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p -ADIC BARRELLEDNESS AND SPACES OF COUNTABLE TYPE

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Several notions of barrelledness for locally convex spaces over non-archimedean valued fields are discussed in this paper and the relation between them is studied. We give examples showing that they are different notions in general. On the other hand, the concepts of elementary and edged set are used to prove that for a wide class of spaces of countable type the various versions of barrelled spaces considered in the paper coincide. We obtain in this way (by a different proof) the non-archimedean counterpart of a well-known result in the theory of barrelled spaces over the real or complex field.

Key Words : p -Adic Analysis; Space of Countable Type; Barrelled Space; Edged Set.

INTRODUCTION

Like in the classical case (i.e., locally convex spaces over the real or complex field see eliminate this (e.g. Bonet and Carreras¹) we consider in this paper the notion of barrelled (resp. \mathcal{N}_0 -barrelled) space over a non-archimedean valued field by requiring that every pointwise bounded family (resp. sequence) of non-archimedean continuous seminorms is equicontinuous. Also, the concept of l^∞ -barrelled space is obtained if one replaces "sequence of continuous seminorms" by "sequence of continuous linear functionals".

But in addition the concept of polar seminorm leads us to consider in the non-archimedean case the polar versions of the above notions by taking a family (resp. sequence) of polar seminorms.

The main purpose of this paper is to study the relationship between these different forms of "barrelledness". It contains all the essentials of Perez-Garcia and Schikhof². Other questions on p -adic

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barrelled spaces have been studied by Pombo³, Schikhof⁴ and van Tiel⁵.

We show in Section 3 that the above notions don't coincide in general. It is interesting to point out that, in contrast to the classical situation, the polar versions of barrelledness are in general different from the ones obtained by considering arbitrary seminorms. On the other hand, the concepts of edged and elementary set are used to prove in Section 3 that for spaces strictly of countable type (of which perfect sequence spaces are a particularly interesting example, see Remark 3.2), l^∞ -barrelled \Leftrightarrow barrelled (Theorem 3.1). We obtain in this way (and with a different proof) the non-archimedean counterpart of a well-known result in the theory of barrelled spaces over the real or complex field (see e.g. Bonnet and Carreras¹, p. 241).

Further, we show (Theorem 4.1) that for spaces of finite type (which are not in general strictly of countable type) we again have that l^∞ -barrelled \Leftrightarrow barrelled.

1. PRELIMINARIES

Throughout this paper K is a non-archimedean valued field that is complete with respect to the metric induced by the non-trivial valuation $|\cdot|$. We set $|K| := \{|\lambda| : \lambda \in K\}$ and $\overline{|K|} :=$ the closure of $|K|$ in \mathbb{R} .

For the basic notions and properties concerning normed and locally convex spaces over K we refer to van Rooij⁶ and Schikhof⁴ respectively. However we recall the following.

1. Let E be a K -vector space and let A be an absolutely convex subset of E (i.e., A is a B_K -module where $B_K = \{\lambda \in K : |\lambda| \leq 1\}$). A is called *edged* if for each $x \in E$, the set $\{|\lambda| : \lambda \in K, \lambda x \in A\}$ is closed in $|K|$. We define A^e to be the smallest edged subset of E that contains A . If the valuation on K is dense we have $A^e = \bigcap_{|\lambda| > 1} \lambda A$ (and $A^e = A$ if the valuation is

discrete). A is called *elementary* if there exist $m \in \mathbb{N}$, $x_1, \dots, x_m \in E$ and absolutely convex subsets C_1, \dots, C_m of K such that $A = C_1 x_1 + \dots + C_m x_m$. The absolutely convex hull of $X \subset E$ is denoted by $\text{co}(X)$ and its linear hull by $[X]$.

A (*non-archimedean*) *seminorm* on E is a map $p : E \rightarrow \mathbb{R}$ satisfying:

- i) $p(x) \in \overline{|K|}$
- ii) $p(\lambda x) = |\lambda| p(x)$
- iii) $p(x + y) \leq \max(p(x), p(y))$

for all $x, y \in E, \lambda \in K$ (see Schikhof⁴). A seminorm p for which $p(x) = 0 \Rightarrow x = 0$ is called a *norm*. If A is an absolutely convex subset of E , then the formula $p_A : [A] \rightarrow \mathbb{R}$, $p_A(x) = \inf\{|\lambda| : x \in \lambda A\}$ ($x \in [A]$) defines its *associated seminorm* p_A on $[A]$.

For a seminorm p on E we denote by E_p the vector space $E/\text{Ker}p$ endowed with the canonical norm. Following Schikhof⁴ we say that p is *polar* if $p = \sup\{|f| : f \in E^*, |f| \leq p\}$, where E^* is the algebraic dual of E .

2. Now assume that E is a Hausdorff locally convex space over K with topological dual E' . A *barrel* in E is an absolutely convex, closed and absorbing subset of E . A set $A \subset E$ is called

polar if A coincides with its bipolar with respect to the dual pair $\langle E, E' \rangle$. Every polar set is edged. If p is a continuous seminorm on E , then p is polar if and only if $\{x \in E : p(x) \leq 1\}$ is a polar set (Schikhof⁴, Proposition 3.4. ii)).

Following Schikhof⁴ we say that E is *polar* if its topology is defined by a family of polar seminorms. Also, E is called *of countable type* (resp. *of finite type*) if for every continuous seminorm p on E the normed space E_p is of countable type (resp. of finite dimension). A normed space is called of countable type if it has a countable subset whose linear hull is dense in the whole space. Recall that E is of finite type if and only if its topology is a weak topology (Schikhof⁷, Theorem 2). Spaces of countable type form the non-archimedean counterpart of the transseparable spaces considered in the case of locally convex spaces over the real or complex field (see e.g. Bonet and Carreras¹, p. 53).

A set A in E is called *compactoid* if for each neighbourhood U of 0 in E there exists a finite set H in E such that $A \subset U + \text{co}(H)$.

In the sequel, E will be a Hausdorff locally convex space over K .

2. BARRELLED-LIKE SPACES

Like in the classical case (i.e. locally convex spaces over the real or complex field, see e.g. Bonet and Carreras¹) we can consider several concepts of barrelledness.

Definition 2.1 — a) E is called *barrelled* if every pointwise bounded family of continuous seminorms on E is equicontinuous.

b) E is called \mathcal{N}_0 -*barrelled* if every pointwise bounded sequence of continuous seminorms on E is equicontinuous.

c) E is called l^α -*barrelled* if every pointwise bounded sequence in E' is equicontinuous.

But, in addition, the concept of polar seminorm leads us to consider in the non-archimedean case the following polar versions of barrelled and \mathcal{N}_0 -*barrelled space*.

Definition 2.2 — E is called *polarly barrelled* (resp. *polarly \mathcal{N}_0 -barrelled*) if every pointwise bounded family (resp. sequence) of polar continuous seminorms on E is equicontinuous.

Obviously every Fréchet space and every inductive limit of a sequence of Fréchet spaces is a barrelled space. Also, by Theorem 9.6 of Schikhof⁴, every reflexive (and hence every Montel) space is polarly barrelled (for several interesting examples of this kind of spaces see de Grande-de Kimpe *et al.*⁸ and Schikhof⁴).

Remark : One can easily see that if $\{U_i\}_{i \in I}$ is a family of absolutely convex zero neighbourhoods in E and $U = \bigcap_{i \in I} U_i$, then U is a barrel (resp. a zero neighbourhood) in E if and

only if $\{Pu_i\}_{i \in I}$ is a pointwise bounded (resp. equicontinuous) family of continuous seminorms on E . Applying this fact, it is very easy to obtain the following description of barrelled spaces in terms of barrels.

E is barrelled if and only if every barrel in E is a zero neighbourhood in E .

Similar descriptions of the other above concepts in terms of barrels can be found in

Proposition 2.3 of Perez-Garcia and Schikhof².

We clearly have the following diagram.

$$\begin{array}{ccc}
 \text{barrelled} & \Rightarrow & \text{polarly barrelled} \\
 \Downarrow & & \Downarrow \\
 \mathcal{N}_0\text{-barrelled} & \Rightarrow & \text{polarly } \mathcal{N}_0\text{-barrelled} \\
 \Downarrow & & \Downarrow \\
 l^\infty\text{-barrelled} & & l^\infty\text{-barrelled}
 \end{array}$$

If every continuous seminorm on E is polar (e.g., when K is spherically complete, or when E is of countable type, see Schikhof⁴), then

$$E \text{ polarly } (\mathcal{N}_0\text{-}) \text{ barrelled} \Leftrightarrow E (\mathcal{N}_0\text{-}) \text{ barrelled.} \quad \dots (1)$$

In the next section, we will see that for a wide class of spaces of countable type the five versions of barrelledness considered in Definitions 2.1 and 2.2 coincide (Theorem 3.1). But we also will show that the converses of the arrows appearing in the above diagram are not true in general (Examples 3.1).

3. BARRELLEDNESS AND SPACES OF COUNTABLE TYPE

One of the purposes of this section is to prove that if E is of countable type then E is (polarly) \mathcal{N}_0 -barrelled if and only if E is l^∞ -barrelled (Proposition 3.1. ii). The proof given for transseparable spaces in the classical case (see e.g. Bonet and Carreras¹, Proposition 8.2.24) can be adapted to the non-archimedean situation (see Remark 3.1 for a different proof of this fact).

Proposition 3.1 — i) If E is metrizable, then E is barrelled (resp. polarly barrelled) if and only if E is \mathcal{N}_0 -barrelled (resp. polarly \mathcal{N}_0 -barrelled).

ii) If E is of countable type, then E is \mathcal{N}_0 -barrelled if and only if E is l^∞ -barrelled.

iii) If E is metrizable and of countable type, then

$$E \text{ is barrelled} \Leftrightarrow E \text{ is } l^\infty\text{-barrelled.} \quad \dots (2)$$

PROOF : i) : Let E be a metrizable \mathcal{N}_0 (resp. polarly \mathcal{N}_0)-barrelled space and suppose there exists a pointwise bounded family $\{p_i\}_{i \in I}$ of (polar) continuous seminorms on E which is not equicontinuous. Then $p(x) := \sup_i p_i(x)$ ($x \in E$) is a seminorm on E which is not continuous. By metrizability, we can find a sequence $(x_n)_n$ in E with $\lim_n x_n = 0$ and an $\varepsilon > 0$ such that $p(x_n) > \varepsilon$ for all n . Hence, for each n there is an $i_n \in I$ for which $p_{i_n}(x_n) > \varepsilon$. Thus, $\{p_{i_n}\}_n$ is a pointwise bounded sequence of (polar) continuous seminorms on E which is not equicontinuous: a contradiction.

ii) : Since E is of countable type, every equicontinuous subset B of E' is weakly*-metrizable (de Grande-de Kimpe⁹, Lemma 2.4) and by Schikhof⁴, Proposition 8.2 it is contained in the

weak*-closed absolutely convex hull of a sequence $(f_n)_n$ in $\pi\text{co}(B)$ (where $\pi \in K$ with $|\pi| > 1$ is fixed). By ℓ^∞ -barrelledness, every pointwise bounded subset of E' which is the union of a sequence of equicontinuous subsets of E' , is equicontinuous. This implies that E is polarly \mathcal{N}_0 -barrelled, and by (1) that E is \mathcal{N}_0 -barrelled.

Property iii) is a direct consequence of i) and ii).

Remark 3.1 : The good behaviour of normed spaces of countable type allows to prove (see Lemma 3.1 of Perez-Garcia and Schikhof²) the following interesting property.

If p is a seminorm on a K -vector space such that E_p is of countable type, then there exists a sequence $(f_n)_n$ in E^ such that $p = \sup_n |f_n|$.*

On the other hand, one can easily see that, applying the Hahn-Banach Theorem, this property also holds when the ground field is the real or complex one and E_p is separable. Hence, this property provides an alternative (in fact shorter) proof of Proposition 3.1. ii) for non-archimedean spaces of countable type and for real or complex transseparable spaces.

In the classical case statement (2) in Proposition 3.1 is satisfied for every separable locally convex space E (see e.g. Bonet and Carreras¹, Corollary 8.2.20). One might expect this property also to hold in the non-archimedean case for spaces of countable type, but this is not true in general (see Example 3.1.2). We therefore consider the following subclass, which includes the metrizable locally convex spaces of countable type.

Definition 3.1 — E is called *strictly of countable type* if there exists a countable set in E whose linear hull is dense in E .

We clearly have that if E is strictly of countable type, then E is of countable type. The converse is true if E is metrizable (Bonet and Carreras¹, p. 53). But there are non-metrizable spaces of countable type which are not strictly of countable type (see Bonet and Carreras¹, Example 2.5.2).

The main result of this section assures that for spaces strictly of countable type, statement (2) of Proposition 3.1 remains true (observe that there are non-metrizable spaces which are strictly of countable type: take c_0 endowed with the weak topology).

Theorem 3.1 — *If E is strictly of countable type, then*

$$E \text{ is barrelled} \Leftrightarrow E \text{ is } \ell^\infty\text{-barrelled.}$$

The proof of Theorem 3.1 differs substantially from the classical one for separable spaces. We need some preliminary machinery for this proof.

Lemma 3.1 — Suppose that E is of countable type. Let T be a closed absolutely convex subset of E . Let D be a finite-dimensional subspace of E and let V be an elementary edged subset of D such that $T \cap D \subset V$. Then there exists a countable set S in E' such that

$$\sup_{f \in S} |f(x)| \leq 1 \text{ for all } x \in T \text{ and } \sup_{f \in S} |f(x)| > 1 \text{ for all } x \in D \setminus V.$$

PROOF : V has the form $C_1 e_1 + \dots + C_n e_n$ (for some $n \in \mathbb{N}$), where $\{e_1, \dots, e_n\}$ is an algebraic basis of D and where C_1, \dots, C_n are edged sets (possibly $\{0\}$) in K (see Schikhof¹⁰, Lemma 2.2 and Proposition 2.10).

1) First, we are going to see that for every $i \in \{1, \dots, n\}$ for which $C_i \neq K$ (or equivalently $\text{diam } C_i < \infty$, where $\text{diam } C_i$ denotes the diameter of the set C_i), there exists a sequence $(g_{im})_m$ in

E' such that, for each m ,

$$|g_{im}| \leq 1 \text{ on } \left(T + \sum_{j \neq i} K e_j \right)^e \text{ and } |g_{im}(\mu_{im} e_i)| > 1, \quad \dots (3)$$

where $(\mu_{im})_m$ is a sequence in K chosen such that :

- a) $|\mu_{i1}| > |\mu_{i2}| > \dots$ and $\lim_m |\mu_{im}| = \text{diam } C_i$, if the valuation on K is dense or if $C_i = \{0\}$,
 b) $\mu_{i1} = \mu_{i2} = \dots$ and $|\mu_{i1}| = \min \{|\lambda| : \lambda \in K, |\lambda| > \text{diam } C_i\}$, otherwise.

To prove the existence of this sequence $(g_{im})_m$ in E' , we first claim that with our choice of $(\mu_{im})_m$ we have that, for each m ,

$$\mu_{im} e_i \notin \left(T + \sum_{j \neq i} K e_j \right)^e. \quad \dots (4)$$

Indeed, suppose that in case a) (4) is not true for some m and choose $\nu \in K$ with $\text{diam } C_i < |\nu| < |\mu_{im}|$. Then, $\mu_{im} e_i \in (\mu_{im}/\nu) \left(T + \sum_{j \neq i} K e_j \right)^e$ and so there exist $\xi_1, \dots, \xi_n \in K$ with

$\xi_i = \nu$, such that $\sum_{j=1}^n \xi_j e_j \in T \cap D \subset V$. In particular, $\nu \in C_i$, which is a contradiction.

Analogously, assume that in case b) (4) is not true. Observe that in this case $\left(T + \sum_{j \neq i} K e_j \right)^e = \left(T + \sum_{j \neq i} K e_j \right)$ so if (4) fails there exist $\xi_1, \dots, \xi_n \in K$ with $|\xi_i| > \text{diam } C_i$ such that $\sum_{j=1}^n \xi_j e_j \in T \cap D \subset V$, again a contradiction.

Now, since $\left(T + \sum_{j \neq i} K e_j \right)$ is closed (Schikhof¹¹, Theorem 1.4. ii)) and edged and E is of countable type we deduce that this set is polar (Schikhof⁴, Theorems 4.4 and 4.7). Applying (4) we derive the existence for each $m \in \mathbb{N}$, of a $g_{im} \in E'$ satisfying (3).

2) Now, we claim that the countable set $S = \{g_{im} : i \in \{1, \dots, n\}, C_i \neq K, m \in \mathbb{N}\} \cup \{0\}$ satisfies the required conditions.

By (3), it is clear that $\sup_{f \in S} |f(x)| \leq 1$ for all $x \in T$ and $g_{im}(e_j) = 0$ for all $j \neq i$ and for all m .

Now, let $x = \xi_1 e_1 + \dots + \xi_n e_n \in D \setminus V$. Then, there is at least one i for which $\xi_i \notin C_i$ (and hence $C_i \neq K$). Also, from the construction of the μ_{im} it is clear that there exists an $m \in \mathbb{N}$ such that $|\mu_{im}| \leq |\xi_i|$. Then, $|g_{im}(x)| = |g_{im}(\xi_i e_i)| = |\xi_i \mu_{im}^{-1} g_{im}(\mu_{im} e_i)| > |\xi_i| |\mu_{im}^{-1}| \geq 1$ and we are done.

The following result is a generalization of the previous lemma.

Proposition 3.2 — Suppose that E is of countable type. Let T be a closed and edged subset of E . Let D be a finite-dimensional subspace of E . Then, there exists a countable set S in E' such that $\sup_{f \in S} |f(x)| \leq 1$ for all $x \in T$ and $\sup_{f \in S} |f(x)| > 1$ for all $x \in D \setminus T$.

PROOF : By Theorem 4.8 of Schikhof¹⁰, there exist countable many elementary edged sets $V_1 \supset V_2 \supset \dots$ in D such that $\bigcap_n V_n = T \cap D$. By Lemma 3.1, for each n , there exists a countable set S_n in E' such that

$$\sup_{f \in S_n} |f(x)| \leq 1 \text{ for all } x \in T \text{ and } \sup_{f \in S_n} |f(x)| > 1 \text{ for all } x \in D \setminus V_n$$

Then the countable set $S = \bigcup_n S_n$ satisfies the conditions.

This is enough material to prove Theorem 3.1.

PROOF OF THEOREM 3.1 : Let E be an l^∞ -barrelled space which is strictly of countable type and let $\{x_1, x_2, \dots\}$ be a countable set in E whose linear hull is dense in E .

Let T be a barrel in E . To prove that T is a zero neighbourhood in E we can assume that T is edged. By Proposition 3.2 we have, for each n , a countable set S_n in E' such that

$$\sup_{f \in S_n} |f(x)| \leq 1 \text{ for all } x \in T \tag{5}$$

and

$$\sup_{f \in S_n} |f(x)| > 1 \text{ for all } x \in D_n \setminus T \tag{6}$$

where D_n is the linear hull of $\{x_1, \dots, x_n\}$.

Then $S := \bigcup_n S_n$ is a countable pointwise bounded subset of E' .

By l^∞ -barrelledness, $U := \{x \in E : |f(x)| \leq 1 \text{ for all } f \in S\}$ is a zero neighbourhood in E . By (5), we clearly have that $T \subset U$. Also, suppose that there exists an $x \in U \setminus T$. Since $U \setminus T$ is open, it contains an element of $\bigcup_n D_n$, which contradicts (6). Hence, $T = U$, which implies that T is a zero neighbourhood in E .

Examples 3.1 — In contrast to the classical situation, there are spaces E for which (1) does not hold (see de Grande-de Kimpe *et al.*¹², Example 1.4.22 and Perez-Garcia and Schikhof, Theorem 2.6). This gives an affirmative answer to the problem raised by the second author in Schikhof⁴, p. 208: Do there exist polar spaces that are polarly barrelled but not barrelled?

Now we are going to show that the converses of the vertical arrows appearing in our diagram (see Section 2) are not true in general.

1. Example of an l^∞ -barrelled space which is not polarly \mathcal{N}_0 -barrelled.

Choose a set I such that $\# I > \# l^\infty$ ($\#$ indicating cardinality). Let $E = l^\infty(I)$ as a K -vector space. Let C be the collection of all countable (pointwise) bounded subsets of $(E, \|\cdot\|_\infty)'$, where $\|\cdot\|_\infty$ is the canonical norm on E . For each $S \in C$ we define a seminorm p_S on E by

$$p_S(x) = \sup \{|f(x)| : f \in S\} \quad (x \in E)$$

and also, for each $m \in \mathbb{N}$ we define a seminorm p_m on E by

$$p_m(x) = \sup \{|x_{(n,i)}| : n \in \{1, 2, \dots, m\}, i \in I\} \quad (x = (x_{(n,i)})_{n,i} \in E).$$

Then, the family of polar seminorms $\{p_S : S \in C\} \cup \{p_m : m \in \mathbb{N}\}$ defines a Hausdorff locally convex topology τ on E for which it is easily seen that (E, τ) is l^∞ -barrelled (observe that $(E, \tau)' = (E, \|\cdot\|_\infty)'$). However, (E, τ) is not polarly \mathcal{N}_0 -barrelled.

Indeed, we shall see that $\|\cdot\|_\infty = \sup_m p_m$ is not τ -continuous. Suppose it was. Then, there would exist an $M > 0$, an $S = \{f_1, f_2, \dots\} \in C$ and an $m \in \mathbb{N}$ such that for all $x = (x_{(n,i)})_{n,i} \in E$,

$$\|x\|_\infty \leq M \max \left\{ \sup_j |f_j(x)|, \sup \{|x_{(n,i)}| : n \leq m, i \in I\} \right\}. \quad \dots (7)$$

Let D be the subspace of E consisting of all $x \in l^\infty(I)$ that vanish on $(\{1\} \times I) \cup (\{2\} \times I) \cup \dots \cup (\{m\} \times I)$. Then, on one hand $(D, \|\cdot\|_\infty)$ is isometrically isomorphic to $(E, \|\cdot\|_\infty)$. But on the other hand we have by (7) that the map $(D, \|\cdot\|_\infty) \rightarrow l^\infty, x \in D \mapsto (f_1(x), f_2(x), \dots)$ is a linear homeomorphism from $(D, \|\cdot\|_\infty)$ onto its image. Hence, $\# I = \#(I \times I) \leq \# l^\infty < I$, a contradiction.

2. *Example of a \mathcal{N}_0 -barrelled (hence l^∞ -barrelled) space of countable type which is not (polarly) barrelled.*

Take an uncountable set I and let $E = c_0(I)$ as a K -vector space. Let τ be the locally convex topology on E generated by the polar seminorms $p_C, (C \subset I, C \text{ countable})$, given by $p_C(x) = \max_{i \in C} |x_i| \quad (x = (x_i)_{i \in I} \in E)$.

It is easy to see that (E, τ) is of countable type. Also, since $(E, \|\cdot\|_\infty)$ is not of countable type (where $\|\cdot\|_\infty$ is the canonical norm on E), we deduce that $\|\cdot\|_\infty = \sup \{p_C, C \subset I, C \text{ countable}\}$ is not τ -continuous. Then, (E, τ) is not polarly barrelled.

To prove \mathcal{N}_0 -barrelledness of (E, τ) , it suffices, by Proposition 3.1ii), to show l^∞ -barrelledness. So, let (f_1, f_2, \dots) be a pointwise bounded sequence in $(E, \tau)'$. Then $M := \sup_n \|f_n\| < \infty$, where $\|f_n\|$ is the norm of f_n as an element of $l^\infty(I)$. Hence, for each $n = 1, 2, \dots$ there exists a countable subset C_n of I for which

$$|f_n(x)| \leq M \max_{m \in C_n} |x_m| \quad \text{for all } x = (x_i)_{i \in I} \in E.$$

Then, for the countable set $C = \bigcup_n C_n \subset I$ we have that $\sup_n |f_n(x)| \leq M P_c(x)$ for each $x = (x_i)_{i \in I} \in E$. Therefore, the sequence (f_1, f_2, \dots) is τ -equicontinuous and we are done.

Remark 3.2 : A particularly interesting class of locally convex spaces that are strictly of countable type is formed by the perfect sequence spaces endowed with the associated normal topology. Recall that a sequence space \mathcal{A} is called *perfect* if $\mathcal{A}^{\times \times} = \mathcal{A}$, where $\mathcal{A}^\times := \{(b_n)_n \in K^{\mathbb{N}} : \lim_n a_n b_n = 0 \text{ for all } (a_n)_n \in \mathcal{A}\}$ is the Köthe-dual of \mathcal{A} . Also, the *normal topology* on \mathcal{A} is the topology $n(\mathcal{A}, \mathcal{A}^\times)$ defined by the family of seminorms $\{p_b : b \in \mathcal{A}^\times\}$, where for each $b = (b_n)_n \in \mathcal{A}^\times$, p_b is defined by $p_b(a) = \sup_n |a_n b_n|$, $a = (a_n)_n \in \mathcal{A}$. Every $a = (a_n)_n \in \mathcal{A}$ can be written uniquely as $a = \sum_n a_n e_n$, where for each n , e_n is the sequence with 1 in the n -th place and 0's elsewhere. Therefore, $(\mathcal{A}, n(\mathcal{A}, \mathcal{A}^\times))$ is strictly of countable type. We always assume that the perfect space \mathcal{A} (resp. \mathcal{A}^\times) is endowed with the corresponding normal topology.

This kind of spaces plays an important role in the set up of a p -adic Quantum Mechanics (see e.g. de Grande-de Kimpe and Khrennikov¹³ and Khrennikov¹⁴). For instance, if B is an infinite matrix consisting of strictly positive real numbers b_n^k ($n, k \in \mathbb{N}$) satisfying $b_n^k \leq b_n^{k+1}$ for all n, k , then the non-archimedean Köthe space $K(B)$ associated with the matrix B and defined by

$$K(B) = \{(\alpha_n)_n \in K^{\mathbb{N}} : \lim_n |\alpha_n| b_n^k = 0, \text{ for all } k = 1, 2, \dots\}$$

is a perfect sequence space which is a Fréchet (and hence barrelled) space (see de Grande-de Kimpe¹⁵). For $b_n^k = k^n$, $K(B)$ is the space of entire functions on K , which is needed for the definition of a non-archimedean Laplace transform in de Grande-de Kimpe and Khrennikov¹³ and Khrennikov¹⁴.

By using Theorem 3.1 we obtain the following characterization of barrelledness (equivalently, ℓ^∞ -barrelledness) for perfect sequence spaces (for other characterizations, see Perez-Garcia and Schikhof², Theorem 4.1).

\mathcal{A} is barrelled if and only if $(\mathcal{A}, \beta(\mathcal{A}, \mathcal{A}^\times))$ is of countable type (where $\beta(\mathcal{A}, \mathcal{A}^\times)$ is the strong topology on \mathcal{A} with respect to the canonical dual pair $(\mathcal{A}, \mathcal{A}^\times)$).

Indeed, clearly if \mathcal{A} is barrelled then $\beta(\mathcal{A}, \mathcal{A}^\times) = n(\mathcal{A}, \mathcal{A}^\times)$ and hence $(\mathcal{A}, \beta(\mathcal{A}, \mathcal{A}^\times))$ is of countable type.

Conversely, assume that $(\mathcal{A}, \beta(\mathcal{A}, \mathcal{A}^\times))$ is of countable type. Let (f_1, f_2, \dots) be a pointwise bounded (and hence bounded) sequence in $\mathcal{A}^\times = (\mathcal{A}, n(\mathcal{A}, \mathcal{A}^\times))'$. It follows from Theorem 8.5 of Schikhof⁴ that $\{f_1, f_2, \dots\}$ is a compactoid subset of \mathcal{A}^\times . By Corollary 2.3 of Perez-Garcia¹⁶, $\{a \in \mathcal{A} : |f_n(a)| \leq 1 \text{ for all } n\}$ is a zero neighbourhood in \mathcal{A} . So, (f_1, f_2, \dots) is equicontinuous. Thus, \mathcal{A} is ℓ^∞ -barrelled and hence barrelled by Theorem 3.1.

4. BARRELLEDNESS AND SPACES OF FINITE TYPE

We show in Theorem 4.1 that for spaces of finite type barrelledness is equivalent to l^∞ -barrelledness. In general, spaces of finite type are not strictly of countable type, hence not metrizable, see Remark 4.1 below, and compare Proposition 3.1 and Theorem 3.1.

Theorem 4.1 — *Let E be a space of finite type. Then,*

$$E \text{ is barrelled} \Leftrightarrow E \text{ is } l^\infty\text{-barrelled}$$

PROOF : Suppose that E is l^∞ -barrelled. Let (f_1, f_2, \dots) be a pointwise bounded sequence in E' , which is equicontinuous by l^∞ -barrelledness. Since E is of finite type, there exist $g_1, \dots, g_m \in E'$ ($m \in \mathbb{N}$) such that

$$\sup_n |f_n(x)| \leq \sup_{i=1}^m |g_i(x)| \text{ for all } x \in E.$$

Hence $\bigcap_{i=1}^m \text{Ker } g_i \subset \bigcap_n \text{Ker } f_n$, and so $[f_1, f_2, \dots] \subset [g_1, \dots, g_m]$ which implies that $[f_1, f_2, \dots]$ is finite-dimensional.

Therefore, every pointwise bounded subset H of E' is finite-dimensional, and hence H is equicontinuous. Then, E is polarly barrelled (Schikhof⁴, Proposition 6.3) and by (1) it is barrelled.

Remark 4.1 : There are barrelled spaces of finite type which are not strictly of countable type.

Example : For each set I , let $E = K^I$ endowed with the product topology (which is of finite type, see Schikhof¹⁷, 4). Let $\{e_i : i \in I\}$ be the unit vectors of K^I . It is well known that $c_{00}(I) := [e_i : i \in I]$ is algebraically isomorphic to E' through the map $c_{00}(I) \rightarrow E' : y = (y_i)_{i \in I} \in c_{00}(I) \mapsto g_y \in E', g_y(x) = \sum_i x_i y_i$ ($x = (x_i)_{i \in I} \in E$). We identify every $y \in c_{00}(I)$ with its image under this map.

First, we are going to see that for every set I , K^I is a barrelled space. For that, take an infinite-dimensional sequence (y_1, y_2, \dots) in $c_{00}(I)$. Every y_s ($s = 1, 2, \dots$) can be written as

$$y_s = \sum_{j=1}^{n_s} \lambda_{s,j} e_j \quad (\lambda_{s,j} \in K, n_s \in \mathbb{N}, \lambda_{s,n_s} \neq 0), \text{ where we can assume that } n_1 < n_2 < \dots$$

We construct inductively a sequence $(\xi_s)_s$ in K such that, for each $s \in \mathbb{N}$,

$$|\xi_1 \lambda_{s,n_1} + \xi_2 \lambda_{s,n_2} + \dots + \xi_s \lambda_{s,n_s}| \geq s \quad \dots (8)$$

and consider $x = (x_i)_{i \in I} \in K^I$ such that $x_i = 0$ if $i \notin \{n_1, n_2, \dots\}$ and $x_i = \xi_m$ if $i = n_m$ for a certain

m . By (8), the sequence $(y_s(x))_s$ is not bounded. Hence, every linearly independent sequence in $c_{00}(I) = E'$ is not pointwise bounded. Thus, applying Theorem 4.1 we conclude that K^I is barrelled.

Now, we are going to see that if $\#I > \#K$, then K^I is not strictly of countable type. Assume that K^I is strictly of countable type and that $\{x_1, x_2, \dots\}$ is a countable subset whose linear hull is dense on K^I . Then, the map $(K^I)' \rightarrow K^{\mathbb{N}} : f \mapsto (f(x_1), f(x_2), \dots)$ is injective, which implies that $\#I \leq \#(K^{\mathbb{N}})' = (\#(K)^{\mathcal{N}_0})' = \#K$ (van Rooij¹⁷, Corollary 3.9).

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