THE COMPLEMENTATION PROPERTY OF ℓ^∞ IN p-ADIC BANACH SPACES

Wim H. Schikhof
Department of Mathematics, K.U. Nijmegen
Toernooiveld, 6525 ED Nijmegen, The Netherlands

ABSTRACT. Let K be a non-archimedean valued field whose valuation is complete and non-trivial. It is shown that ℓ^∞ is complemented in any polar K-Banach space (Theorem 1.2) which opens the way for various descriptions of complemented subspaces of ℓ^∞ (Theorem 2.3).

INTRODUCTION. (For unexplained terms see below.) Let K be as above. If K is spherically complete and E is a K-Banach space then, like in the 'classical' theory, each closed subspace is weakly closed, each subspace D of E has the weak extension property (WEP) i.e. each $f \in D'$ can be extended to an $\widetilde{f} \in E'$. It is also well known that these statements are false if K is not spherically complete. In fact, the following questions are posed in [1].

Q.1. Does every weakly closed subspace have the WEP ?

Q.2. Is every closed subspace having the WEP weakly closed?

It was a close look at these questions (for the (negative) answers, see the Remark following Proposition 1.5 and Example 3.3) that revealed the above-mentioned complementation property of ℓ^∞. Again, for spherically complete K this is nothing new as it follows directly from the spherical completeness of ℓ^∞. However, for nonspherically complete K the results came as a surprise.

PRELIMINARIES. Let K be as above. From now on until §5 we suppose that K is NOT spherically complete. We use the terminology of [2].

P.1. For an absolutely convex subset A of a K-vector space we set $A^e := \cap\{\lambda A : \lambda \in K, |\lambda| > 1\}$. A is *edged* if $A = A^e$.

P.2. Let E, F be K-Banach spaces. $\mathcal{L}(E,F)$ is the Banach space of all continuous linear maps $E \to F$ with the operator norm, $E' := \mathcal{L}(E,K)$. E and F are *isomorphic* if there exists a surjective linear isometry $E \to F$. We write $E \sim F$ if E, F are linearly homeomorphic. We shall say that E and F are *almost isomorphic* if for each $\varepsilon > 0$ there exists a linear homeomorphism $T : E \to F$ with $\|T\| \le 1 + \varepsilon$, $\|T^{-1}\| \le 1 + \varepsilon$.

$T \in \mathcal{L}(E,F)$ is a *quotient map* if T maps $\{x \in E : \|x\| < 1\}$ onto $\{x \in F : \|x\| < 1\}$. The closed unit ball $\{x \in E : \|x\| \leq 1\}$ is denoted B_E. The norm closure of a set $X \subset E$ is $\overline{X}$. Its weak closure, i.e. the closure with respect to the weak topology (w-topology) $\sigma(E,E')$, is $\overline{X}^w$. Similarly, if $Y \subset E'$ then $\overline{Y}^{w'}$ is the w'-closure of Y i.e. the closure with respect to the w'-topology $\sigma(E',E)$.

For $X \subset E$ we set $X^0 := \{f \in E' : |f(x)| \leq 1 \text{ for all } x \in X\}$
For $Z \subset E'$ we set $X_0 := \{x \in E : |f(x)| \leq 1 \text{ for all } f \in Z\}$

X is a *polar set* if $(X^0)_0 = X$.
Let $A \in \mathcal{L}(E,F)$. If $\overline{AB_E}$ is a neighbourhood of 0 in F then $AB_E = \overline{AB_E}$. (The 'classical' proof of (one half) of Banach's Open Mapping Theorem yields that AB_E is a zero neighbourhood. By absolute convexity AB_E is open, hence closed.)

P.3. Let E, F be K-Banach spaces. The natural map $j_E : E \to E''$ satisfies $\|j_E\| \leq 1$. We call E a *polar Banach space* (and the norm a *polar norm*) if j_E is an isometry into E''. Following [2], E is *topologically pseudoreflexive* if j_E is a linear homeomorphism into E''. E is *reflexive* if j_E is an isomorphism. Recall ([2], 4.17) that c_0 and ℓ^∞ are reflexive and duals of one another via the pairing $\langle x,y\rangle = \sum x_n y_n$ ($x \in c_0, y \in \ell^\infty$). In the same spirit, $c_0(I)$ and $\ell^\infty(I)$ are reflexive if the cardinality of I is nonmeasurable ([2], 4.21).

For $A \in \mathcal{L}(E,F)$ we define its *adjoint* $A' \in \mathcal{L}(F',E')$ by $A'(f) = f \circ A$ ($f \in F'$). We have $\|A'\| \leq \|A\|$. The diagram

$$\begin{array}{ccc} E & \xrightarrow{A} & F \\ \downarrow{\scriptstyle j_E} & & \downarrow{\scriptstyle j_F} \\ E'' & \xrightarrow[A'']{} & F'' \end{array}$$

commutes. A subspace D of E has the WEP in E if the adjoint $E' \to D'$ of the inclusion map $D \to E$ is surjective.

P.4. The following two statements are not hard to prove.

1. Let $(E, \|\ \|)$ be a K-Banach space. Let $\varepsilon_1 > \varepsilon_2 > \ldots, \lim_{n\to\infty} \varepsilon_n = 0$. If $p_1, p_2, \ldots$ are polar norms on E such that $\|x\| \leq p_n(x) \leq (1+\varepsilon_n)\|x\|$ ($n \in \mathbb{N}, x \in E$) then $\|\ \|$ is a polar norm.
2. If E is a polar K-Banach space and $E \sim \ell^\infty$ then E is almost isomorphic to ℓ^∞. (By reflexivity, $E' \sim c_0$. By choosing t-orthogonal bases of E', where t is close to 1 one proves that c_0 and E' are almost isomorphic, hence so are E and ℓ^∞.)

P.5. Let A be a complete metrizable absolutely convex compactoid in a Hausdorff locally convex space E over K. Let $\lambda \in K$, $|\lambda| > 1$. Then there exist $e_1, e_2, \ldots$ in λA with $\lim_{n\to\infty} e_n = 0$ such that, with B the closed absolutely convex hull of $\{e_1, e_2, \ldots\}$,

(i) $A \subset B \subset \lambda A$.
(ii) Each $x \in B$ has a unique representation $x = \sum_{n=1}^\infty \lambda_n e_n$, where $\lambda_n \in K$, $|\lambda_n| \leq 1$ for each n.
(iii) If $\lambda_n \in K$, $|\lambda_n| \leq 1$ for each n then $\sum_{n=1}^\infty \lambda_n e_n \in B$.
(Proof. By [4], 6.1 A, as a topological module over $\{\lambda \in K : |\lambda| \leq 1\}$, is isomorphic to a closed compactoid of c_0. Then apply [2], 4.37 $(\alpha) \Rightarrow (\varepsilon)$.)

P.6. (p-adic Alaoglu Theorem) Let E be a K-Banach space. Then $B_{E'}$ is, for the w'-topology, a complete compactoid. (The proof is standard.)

§1 THE COMPLEMENTATION PROPERTY OF ℓ^∞.

Proposition 1.1. *Let E be a polar K-Banach space, let D be a closed subspace with inclusion map $i : D \to E$. Then $(\overline{i'(B_{E'})}^{w'})^e = B_{D'}$.*

Proof. Since $\|i'\| \le 1$ and $B_{D'}$ is w'-closed and edged we have $S := (\overline{i'(B_{E'})}^{w'})^e \subset B_{D'}$. Suppose we had an $f \in B_{D'}$, $f \notin S$. As the w'-topology on D' is of countable type and S is w'-closed and edged, S and $\{f\}$ can be separated by a w'-continuous linear function $D' \to K$ ([3], 4.4 and 4.7). But such functions are evaluations ([5], Th. 4.10) so there is an $x \in D$ with $|f(x)| > 1$ and $|h(x)| \le 1$ for all $h \in S$. We have $1 < |f(x)| \le \|f\|\,\|x\| \le \|x\|$ so $\|x\| > 1$. On the other hand, by polarity,

$$\begin{aligned}\|x\| = \|i(x)\| &= \sup\{|g \circ i(x)| : g \in B_{E'}\}\\ &= \sup\{|i'(g)(x)| : g \in B_{E'}\}\\ &\le \sup\{|h(x)| : h \in S\} \le 1,\end{aligned}$$

which is a contradiction.

Theorem 1.2. *Let E be a subspace of some polar K-Banach space X. Let $E \sim \ell^\infty$. Then, for each $t \in (0,1)$, E has a t-orthogonal complement in X.*

Proof. Let $i : E \to X$ be the inclusion map.

I. We show that the adjoint $i' : X' \to E'$ is a quotient map. By reflexivity the w'-topology on E' equals the weak topology. The norm topology on E' is strongly polar ([3], 4.4) so every norm-closed absolutely convex edged set is weakly closed. ([3], 4.9). Together with Proposition 1.1 this leads to

$$B_{E'} = \overline{i'(B_{X'})}^{w'})^e = (\overline{i'(B_{X'})}^{w})^e = (\overline{i'(B_{X'})})^e$$

implying that $\overline{i'(B_{X'})}$ is a norm neighbourhood of 0 in E'. Then (see P.2) $i'(B_{X'})$ is closed so we find $B_{E'} = i'(B_{X'})^e$, which is what we wanted to prove.

II. Let $t \in (0,1)$. The space E' is of countable type so it has a $\sqrt{t}$-orthogonal base $e_1, e_2, \ldots$. Now $i' : X' \to E'$ is a quotient map so we can choose $z_1, z_2, \ldots \in X'$ such that $i'(z_n) = e_n, \|z_n\| \le t^{-\frac{1}{2}}\|e_n\|$ for each n. The formula

$$T(\sum_{n=1}^{\infty} \lambda_n e_n) = \sum_{n=1}^{\infty} \lambda_n z_n \quad (\lambda_n \in K, \lim_{n\to\infty} \|\lambda_n e_n\| = 0)$$

defines a $T \in \mathcal{L}(E', X')$ for which $\|T\| \le t^{-1}$ and $i' \circ T$ is the identity on E'. Then $T' \circ i''$ is the identity on E'' and it is easily seen from the diagram

$$\begin{array}{ccccc} E'' & \xrightarrow{i''} & X'' & \xrightarrow{T'} & E'' \\ \uparrow j_E & & \uparrow j_X & & \downarrow j_E^{-1} \\ E & \underset{i}{\longrightarrow} & X & & E \end{array}$$

that $P := j_E^{-1} \circ T' \circ j_X$ is in $\mathcal{L}(X, E)$, has norm $\leq t^{-1}$ and is a projection onto E.

Remark. The space E of Theorem 1.2 is almost isomorphic to ℓ^∞ (see P.4.2).

Corollary 1.3. *Let E be a subspace of some topologically pseudoreflexive K-Banach space X. Let $E \sim \ell^\infty$. Then E is complemented in X.*

Corollary 1.4.
(i) *Let E, X be as in* 1.2. *Then, for each $\varepsilon > 0$, each $f \in E'$ can be extended to an $\tilde{f} \in X'$ with $\|\tilde{f}\| \leq (1+\varepsilon)\|f\|$.*
(ii) *Let E, X be as in* 1.3. *Then E has the WEP in X.*

Corollary 1.4 yields the following result that may look bizarre at first sight.

Proposition 1.5. *Suppose $\#K$ is nonmeasurable. Let E be a K-Banach space such that E' separates the points of E. Suppose $E' \sim \ell^\infty$. Then $E \sim c_0$.*
Proof. $j_E : E \to E''$ is injective, so $\#E \leq \#E'' = \#c_0 = \#K$ and $\#E$ is nonmeasurable. Thus we can, in a standard way, construct a quotient map $\pi : c_0(I) \to E$ where $\#I$ is nonmeasurable. Then $\pi' : E' \to c_0(I)'$ is an isometry. By Corollary 1.4 $\pi'(E')$ has the WEP in $c_0(I)'$ so $\pi'' : c_0(I)'' \to E''$ is surjective. From the reflexivity of $c_0(I)$ and the commutative diagram

$$\begin{array}{ccc} c_0(I) & \xrightarrow{\pi} & E \\ \big\downarrow{\scriptstyle j_{c_0(I)}} & & \big\downarrow{\scriptstyle j_E} \\ c_0(I)'' & \xrightarrow[\pi'']{} & E'' \end{array}$$

it follows that j_E is surjective. So, $E \sim E'' \sim c_0$.

Remark. It may be worth noticing that the 'dual' statement of Proposition 1.5 is false! In fact, in [2], 4.J a closed subspace D of ℓ^∞ is constructed for which $D' \sim c_0$ but $D \not\sim \ell^\infty$. This D furnishes also a negative answer to the question 2 in the Introduction: it is easily seen that each $f \in D'$ has a unique extension $\tilde{f} \in (\ell^\infty)'$, so D has the WEP in ℓ^∞, but also D is weakly dense so that D is not weakly closed.

§2 COMPLEMENTED SUBSPACES OF ℓ^∞.

We first prove two useful lemmas. Let E, F, G be K-Banach spaces, let $i \in \mathcal{L}(E, F)$, $\pi \in \mathcal{L}(F, G)$. Suppose Im i = Ker π, π is surjective. A standard application of the Open Mapping Theorem shows that for the adjoint sequence $G' \xrightarrow{\pi'} F' \xrightarrow{i'} E'$ we have Im π' = Ker i'. If i' is surjective we can apply the same argument to $G' \xrightarrow{\pi'} F' \xrightarrow{i'} E'$ yielding the following.

Lemma 2.1. *Let D be a closed subspace of a K-Banach space E with inclusion map $i : D \to E$ and quotient map $\pi : E \to E/D$. Suppose D has the WEP in E. Then in the commutative* *diagram*

$$\begin{array}{ccccc} D & \xrightarrow{i} & E & \xrightarrow{\pi} & E/D \\ \downarrow j_D & & \downarrow j_E & & \downarrow j_{E/D} \\ D'' & \xrightarrow[i'']{} & E'' & \xrightarrow[\pi'']{} & (E/D)'' \end{array}$$

we have $\mathrm{Im}\, i'' = \mathrm{Ker}\, \pi''$, *and* i'' *is injective.*

Lemma 2.2. (Compare also Example 3.3) *Let D be a closed subspace of a polar K-Banach space E, and let D have the* WEP *in E.*

(i) *If D is reflexive then D is weakly closed.*

(ii) *If E is reflexive, D is weakly closed then D is reflexive.*

Proof. Straightforward from Lemma 2.1 and 'diagram chasing'. (Observe that D is weakly closed if and only if $j_{E/D}$ is injective.)

Remark. The following corollary is a partial positive answer to question 2 (see also the remark following Proposition 1.5). *Let E be a K-Banach space with a base whose cardinality is nonmeasurable. Then every closed subspace with the WEP in E is weakly closed.* (**Proof.** Without loss, assume $E = c_0(I)$ where $\#I$ is nonmeasurable. By Gruson's Theorem [2], 5.9 any closed subspace is isomorphic to $c_0(J)$ for some set J. Then $\#J \leq \#I$ so $c_0(J)$ is reflexive. Now apply Lemma 2.2.(i).)

Theorem 2.3. *For a closed subspace D of ℓ^∞ the following are equivalent.*

(a) *For each $t \in (0,1)$, D has a t-orthogonal complement.*

(b) *D is complemented.*

(c) *D is weakly closed.*

(d) *Either D is finite dimensional or $D \sim \ell^\infty$.*

(e) *Either ℓ^∞/D is finite dimensional or $\ell^\infty/D \sim \ell^\infty$.*

(f) *ℓ^∞/D is a polar K-Banach space.*

(g) *$(\ell^\infty/D)'$ separates the points of ℓ^∞/D.*

(h) *D is reflexive and has the WEP in ℓ^∞.*

(i) *B_D is weakly closed in ℓ^∞.*

Proof. We first prove (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a). Trivially (a) $\Rightarrow$ (b) $\Rightarrow$ (c); (d) $\Rightarrow$ (a) is Theorem 1.2. To establish (c) $\Rightarrow$ (d), let D be weakly closed, D infinite dimensional. By reflexivity, $D = A^0$ where A is a closed subspace of c_0. Then A has an infinite dimensional complement B in c_0 ([2], 3.16(v)) so $B \sim c_0$ and $D \sim B' \sim \ell^\infty$.

Next, we prove (a) $\Rightarrow$ (f) $\Rightarrow$ (g) $\Rightarrow$ (c) of which only (a) $\Rightarrow$ (f) needs some attention. Let $0 < t_1 < t_2 < \ldots < 1$, $\lim_{n\to\infty} t_n = 1$. For each $n \in \mathbb{N}$ there is a projection $P_n : \ell^\infty \to D$ with $\|P_n\| \leq t_n^{-1}$. Then $Q_n := I - P_n$ is a projection with norm $\leq t_n^{-1}$ and kernel D. So there is a bijection $A_n \in \mathcal{L}(\ell^\infty/D, Q_n\ell^\infty)$ making

$$\begin{array}{ccc} \ell^\infty & \xrightarrow{\pi} & \ell^\infty/D \\ Q_n \searrow & & \swarrow A_n \\ & Q_n\ell^\infty & \end{array}$$

commute. For each n the norm $z \mapsto \|A_n z\|$ $(z \in \ell^\infty/D)$ is polar. Also we have

$$\|z\| \le \|A_n z\| \le t_n^{-1}\|z\| \qquad (z \in \ell^\infty/D)$$

so that (see P.4.1) the quotient norm on ℓ^∞/D is polar. We have also (b) $\Rightarrow$ (e) (E/D is linearly homeomorphic to a complemented subspace of ℓ^∞) and (e) $\Rightarrow$ (g). So at this stage we have proved that (a) - (g) are equivalent. The implication (a) $\Rightarrow$ (h) is easy, (h) $\Rightarrow$ (c) is Lemma 2.2 (i) and (c) $\Rightarrow$ (i) is obvious. So we shall complete the proof by showing (i) $\Rightarrow$ (d). Let D be infinite dimensional. The 'closed' unit ball of ℓ^∞ is for the w'-topology $\sigma(\ell^\infty, c_0)$ (which equals the weak topology) a metrizable ([3],8.3), complete (P.6), edged compactoid, hence so is B_D. Let $\lambda \in K, |\lambda| > 1$. By P.5 there exist $f_1, f_2, ... in \lambda B_D$ with $\lim_{n\to\infty} f_n = 0$ weakly such that

$$B_D \subset \overline{co}^w\{f_1, f_2, ...\} \subset \lambda B_D$$

and such that each element of $\overline{co}^w\{f_1, f_2, ...\}$ has a unique representation $\sum_{n=1}^{\infty} \lambda_n f_n$ where $\lambda_n \in K, |\lambda_n| \le 1$ (the summation is with respect to the weak topology of ℓ^∞). The formula

$$(\lambda_1, \lambda_2, ...) \overset{T}{\mapsto} \sum_{n=1}^{\infty} \lambda_n f_n$$

defines therefore a linear bijection $T : \ell^\infty \to D$. It is easily seen that $\|x\| \le \|Tx\| \le |\lambda|\ \|x\|$ $(x \in \ell^\infty)$. Thus, D is linearly homeomorphic to ℓ^∞.

Remark. The implication (i) $\Rightarrow$ (c) is an ultrametric version of the classical Banach-Dieudonné Theorem which states that a subspace D of the dual of a complex Banach space is w'-closed as soon as the closed unit ball of D is w'-closed. In the Appendix we shall prove a stronger version (Krein-Šmulian Theorem) for Banach spaces over a spherically complete ground field.

§3 HOW ABOUT $\ell^\infty(I)$?

It is natural to ask to what extent the previous results can be generalized to $\ell^\infty(I)$. The only positive result we have is in fact Proposition 3.1. The examples 3.2 and 3.3 show that several implications in Theorem 2.3 fail if we replace ℓ^∞ by $\ell^\infty(I)$.

Proposition 3.1. *Let I be a set whose cardinality is nonmeasurable. For a closed subspace D of $\ell^\infty(I)$ the following are equivalent.*
(a) *D is complemented.*
(b) *$D \sim \ell^\infty(J)$ for some set J where $\#J$ is nonmeasurable. D has the WEP in $\ell^\infty(I)$.*
Proof.
(a) $\Rightarrow$ (b). Clearly D has the WEP, is weakly closed, so D is reflexive by Lemma 2.2 (ii). ($\ell^\infty(I)$ is reflexive). Let $P : \ell^\infty(I) \to D$ be a linear continuous surjection. Then $P' : D' \to c_0(I)$ is a norm homeomorphism into $c_0(I)$ so by Gruson's Theorem $D' \sim c_0(J)$ where $\#J \le \#I$ is nonmeasurable. So $D \sim D'' \sim \ell^\infty(J)$.

(b) $\Rightarrow$ (a). Let $i : D \hookrightarrow \ell^\infty(I)$ be the inclusion map. By (b) the adjoint $c_0(I) \xrightarrow{i'} D'$ is surjective. Now $D \sim \ell^\infty(J)$, where $\#J$ is nonmeasurable so $D' \sim c_0(J)$ and there is a map $T \in \mathcal{L}(D', c_0(I))$ such that $i' \circ T$ is the identity on D'. Then $T' \circ i''$ is the identity on D'' and $j_D^{-1} \circ T' \circ j_{\ell^\infty(I)}$ is a projection of $\ell^\infty(I)$ onto D.

Example 3.2. *Let $\#I = \#K$ be nonmeasurable. Then there exists an infinite dimensional closed subspace A_1 of $\ell^\infty(I)$ that has the WEP in $\ell^\infty(I)$ and is of countable type.* (Hence, D is weakly closed, reflexive, but not complemented (Lemma 2.2 (i) and Proposition 3.1).)
Proof. We can make, in a standard way, a quotient map $c_0(I) \xrightarrow{\pi} \ell^\infty$. By reflexivity π'' is surjective, so $A_1 := \pi'((\ell^\infty)')$ has the WEP in $c_0(I)'$ and is of countable type.

Example 3.3. (Negative answer to question 1) *Let I, K be as above. Then there exists a weakly closed subspace A_2 of $\ell^\infty(I)$ such that A_2 is of countable type, but A_2 does not have the WEP in $\ell^\infty(I)$.*
Proof. Let D be as in the Remark following Proposition 1.5. Again, make a quotient map $\pi : c_0(I) \to D$. It is easily seen that $A_2 := \pi'(D')$ is weakly closed, of countable type. If A_2 had the WEP then π'' would be surjective. Then, by reflexivity of $c_0(I)$, j_D would be surjective conflicting the nonreflexivity of D.

§4 SOME CONSEQUENCES FOR STRONGLY POLAR SPACES.

Recall that a K-Banach space E is *strongly polar* ([3], 3.5) if $\sup\{|f| : f \in E', |f| \le p\} = p$ for each continuous seminorm p on E. It is proved in [3], 4.2 that E is strongly polar if and only if for each continuous seminorm p, for each subspace $D \subset E$, for each $f \in D'$ with $|f| \le p$, for each $\varepsilon > 0$, there exists an extension $\tilde{f} \in E'$ such that $|\tilde{f}| \le (1+\varepsilon)p$. It is still an intruiging open problem whether each strongly polar K-Banach space is of countable type. The previous theory yields the following results.

Proposition 4.1. *If E is reflexive and strongly polar then each closed subspace is reflexive.*
Proof. Any closed subspace is weakly closed. Now apply Lemma 2.2(ii).

Proposition 4.2. *Let E be reflexive and strongly polar. Let E' be a subspace of some polar K-Banach space X. Then E' has the* WEP *in X and E' is weakly closed in X.*
Proof. The first statement follows from Part I of the proof of Theorem 1.2. For the second statement apply Lemma 2.2(i).

Proposition 4.3. *Let E be a reflexive strongly polar space. If E' is linearly homeomorphic to a subspace of ℓ^∞ then E is of countable type.*
Proof. Assume $\dim E = \infty$. By Proposition 4.2 and Theorem 2.3 (h) $\Rightarrow$ (d), $E' \sim \ell^\infty$. Then $E \sim E'' \sim c_0$.

§5 APPENDIX: THE p-ADIC KREIN-ŠMULIAN THEOREM.

Throughout §5 K is spherically complete. By modifying 'classical' techniques we shall prove:

Theorem 5.1. *Let E be a K-Banach space. A convex subset C of E' is w'-closed if and only if for each $n \in \mathbb{N}$ the set $C \cap \{f \in E' : ||f|| \le n\}$ is w'-closed.*

Proof. We only need to prove the 'if' part.

(1) From the assumption on C one easily deduces that $C \cap B$ is w'-closed in B for every bounded set $B \subset E'$.

(2) Let bw' be the topology on E' of uniform convergence on compact subsets of E. Then bw' is locally convex, stronger than w', but coincides with w' on bounded subsets of E'. As $j_E(E) = (E', w')'$ and bw' is admissible we have $(E', bw')' = j_E(E)$ so a convex subset of E' is w'-closed if and only if it is bw'-closed. (See [5] for details)

(3) Let $a \in E' \backslash C$; it suffices by (2) to find a bw'-neighbourhood U of a for which $U \subset E' \backslash C$. We may assume $a = 0$, see (1). For each $r > 0$ set $B_r := \{x \in E : ||x|| \le r\}$, $B'_r := \{f \in E' : ||f|| \le r\}$. We shall find finite subsets $F_0, F_1, \ldots$ of E such that $F_n \subset B_{1/n}$ for each $n \in \{1, 2, 3, \ldots\}$ and $F_0^0 \cap F_1^0 \cap \ldots \cap F_n^0 \cap B'_{n+1} \subset E' \backslash C$ for each $n \in \{0, 1, 2, \ldots\}$. (Then $X := \bigcup_n F_n \cup \{0\}$ is compact so $U := X^0$ is a bw'-zero neighbourhood, $U \subset E' \backslash C$.) As $C \cap B'_1$ is w'-closed there is a finite set $F_0 \subset E$ for which $F_0^0 \cap B'_1 \subset E' \backslash C$. Suppose we have chosen $F_0, F_1, \ldots, F_{n-1}$ with the required properties, in particular

$$(*) \qquad F_0^0 \cap F_1^0 \cap \ldots \cap F_{n-1}^0 \cap B'_n \subset E' \backslash C$$

and suppose there is no F_n that meets the requirements. Then, for each finite subset F of $B_{1/n}$ we have

$$A_F := F^0 \cap F_0^0 \cap F_1^0 \cap \ldots \cap F_{n-1}^0 \cap B'_{n+1} \cap C \ne \emptyset$$

The sets A_F, where F is a finite subset of $B_{1/n}$, are c-compact in the w'-topology and have the finite intersection property. So there is an $f \in \bigcap_F A_F$. Then $f \in C$ and $|f| \le 1$ on each finite subset of $B_{1/n}$, so $||f|| \le n$ i.e. $f \in B'_n$. Then, by (*),

$$f \in F_0^0 \cap F_1^0 \cap \ldots \cap F_{n-1}^0 \cap B'_n \subset E' \backslash C$$

contradicting $f \in C$.

Corollary 5.2. *A subspace D of E' is w'-closed if and only if B_D is w'-closed.*

Proof. Suppose B_D is w'-closed. Let $\lambda \in K, |\lambda| > 1$. For each $n \in \mathbb{N}$ the set $D \cap \{f \in E' : ||f|| \le |\lambda|^n\} = \lambda^n B_D$ is w'-closed. Let $r > 0$. For large n we have $|\lambda|^n \ge r$ so that $D \cap \{f \in E' : ||f|| \le r\} = \lambda^n B_D \cap \{f \in E' : ||f|| \le r\}$ is w'-closed. Now apply Theorem 5.1.

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