \) let \(s \in \sigma\) be a representative. We first prove (1) and (2) below.

1. If \(\text{Stab}(s) = \{g \in G : gs = s\}\) is not a maximal proper convex subgroup of \(G\) then \(\dim F_{\sigma} < \infty.\)

2. If \(\text{Stab}(s)\) is a maximal convex subgroup then \(F_{\sigma}\) is spherically complete.

Proof of (1). If \(\dim F_{\sigma} = \infty\) it would have an orthogonal base \(f_1, f_2, \ldots\) with \(||f_n|| = s = 1\) for all \(n. \) We have rank \(G/\text{Stab}(s) > 1,\) so there is a sequence \(v_1 > v_2 > \cdots\) in \(G/\text{Stab}(s)\) with \(v_n > 1\) for all \(n. \) Choose \(\lambda_1, \lambda_2, \ldots \in K\) with \(\pi(|\lambda_n|) = v_n\) for each \(n\) where \(\pi : G \to G/\text{Stab}(s)\) is the canonical map). Then \(|\lambda_1| > |\lambda_2| > \cdots\) so that \(|\lambda_1 f_1| > ||\lambda_2 f_2\| \geq \ldots. \) If, for some \(n, ||\lambda_n f_n|| = ||\lambda_{n+1} f_{n+1}||\) then \(|\lambda_{n+1}^{-1} \lambda_n| s = s, so |\lambda_{n+1}^{-1} \lambda_n| \in \text{Stab}(s)\) implying \(v_{n+1} = v_n, a contradiction. Hence \(|\lambda_1 f_1| > ||\lambda_2 f_2\| > \ldots and therefore \lim_{n \to \infty} \lambda_n f_n = 0,\) i.e. \(\lambda_n \to 0\) or \(v_n = \pi(|\lambda_n|) \to 0,\) a contradiction.

Proof of (2). We prove (Lemma 4.1.5 (ii)) that every strictly decreasing sequence of norm values in \(F_{\sigma}\) tends to 0. Now, since \(||F_{\sigma}|| = Gs \cup \{0\}\) such a sequence has the form \(||\lambda_1| s > |\lambda_2| s > \cdots. \) Letting \(\pi : G \to G/\text{Stab}(s)\) be the canonical map we have \(\pi(|\lambda_1|) > \pi(|\lambda_2|) > \cdots. \) By (iii), \(G/\text{Stab}(s) \simeq \mathbb{Z},\) so that \(\lim_{n \to \infty} \pi(|\lambda_n|) = 0\) implying \(|\lambda_n| s \to 0. \) With (1), (2) being proved, let \(\Sigma = \{\sigma_1, \sigma_2, \ldots\}, \) let \(H_n := F_{\sigma_1} + F_{\sigma_2} + \cdots + F_{\sigma_n} (n \in \mathbb{N}). \) \(H_n\) is the orthogonal direct sum of a spherically complete space \(A\) (the sum of all \(F_{\sigma}\) which are spherically complete, 4.1.5 (i)) and a finite-dimensional space (the sum of the other \(F_{\sigma}\)). By 2.4.4, \(A\) has an orthogonal complement in \(E. \) We have \(A \subset H_n\) and \(H_n/A\) is finite-dimensional so by the proof
of 3.3.1 also $H_n$ is orthocomplemented. Now let $x \in E$. To prove $x \in F$ we may assume that $x$ is not in the union of the $H_n$. There exist $h_n \in H_n \ (n \in \mathbb{N})$ such that $\|x - h_n\| = \text{dist}(x, H_n)$. Then $x - h_n \perp H_n$ for each $n$. Now $F_{a_h}$ has an orthogonal base which is a maximal orthogonal set in \{ $x \in E : \|x\| \in G_{S_h}$ \} for each $k \ (2.4.13)$, hence $\|x - h_n\| \not\in G_{S_1 \cup S_2 \cup \cdots \cup S_n}$. Thus the sequence $\|x - h_1\| \geq \|x - h_2\| \geq \cdots$ has a subsequence $i \mapsto \|x - h_{i,n}\|$ for which $\|x - h_{i,n}\| \not\in G \|x - h_{j,n}\|$ whenever $i \neq j$. Then this subsequence is strictly decreasing. Orthogonality of $x - h_{1,n}, x - h_{2,n}, \ldots$ follows from 3.2.8. By (ii), $\lim_{i \to \infty} \|x - h_{i,n}\| = 0$ i.e. $x \in \left[ h_1, h_2, \ldots \right] \subset F$.

4.2 Examples of norm-Hilbert spaces

We will present two groups of examples, namely in cases where $G$ has or has not a maximal proper convex subgroup.

EXAMPLE 4.2.1 An infinite-dimensional norm-Hilbert space over a field whose value group has a maximal proper convex subgroup. Let $G_1$ be a linearly ordered abelian group satisfying the countability conditions of Proposition 1.4.4 (e.g., $G_1 = (0, \infty)$), let $G := G_1 \times \mathbb{Z}$, where $\mathbb{Z}$ is written multiplicatively with generator $a > 1$, ordered antilexicographically and let $K$ be a complete valued field with value group $G$. Choose $b_1, b_2, \ldots \in \mathbb{R}$ such that $1 < b_1 < b_2 < \cdots < a$ and put

$$X := \{(r, a^n b_m) : r \in G_1, \ n \in \mathbb{Z}, \ m \in \mathbb{N}\}.$$ 

With the ordering inherited from the antilexicographic ordering on $G_1 \times (0, \infty)$ and the structure map given by the formula

$$(s, a^k) \cdot (r, a^n b_m) = (sr, a^{n+k} b_m)$$

$X$ becomes a $G$-module. Let $E := c_0$ but with the norm given by

$$\|x\| = \max_n |\xi_n|(1, b_n) \quad (x = (\xi_1, \xi_2, \ldots))$$

(Indeed, since $(1, 1) < (1, b_n) < (1, a)$ for all $n$ we have that $\| \cdot \|$ is equivalent to the usual norm). We prove that $(E, \| \cdot \|)$ is a norm-Hilbert space by verifying (β) of Theorem 4.1.3. Clearly $G_1 \times \{0, 1, 0, 0, \ldots\}$ is an orthogonal base for $E$; $G_1 \times \{1\}$ is the maximal proper convex subgroup and $G/G_1 \times \{1\} \simeq \mathbb{Z}$, so it suffices to check (ii). To this end, let $f_1, f_2, \ldots$ be an orthogonal sequence such that, with $\|f_m\| = (r_m, a^{nm} b_{sm}) \ (m \in \mathbb{N})$ we have

$$(r_1, a^{n_1} b_{s_1}) > (r_2, a^{n_2} b_{s_2}) > \cdots$$

In the canonical orthogonal decomposition $E = \bigoplus_{\sigma} E_\sigma$ of $E$ each $E_\sigma$ is one-dimensional; therefore we must have that

$$a^{nm} b_{sm} \neq a^{nm+1} b_{sm+1}$$

hence $a^{nm} b_{sm} > a^{nm+1} b_{sm+1}$

for each $m$. We see that $a^{n_m-n_{m+1}} > b_{sm+1}/b_{sm} > a^{-1}$, hence $n_m - n_{m+1} > -1$ or $n_m \geq n_{m+1}$. If $n_k = n_{k+1} = \ldots$ for some $k$ we would have $b_{sk} > b_{sk+1} > \cdots$.
which is impossible since the set \( \{ b_n : n \in \mathbb{N} \} \) is well-ordered. So \( \lim_{k \to \infty} n_k = -\infty \) proving that \( \lim_{m \to \infty} \| f_m \| = 0 \).

**REMARK** In the above example \( G \) is of infinite rank, finite rank, rank 1 accordingly as \( G_1 \) is of infinite rank, of finite rank, \( \{ 1 \} \) respectively. We will see in 4.4.6 that there exist no infinite-dimensional form-orthogonal Hilbert spaces when \( G \) has a maximal convex proper subgroup.

**EXAMPLE 4.2.2** An infinite-dimensional norm-Hilbert space over a field whose value group does not have maximal proper convex subgroups. Let \( G \) be the union of a strictly increasing sequence of convex subgroups \( \{ 1 \} = H_1 \subset H_2 \subset \ldots \). Set \( s_n := \sup_{H_n} H_n, t_n := \inf_{H_n} H_n \) \( (n \in \mathbb{N}) \) and let \( E \) be the set of all \( x = (\xi_1, \xi_2, \ldots) \in K^\mathbb{N} \) for which \( \lim_{n \to \infty} |\xi_n|s_n = 0 \) and where \( \| x \| = \max_n |\xi_n|s_n. \) Then clearly \( E \) is a \( G^\# \)-normed Banach space with orthogonal base \((1,0,0,\ldots), (0,1,0,\ldots), \ldots\). To prove \( E \) to be norm-Hilbertian, let \( f_1, f_2, \ldots \in K \) be such that \( n |f_n|s_n \) is strictly decreasing; we show \( \lim_{n \to \infty} |\xi_n|s_n = 0 \). Let \( \varepsilon \in G; \) there is an \( m \) such that \( t_m < \varepsilon \) and \( |\xi_1|s_1 < s_m. \) We claim that \( |\xi_n|s_n \leq t_m \) for all \( n > m \). In fact, let \( n > m \). If \( |\xi_n| \in H_n \) then \( |\xi_n|s_n = \sup_{H_n} |\xi_n|h \geq s_n > s_m \), a contradiction. If \( |\xi_n| > s_n \) then \( |\xi_n|s_n \geq s_n > s_m \) which is again a contradiction. We see that \( |\xi_n| < t_n \) for \( n > m \). Then \( |\xi_n|s_n = \sup_{H_n} |\xi_n|h \leq t_n \leq t_m \) and we are done.

**REMARK** We are particularly interested in norm-Hilbert spaces over fields whose value group does not have maximal proper convex subgroups for two reasons. Firstly, because there do exist form-Hilbert spaces over such fields (see 4.4.9), secondly because such spaces have particular properties such as: each bounded set is a compactoid! (See 4.3.7). We devote the next section to the study of these so-called 'Keller spaces' named after the inventor of the first non-classical form-Hilbert space [5].

### 4.3 The Keller spaces

**LEMMA 4.3.1** The following statements on \( K \) are equivalent.

(\( \alpha \)) The value group \( G \) does not have maximal proper convex subgroups.

(\( \beta \)) \( G \) is the union of a strictly increasing sequence of convex subgroups.

(\( \gamma \)) There is a \( G \)-module \( X \) and a sequence \( s_1, s_2, \ldots \) in \( X \) satisfying the type condition. (See 1.6.4)

Proof. (\( \alpha \)) \( \Rightarrow \) (\( \beta \)). Let \( g_1 < g_2 < \cdots \) be a cofinal sequence in \( G \), let \( H_1 \) be the smallest convex subgroup containing \( g_1 \). If \( H_1 \) were equal to \( G \) then by [9], Prop. 3, page 14, \( G \) would have a maximal proper subgroup, so \( H_1 \neq G \), and there exists an \( n_2 > n_1 := 1 \) such that \( g_{n_2} \not\in H_1 \). Then the convex subgroup generated by \( g_{n_2} \) contains \( H_1 \) properly and is not equal to \( G \) for the same reason as above for \( H_1 \), etc.. We obtain a strictly increasing sequence \( H_1 \subset H_2 \subset \ldots \) of convex subgroups, their union is cofinal so it must be equal to \( G \). (\( \beta \)) \( \Rightarrow \) (\( \gamma \)). We proved in 4.2.2 that the sequence \( s_1, s_2, \ldots \) in \( G^\# \) satisfies the type condition. (In fact, the conclusion can be drawn just from the assumption that \( n \mapsto |\xi|s_n \) is bounded above, choose \( m \) such that \( t_m < \varepsilon \) and \( |\xi_n|s_n < s_m \) for each \( n \)) (\( \gamma \)) \( \Rightarrow \) (\( \alpha \)). This is the first part of
the proof of 1.6.6, \((\alpha) \Rightarrow (\beta)\).

**DEFINITION 4.3.2** Varying on 1.6.4 a sequence of non-zero vectors \(x_1, x_2, \ldots \) in a normed space is said to satisfy the type condition if, for each sequence \(\alpha_1, \alpha_2, \ldots \) in \(K\), boundedness above of \(\{\alpha_n x_n : n \in \mathbb{N}\}\) implies \(\lim_{n \to \infty} \alpha_n x_n = 0\).

It is not hard to see that \(x_1, x_2, \ldots \) satisfies the type condition in the sense of 4.3.2 if and only if \(\|x_1\|, \|x_2\|, \ldots \) satisfies the type condition in the sense of 1.6.4.

**DEFINITION 4.3.3** A Banach space \(E\) over \(K\) is called a Keller space if \(K\) satisfies \((\alpha), (\beta), (\gamma)\) of 4.3.1 and for each closed subspace \(D\) of \(E\) there is a linear surjective projection \(P : E \to D\) for which \(\|P x\| \leq \|x\|\) (\(x \in E\)).

**FROM NOW ON IN 4.3 WE ASSUME \(K\) TO SATISFY (\(\alpha\) – (\(\gamma\)) OF 4.3.1.**

Before proving our Main Theorem we first prove that \(c_0\) is not a Keller space (4.3.4) and that a Keller space is of countable type.

**LEMMA 4.3.4** \(c_0\) has a closed subspace without closed complement.

**Proof.** Let \(E\) be the Keller space constructed in 4.2.2. By Theorem 3.2.6 it is a quotient of \(c_0\), so let \(\pi : c_0 \to E\) be a quotient map. If \(\text{Ker} \ \pi\) had a closed complement \(D\) then it would be linearly homeomorphic to \(E\) by Banach's Open Mapping Theorem. It follows that \(D\) has a Schauder base \(e_1, e_2, \ldots\) satisfying the type condition. But on the other hand, for each Schauder base \(f_1, f_2, \ldots\) of \(D\) we can arrange that \(\|f_n\| = 1\) for all \(n\) implying that no Schauder base of \(D\) can satisfy the type condition, a contradiction.

**COROLLARY 4.3.5** A Keller space is of countable type. In particular, it is a norm-Hilbert Space.

**Proof.** Let \(E\) be an \(X\)-normed Keller space. \(X\) has a cofinal sequence \(t_1 > t_2 > \cdots\) and a cofinal sequence \(s_1 < s_2 < \cdots\). Then \(B_n := \{x \in X : t_n < x < s_n\}\) is bounded above and below in \(X\) for each \(n \in \mathbb{N}\). Now let \(\{e_i : i \in I\}\) be a maximal orthogonal set of nonzero vectors and suppose \(I\) is uncountable; it suffices to derive a contradiction. For each \(i \in I\) there is an \(n(i) \in \mathbb{N}\) such that \(\|e_i\| \in B_{n(i)}\). By uncountability there exists an \(m \in \mathbb{N}\) for which \(S := \{i \in I : n(i) = m\}\) is infinite, so assume \(S \supset \mathbb{N}\). Then \(t_m \leq \|e_n\| \leq s_m\) for all \(n \in \mathbb{N}\), so

\[
(\lambda_1, \lambda_2, \ldots) \mapsto \sum_{n=1}^{\infty} \lambda_n e_n
\]

is a linear homeomorphism of \(c_0\) onto a closed subspace of \(E\). But a closed subspace of \(E\) is a Keller space and cannot be isomorphic to \(c_0\) according to the previous lemma, a contradiction.

**LEMMA 4.3.6** Let \(E\) be an \(X\)-normed space for some \(G\)-module \(X\), let \(x_1, x_2, \ldots\) be a sequence in \(E\) satisfying the type condition. Let \(M \in X\), let \(\Lambda_M = \{(\lambda_1, \lambda_2, \ldots) \in \)
$K^N : \|\lambda_n x_n\| \leq M$ for all $n \in \mathbb{N}$. Then $\lim_{n \to \infty} \lambda_n x_n = 0$ uniformly on $(\lambda_1, \lambda_2, \ldots) \in \Lambda_M$.

Proof. Suppose not. Then there would be an $\varepsilon \in X$ such that for all $n \in \mathbb{N}$ we could find a $(\lambda_1, \lambda_2, \ldots) \in \Lambda_M$ and $i > n$ such that $\|\lambda_i x_i\| > \varepsilon$, i.e. $\varepsilon < \|\lambda_i x_i\| \leq M$. This would imply that some subsequence of $x_1, x_2, \ldots$ does not satisfy the type condition which conflicts 1.6.5 (i).

We now formulate the Main Theorem characterizing Keller spaces in several ways.

**THEOREM 4.3.7** Let $G$ be the union of a strictly increasing sequence of convex subgroups. Then, for an infinite-dimensional $K$-Banach space $E$ with an orthogonal base $e_1, e_2, \ldots$ the following are equivalent.

1. $E$ is a Keller space.
2. $E$ is a norm-Hilbert space.
3. $e_1, e_2, \ldots$ satisfies the type condition.
4. Each orthogonal sequence in $E$ satisfies the type condition.
5. Each bounded subset of $E$ is a compactoid in $E$.
7. Every closed subspace has a closed complement.
8. No subspace of $E$ is linearly homeomorphic to $c_0$.

Proof. Let us suppose that $E$ is $X$-normed for some $G$-module $X$. Since $E$ is of countable type, $(\alpha) \iff (\beta)$ is obvious. We prove $(\gamma) \Rightarrow (\varepsilon) \Rightarrow (\delta) \Rightarrow (\zeta) \Rightarrow (\beta) \Rightarrow (\eta) \Rightarrow (\gamma)$. $(\gamma) \Rightarrow (\varepsilon)$. We prove that $B_E(0, r)$ is a compactoid in $E$ for each $r \in X$. Let $A := \{\lambda_1, \lambda_2, \ldots \} \in K^N : \lim_{n \to \infty} \lambda_n e_n = 0$ and $\|\sum_{n=1}^{\infty} \lambda_n e_n\| \leq r$.

By orthogonality and the type condition we have $A = \{(\lambda_1, \lambda_2, \ldots) \in K^N : ||A^\infty\| < r \}$ for each $n$. Now let $\varepsilon \in X$. By 4.3.6 there is an $N$ such that $||\lambda_n e_n|| < \varepsilon$ for all $n > N$ and all $(\lambda_1, \lambda_2, \ldots) \in A$. Now, let $x \in B_E(0, r)$ have expansion $x = \sum_{n=1}^{\infty} \lambda_n e_n$. Then $x \in \sum_{i=1}^{N} \lambda_i e_i + \sum_{n=N+1}^{\infty} \lambda_n e_n + B_E(0, \varepsilon)$. Choose $\mu \in K$ such that $|\mu| ||e_i|| > r$ for all $i \in \{1, \ldots, N\}$. Then $||\mu|| ||e_i|| > |\lambda_i| ||e_i||$ hence $|\mu| > |\lambda_i|$ for $i \in \{1, \ldots, N\}$ and therefore $\lambda_1 e_1 + \cdots + \lambda_N e_N \in co\{\mu e_1, \ldots, \mu e_N\}$. So $B_E(0, r) \subset co\{\mu e_1, \ldots, \mu e_N\} + B_E(0, \varepsilon)$ proving compactoidity. To prove $(\varepsilon) \Rightarrow (\delta)$, let $f_1, f_2, \ldots$ be an orthogonal sequence in $E$, let $\lambda_1, \lambda_2, \ldots \in K$ be such that $\{\lambda_n f_n : n \in \mathbb{N}\}$ is bounded (above). Then this set is a compactoid by assumption and orthogonality implies by 3.5.5 that $\lim_{n \to \infty} \lambda_n f_n = 0$. The implication $(\delta) \Rightarrow (\gamma)$ being trivial we have established the equivalence of $(\gamma)$, $(\varepsilon)$, $(\delta)$. We now prove $(\delta) \Rightarrow (\zeta)$. The space $[A]$ satisfies $(\delta)$ (with $E$ replaced by $[A]$) and, by the equivalence of above, also $(\varepsilon)$ (with $E$ replaced by $[A]$) which is $(\zeta)$. We proceed to prove $(\zeta) \Rightarrow (\beta)$. By 4.1.3 it suffices to show that every strictly decreasing orthogonal sequence tends to 0. But this is clear from compactoidity and 3.5.5.

The implication $(\beta) \Rightarrow (\eta)$ is obvious. We continue with $(\eta) \Rightarrow (\theta)$. Let $D$ be a closed subspace of $E$. Let $F$ be a closed subspace of $D$. By $(\eta)$ $F$ has a closed complement $C$ in $E$. Then $C \cap D$ is a closed complement of $F$ in $D$, proving that $D$ satisfies $(\eta)$, and from 4.3.4 it follows that $D$ cannot be linearly homeomorphic to $c_0$. Finally we prove $(\theta) \Rightarrow (\gamma)$. Let $\lambda_1, \lambda_2, \ldots \in K$ be such that $\{||\lambda_n e_n|| : n \in \mathbb{N}\}$ is bounded above, say by $M \in X$. If not $\lambda_n e_n \to 0$ we would have a $\delta \in X$, a
subsequence \( n_1 < n_2 < \cdots \) of 1, 2, 3, \ldots such that
\[
\delta \leq \| \lambda_n e_n \| \leq M \quad (i \in \mathbb{N}).
\]
But then the formula
\[
T((\xi_1, \xi_2, \ldots)) = \sum_{i=1}^{\infty} \xi_i \lambda_n e_n,
\]
defines a linear homeomorphism of \( c_0 \) onto \( Tc_0 \subset E \) conflicting \((\theta)\).

**COROLLARY 4.3.8** Closed subspaces and quotients of Keller spaces are Keller spaces. If two Banach spaces with an orthogonal base are linearly homeomorphic and one is a Keller space then so is the other.

The canonical decomposition (see 3.4.4) is suited to characterizing spaces (containing subspaces that are) isomorphic to \( c_0 \) or a Keller space. Recall that the topological type of an element of a \( G \)-module \( X \) depends only on the algebraic type, i.e. (see the introduction of 1.6 and 1.6.1) \( \tau(gs) = \tau(s) \) for all \( s \in X, g \in G \). For the next theorem it is more convenient to define the type function as a map defined on the collection \( \Sigma \) of all algebraic types (with values in the collection of all proper convex subgroup of \( G \)) via the formula
\[
Gs \mapsto \tau(s).
\]
We shall denote this type function by \( \bar{\tau} \). We will say that \( \lim_{\sigma} \bar{\tau}(\sigma) = \infty \) if for each proper convex subgroup \( H \) of \( G \) we have \( \bar{\tau}(\sigma) \subset H \) for only finitely many \( \sigma \in \Sigma \). (This ties in with Definition 1.6.4 (ii)). We will say that \( \bar{\tau} \) is bounded if there is a proper convex subgroup \( H \) of \( G \) such that \( \bar{\tau}(\sigma) \subset H \) for all \( \sigma \in \Sigma \).

We will say that a Banach space \( E \) contains a Banach space \( F \) if there exists a linear homeomorphism of \( F \) onto a subspace of \( E \).

**THEOREM 4.3.9** Let \( E \) be an infinite-dimensional Banach space of countable type with an orthogonal base. Then
\begin{itemize}
  \item[(i)] \( E \) is a Keller space if and only if it does not contain \( c_0 \),
  \item[(ii)] \( E \) is linearly homeomorphic to \( c_0 \) if and only if it does not contain an infinite-dimensional Keller space.
\end{itemize}

Proof. (i) Follows from 4.3.7 \((\alpha) \iff (\theta)\); (ii) is a consequence of the next theorem.

**THEOREM 4.3.10** Let \( E \) be an infinite-dimensional Banach space of countable type with an orthogonal base, let \( \Sigma \) be the set of algebraic types of \( Y := \{ \|x\| : x \in E, x \neq 0 \} \), let
\[
E = \bigoplus_{\sigma \in \Sigma} E_{\sigma}
\]
be the canonical decomposition of \( E \). Then we have the following.
\begin{itemize}
  \item[(i)] \( E \) contains \( c_0 \) if and only if \( E \) is bounded or \( \dim E_{\sigma} = \infty \) for some \( \sigma \in \Sigma \).
  \item[(ii)] \( E \) contains an infinite-dimensional Keller space if and only if \( \bar{\tau} \) is unbounded.
  \item[(iii)] \( E \) is linearly homeomorphic to \( c_0 \) if and only if \( \bar{\tau} \) is bounded.
\end{itemize}
(iv) E is a Keller space if and only if each $E_\sigma$ is finite-dimensional and $\lim_\sigma \overline{\tau}(\sigma) = \infty$.

Proof. We prove (ii), (iii) and (iv) (i) follows from (iv) and 4.3.9 (i)). If $E$ contains an infinite-dimensional Keller space it contains an orthogonal sequence $e_1, e_2, \ldots$ satisfying the type condition; so by Theorem 1.6.6 we have $\lim_n \tau(||e_n||) = \infty$ i.e. $\lim_n \overline{\tau}(\sigma_n) = \infty$ for some sequence $\sigma_1, \sigma_2, \ldots \in \Sigma$. We see that $\overline{\tau}$ is unbounded.

If, conversely, $\overline{\tau}$ is unbounded we can find mutually distinct $\sigma_1, \sigma_2, \ldots \in \Sigma$ with $\lim_n \overline{\tau}(\sigma_n) = \infty$. Choose, for each $n$, a vector $x_n \in E$ for which $||x_n|| \in \sigma_n$. Then $x_1, x_2, \ldots$ is orthogonal (3.2.8) and satisfies the type condition by 1.6.6. So $[x_1, x_2, \ldots]$ is a Keller space. This proves (ii). To prove (iii), let $E$ be linearly homeomorphic to $c_0$. If it contained an infinite-dimensional Keller space $D$ then by 4.3.9 (i) $D$ does not contain $c_0$, in particular, $D$ is not linearly homeomorphic to $c_0$, a contradiction. By (ii), $\overline{\tau}$ is not unbounded, i.e. bounded. Conversely, if $\overline{\tau}$ is bounded, say $\overline{\tau}(\sigma) \subset H$ for all $\sigma \in \Sigma$ and some proper convex subgroup $H$, then take $g_1, g_2 \in G$ for which $g_1 < h < g_2$ for all $h \in H$. Then for each $s \in Y$, $Gs$ intersects $\text{conv}(Hs_0)$ (1.6.2), so, if $e_1, e_2, \ldots$ is an orthogonal base of $E$ there are $\lambda_1, \lambda_2, \ldots \in K$ such that $g_1s_0 \leq |\lambda_n||e_n|| \leq g_2s_0$ for all $n$. Then $(\xi_1, \xi_2, \ldots) \mapsto \sum_{n=1}^{\infty} \xi_n\lambda_n e_n$ is a linear homeomorphism of $c_0$ onto some subspace of $E$. Finally we prove (iv). If $E$ is a Keller space then so is its subspace $E_\sigma$, so by Remark 3.4.6 and 4.3.9 (i), $\dim E_\sigma < \infty$. If not $\lim_\sigma \overline{\tau}(\sigma) = \infty$ there were mutually distinct $\sigma_1, \sigma_2, \ldots \in \Sigma$ such that $n \mapsto \overline{\tau}(\sigma_n)$ is bounded. Choose $e_n \in E$ with $||e_n|| \in \sigma_n$. Then $e_1, e_2, \ldots$ is orthogonal, so it satisfies the type condition by 4.3.7 (a) $\iff (\delta)$. But then $\lim_n \tau(||e_n||) = \lim_n \overline{\tau}(\sigma_n) = \infty$ by 1.6.6, a contradiction. Hence, $\lim_\sigma \overline{\tau}(\sigma) = \infty$.

Conversely suppose that each $E_\sigma$ is finite-dimensional and that $\lim_\sigma \overline{\tau}(\sigma) = \infty$. By choosing an orthogonal base of $E_\sigma$ for every $\sigma \in \Sigma$ and by taking the union we obtain an orthogonal base $e_1, e_2, \ldots$ of $E$. By finite-dimensionality, $\{n : ||e_n|| \in \sigma \}$ is finite for each $\sigma \in \Sigma$, so $\{||e_n|| : n \in \mathbb{N}\}$ meets infinitely many $\sigma \in \Sigma$, so since $\lim_\sigma \overline{\tau}(\sigma) = \infty$ we have $\lim_n \tau(||e_n||) = \infty$. Applying 1.6.6 we obtain that $e_1, e_2, \ldots$ satisfies the type condition i.e. that $E$ is a Keller space.

We like to end this section with a discussion on reflexivity of Keller spaces. First some remarks on duality for general normed spaces $E$. If $E$ is $X$-normed but $G \not\subset X$ then the Lipschitz norm (see 2.2) $||f|| = \inf\{g \in G : |f(x)| \leq g||x||$ for all $x \in E\}$ is meaningless for $f \in E'$. The topology on $E'$ of uniform convergence on bounded sets is perfectly defined but again, there is no canonical norm that describes this topology: for each $\delta \in X$ one may take $||f||_d = \sup\{|f(x)| : x \in B_E(0, \delta)\} (\text{see 2.2})$. Clearly, $E'$ is always a normable space (if $G \subset X$ the Lipschitz norm is equivalent to $||f||_d$ for each $\delta$, according to 2.5.5) but there is no natural device to define a norm on $E'$, that is valid for each $E$. In any case, the bidual $E''$ is also well-defined as a normable space and we can define the following concept.

**DEFINITION 4.3.11** A normed space is called (topologically) reflexive if the natural map $j_E : E \rightarrow E''$ (given by $j_E(x)(f) = f(x)$ (for $f \in E'$, $x \in E$)) is a linear homeomorphism.

**LEMMA 4.3.12** Let $E$ be an $X$-normed Keller space where $X$ is some $G$-module. Then there exists an equivalent $G^\#$-norm on $E$ for which it is again a Keller space.
Proof. Define, like in 1.5.6 and 2.1.9, a map \( \phi : X \cup \{0\} \to G^\# \cup \{0\} \) by \( \phi(0) := 0 \) and \( \phi(s) := \inf_{G^\#} \{ g \in G : g s_0 \geq s \} \) (\( s \in X \)) where \( s_0 \in X \) is fixed. Then \( N : x \mapsto \phi(\|x\|) \) (\( x \in E \)) is a \( G^\# \)-norm equivalent to \( \| \| \) by 2.1.9. Let \( e_1, e_2, \ldots \) be an orthogonal base in \( (E, \| \|) \). To show that it is an orthogonal base in \( (E, N) \) it suffices to prove orthogonality. Let \( \lambda_1, \ldots, \lambda_n \in K \). Then \( N(\sum_{i=1}^n \lambda_i e_i) = \phi(\sum_{i=1}^n \lambda_i e_i) = \phi(\max_i \|\lambda_i e_i\|) = \max_i \|\lambda_i N(e_i)\| \) where we have used increasingness of \( \phi \). Thus, \( (E, N) \) has an orthogonal base and is linearly homeomorphic to a Keller space. Then it is itself a Keller space by 4.3.8.

Thanks to the above Lemma, to prove that Keller spaces are reflexive it suffices to show that \( G^\# \)-normed Keller spaces are. To be able to describe the reflexivity of the first Keller space in history (see [5]) in a more geometric way we shall prove slightly more. For topological reflexivity only the reader may take \( \Gamma = G \) in the next Proposition and Theorem.

**PROPOSITION 4.3.13** Let \( E \) be a \( G^\# \)-normed Keller space where \( \Gamma \) is a linearly ordered group containing \( G \) as a cofinal subgroup. For \( f \in E' \) set

\[
\|f\| := \inf\{ g \in \Gamma : |f(x)| \leq g\|x\| \text{ for all } x \in E \}.
\]

Let \( e_1, e_2, \ldots \) be an orthogonal base of \( E \) and let \( f_1, f_2, \ldots \in E' \) be the coordinate functions given by

\[
f_n(\sum_{m=1}^\infty \lambda_m e_m) = \lambda_n.
\]

Then \( \| \| \) induces the topology of uniform convergence on bounded sets and \( E' = (E', \| \|) \) is a Keller space with orthogonal base \( f_1, f_2, \ldots \). We have \( \|f_n\| = \omega(\|e_n\|) \) for each \( n \) (where \( \omega \) is the antipode \( \Gamma^\# \to \Gamma^\# \), defined in 1.3.1).

Proof. It is easily seen that \( \| \| \) is equivalent to the 'ordinary' Lipschitz norm \( f \mapsto \inf\{ g \in G : |f(x)| \leq g\|x\| \text{ for all } x \in E \} \) (by using the fact that, if \( g_1, g_2, \ldots \in \Gamma \), \( \inf_n g_n = 0 \), there exist, by coinitiality, \( h_1, h_2, \ldots \in G \) for which \( h_n < g_n \) for each \( n \) and so \( \inf_n h_n = 0 \)). Then 2.5.5 shows that \( \| \| \) induces the usual topology on \( E' \).

We now prove that \( \|f_n\| = \omega(\|e_n\|) \) for all \( n \in \mathbb{N} \). By definition we have

\[
\|f_n\| = \inf\{ g \in \Gamma : |f_n(x)| \leq g\|x\| \text{ for all } x \in E \}.
\]

Now the expression \(|f_n(x)| \leq g\|x\| \) for all \( x \in E' \) is equivalent to \(|f_n(e_m)| \leq g\|e_m\| \) for all \( m \in \mathbb{N} \) which is in turn equivalent to \(|f_n(e_n)| \leq g\|e_n\| \) i.e. to \( 1 \leq g\|e_n\| \). We see that \( \|f_n\| = \inf\{ g \in \Gamma : 1 \leq g\|e_n\| \} = \omega(\|e_n\|) \) by 1.3.1 (i). From 1.6.7 it follows that \( f_1, f_2, \ldots \) satisfies the type condition. So, it remains to be shown that \( f_1, f_2, \ldots \) is an orthogonal base for \( E' \). To prove orthogonality, let \( \lambda_1, \ldots, \lambda_n \in K \); we show that \( \|\sum_{i=1}^n \lambda_i f_i\| \geq \max_i |\lambda_i| f_i \| \). Writing \( f = \sum_{i=1}^n \lambda_i f_i \) we have \( f(e_i) = \lambda_i \) for each \( i \in \{1, \ldots, n\} \). If \( g \in \Gamma, g \geq \|f\| \) then by definition \( |f(x)| \leq g\|x\| \) for all \( x \in E \), so in particular for \( x = e_1, e_2, \ldots, e_n \) yielding \( \lambda_i \leq g\|e_i\| \) for \( i \in \{1, \ldots, n\} \). If \( \lambda_i \neq 0 \) we have \( 1 \leq |\lambda_i|^{-1} g\|e_i\| \) implying \( |\lambda_i|^{-1} g \geq \omega(\|e_i\|) = \|f_i\| \) i.e. \( |\lambda_i| \|f_i\| \leq g \). The latter formula is also trivially valid for \( \lambda_i = 0 \) and we find \( \max_i |\lambda_i| \|f_i\| \leq g \) and orthogonality is proved. Now let \( f \in E' \); we prove that \( f = \sum_{n=1}^\infty \lambda_n f_n \) where \( \lambda_n := f(e_n) \) for each \( n \). In fact, we have for each \( n \) that \( |\lambda_n| \leq g\|e_n\| \) for all \( g \in \Gamma \),
$g \geq ||f||$. So, $|\lambda_n|^{-1} g \geq \omega(||e_n||) = ||f_n||$ i.e. $|\lambda_n||f_n|| \leq g$ for all $g \in \Gamma, g \geq ||f||$, for all $n \in \mathbb{N}$. Thus, $\{||\lambda_n f_n|| : n \in \mathbb{N}\}$ is bounded (above) and by the type condition $||\lambda_n f_n|| \to 0$. Therefore $\tilde{f} := \sum_{n=1}^{\infty} \lambda_n f_n$ makes sense ($E'$ is complete by 2.3.7). But $\tilde{f}(e_n) = f(e_n)$ for each $n$ so $f = \tilde{f}$ and we are done.

**Theorem 4.3.14** A Keller space is reflexive. In particular, let $E, \Gamma, ||\cdot||$ be as in 4.3.13 and define on $E''$ the norm

$$\theta \mapsto \inf\{g \in \Gamma : |\theta(f)| \leq g||f|| \text{ for all } f \in E'\}.$$  

Then the natural map $j_E : E \to E''$ is a surjective isometry.

Proof. Thanks to 4.3.12 it suffices to prove the second statement. Let $e_1, e_2, \ldots$ be an orthogonal base of $E$. From 4.3.13 we obtain that $(E', ||\cdot||)$ is a Keller space with orthogonal base $f_1, f_2, \ldots$ where the $f_n$ are the coordinate functions, and where $||f_n|| = \omega(||e_n||)$ for each $n$. By the same token $E''$ is a Keller space with the coordinate functions $\delta_n: \Sigma \lambda_m f_m \mapsto \lambda_n$ as an orthogonal base where $||\delta_n|| = \omega(||f_n||) = \omega^2(||e_n||) = ||e_n||$ (1.3.1) for each $n$. We complete the proof by showing that $j_E$ maps $e_n$ into $\delta_n$ for each $n$. But that is clear as for each $m$ we have $j_E(e_n)( f_m ) = f_m(e_n) = \delta_{mn} = \delta_n(f_m)$.

### 4.4 Form-Hilbert spaces

Throughout 4.4, let $\lambda \mapsto \lambda^*$ be an isometrical involution in $K$ (that is allowed to be the identity). Also, assume $|2| = 1$; this technicality is needed for 4.4.2. A **Hermitean form** on a $K$-vector space $E$ is a map $(\cdot, \cdot): E \times E \to K$ satisfying

$$(x + y, z) = (x, z) + (y, z)$$

$$(\lambda x, y) = \lambda (x, y)$$

$$(x, y) = (y, x)^*$$

for all $x, y \in E$, $\lambda \in K$.

Let $\Gamma$ be the divisible hull of $G$. It is known ([13]) that there is precisely one way to extend the ordering of $G$ so as to let $\Gamma$ become an linearly ordered group. Let $\sqrt{G} = \{s \in \Gamma : s^2 \in G\}$. Then $\sqrt{G}$ is a linearly ordered group; we consider it as a $G$-module.

**Definition 4.4.1** ([3], Def. 15). A normed space $E$ is called a **definite space** if there is a Hermitean form $(\cdot, \cdot)$ such that $||x|| = \sqrt{|(x, x)|}$ for all $x \in E$.

A definite space has its norm values in $\sqrt{G} \cup \{0\}$.

**Proposition 4.4.2** Let $E$ be a definite Banach space of countable type. Then we have the following.

(i) If $x, y \in E$, $(x, y) = 0$ then $||x + y|| = \max(||x||, ||y||)$ (Form-orthogonality implies norm-orthogonality).
(ii) \(|(x, y)| \leq \|x\| \cdot \|y\|\) \((x, y) \in E\) (Cauchy-Schwarz).

(iii) \(E\) has a form-orthogonal base \(e_1, e_2, \ldots\). For \(x \in E\) we have
\[
x = \sum_{i=1}^{\infty} (x, e_i) (e_i, e_i)^{-1} e_i, \quad \|x\| = \max_{i} |(x, e_i)| \cdot \|e_i\|^{-1}.
\]

Proof. For (i), (ii), see [3], Lemma 14. To prove (iii), let \(x_1, x_2, \ldots\) be a linearly independent sequence whose linear hull is dense in \(E\). The well-known Gram-Schmidt process
\[
e_1 := x_1
\]
\[
e_2 := x_2 - \frac{(x_2, e_1)}{(e_1, e_1)} e_1
\]
\[
e_3 := x_3 - \frac{(x_3, e_2)}{(e_2, e_2)} e_2 - \frac{(x_3, e_1)}{(e_1, e_1)} e_1
\]
\[\vdots
\]
leads to a form orthogonal sequence \(e_1, e_2, \ldots\) for which \([e_1, e_2, \ldots, e_n] = [x_1, x_2, \ldots, x_n]\) for each \(n\). Thus, \(e_1, e_2, \ldots\) is norm orthogonal with dense linear hull and therefore norm orthogonal base by 2.4.17. For \(x \in E\), let \(x = \sum \xi_i e_i\) be its expansion. Then for each \(j\) we have \((x, e_j) = (\xi_j e_j, e_j)\) and (iii) follows.

DEFINITION 4.4.3. A definite Banach space of countable type is called a form-Hilbert space if for every closed subspace \(D \subset E\) we have \(D + D^\perp = E\) where \(D^\perp := \{x \in E : (x, d) = 0\) for all \(d \in D\).

Thus, norm-(form-)Hilbert spaces are characterized by the fact that every closed subspace has a norm-(form-)orthogonal complement. Form Hilbert spaces (also called GKK-spaces in [11]) have been extensively studied in [3] and [11].

An immediate consequence of the Definition is

PROPOSITION 4.4.4 Every form-Hilbert space is a norm-Hilbert space.

Proof. 4.4.2 (i).

However we can say more.

THEOREM 4.4.5. Let \(E\) be an infinite-dimensional definite space of countable type. The following are equivalent.

(a) \(E\) is a form-Hilbert space.

(b) \(E\) is a norm-Hilbert space and \(\sqrt{G}\) has a sequence with the type condition.

(c) A subspace \(D\) is closed if and only if \(D^\perp = D\).

(d) \(E\) is a Keller space.

Proof. The equivalence of (a), (c) and (d) follows from [3], Th. 28 and Theorem 4.3.7 \((a) \iff (c)\). If \(\sqrt{G}\) admits a sequence with the type condition we have by 4.3.1 that \(G\) has no maximal convex proper subgroups. So we have \((b) \Rightarrow (d)\). Conversely if \((d)\) holds then \(\|E\| \subset \sqrt{G} \cup \{0\}\) by definiteness and each orthogonal base of \(E\) has the type condition \((4.3.7 (a) \Rightarrow (c))\). So we have \((b)\).
COROLLARY 4.4.6 If $G$ admits a maximal proper convex subgroup (in particular, if the valuation of $K$ has finite rank) then there do not exist infinite-dimensional form-Hilbert spaces over $K$.

Proof. See the previous proof.

COROLLARY 4.4.7 If $G$ has no maximal proper convex subgroup then a definite Banach space of countable type is norm-Hilbert space if and only if it is a form-Hilbert space.

Proof. 4.4.4 and 4.4.5 ($\delta \Rightarrow \alpha$).

COROLLARY 4.4.8 Let $G$ have no proper maximal convex subgroup. Let $E$ be a norm-Hilbert space with canonical decomposition $\bigoplus_{\sigma \in \Sigma} E_\sigma$. Then $E$ is a form-Hilbert space if and only if each $E_\sigma$ is one. In particular, if each $E_\sigma$ is onedimensional then $E$ is a form Hilbert space if and only if $\|x\| \in \sqrt{G}$ for each nonzero $x \in E$.

Proof. The "only if" parts are clear. To prove the first "if" part, suppose that each $E_\sigma$ is a form-Hilbert space with Hermitean form $(\ ,\ )_\sigma$. It suffices to define a Hermitean form $(\ ,\ )$ on $E$ for which $\|x\|^2 = |(x,x)|$ ($x \in E$) (4.4.7). To this end, let $x \in E$, $x = \sum_{\sigma \in \Sigma} x_\sigma$ where $x_\sigma \in E_\sigma$ for each $\sigma \in \Sigma$: similarly, let $y \in E$, $y = \sum_{\sigma \in \Sigma} y_\sigma$. We have $x_\sigma \to 0$, $y_\sigma \to 0$ so $|\langle x_\sigma, y_\sigma \rangle| \leq \|x_\sigma\| \|y_\sigma\| \to 0$ and the definition

$$
\langle x, y \rangle := \sum_\sigma \langle x_\sigma, y_\sigma \rangle
$$

makes sense. If $x_\sigma \neq 0$, $x_\tau \neq 0$ for some $\sigma, \tau \in \Sigma$, $\sigma \neq \tau$ we have $\|x_\sigma\| \notin G\|x_\tau\|$, so $\|x_\sigma\| \neq \|x_\tau\|$ and therefore $\|x_\sigma\|^2 \neq \|x_\tau\|^2$. Thus, if $\sigma \neq \tau$ and $(x_\sigma, y_\sigma)_\sigma$ and $(x_\tau, y_\tau)_\tau$ are not both 0 then $|\langle x_\sigma, x_\sigma \rangle| \neq |\langle x_\tau, x_\tau \rangle|$ so that for $x \in E$ we have

$$
|\langle x, x \rangle| = \big| \sum_\sigma \langle x_\sigma, x_\sigma \rangle \big| = \max_\sigma |\langle x_\sigma, x_\sigma \rangle| = \max_\sigma \|x_\sigma\|^2 = \|x\|^2.
$$

Now suppose that each $E_\sigma$ is onedimensional, say $E_\sigma = Ka_\sigma$. By assumption there is a $c_\sigma \in K$ such that $|c_\sigma| = \|a_\sigma\|^2$. Then the formula $(\lambda a_\sigma, \mu a_\sigma) = \lambda \mu c_\sigma$ defines a form on $K a_\sigma$ making it into a form-Hilbert space. Now apply the first part of the proof to conclude that $E$ is a form-Hilbert space.

COROLLARY 4.4.9 The following conditions on $K$ are equivalent.

$(\alpha)$ There exists an infinite-dimensional form-Hilbert space over $K$.

$(\beta)$ $G$ has no maximal proper convex subgroups. The $G$-module $\sqrt{G}$ admits a sequence having the type condition.

Proof. $(\alpha) \Rightarrow (\beta)$. Corollary 4.4.6 furnishes the first part of $(\beta)$. Then by 4.3.7 every orthogonal base has the type condition which yields a sequence in $\sqrt{G}$ with the type condition. Conversely, let $s_1, s_2, \ldots \in \sqrt{G}$ satisfy the type condition. By taking a suitable subsequence we may assume $s_n \notin Gs_m$ whenever $n \neq m$. Let $E :=$
\{(\xi_1, \xi_2, \ldots) \in K^N : \lim_n |\xi_n| s_n = 0\} and for \(x = (\xi_1, \xi_2, \ldots), y = (\eta_1, \eta_2, \ldots) \in E\),

\[(x, y) = \sum_{n=1}^{\infty} \xi_n \eta_n a_n\]

where \(a_n \in K\) are such that \(|a_n| = s_n^2\). We see that \(|(x, x)| = |\sum_n \xi_n^2 a_n| = \max_n (|\xi_n| s_n)^2\). So \(E\) is a Keller space and definite hence form-Hilbert.

REFERENCES


