AN APPROACH TO $p$-ADIC ALMOST PERIODICITY
BY MEANS OF COMPACTOIDS

by

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ABSTRACT. A bounded function $f$ from a group $G$ into a nonarchimedean valued field $K$ is almost periodic ($f \in AP(G \rightarrow K)$) if its left translates form a compactoid in the uniform topology. In Chapter I uniqueness of a $K$-valued invariant mean is proved (Theorem 7.5). Chapter II is a technical preparation to Chapter III in which, for abelian $G$, all pairs $G, K$ are determined for which there exists an invariant mean on $AP(G \rightarrow K)$ (Theorems 14.3 and 15.7). If $G, K$ is such a pair and $K$ is algebraically closed the characters form an orthonormal base of $AP(G \rightarrow K)$ (Theorems 14.2 and 15.6). As an example, almost periodic functions on $\mathbb{Q}_p$ are considered (§17).

INTRODUCTION. The following concept, which is directly borrowed from the theory of complex valued almost periodic functions, is studied in [3], [6]. Let $G$ be a group, let $K$ be a nonarchimedean valued field. A function $f : G \rightarrow K$ is almost periodic if its left translates form a precompact set in the $K$-Banach space $B(G \rightarrow K)$ of all bounded functions $G \rightarrow K$ with the supremum norm $\| \cdot \|_\infty$. However, one might argue that this definition is too restrictive as even characters may not be almost periodic (see §1, Example 3). Thus, in this paper we shall study the consequences of replacing in the above 'precompact' by 'compactoid'.

PRELIMINARIES. Throughout this paper $G$ is a group with unit element $e$ (if $G$ is abelian and additively written the unit element is 0), and $K$ is a nonarchimedean valued field which is complete under the metric induced by the valuation $| \cdot |$. We shall admit trivial valuations.

For a prime number $p$, $\mathbb{Q}_p$ is the valued field of the $p$-adic numbers, $\mathbb{Z}_p := \{ \lambda \in \mathbb{Q}_p :$
$|\lambda| \leq 1$ and $C_p$ is the completion of the algebraic closure of $Q_p$.

A subset $A$ of a $K$-vector space $E$ is absolutely convex if $0 \in A$ and if $x, y \in A$, $\lambda, \mu \in K$, $|\lambda| \leq 1$, $|\mu| \leq 1$ implies $\lambda x + \mu y \in A$. For a subset $Y$ of $E$ we denote the smallest $K$-subspace containing $Y$ by $[Y]$, the smallest absolutely convex subset containing $Y$ by $\text{co} Y$. Observe that, if $K$ is trivially valued, $\text{co} Y = [Y]$. If $A \subset E$ is absolutely convex we set $A^e := A$ if the valuation of $K$ is discrete, $A^e := \bigcap \{ \lambda A : \lambda \in K, |\lambda| > 1 \}$ if the valuation of $K$ is dense.

Let $E$ be a normed space over $K$. For $a \in E$ and $\varepsilon > 0$ we set $B(a, \varepsilon) := \{ x \in E : \| x - a \| \leq \varepsilon \}$. The closure of a set $Y \subset E$ is $\overline{Y}$. Instead of $\text{co} Y$ we write $\overline{co} Y$.

The following crucial notion is a 'convexification' of precompactness, useful if $K$ is not locally compact. A subset $Y$ of $E$ is a compactoid if for each $\varepsilon > 0$ there exists a finite set $F \subset E$ such that $Y \subset B(0, \varepsilon) + \text{co} F$ (rather than $Y \subset B(0, \varepsilon) + F$). For fundamentals on compactoids we refer to [6], [5], [10]. The (topological) dual space of $E$ is $E'$, the space of all linear continuous operators $E \to E$ is $\mathcal{L}(E)$. For the notion of $c$-orthogonality (where $0 < c \leq 1$) and the symbol $\perp$ we refer to [6]. $E$ is of countable type if there exists a countable set $Y \subset E$ with $[Y] = E$.

For a set $X$ let $B(X \to K)$ be the set of all bounded functions $X \to K$. With respect to pointwise operations and the supremum norm $\| \|_\infty$, $B(X \to K)$ is a $K$-Banach algebra. Observe that, if $K$ is trivially valued then $\| \|_\infty$ is also trivial so that $Y \subset B(X \to K)$ is a compactoid if and only if $[Y]$ is finite dimensional.

For $f : G \to K$ and $s \in G$ we set

\[ f_s(x) := f(sx) \quad (x \in G) \]
\[ f^*(x) := f(xs) \quad (x \in G) \]
\[ f^y(x) := f(x^{-1}) \quad (x \in G) \]
\[ f_0 := \{ f_s : s \in G \} \]
\[ f^G := \{ f^* : s \in G \} \]

A character is a map $\alpha : G \to T_K := \{ \lambda \in K : |\lambda| = 1 \}$ for which $\alpha(xy) = \alpha(x)\alpha(y)$ for all $x, y \in G$. The set of all characters is an abelian group $G_K^\times$ under pointwise multiplication.
CHAPTER I: GENERAL THEORY

§1. ALMOST PERIODIC FUNCTIONS

DEFINITION 1.1. A function $f : G \to K$ is almost periodic if $f_G$ is a compactoid in $B(G \to K)$. The set of all almost periodic functions $G \to K$ is denoted $AP(G \to K)$.

Examples.
1. For a character $\alpha$ we have $\alpha_G \in \text{co}\{\alpha\}$. Thus, trivially, characters are almost periodic.
2. A (left) representative function on $G$ is an $f \in B(G \to K)$ for which $[f_G]$ is finite dimensional. Since $f_G$ is also bounded it is a compactoid. Thus, the collection $\mathcal{R}(G \to K)$ of all representative functions is in $AP(G \to K)$.

Let $E$ be a finite dimensional normed space over $K$. A bounded representation of $G$ in $E$ is a homomorphism $s \mapsto U_s$ of $G$ into the group of all invertible elements of $L(E)$ such that $\sup\{\|U_s x\| : s \in G\} < \infty$ for each $x \in E$. If $s \mapsto U_s$ is a bounded representation and $x \in E$, $f \in E'$ then $s \mapsto f(U_s x)$ is in $\mathcal{R}(G \to K)$.

3. We shall say (see the Introduction) that an $f \in B(G \to K)$ is strictly almost periodic ($f \in SAP(G \to K)$) if $f_G$ is precompact. Obviously, $SAP(G \to K) \subset AP(G \to K)$. If each closed bounded subset of $K$ is compact then 'compactoidity' and 'precompactness' are identical notions in $B(G \to K)$ so $SAP(G \to K) = AP(G \to K)$. To see that equality does not hold in general, let $K$ be algebraically closed and let $G$ be the group of roots of unity in $K$. Then $G$ is a discrete infinite subgroup of $T_K$ ([9], Exercise 32.G). The identity $\alpha : G \to K$ is a character (hence, $\alpha \in AP(G \to K)$) but $\alpha(G)$, hence $\alpha_G$, is not precompact (so, $\alpha \notin SAP(G \to K)$).

4. If the valuation of $K$ is trivial then $\mathcal{R}(G \to K) = AP(G \to K)$.

§2. GENERAL PROPERTIES OF $AP(G \to K)$

In this section we prove the non-archimedean counterparts of some basic facts from the theory of complex almost periodic functions.

First we show that 'left almost periodicity' and 'right almost periodicity' are identical notions.

THEOREM 2.1. If $f \in AP(G \to K)$ then $f^G$ is a compactoid.
Proof. Let $\varepsilon > 0$ By compactoidity of $f_G$ there exist $e_1, \ldots, e_n \in B(G \to K)$ and functions $\lambda_1, \ldots, \lambda_n : G \to K$ with $\|\lambda_i\|_{\infty} \leq 1$ for each $i \in \{1, \ldots, n\}$ such that for all $s \in G$

$$\|f_s - \sum_{i=1}^{n} \lambda_i(s)e_i\|_{\infty} \leq \varepsilon$$

i.e.

$$|f(st) - \sum_{i=1}^{n} \lambda_i(s)e_i(t)| \leq \varepsilon \quad (s, t \in G)$$

or

$$\|f^t - \sum_{i=1}^{n} e_i(t)\lambda_i\|_{\infty} \leq \varepsilon \quad (t \in G).$$

We see that the bounded set $f^G$ is contained in $B(0, \varepsilon) + [\lambda_1, \ldots, \lambda_n]$. Hence, $f^G$ is a compactoid.

Remark By taking $\varepsilon = 0$ in the above proof (and by making the obvious modifications) we find that 'left representative function' equals 'right representative function'. If $f \in \mathcal{R}(G \to K)$ then $\dim[f_G] = \dim[f^G]$.

**PROPOSITION 2.2.** Let $f \in AP(G \to K)$. Then $f^G := \{f_s^t : s, t \in G\}$ is a compactoid.

Proof. Let $\varepsilon > 0$. There exist $e_1, \ldots, e_n \in AP(G \to K)$ such that $f_G \subset B(0, \varepsilon) + \text{co}\{e_1, \ldots, e_n\}$. For each $i \in \{1, \ldots, n\}$ we can find, by Theorem 2.1, a finite set $F_i \subset AP(G \to K)$ such that $e_i^G \subset B(0, \varepsilon) + \text{co} F_i$. Set $F := \bigcup_{i=1}^{n} F_i$. Then $F$ is finite and for each $s, t \in G$ we have

$$f_s^t \in B(0, \varepsilon)^t + \text{co}\{e_1^t, \ldots, e_n^t\} \subset B(0, \varepsilon) + B(0, \varepsilon) + \text{co} F = B(0, \varepsilon) + \text{co} F.$$

**PROPOSITION 2.3.** Let $s \in G$. If $f : G \to K$ is almost periodic then so are $f_s$, $f^s$ and $f^v$.

Proof. We only prove almost periodicity of $f^v$. For any $t \in G$ we have $(f^v)_t = (f^t)^v$ so that $(f^v)_G = (f^G)^v$, which is the image of a compactoid under the continuous linear map $g \mapsto g^v$, hence a compactoid. Thus, $f^v \in AP(G \to K)$.

**LEMMA 2.4.** Let $X$ be a set and let $Y, Z$ be compactoids in $B(X \to K)$. Then so is $Y \cap Z := \{fg : f \in Y, g \in Z\}$.

Proof. $Y, Z$ are bounded; without loss we may assume their diameter to be $\leq \frac{1}{2}$. Let $0 < \varepsilon < 1$. By Katsaras' Theorem (see [10]) there exist $e_1, \ldots, e_n$ in the unit
ball of $B(X \to K)$ such that $Y \subset B(0, \varepsilon) + \text{co}\{e_1, \ldots, e_n\}$. Similarly, there exist $f_1, \ldots, f_m$ in the unit ball of $B(X \to K)$ such that $Z \subset B(0, \varepsilon) + \text{co}\{f_1, \ldots, f_m\}$. Since $B(0, \varepsilon) \cdot \text{co}\{e_1, \ldots, e_n\}$ and $B(0, \varepsilon) \cdot \text{co}\{f_1, \ldots, f_m\}$ are contained in $B(0, \varepsilon)$ we have

$$X \cdot Y \subset (B(0, \varepsilon) + \text{co}\{e_1, \ldots, e_n\})(B(0, \varepsilon) + \text{co}\{f_1, \ldots, f_m\})$$

$$\subset B(0, \varepsilon) + \text{co}\{e_1 f_1, e_1 f_2, e_2 f_1, \ldots, e_n f_m\}$$

which proves the lemma.

**THEOREM 2.5.** $AP(G \to K)$ is a closed unitary $K$-subalgebra of $B(G \to K)$.

**Proof.** Clearly the function $s \mapsto 1$ ($s \in G$) is almost periodic. Let $f, g \in AP(G \to K)$, $\lambda \in K$. Then

(i) $(\lambda f) \sigma = \lambda f \sigma$

(ii) $(f + g) \sigma \subset f \sigma + g \sigma$

(iii) $(f \cdot g) \sigma \subset f \sigma \cdot g \sigma$

The sets at the right hand sides of (i) and (ii) are images of the compactoids $f \sigma$, $f \sigma \times g \sigma$ respectively under a linear continuous map, proving that $AP(G \to K)$ is a $K$-vector space. Lemma 2.4 and (iii) show that $AP(G \to K)$ is multiplicatively closed. To prove topological closedness, let $f \in \overline{AP(G \to K)}$, let $\varepsilon > 0$. There is a $g \in AP(G \to K)$ with $\|f - g\|_{\infty} \leq \varepsilon$. There is a finite set $F \subset B(G \to K)$ such that $g \sigma \subset B(0, \varepsilon) + \text{co} F$. Since $\|f_s - g_s\|_{\infty} \leq \varepsilon$ for all $s \in G$ we also have $f \sigma \subset B(0, \varepsilon) + \text{co} F$ implying $f \in AP(G \to K)$.

**COROLLARY 2.6.**

(i) The uniform closure of $R(G \to K)$ is in $AP(G \to K)$.

(ii) If $f \in AP(G \to K)$ and $P : K \to K$ is a polynomial function then $P \circ f \in AP(G \to K)$.

**PROBLEM.** Do we have $\overline{R(G \to K)} = AP(G \to K)$? (For a partial answer, see Theorems 14.2 and 15.6.)

**PROBLEM.** Let $f \in AP(G \to K)$ such that $\inf\{|f(x)| : x \in G\} > 0$. Does it follow that $1/f \in AP(G \to K)$?

**PROPOSITION 2.7.** Let $f \in AP(G \to K)$ and let $\sigma : K \to K$ be a continuous automorphism of $K$. Then $\sigma \circ f \in AP(G \to K)$.

**Proof.** As $\sigma$ is a homeomorphism ([9], Exercise 9.D) the valuation $x \mapsto |\sigma(x)|$ ($x \in K$) is equivalent to $|\ |$, so there exists a $c > 0$ such that $|x| = |\sigma(x)|^c$ for all $x \in K$. Now let $\varepsilon > 0$. There exist $e_1, \ldots, e_n \in B(G \to K)$ such that

$$f \sigma \subset B(0, \varepsilon^c) + \text{co}\{e_1, \ldots, e_n\}.$$
It follows that
\[(\sigma \circ f)_G = \sigma \circ f_G \subset B(0, \varepsilon) + \text{co}\{\sigma \circ e_1, \ldots, \sigma \circ e_n\}\]
implying \(\sigma \circ f \in AP(G \to K)\).

§3. EXTENSION OF THE BASE FIELD

We need the following elementary lemma on compactoids.

**Lemma 3.1.** Let \(E\) be a normed space over \(K\). A subset \(Y\) of \(E\) is a compactoid if and only if \(Y\) is bounded and for each \(\varepsilon > 0\) there exists a finite set \(F \subset Y\) such that \(Y \subset B(0, \varepsilon) + [F]\).

**Proof.** Suppose \(Y\) is a compactoid. Then \(Y\) is bounded. By Katsaras' Theorem there exists, for each \(\varepsilon > 0\), a finite set \(H \subset [Y]\) such that \(Y \subset B(0, \varepsilon) + \text{co} H\). For each \(z \in H\) there exists a finite set \(F_z \subset Y\) such that \(z \in [F_z]\). Then \(Y \subset B(0, \varepsilon) + [F]\) where \(F\) is the finite set \(\bigcup_{z \in H} F_z\). The converse: \(Y\) is a bounded local compactoid, hence a compactoid by [5], Theorem 6.7, \((\alpha) \iff (\gamma)\) and 6.8.

In this section \((L, |\ |)\) is a complete valued field extending \((K, |\ |)\). Our purpose is to compare \(AP(G \to K)\) and \(AP(G \to L)\). First some terminology. For a subset \(Y\) of a Banach space \(E\) over \(L\) we say that \(Y\) is an \(L\)-compactoid (\(L\)-compactoid) if it is a compactoid in \(E\) considered as a Banach space over \(K(L)\). In the same spirit we use expressions like \(\text{co}_L Y\), \(\text{co}_K Y\), \([Y]_K\), \([Y]_L\), etc.

**Proposition 3.2.** Let \(X\) be a set, let \(Y \subset B(X \to K)\). Then \(Y\) is a \(K\)-compactoid in \(B(X \to K)\) if and only if \(Y\) is an \(L\)-compactoid in \(B(X \to L)\).

**Proof.** Trivially, if \(Y\) is a \(K\)-compactoid then it is an \(L\)-compactoid. Conversely, let \(Y\) be an \(L\)-compactoid, let \(\varepsilon > 0\). By Lemma 3.1 there exists a finite subset \(\{e_1, \ldots, e_n\}\) of \(Y\) such that
\[Y \subset B(0, \varepsilon) + [e_1, \ldots, e_n]_L\]

Let \(y \in Y\). There exist \(\lambda_1, \ldots, \lambda_n \in L\) such that
\[(*) \quad y(x) = \delta(x) + \sum_{i=1}^n \lambda_i e_i(x) \quad (x \in X)\]

where \(\delta \in B(X \to L), \|\delta\|_\infty \leq \varepsilon\). Now set
\[D := [1, \lambda_1, \ldots, \lambda_n]_K \subset L\]
Since \( e_i(x) \in K \) for each \( i \) we have \( \delta(x) \in D \) for each \( x \in X \). There exists a \( K \)-linear map \( \varphi : D \to K \) with \( \varphi(1) = 1 \) and \( |\varphi(d)| \leq 2|d| \) for each \( d \in D \). Application of \( \varphi \) to (*) leads to

\[
y(x) = \varphi(\delta(x)) + \sum_{i=1}^{n} \varphi(\lambda_i)e_i(x) \quad (x \in X)
\]

yielding

\[
y \in B(0,2\varepsilon) + [e_1, \ldots, e_n]_K
\]

This, together with the boundedness of \( Y \), implies by Lemma 3.1 that \( Y \) is a \( K \)-compactoid.

**Remark.** By taking \( \varepsilon = 0 \) in the above proof we obtain that, if \( \text{dim}_L[Y]_L < \infty \) then \( \text{dim}_K[Y]_K = \text{dim}_L[Y]_L \).

**THEOREM 3.3.** \( \text{AP}(G \to K) = \text{AP}(G \to L) \cap B(G \to K) \).

**Proof.** The inclusion \( \text{AP}(G \to K) \subset \text{AP}(G \to L) \cap B(G \to K) \) is obvious. Conversely, if \( f \in \text{AP}(G \to L) \cap B(G \to K) \) then \( f_G \) is an \( L \)-compactoid and is in \( B(G \to K) \). By Proposition 3.2 it is a \( K \)-compactoid, so \( f \in \text{AP}(G \to K) \).

**Remark.** \( \mathcal{R}(G \to K) = \mathcal{R}(G \to L) \cap B(G \to K) \) (see the Remark above).

**Remark.** In general we do not have \( [\text{AP}(G \to K)]_L = \text{AP}(G \to L) \) (see the Remark following Theorem 16.1).

§4. INTERMEZZO: A COMPARISON OF \( \text{SAP}(G \to K) \) AND \( \text{AP}(G \to K) \)

Recall (see §1, Example 3 and Introduction) that, by definition, an \( f : G \to K \) is in \( \text{SAP}(G \to K) \) if \( f_G \) is precompact in \( B(G \to K) \). Let \( P(G \to K) \) (see [6], Example 3.E) be the \( K \)-Banach algebra of all \( f \in B(G \to K) \) for which \( \overline{f(G)} \) is a compact subset of \( K \). In this section we prove the following theorem.

**THEOREM 4.1.**

(i) \( \text{SAP}(G \to K) = P(G \to K) \cap \text{AP}(G \to K) \).

(ii) \( \text{SAP}(G \to K) \) is the closure of the \( K \)-linear span of the idempotents in \( \text{AP}(G \to K) \).

We shall prove three lemmas which together form a proof of Theorem 4.1.

**LEMMA 4.2.** \( \text{SAP}(G \to K) \subset P(G \to K) \cap \text{AP}(G \to K) \).
Proof. Only $SAP(G \rightarrow K) \subseteq P(G \rightarrow K)$ may need a proof. So, let $f \in SAP(G \rightarrow K)$. Then $f(G)$ is the image of the precompact set $f_G$ under the map $g \mapsto g(e)$. Hence, $f(G)$ is precompact.

**Lemma 4.3.** If $h$ is an idempotent in $AP(G \rightarrow K)$ then $h_G$ is finite, in particular $h \in SAP(G \rightarrow K)$.

**Proof.** We have $h(G) \subseteq \{0, 1\}$ so, by Theorem 3.3, $h \in AP(G \rightarrow K_0)$ where $K_0$ is the closure of the prime field of $K$. We have two cases.

(i) $K_0$ is isomorphic to $Q_p$ for some prime $p$, or is a finite field. Then each bounded subset of $K_0$ is compact so $h_G$ is precompact (see §1, Example 3). If $h_s \neq h_t$ for some $s, t \in G$ then $\|h_s - h_t\|_\infty = 1$.

It follows that $h_G$ is finite.

(ii) $K_0$ is isomorphic to $Q$ with the trivial valuation. Then $h \in R(G \rightarrow Q)$ (see §1, Example 4) so $n := \dim[h_G]_Q$ is finite. There exist $x_1, \ldots, x_n \in G$ such that

$$
\Phi : g \mapsto (g(x_1), g(x_2), \ldots, g(x_n))
$$

is a $Q$-linear isomorphism $[h_G]_Q \rightarrow Q^n$. For each $s \in G$ we have

$$
\Phi(h_s) = (h(sx_1), h(sx_2), \ldots, h(sx_n))
$$

which is a sequence of zeros and ones. Then, $\Phi(h_G)$ is finite and so is $h_G$.

**Lemma 4.4.** Let $f \in AP(G \rightarrow K) \cap P(G \rightarrow K)$. For each $\varepsilon > 0$ there exist idempotents $e_1, \ldots, e_n \in AP(G \rightarrow K)$ and $\lambda_1, \ldots, \lambda_n \in K$ such that $\|f - \sum_{i=1}^n \lambda_i e_i\|_\infty \leq \varepsilon$.

**Proof.** By compactness of $\overline{f(G)}$ the equivalence relation defined by $x \sim y$ if $|x - y| \leq \varepsilon$ yields a partition of $\overline{f(G)}$ into say, $n$ balls $B_1, B_2, \ldots, B_n$ (relative to $\overline{f(G)}$) of radius $\varepsilon$.

For each $i \in \{1, \ldots, n\}$ choose an $x_i \in G$ with $f(x_i) \in B_i$ and define $h : G \rightarrow K$ by

$$
h = \sum_{i=1}^n f(x_i) \xi_{f^{-1}(B_i)}.
$$

(Here for a subset $Y$ of $G$, $\xi_Y$ denotes the $K$-valued characteristic function of $Y$.) By construction $\|f - h\|_\infty \leq \varepsilon$. We complete the proof by showing that for each $i \in \{1, \ldots, n\}$ the idempotent $\xi_{f^{-1}(B_i)}$ is in $AP(G \rightarrow K)$. By the Kaplansky-Weierstrass Theorem ([9], Theorem 43.3) there exist polynomial functions $P_1, P_2, \ldots : K \rightarrow K$ such that $P_n \rightarrow \xi_{B_i}$ uniformly on $\overline{f(G)}$. Then $\lim_{n \rightarrow \infty} P_n \circ f = \xi_{B_i} \circ f = \xi_{f^{-1}(B_i)}$ uniformly on $G$. By Corollary 2.6(ii) and Theorem 2.5, $\xi_{f^{-1}(B_i)} \in AP(G \rightarrow K)$.
§5. SEPARATION OF VARIABLES

In this section we study the formula

$$f(xy) = \sum_{n=1}^{\infty} a_n(x)b_n(y) \quad (x, y \in G)$$

for $f \in AP(G \to K)$ and suitable functions $a_n$ and $b_n$.

First, a technicality. Recall ([6]) that, if $K$ is nontrivially valued and $c \in (0,1]$, a sequence $q_1, q_2, \ldots$ in $B(G \to K)$ is (by definition) $c$-orthogonal if for each $n \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_n \in K$

$$\| \sum_{i=1}^{n} \lambda_i q_i \|_\infty \geq c \max_{1 \leq i \leq n} \| \lambda_i q_i \|_\infty.$$ 

In this paper we extend this definition by allowing $K$ to have a trivial valuation. It is not hard to see that if $K$ is trivially valued and $q_1, q_2, \ldots \in B(G \to K)$ are nonzero then $q_1, q_2, \ldots$ is $c$-orthogonal for some $c \in (0,1]$ if and only if the $q_1, q_2, \ldots$ are $K$-linearly independent.

**Lemma 5.1.** Let $f \in AP(G \to K)$, let $\lambda \in K$, $\lambda = 1$ if the valuation of $K$ is discrete, $|\lambda| > 1$ if the valuation of $K$ is dense. Then there exist a $|\lambda|^{-1}$-orthogonal sequence $b_1, b_2, \ldots$ in $\overline{K} f_G$ with $\lim_{n \to \infty} \|b_n\|_\infty = 0$, and functions $a_1, a_2, \ldots : G \to K$ with $\|a_n\|_\infty \leq 1$ for each $n$ such that

$$f(xy) = \sum_{n=1}^{\infty} a_n(x)b_n(y) \quad (x, y \in G)$$

**Proof.** First assume that the valuation of $K$ is not trivial. By compactoid of $f_G$ there exists ([6], 4.36A and 4.37) a $|\lambda|^{-1}$-orthogonal sequence $b_1, b_2, \ldots$ in $\overline{K} f_G$ such that $\lim_{n \to \infty} \|b_n\|_\infty = 0$ and $f_G \subset \overline{K}\{b_1, b_2, \ldots\}$. So, for each $x \in G$ there exist (unique) elements $a_1(x), a_2(x), \ldots$ of $B(0,1) \subset K$ for which $f_x = \sum_{n=1}^{\infty} a_n(x)b_n$ i.e.

$$f(xy) = \sum_{n=1}^{\infty} a_n(x)b_n(y) \quad (x, y \in G)$$

If the valuation of $K$ is trivial then $[f_G]$ is finite-dimensional, let $b_1, \ldots, b_n$ be a base. In the above spirit we find $a_1, \ldots, a_n : G \to K$ such that $f(xy) = \sum_{i=1}^{n} a_i(x)b_i(y) \quad (x, y \in G).$ To arrive at the conclusion, take $a_i = b_i = 0$ if $i > n.$
Remark. By Theorem 2.1 we obtain also a true statement, if in the formulation of Lemma 5.1 we replace \( f_G \) by \( f^G \) and \( f(xy) \) by \( f(yx) \).

**Proposition 5.2.** Let \( f \in AP(G \to K) \). Let \( S := \overline{\text{co}} f^G \), let \( E := [f_G] \). If \( \phi \in E' \), \( \|\phi\| \leq 1 \) then \( s \mapsto \phi(f_s) \) is an element of \( S^c \). (In particular, it is almost periodic, and these conclusions hold for any \( \phi \in AP(G \to K)' \) with \( \|\phi\| \leq 1 \).)

**Proof.** Choose \( \lambda \in K, |\lambda| > 1 \) if the valuation of \( K \) is dense, \( \lambda = 1 \) otherwise; we prove that \( s \mapsto \phi(f_s) \) is in \( \lambda^2 S \). By the above remark there exists a sequence \( a_1, a_2, \ldots \in \lambda S \) with \( \lim_{n \to \infty} \|a_n\|_\infty = 0 \) and \( b_1, b_2, \ldots : G \to B(0,1) \subset K \) such that \( f(xy) = \sum_{n=1}^{\infty} a_n(x)b_n(y) \) \( (x, y \in G) \), i.e.

\[
(*) \quad f_x = \sum_{n=1}^{\infty} a_n(x)b_n \quad (x \in G)
\]

Let \( E_1 \) be the Banach space generated by \( E \cup \{b_1, b_2, \ldots\} \). Then \( \phi \) can be extended to an element \( \tilde{\phi} \in E_1' \) for which \( \|\tilde{\phi}\| \leq |\lambda| \). (If the valuation is discrete then \( K \) is spherically complete and we have the Hahn-Banach Theorem ([6], Theorem 4.8), if the valuation is dense, observe that \( E_1 \) is of countable type and apply [6], Theorem 3.16 (vi).) Application of \( \tilde{\phi} \) to \( (*) \) yields

\[
\phi(f_x) = \sum_{n=1}^{\infty} a_n(x)\tilde{\phi}(b_n)
\]

Thus, our function \( s \mapsto \phi(f_s) \) equals

\[
\sum_{n=1}^{\infty} a_n\tilde{\phi}(b_n)
\]

which is in \( \lambda^2 S \) since for each \( n \in \mathbb{N} \), \( a_n \in \lambda S \) and \( |\tilde{\phi}(b_n)| \leq |\lambda| \|b_n\|_\infty \leq |\lambda| \).

**Proposition 5.3.** For a function \( f : G \to K \) the following statements are equivalent.

\( (a) \) \( f \) is almost periodic.
\( (\beta) \) There exist sequences \( n \mapsto a_n \) and \( n \mapsto b_n \) in \( B(G \to K) \) with \( \lim_{n \to \infty} \|a_n\|_\infty \|b_n\|_\infty = 0 \) and \( f(xy) = \sum_{n=1}^{\infty} a_n(x)b_n(y) \) \( (x, y \in G) \).
\( (\gamma) \) There exist sequences \( n \mapsto a_n \) and \( n \mapsto b_n \) in \( AP(G \to K) \) with \( \lim_{n \to \infty} \|a_n\|_\infty \|b_n\|_\infty = 0 \) and \( f(xy) = \sum_{n=1}^{\infty} a_n(x)b_n(y) \) \( (x, y \in G) \).
(δ) Let \( \lambda \in K, \lambda = 1 \) if the valuation of \( K \) is discrete, \(|\lambda| > 1 \) if the valuation of \( K \) is dense. Then there exist sequences \( n \mapsto a_n \in \overline{\mathbb{C} f^G}, n \mapsto b_n \in \overline{\mathbb{C} f^G}, n \mapsto c_n \in K \) with

\[
\lim_{n \to \infty} \|a_n\|_\infty = \lim_{n \to \infty} \|b_n\|_\infty = 0, \quad \|c_n a_n\|_\infty \leq |\lambda|, \quad \|c_n b_n\|_\infty \leq |\lambda|
\]

for each \( n \in \mathbb{N} \), and \( f(x y) = \sum_{n=1}^{\infty} c_n a_n(x) b_n(y) \) (\( x, y \in G \)).

**Proof.** (a) \( \Rightarrow \) (δ). Choose \( \mu \in K, 1 < |\mu|^2 \leq |\lambda| \) if the valuation of \( K \) is dense, \( \mu = 1 \) otherwise. By Lemma 5.1 there exists a \(|\mu|^{-1}\)-orthogonal sequence \( q_1, q_2, \ldots \) in \( \mu \overline{\mathbb{C} f^G} \) with \( \lim_{n \to \infty} \|q_n\|_\infty = 0 \) and functions \( p_1, p_2, \ldots : G \to B(0,1) \subset K \) such that

\[
f(x) = \sum_{n=1}^{\infty} p_n(x) q_n \quad (x \in G).
\]

Now set

\[
a_n = \mu^{-1} d_n p_n, \quad b_n = \mu^{-1} q_n, \quad c_n = \mu^2 d_n^{-1} \quad \text{if } q_n \neq 0
\]

\[
a_n = 0, \quad b_n = 0, \quad c_n = 0 \quad \text{if } q_n = 0
\]

where \( d_n \in K \) is chosen such that

\[
|\mu|^{-1} \|q_n\|_\infty \leq |d_n| < \|q_n\|_\infty \quad \text{if the valuation is dense}
\]

\[
|d_n| = \|q_n\|_\infty \quad \text{if the valuation is discrete}.
\]

We prove that \( a_n \in \overline{\mathbb{C} f^G} \) (the other statements of (δ) are easily checked). To this end we may suppose that \( q_n \neq 0 \). Let \( \phi \) be the coordinate map

\[
\sum_{i=1}^{\infty} \lambda_i q_i \mapsto \lambda_n
\]

defined on the Banach space generated by \( \{q_1, q_2, \ldots\} \). Observe that this space contains \([f_G]\). By \(|\mu|^{-1}\)-orthogonality

\[
|\phi(\sum_{i=1}^{\infty} \lambda_i q_i)| = \|q_n\|_\infty^{-1} \|\lambda_n q_n\|_\infty \leq |\mu| \|q_n\|_\infty^{-1} \|\sum_{i=1}^{\infty} \lambda_i q_i\|_\infty
\]

so that \( \|\phi\| \leq |\mu| \|q_n\|_\infty^{-1} \). By our choice of \( d_n \) the map \( \mu^{-1} d_n \phi \) has norm \( \leq 1 \) if the valuation is discrete, \( < 1 \) if the valuation is dense. Proposition 5.2 then tells us that

\[
x \mapsto \mu^{-1} d_n \phi(f_x)
\]

is in \( \overline{\mathbb{C} f^G} \). But since

\[
\phi(f_x) = \phi(\sum_{i=1}^{\infty} p_i(x) q_i) = p_n(x) \quad (x \in G)
\]

we have that \( a_n = \mu^{-1} d_n p_n \in \overline{\mathbb{C} f^G} \).
The implications $(\delta) \Rightarrow (\gamma) \Rightarrow (\beta)$ are trivial. We prove $(\beta) \Rightarrow (\alpha)$. We may assume in $(\beta)$ that $\|a_n\|_\infty \leq 1$ for each $n$ and $\lim_{n \to \infty} \|b_n\|_\infty = 0$. Then $(\beta)$ implies that for each $x \in G$

$$f_x = \sum_{n=1}^{\infty} a_n(x)b_n \in \overline{\ell_1}\{b_1, b_2, \ldots\}$$

and $f_G$ is a compactoid.

§6. CONVOLUTION ON THE DUAL SPACE

Throughout §6, $E$ is a closed subspace of $AP(G \to K)$ that is shift invariant (i.e. $f \in E$ implies $f_G \subset E$ and $f^G \subset E$). We shall define a convolution product $*$ on its dual $E'$.

First a notation. For $\mu(f)$ ($f \in E$, $\mu \in E'$) we sometimes shall write $\int f(x)d\mu(x)$ (just for convenience, no 'measure' is involved).

Let $\mu, \nu \in E'$ and $f \in E$. By Proposition 5.3 $(\alpha) \Rightarrow (\delta)$ we may write

$$f(xy) = \sum_{n=1}^{\infty} a_n(x)b_n(y) \quad (x, y \in G)$$

where $a_n$ and $b_n$ are again in $E$ and $\lim_{n \to \infty} \|a_n\|_\infty \|b_n\|_\infty = 0$. Then

$$x \mapsto \nu(f_x) = \int f(xy)d\nu(y) = \sum_{n=1}^{\infty} a_n(x)\nu(b_n)$$

is also in $E$ and we can apply $\mu$ to it obtaining

$$(\mu \ast \nu)(f) := \int (\int f(xy)d\nu(y))d\mu(x) = \sum_{n=1}^{\infty} \mu(a_n)\nu(b_n)$$

and we have defined a map $\mu \ast \nu : E \to K$. Before studying it we observe that $(\mu \ast \nu)(f)$ can also be obtained by applying $\nu$ to the function $y \mapsto \nu(fy)$. Thus we have the following

**PROPOSITION 6.1. (‘FUBINI THEOREM’)** Let $\mu, \nu \in E'$, $f \in E$. Then

$$\int (\int f(xy)d\nu(y))d\mu(x) = \int (\int f(xy)d\mu(x))d\nu(y).$$

Henceforth we shall omit unnecessary brackets and write

$$(\mu \ast \nu)(f) = \int \int f(xy)d\nu(y)d\mu(x) = \int \int f(xy)d\mu(x)d\nu(y).$$
PROPOSITION 6.2. With convolution as multiplication the space $E'$ (in particular, $AP(G \to K)'$) is a $K$-Banach algebra with unit. If $G$ is commutative then so is $E'$.

Proof. Let $\mu, \nu \in E'$. From the definition it follows that $\mu * \nu$ is a $K$-linear map $E \to K$. From

$$||((\mu * \nu)(f))|| \leq ||\mu|| \sup_{x \in G} |\nu(f_x)| \leq ||\mu|| ||\nu|| \|f\|_\infty \quad (f \in E)$$

we obtain $\mu * \nu \in E'$ and $||\mu * \nu|| \leq ||\mu|| ||\nu||$. The unit element of $E'$ is $\delta_e$ define by $\delta_e(f) := f(e)$ $(f \in E)$. The rest of the proof is also straightforward.

§7. INVARIANT MEANS

Let us call a closed shift invariant subspace $E$ of $AP(G \to K)$ a special $G$-module if $E$ contains the constant functions and if $f \in E$ implies $f^\vee \in E$. Examples of such special $G$-modules are $AP(G \to K)$, $SAP(G \to K)$, $\overline{R(G \to K)}$, the set of all almost periodic functions that are continuous with respect to some group topology on $G$.

DEFINITION 7.1. Let $E$ be a special $G$-module. An element $\mu$ of $E'$ is a left invariant mean on $E$ if

(i) $\mu(f_s) = \mu(f) \quad (f \in E, \ s \in G)$
(ii) $\mu(1) = 1$
(iii) $\|\mu\| = 1$.

(Here, the symbol $1$ is used for both the constant function one and the unit element of $K$.) Similarly one defines right invariant mean. If $\mu$ is both a left invariant mean and a right invariant mean it is an invariant mean.

LEMMA 7.2. If $\mu \in E'$ is a left invariant mean then $\mu^\vee$, defined by

$$\mu^\vee(f) = \mu(f^\vee),$$

is a right invariant mean.

Proof. We have $\mu^\vee \in E'$, $\mu^\vee(1) = 1$, $\|\mu^\vee\| = \|\mu\|$. For $s \in G$, $f \in E$ we have $\mu^\vee(f^s) = \mu((f^s)^\vee) = \mu((f^\vee)^{s^{-1}}) = \mu(f^\vee) = \mu^\vee(f)$.

LEMMA 7.3. Each left invariant mean on $E$ is a right invariant mean.

Proof. Let $\mu \in E'$ be a left invariant mean. By Lemma 7.2, $\mu^\vee$ is a right invariant mean. Let $f \in E$. We have

$$(\mu^\vee * \mu) = \int \int f(xy)d\mu(y)d\mu^\vee(x) = \mu(f)\mu^\vee(1) = \mu(f).$$
But also, with Proposition 6.1,

\[ (\mu^\vee \ast \mu)(f) = \int \int f(xy) d\mu^\vee(x) d\mu(y) = \mu^\vee(f) \mu(1) = \mu^\vee(f). \]

Hence, \( \mu = \mu^\vee \) and \( \mu \) is right invariant.

The following corollary obtains.

**THEOREM 7.4.** Let \( E \) be a special \( G \)-module.

(i) If \( \mu_1 \) and \( \mu_2 \) are left (right) invariant means on \( E \) then \( \mu_1 = \mu_2 \).

(ii) If \( \mu \) is a left (right) invariant mean on \( E \) then \( \mu \) is an invariant mean and \( \mu = \mu^\vee \).

**Proof.** (i) Let \( \mu_1, \mu_2 \) be left invariant means. By Lemma 7.3 \( \mu_1 \) is right invariant. Like in the previous proof one has \( \mu_1 \ast \mu_2 = \mu_2 \) and \( \mu_1 \ast \mu_2 = \mu_1 \).

(ii) Obvious from Lemma 7.3.

We state two further corollaries.

**THEOREM 7.5.** (UNIQUENESS THEOREM FOR INVARIANT MEANS)

There exists at most one invariant mean on \( AP(G \rightarrow K) \).

**THEOREM 7.6.** Let \( \tau \) be a group topology on \( G \). Then, on the set of all \( \tau \)-continuous almost periodic functions \( G \rightarrow K \), there exists at most one invariant mean.

**§8. A CRITERION FOR EXISTENCE OF INVARIANT MEANS**

In contrast to the theory of complex valued almost periodic functions, invariant means on \( AP(G \rightarrow K) \) do not always exist. For example, this happens if \( G = \mathbb{Z}_p \) and \( K = \mathbb{Q}_p \).

**Proof.** Let \( f : \mathbb{Z}_p \rightarrow \mathbb{Q}_p \) be the function \( x \mapsto x \). Then for each \( s \in \mathbb{Z}_p \) we have \( f_s(x) = s + x \) \((x \in \mathbb{Z}_p)\) whence \( f_s = s.1 + f \in [1, f] \), so \( f \in R(\mathbb{Z}_p \rightarrow \mathbb{Q}_p) \). If \( \mu \in AP(\mathbb{Z}_p \rightarrow \mathbb{Q}_p)' \) is shift invariant then from \( f_1 = 1 + f \) we obtain \( \mu(1) = \mu(f_1) - \mu(f) = 0 \neq 1 \).

In Theorem 8.2 we shall give a criterion of existence of invariant means. The proof is somewhat involved as \( K \) may not be spherically complete so we do not have the 'full' Hahn-Banach Theorem. For that very reason we first consider a slight extension of the results of §7. Let \( E \) be a special \( G \)-module. Let us say that \( \mu \in E' \) is a topological left invariant mean if \( \mu \) satisfies (i) and (ii) of Definition 7.1 (i.e., the condition \( ||\mu|| = 1 \) is dropped). Similarly we introduce 'topological right invariant mean' and 'topological invariant mean'. The following extension of Theorem 7.4 emerges directly from the proofs of the Lemmas 7.2 and 7.3.
PROPOSITION 8.1. Let \( E \) be a special \( G \)-module.

(i) If \( \mu_1 \) and \( \mu_2 \) are left (right) topological means on \( E \) then \( \mu_1 = \mu_2 \).

(ii) If \( \mu \) is a left (right) topological invariant mean on \( E \) then \( \mu \) is a topological invariant mean and \( \mu = \mu^\vee \).

For any special \( G \)-module \( E \subset AP(G \to K) \) we introduce

\[
H^l := \{ f_s - f : f \in E, s \in G \} \\
H^r := \{ f_s - f : f \in E, s \in G \} \\
H := H^l + H^r.
\]

THEOREM 8.2. For a special \( G \)-module \( E \subset AP(G \to K) \) the following are equivalent.

(\( \alpha \)) There exists an invariant mean on \( E \).

(\( \beta \)) \( 1 \perp H^l \).

(\( \gamma \)) \( 1 \perp H^r \).

(\( \delta \)) \( 1 \perp H \).

If (\( \alpha \)) and (\( \delta \)) are true then \( H^l = H^r = H \) and \( \dim E/H = 1 \).

Proof. For trivially valued \( K \) the purely algebraic proof (\( 1 \perp H^l \) if and only if \( 1 \not\in H^l \) etc.) is left to the reader. So assume the valuation of \( K \) is not trivial. For the equivalence of (\( \alpha \)) and (\( \delta \)) it suffices, by symmetry, to prove (\( \alpha \)) \( \Rightarrow \) (\( \delta \)) \( \Rightarrow \) (\( \beta \)) \( \Rightarrow \) (\( \alpha \)).

(\( \alpha \)) \( \Rightarrow \) (\( \delta \)). Let \( M \) be an invariant mean on \( E \). Then \( M \) vanishes on \( H \) so that for all \( g \in H \)

\[
1 = |M(1)| = |M(1 - g)| \leq ||1 - g||_\infty.
\]

We conclude that \( 1 \perp H \).

(\( \delta \)) \( \Rightarrow \) (\( \beta \)). Trivial.

(\( \beta \)) \( \Rightarrow \) (\( \alpha \)). First, let \( F \subset E \) be a special \( G \)-module which is of countable type as a Banach space. Set \( H^l_F := [f_s - f : f \in F, s \in G] \). Then \( 1 \perp H^l_F \). The map

\[
\phi : \lambda \mapsto \lambda + g \quad (\lambda \in K, g \in H^l_F),
\]

defined on \( K \cdot 1 + H^l_F \) is \( K \)-linear and \( ||\phi|| = 1 \), by orthogonality. Thanks to [6], Theorem 3.16 (vi), for each \( \varepsilon > 0 \) we can choose an extension \( M_\varepsilon \in F' \) of \( \phi \) with \( ||M_\varepsilon|| \leq 1 + \varepsilon \). By construction, each \( M_\varepsilon \) is a topological left invariant mean on \( F \). From Proposition 8.1 we infer that \( M := M_\varepsilon \) does not depend on \( \varepsilon \). But then \( ||M|| \leq 1 + \varepsilon \) for each \( \varepsilon > 0 \) so \( M \) is an invariant mean on \( F \). Thus, at this stage of the proof we have shown that on each special \( G \)-module \( F \subset E \), \( F \) of countable type, there exists a (unique) invariant mean \( M_F \). As each element \( f \) of \( E \) lies in such a \( G \)-module viz. \([\{1\} \cup f_G \cup (f^\vee)_G] \) we can,
by uniqueness, glue the $M_F$'s together obtaining a map $M : E \to K$. A straightforward argument yields that $M$ is an invariant mean on $E$.

For the second part of the Theorem, assume (α) – (δ); we prove that $K.1 + H^1 = E$.

Suppose we had an $f \in E$, $f \notin K.1 + H^1$. Let $F$ be the smallest special $G$-module in $E$ containing $f$. Then $F$ is of countable type and (with $H^1_F$ as above) $f \notin K.1 + H^1_F$. As the latter set is closed the maps

$$
\phi_1 : \lambda f + \mu.1 + h \mapsto \lambda + \mu \\
\phi_2 : \lambda f + \mu.1 + h \mapsto \mu
$$

are in $(Kf + K.1 + H^1_F)$. They extend to elements $\tilde{\phi}_1, \tilde{\phi}_2$ of $F'$. By construction $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are topological left invariant means on $F$, so $\phi_1 = \phi_2$ (Proposition 8.1), a contradiction.

We state a few corollaries.

**THEOREM 8.3.** Let $(L, | 1 |)$ be a complete valued field extending $(K, | 1 |)$. If $M : AP(G \to L) \to L$ is an (L-linear) invariant mean on $AP(G \to L)$ then $M|AP(G \to K)$ is a (K-linear) invariant mean : $AP(G \to K) \to K$.

**Proof.** Let

$$
H_1 := [f_s - f : f \in AP(G \to L), s \in G]_L \\
H_2 := [f_s - f : f \in AP(G \to K), s \in G]_K
$$

Then $H_2 \subset H_1$. By Theorem 8.2 $M$ has the form

$$
M : \lambda.1 + h \mapsto \lambda \quad (\lambda \in L, h \in H_1)
$$

From $1 \perp H_1$ it follows that $1 \perp H_2$. So, again by Theorem 8.2, $AP(G \to K) = K.1 + H_2$ and

$$
N : \lambda.1 + h \mapsto \lambda \quad (\lambda \in K, h \in H_2)
$$

is the (unique) invariant mean on $AP(G \to K)$. We see that $N$ is the restriction of $M$ to $AP(G \to K)$.

**PROPOSITION 8.4.** Let $f \in AP(G \to K)$ and let $K_1$ be the smallest closed subfield of $K$ containing $f(G)$. If $M$ is an invariant mean on $AP(G \to K)$ then $M(f) \in K_1$.

**Proof.** By Theorem 3.3 we have $f \in AP(G \to K_1)$. Now apply Theorem 8.3.

**PROPOSITION 8.5.** If $M$ is an invariant mean on $AP(G \to K)$ and $\sigma : K \to K$ is a continuous automorphism then

$$
\sigma(M(f)) = M(\sigma \circ f) \quad (f \in AP(G \to K)).
$$
Proof. Proposition 2.7 and continuity of $\sigma^{-1}$ yield that 

$$N : f \mapsto \sigma^{-1}(M(\sigma \circ f))$$

is a $K$-linear continuous map $AP(G \to K) \to K$. As $\sigma \circ f_s = (\sigma \circ f)_s$ for all $s \in G$, $N$ is shift invariant. Since also $N1 = 1$, $N$ is a topological left invariant mean so that $N = M$ by Proposition 8.1.

The following observation is simple.

**Proposition 8.6.** If there exists an invariant mean on $AP(G \to K)$ and $H$ is a normal subgroup of $G$ then there exists an invariant mean on $AP(G/H \to K)$.

**Proof.** Let $\pi : G \to G/H$ be the quotient map, let $M$ be an invariant mean on $AP(G \to K)$. One verifies immediately that $f \mapsto M(f \circ \pi)$ is an invariant mean on $AP(G/H \to K)$.

**Problem.** Characterize the pairs $G, K$ for which there exists an invariant mean on $AP(G \to K)$.

**Comment.** For abelian $G$ the problem will be solved in Chapter III (Theorems 14.3 and 15.7). For the general case so far we only have the modest results below (obtained by brute force).

**Proposition 8.7.** Suppose each finite subset of $G$ generates a finite subgroup. Let $k$ be the residue class field of $K$.

(i) If the characteristic of $k$ is 0 then there exists an invariant mean on $AP(G \to K)$.

(ii) If the characteristic of $k$ is $p \neq 0$ and $G$ is $p$-free (i.e. if $H_1 \subset H_2$ are subgroups then the index $[H_2 : H_1]$, whenever finite, is not divisible by $p$) then there exists an invariant mean on $AP(G \to K)$.

**Proof.** There exists a nontrivially valued spherically complete field $(L, | |)$ extending $(k, | |)$. By [7], Theorem 3.6 there exists an invariant mean on $B(G \to L)$, hence on $AP(G \to L)$. Now apply Theorem 8.3.

**Proposition 8.8.** Let the residue class field of $K$ have characteristic $p \neq 0$. If there exists an invariant mean on $AP(G \to K)$ then no finite quotient group of $G$ has elements of order $p$.

**Proof.** Let $\pi : G \to G'$ be a surjective homomorphism while $G'$ is finite with cardinality $n$. There exists an invariant mean $M$ on $AP(G' \to K) = K^{G'}$ (Proposition 8.6). By translation invariance $nM(\xi_{\{e\}}) = M(\xi_{G'}) = 1$. Hence

$$1 = |nM(\xi_{\{e\}})| \leq |n|.$$

We see that $n$ is not divisible by $p$. 

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CHAPTER II: PERTURBATION THEORY

The purpose of this Chapter is to develop a technique and to prove the Focussing Theorem 9.4 yielding Theorem 13.1 which is an essential stepping stone to the abelian theory in Chapter III.

§9. EPSILON CHARACTERS AND BLURRED REPRESENTATIONS

We start off by quoting [12], §1.

DEFINITION 9.1 Let $G$ be an abelian group, let $0 < \varepsilon < 1$. A function $f : G \to T_K$ ($= \{\lambda \in K : |\lambda| = 1\}$) is an $\varepsilon$-character if

$$|f(x + y) - f(x)f(y)| < \varepsilon \quad (x, y \in G).$$

The following result is proved in [12], Corollary 1.2.

THEOREM 9.2 (ONEDIMENSIONAL FOCUSSING THEOREM)

Let $K$ be algebraically closed with residue class field $k$, let $G$ be an abelian group. Assume

(i) if the characteristic of $k$ is zero then $G$ is a torsion group,
(ii) if the characteristic of $k$ is $p \neq 0$ then $x \mapsto px$ ($x \in G$) is a bijection $G \to G$.

Then, for each $\varepsilon \in (0,1)$ and $\varepsilon$-character $f : G \to T_K$ there exists a (unique) character $\alpha : G \to T_K$ such that $|f(x) - \alpha(x)| \leq \varepsilon$ for all $x \in G$.

We now extend the notion of $\varepsilon$-character to higher dimensions. For convenience let us say that a bi-normed space is a triple $(E, ||_1, ||_2)$ where $E$ is a nonzero finite dimensional space over $K$ and where $||_1, ||_2$ are norms on $E$ satisfying

$$r_E := \sup \left\{ \frac{||x||_2}{||x||_1} : x \in E, x \neq 0 \right\} < 1.$$

DEFINITION 9.3. Let $G$ be an abelian group, let $(E, ||_1, ||_2)$ be a bi-normed space. A map $s \mapsto A_s$ of $G$ into $L(E)$ is a blurred representation of $G$ in $(E, ||_1, ||_2)$ if

(i) Each $A_s$ is an isometry for $||_1$ and for $||_2$.
(ii) For all $s, t \in G$

$$||A_{s+t}x - A_sA_tx||_1 \leq ||x||_2 \quad (x \in E)$$

(Example: $E = K, ||_1 = | |, ||_2 = \varepsilon | |$ where $0 < \varepsilon < 1$, $A_s\lambda = f(s)\lambda$ ($\lambda \in K, s \in G$), where $f$ is an $\varepsilon$-character.)
We shall devote sections 9, 10, 11, 12 to proving the following extension of Theorem 9.2.

**THEOREM 9.4. (FOCUSSING THEOREM)** Let $K$ be nontrivially valued, algebraically closed and spherically complete, with residue class field $k$, let $G$ be an abelian group. Assume

(i) if the characteristic of $k$ is 0 then $G$ is a torsion group,
(ii) if the characteristic of $k$ is $p \neq 0$ then $x \mapsto px (x \in G)$ is a bijection $G \rightarrow G$.

Then, for each blurred representation $s \mapsto A_s$ of $G$ in a bi-normed space $(E, \| \cdot \|_1, \| \cdot \|_2)$ and for each $\rho > 1$ there exists a representation $s \mapsto U_s$ of $G$ in $E$ such that

$$\|A_s x - U_s x\|_1 \leq \rho \|x\|_2 \quad (x \in E)$$

(Unfortunately, the techniques used to prove Theorem 9.2 do not directly carry over to the $n$-dimensional case.)

**FROM NOW ON IN THIS CHAPTER $G$ IS AN ABELIAN GROUP. $K$ IS A NONTRIVIALLY VALUED FIELD THAT IS ALGEBRAICALLY CLOSED AND SPHERICALLY COMPLETE. THE RESIDUE CLASS FIELD OF $K$ IS $k$, WITH CHARACTERISTIC $\operatorname{char} k$. $E = (E, \| \cdot \|_1, \| \cdot \|_2)$ IS A BI-NORMED SPACE IN THE SENSE OF ABOVE. WE SET**

$$r_E := \sup \left\{ \frac{\|x\|_2}{\|x\|_1} : x \in E, x \neq 0 \right\},$$

$$\varepsilon_E := \inf \left\{ \frac{\|x\|_2}{\|x\|_1} : x \in E, x \neq 0 \right\}$$

We conclude this section with a few basic facts on blurred representations.

**PROPOSITION 9.5.** Let $s \mapsto A_s$ ($s \in G$) be a blurred representation in $(E, \| \cdot \|_1, \| \cdot \|_2)$.

(i) If $s_1, \ldots, s_n \in G$ then

$$\|A_{s_1} + \ldots + A_{s_n} x - A_{s_1} A_{s_2} \ldots A_{s_n} x\|_1 \leq \|x\|_2 \quad (x \in E)$$

(ii) $\|A_0 x - x\|_1 \leq \|x\|_2 \quad (x \in E)$.

(iii) If $s \mapsto X_s$ is any map $G \rightarrow \mathcal{L}(E)$ for which

$$\|X_s x\|_1 \leq \|x\|_2 \quad (s \in G, x \in E)$$

then $s \mapsto A_s + X_s$ is also a blurred representation of $G$ in $(E, \| \cdot \|_1, \| \cdot \|_2)$.

**Proof.** Immediate verification.
§10. A SPECIAL CASE OF THE FOCUSSING THEOREM

PROPOSITION 10.1. Let char \( k = 0 \), let \( G \) be a torsion group, let \( s \mapsto A_s \) be a blurred representation of \( G \) in \((E, \| \cdot \|_1, \| \cdot \|_2)\). Suppose

\[
s \in G, \; \lambda, \mu \text{ eigenvalues of } A_s \implies |\lambda - \mu| < 1.
\]

Then there exists a representation \( s \mapsto U_s \) of \( G \) in \( E \) such that \( \|U_s x - A_s x\|_1 \leq \|x\|_2 \) \((s \in G, x \in E)\). In fact one can choose

\[
U_s x = \alpha(s)x \quad (s \in G, x \in E)
\]

where \( \alpha \) is a suitable character of \( G \).

Proof. Let \( s \in G \). We first shall find a \( \lambda(s) \in K \) such that \( \|A_s x - \lambda(s)x\|_1 \leq \|x\|_2 \) \((x \in E)\). There exists an \( n \in \mathbb{N} \) such that \( ns = 0 \). By Proposition 9.5 (i) we have

\[
\|A^n_s x - A_0 x\|_1 = \|A^n_s x - A_n x\|_1 \leq \|x\|_2 \quad (x \in E)
\]

and by Proposition 9.5(ii)

\[
\|A_0 x - x\|_1 \leq \|x\|_2 \quad (x \in E)
\]

so that

\[(1) \quad \|A^n_s x - x\|_1 \leq \|x\|_2 \quad (x \in E)\]

Now, by the algebraic closedness of \( K \), the polynomial \( X^n - 1 \in K[X] \) decomposes into \((X - \vartheta_1)(X - \vartheta_2)\ldots(X - \vartheta_n)\) where \( \vartheta_1, \ldots, \vartheta_n \) are the \( n \)th roots of unity. Since char \( k = 0 \) we have (see, for example, [9], Exercise 32.G)

\[i \neq j \implies |\vartheta_i - \vartheta_j| = 1.\]

Thus in

\[(2) \quad A^n_s - I = (A_s - \vartheta_1 I)(A_s - \vartheta_2 I)\ldots(A_s - \vartheta_n I)\]

we may suppose that for all eigenvalues \( \lambda \) of \( A_s \),

\[|\lambda - \vartheta_2| = |\lambda - \vartheta_3| = \ldots = |\lambda - \vartheta_n| = 1.\]

Let \( i \in \{2, 3, \ldots, n\} \); we prove that \( A_s - \vartheta_i I \) is an isometry with respect to \( \| \cdot \|_1 \). In fact, its norm is obviously \( \leq 1 \), by considering an eigenvector we obtain \( \|A - \vartheta_i I\| = 1 \).
Further, $|\det(A_s - \vartheta I)| = 1$ (the product of its eigenvalues has absolute value 1). By [4], Corollary 1.7, $A_s - \vartheta I$ is an isometry with respect to $\| \cdot \|_1$. Hence, for $x \in E$ we have by (1) and (2)

$$\|A_s x - \vartheta x\|_1 = \|A_s^n x - x\|_1 \leq \|x\|_2.$$ 

By repeating this first part of the proof for every $s \in G$ we obtain a map $s \mapsto \lambda(s)$ of $G$ into $T_K$ such that

$$(3) \quad \|A_s x - \lambda(s)x\|_1 \leq \|x\|_2 \quad (s \in G, x \in E)$$

From $\|A_{s+t} x - A_s A_t x\|_1 \leq \|x\|_2$ we obtain

$$|\lambda(s + t) - \lambda(s)\lambda(t)||x||_1 \leq \|x\|_2 \quad (s, t \in G, x \in E)$$

Thus, $\lambda$ is an $\varepsilon_E$-character (see §9). By Theorem 9.2 there is a character $\alpha$ of $G$ such that $\|\lambda - \alpha\|_\infty \leq \varepsilon_E$. Hence

$$|\lambda(s) - \alpha(s)||x||_1 \leq \|x\|_2 \quad (s \in G, x \in E)$$

This, combined with (3) yields

$$\|A_s x - \alpha(s)x\|_1 \leq \|x\|_2 \quad (s \in G, x \in E)$$

and the formula $U_s x = \alpha(s)x$ defines the desired representation.

Proposition 10.1 has an analog for the case $\text{char} \ k = p \neq 0$, but the conditions are slightly stronger. First a notation. If, for some prime $p$, the map $x \mapsto px$ is a bijection $G \to G$ we shall write $p^{-1}x = y$ if $x = py$. Similarly, for each $n \in \mathbb{N}$, the inverse of $x \mapsto p^n x$ is denoted $y \mapsto p^{-n} y$.

**PROPOSITION 10.2.** Let $\text{char} \ k = p \neq 0$, let $G$ be a group for which $x \mapsto px$ is a bijection, let $s \mapsto A_s$ be a blurred representation of $G$ in $(E, \| \cdot \|_1, \| \cdot \|_2)$. Suppose there exists a $c \in (0,1)$ such that

$$s \in G, \lambda, \mu \text{ eigenvalues of } A_s \implies |\lambda - \mu| \leq c.$$ 

Then there is a representation $s \mapsto U_s$ of $G$ in $E$ such that $\|U_s x - A_s x\|_1 \leq \|x\|_2$ ($s \in G, x \in E$). In fact one can choose

$$U_s x = \alpha(s)x \quad (s \in G, x \in E)$$
where $\alpha$ is a suitable character of $G$.

Proof. Let $s \in G$. We shall prove the existence of a $\lambda(s) \in K$ such that $||A_s x - \lambda(s) x||_1 \leq ||x||_2$ ($x \in E$). (The second part of the proof is identical to the one of Proposition 10.1.) Let $m := \dim E$, choose $k_m \in \mathbb{N}$ such that $(s_1^{k_m}), (s_2^{k_m}), \ldots, (s_{m-1}^{k_m})$ are all divisible by $p$. Let $\omega \in \mathbb{N}$ be such that (for $\varepsilon_E$ see §9)

$$\max(c, |p|)^\omega < \frac{1}{2} \varepsilon_E$$

and set $l = k_m \omega$. Consider the operator $T := A_{p^{-1} s}$. By Proposition A.1.1 (see Appendix) there is an $||_1$-orthonormal base $e_1, \ldots, e_m$ of $E$ such that $2 \geq ||e_1||_1 \geq ||e_2||_1 \geq \ldots \geq ||e_m||_1 \geq 1$ and such that the matrix of $T$ with respect to $e_1, \ldots, e_m$ has the upper triangular form

$$\beta = \begin{pmatrix} \lambda_1 & * \\ \vdots & \ddots \\ \Theta & \lambda_m \end{pmatrix}$$

By assumption, $|\lambda_i - \lambda_j| \leq c$ for all $i, j \in \{1, \ldots, m\}$. From Proposition A.1.2 it follows that the entries of $\beta$ are all in $B(0, 1) \subset K$. Thus, $\beta$ is an element of $M_c$ in the sense of Definition A.2.3. With $p$ as in §A.2 we then have by Proposition A.2.4

$$\rho(\beta^{p'}) \leq \tau^\omega \rho(\beta) \leq \tau^\omega$$

where $\tau = \max(c, |p|)$. Hence

$$\rho(\beta^{p'}) < \frac{1}{2} \varepsilon_E$$

By the definition of $\rho$ there exists a $\lambda \in K$ such that

$$|\beta^{p'} - \lambda| < \frac{1}{2} \varepsilon_E$$

(Here $u$ is the identity matrix and $| \cdot |$ is as in §A.2.) By Proposition A.1.2 (i) we have for any $x \in E, x \neq 0$

$$\frac{||T^{p'} x - \lambda x||_1}{||x||_1} \leq 2|\beta^{p'} - \lambda|$$

This, combined with (*) yields (recall the definition of $\varepsilon_E$)

$$||T^{p'} x - \lambda x||_1 \leq ||x||_2 \quad (x \in E)$$

But also, $s \mapsto A_s$ is a blurred representation, so

$$||T^{p'} x - A_s x||_1 \leq ||x||_2 \quad (x \in E)$$
and we find the announced result

\[ \|A_\ast x - \lambda x\|_1 \leq \|x\|_2 \quad (x \in E). \]

**COROLLARY 10.3.** The conclusion of the Focussing Theorem 9.4 holds if \( \text{dim} \ E = 1 \) (even with \( \rho := 1 \)).

**Proof.** Theorem 9.2 or Proposition 10.1 and 10.2 and the observation that \( A_\ast \) has only one eigenvalue.

**Note.** It follows from [12], Proposition 3.4 that Proposition 10.2 becomes a falsity if we replace the condition of bijectivity of \( x \mapsto px \) by just surjectivity.

§11. c-DIAGONALIZABILITY

**DEFINITION 11.1.** Let \( c \in (0,1] \). A blurred representation \( s \mapsto A_\ast \) of \( G \) in \((E, \| \|_1, \| \|_2)\) is \( c\)-diagonalizable if for each \( s \in G \) there exists a base of \( E \) that is \( c\)-orthogonal with respect to \( \| \|_1 \) and to \( \| \|_2 \) such that the matrix of \( A_\ast \) with respect to this base has the diagonal form.

In this section we shall prove that blurred representations are 'close' to \( c\)-diagonalizable ones.

**PROPOSITION 11.2.** Let \( \text{char} \ k = 0 \), let \( G \) be a torsion group and let \( s \mapsto A_\ast \) be a blurred representation of \( G \) in \((E, \| \|_1, \| \|_2)\). Then there exists a 1-diagonalizable blurred representation \( s \mapsto B_\ast \) of \( G \) in \((E, \| \|_1, \| \|_2)\) such that \( \|A_\ast x - B_\ast x\|_1 \leq \|x\|_2 \) \((s \in G, x \in E)\).

**Proof.** Let \( s \in G \). By Proposition 9.5 (iii) it is enough to find a \( B_\ast \in \mathcal{L}(E) \) such that \( \|A_\ast x - B_\ast x\|_1 \leq \|x\|_2 \) \((x \in E)\) and such that the matrix of \( B_\ast \) with respect to some bi-orthogonal base of \( E \) is diagonal. To this end we first modify \( t \mapsto A_t \) as follows. Let the group \( H \) generated by \( s \) have \( r \) elements. Set

\[ \tilde{A}_t := \begin{cases} A_t & \text{if } t \in G \setminus H \\ A_j^i & \text{if } t = js \text{ for some } j \in \{0,1,\ldots,r-1\} \end{cases} \]

Then clearly \( \|\tilde{A}_t x - A_\ast x\|_1 \leq \|x\|_2 \) for all \( t \in G, x \in E \) so that, to prove the Proposition we may assume that \( \tilde{A}_t = A_t \) i.e. that

\[ A_{js} = A_j^i \quad (j \in \{0,1,\ldots,r-1\}) \]
We now apply Proposition A.1.3 to $T := A_s$ with $c = 1$ for $(E, ||_1)$ and $(E, ||_2)$. We find a decomposition of $E$

$$E = D_1 \oplus D_2 \oplus \ldots \oplus D_n$$

into subspaces that are orthogonal with respect to both norms $||_1$ and $||_2$, such that $A_s D_i \subset D_i$ and the eigenvalues of $A_s | D_i$ have distances strictly less than 1 for each $i \in \{1, \ldots, n\}$. By (1) these conclusions hold for $A_s^j = A_{j_s}$ ($j \in \{0,1, \ldots r-1\}$). Let $i \in \{1, \ldots, n\}$. Then the map

$$t \mapsto A_s | D_i \quad (t \in H)$$

is a blurred representation of $H$ in $(D_i, ||_1, ||_2)$ satisfying the conditions of Proposition 10.1 and we may conclude that there exists a $\lambda_i \in K$ such that

$$||A_s x - \lambda_i x||_1 \leq ||x||_2 \quad (x \in D_i)$$

Now define $B_s \in \mathcal{L}(E)$ by

$$B_s(d_1 + \ldots + d_n) = \sum_{i=1}^{n} \lambda_i d_i \quad (d_i \in D_i \text{ for each } i)$$

By $||_2$-orthogonality of the $D_i$ we have, with $x = d_1 + \ldots + d_n \in E$

$$||A_s x - B_s x||_1 \leq \max_{1 \leq i \leq n} ||A_s d_i - \lambda_i d_i||_1 \leq \max_{1 \leq i \leq n} ||d_i||_2 = ||x||_2.$$ 

By choosing in each $D_i$ a base that is orthogonal with respect to both $||_1$ and $||_2$ (Proposition A.1.4) we obtain a base $e_1, \ldots, e_m$ of $E$, orthogonal with respect to both norms, such that the matrix of $B_s$ with respect to $e_1, \ldots, e_m$ has the diagonal form.

**Proposition 11.3.** Let $\text{char } k = p \neq 0$, let $x \mapsto px$ be a bijection of $G$ and let

$$s \mapsto A_s$$

be a blurred representation of $G$ in $(E, ||_1, ||_2)$. Then for each $c \in (0,1)$ there exists a $c$-diagonalizable blurred representation $s \mapsto B_s$ in $(E, ||_1, ||_2)$ such that $||A_s x - B_s x||_1 \leq ||x||_2$ ($s \in G$, $x \in E$).

**Proof.** Let $s \in G$ and $c \in (0,1)$. We shall prove that there exist a decomposition of $E$

$$E = D_1 \oplus \ldots \oplus D_n$$

into subspaces that are $c$-orthogonal with respect to $||_1$ and $||_2$ and $\lambda_1, \ldots, \lambda_n \in K$ such that the map $B_s \in \mathcal{L}(E)$ given by

$$B_s(d_1 + \ldots + d_n) = \sum_{i=1}^{n} \lambda_i d_i \quad (d_i \in D_i \text{ for each } i)$$

$$24$$
satisfies \( \| A_2 x - B_2 x \|_1 \leq \| x \|_2 \) (\( x \in E \)). (Like in the previous proof we then can find the required base of \( E \) by choosing in each \( D_i \) a base that in orthogonal with respect to both norms.) Let \( m := \dim E \), let \( k_m \in \mathbb{N} \) be such that \( (p_1^m), (p_2^m), \ldots, (p_{m-1}^m) \) are all divisible by \( p \). Let \( \xi := c^{4/m^2} \) and choose \( \omega \in \mathbb{N} \) such that

\[
\max(\xi, |p|)^\omega < \frac{1}{2} c \epsilon E
\]

and set \( l := k_m \omega \). We apply Proposition A.1.3 to \( T := A_{p-1} \), with \( \xi \) in place of \( c \), for \((E, \| \cdot \|_1)\) and \((E, \| \cdot \|_2)\). We find a decomposition of \( E \)

\[
E = D_1 \oplus \cdots \oplus D_n
\]

into subspaces that are \( c \)-orthogonal (since \( \xi^{4/m^2} = c \)) with respect to both norms \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) such that \( TD_i \subseteq D_i \) and the eigenvalues of \( T|D_i \) have distances \( < \xi \) for each \( i \in \{1, \ldots, n\} \). Now let \( i \in \{1, \ldots, n\} \). By Proposition A.1.1 there exists a base of \( D_i \), say \( e_1, \ldots, e_r \), which is \( \| \cdot \|_1 \)-orthogonal such that \( 2 \geq \| e_1 \|_1 \geq \| e_2 \|_1 \geq \cdots \geq \| e_r \|_1 \geq 1 \)

and such that the matrix \( \beta \) of \( T|D_i \) with respect to \( e_1, \ldots, e_r \) has the upper triangular form

\[
\beta = \begin{pmatrix}
\mu_1 & * \\
\vdots & \ddots & \ddots \\
0 & \cdots & \mu_r
\end{pmatrix}
\]

where the entries are all in \( B(0,1) \subseteq K \) and the eigenvalues have distances \( < \xi \). Thus, the matrix \( \beta \) is in \( M_r \) in the sense of Definition A.2.3 (only the dimension is \( r \) instead of \( m \)). Of course, \( (p_1^m), \ldots, (p_{m-1}^m) \) are divisible by \( p \) so that, with \( \rho \) as in §A.2, we have by Proposition A.2.4

\[
\rho(\rho^i) \leq \max(\xi, |p|)^\omega \rho(\beta) \leq \max(\xi, |p|)^\omega \rho(\beta) < \frac{1}{2} c \epsilon E
\]

By the definition of \( \rho \) there exists a \( \lambda_i \in K \) such that

\[
|\beta^i - \lambda_i u| \leq \frac{1}{2} c \epsilon E
\]

By Proposition A.1.2(i) we find

\[
\| T^i x - \lambda_i x \|_1 \leq c \| x \|_2 \quad (x \in D_i)
\]

As announced we define \( B_\ast \in \mathcal{L}(E) \) by

\[
B_\ast(d_1 + \cdots + d_n) = \sum_{i=1}^n \lambda_i d_i \quad (d_i \in D_i \text{ for each } i)
\]

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Then, by c-orthogonality of $D_1, \ldots, D_n$ with respect to $\| \cdot \|$ we have for $x = d_1 + \ldots + d_n \in E$
\[ \|T^p x - B_s x\|_1 \leq \max_{1 \leq i \leq n} \|T^p d_i - \lambda_i d_i\|_1 \leq c \max_{1 \leq i \leq n} \|d_i\|_2 \leq \|x\|_2. \]
Since also $t \to A_t$ is a blurred representation we have
\[ \|T^p x - A_s x\|_1 \leq \|x\|_2 \]
and we find
\[ \|A_s x - B_s x\|_1 \leq \|x\|_2 \quad (x \in E) \]

§12. PROOF OF THE FOCUSSING THEOREM

We now are ready to give a

Proof of the Focussing Theorem 9.4. Without loss of generality we may assume that $\rho < r_E^{-1}$ (for $r_E$ see §9). The proof runs by induction on $m := \dim E$. If $m = 1$, apply Corollary 10.3. Suppose the conclusion of the Focussing Theorem holds for all blurred representations of $G$ in bi-normed spaces of dimension $< m$. Now let $s \mapsto A_s$ be a blurred representation in $(E, \| \cdot \|_1, \| \cdot \|_2)$ where $\dim E = m$. Let $c := \sqrt{r_E}$. By Propositions 11.2 and 11.3 we may (and shall) assume that $s \mapsto A_s$ is $c$-diagonalizable. If for all $s \in G$ and eigenvalues $\lambda, \mu$ of $A_s$ we had $|\lambda - \mu| \leq c$ we obtain the conclusion of the Focussing Theorem from Propositions 10.1 and 10.2. Therefore we may (and shall) also assume that there exists an $s \in G$ and eigenvalues $\lambda, \mu$ of $A_s$ with $|\lambda - \mu| > c$. Thus we can order the eigenvalues $\lambda_1, \ldots, \lambda_m$ of $A_s$ (counted with multiplicity) in such a way that for some $n \in N$, $1 \leq n < m$
\[ i \in \{1, \ldots, n\}, \quad j \in \{n + 1, \ldots, m\} \implies |\lambda_i - \lambda_j| > c. \]
Now $A_s$ is $c$-diagonalizable so there is a base $e_1, \ldots, e_m$ of $E$ which is $c$-orthogonal with respect to both norms on which the matrix of $A_s$ has the diagonal form
\[ \begin{pmatrix}
\lambda_1 & \Theta \\
\Theta & \ddots \\
& & \lambda_m
\end{pmatrix} \]
By the above, $D_1 := [e_1, \ldots, e_n]$ and $D_2 := [e_{n+1}, \ldots, e_m]$ are proper subspaces of $E$. Let $P_1 : [e_1, \ldots, e_m] \to [e_1, \ldots, e_n]$ be the obvious projection $E \to D_1$ and consider the map $t \mapsto Q_t$ of $G$ into $\mathcal{L}(D_1)$ defined by
\[ Q_t = P_1 A_t |D_1 \quad (t \in G) \]
We shall prove that \( t \mapsto Q_t \) is a blurred representation of \( G \) in \((D_1, \| \cdot \|_1, c^{-3}\| \cdot \|_2)\). (Observe that for \( x \in D_1 \) we have \( c^{-3}\|x\|_2 \leq c^{-3}r_E\|x\|_1 = c^{-3}c^5\|x\|_1 = c^2\|x\|_1 \), so that \((D_1, \| \cdot \|_1, c^{-3}\| \cdot \|_2)\) is, indeed, a bi-normed space in the sense of §9). This proof runs in several steps.

(i) For all \( t \in G, d \in D_1 \) we have \( \|Q_t d - A_t d\|_1 \leq c^{-3}\|d\|_2 \).

Proof. Let \( i \in \{1, \ldots, n\} \), let \( t \in G \). About \( A_t e_i \) we can, so far, only say that it lies in \( E \): there exist \( \xi_1, \ldots, \xi_m \in K \) such that

\[
A_t e_i = \sum_{j=1}^m \xi_j e_j
\]

Then

\[
A_s A_t e_i = \sum_{j=1}^m \xi_j \lambda_j e_j
\]

On the other hand we have \( A_s e_i = \lambda_i e_i \) so

\[
A_t A_s e_i = \sum_{j=1}^m \xi_j \lambda_i e_j
\]

Hence,

\[
\|e_i\|_2 \geq \|A_s A_t e_i - A_s A_s e_i\|_1 = \sum_{j=1}^m |\xi_j| |\lambda_i - \lambda_j| e_j\|_1 = c \sum_{n<j \leq m} |\xi_j| |\lambda_i - \lambda_j| e_j\|_1
\]

\[
\geq c^2 \sum_{n<j \leq m} \|\xi_j e_j\|_1 \geq c^2 \sum_{i=n+1}^m \|\xi_i e_i\|_1 = c^2 \|A_t e_i - P_1 A_t e_i\|_1 = c^2 \|A_t e_i - e_i\|_1
\]

For \( d \in D_1, d = \sum_{i=1}^n \eta_i e_i \), where \( \eta_i \in K \) we find

\[
\|Q_t d - A_t d\|_1 \leq \max_{1 \leq i \leq n} |\eta_i| \|Q_t e_i - A_t e_i\|_1 \leq c^{-2} \max_{1 \leq i \leq n} |\eta_i| \|e_i\|_2
\]

\[
\leq c^{-3} \|\sum_{i=1}^n \eta_i e_i\|_2 = c^{-3} \|d\|_2.
\]

(ii) Each \( Q_t \) is an isometry with respect to \( \| \cdot \|_1 \).

Proof. Let \( d \in D_1, d \neq 0 \). Since, by (i),

\[
\|Q_t d - A_t d\|_1 \leq c^{-3}\|d\|_2 \leq c^2\|d\|_1 < \|d\|_1
\]

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we have \( \| Q_t d \|_1 = \| A_t d \|_1 = \| d \|_1 \).

(iii) Each \( Q_t \) is an isometry with respect to \( \| \cdot \|_2 \).

Proof. Let \( d \in D_1, d \neq 0 \). Since, by (i)

\[
\| Q_t d - A_t d \|_2 \leq r_E \| Q_t d - A_t d \|_1 \leq c^{-3} r_E \| d \|_2 = c^2 \| d \|_2 < \| d \|_2
\]

we have \( \| Q_t d \|_2 = \| A_t d \|_2 = \| d \|_2 \).

(iv) For \( t, u \in G, d \in D_1 : \| Q_{t+u} d - Q_t Q_u d \|_1 \leq c^{-3} \| d \|_2 \).

Proof. We have, by (i),

\[
\| Q_{t+u} d - A_{t+u} d \|_1 \leq c^{-3} \| d \|_2
\]

Also, by (i) and (ii),

\[
\| Q_t Q_u d - A_t A_u d \|_1 \leq \max \{ \| Q_t Q_u d - A_t Q_u d \|_1, \| A_t Q_u d - A_t A_u d \|_1 \}
\]

\[
\leq c^{-3} \| d \|_2 \lor \| Q_u d - A_u d \|_1
\]

\[
\leq c^{-3} \| d \|_2 \lor c^{-3} \| d \|_2 = c^{-3} \| d \|_2
\]

This, combined with (1) and \( \| A_{t+u} d - A_t A_u d \|_1 \leq \| d \|_2 \leq c^{-3} \| d \|_2 \) proves (iv).

Thus, we have proved ((ii), (iii), (iv)) the announced result: \( t \mapsto Q_t \) is a blurred representation of \( G \) in \( (D_1, \| \cdot \|_1, c^{-3} \| \cdot \|_2) \). Similarly we have a projection \( P_2 : E \rightarrow D_2 \) and \( t \mapsto R_t \), where

\[
R_t := P_2 A_t | D_2,
\]

is a blurred representation of \( G \) in \( (D_2, \| \cdot \|_1, c^{-3} \| \cdot \|_2) \). By the induction hypothesis (recall that \( D_1, D_2 \) are proper subspaces of \( E \)) there exist representations \( t \mapsto V_t \) of \( G \) in \( D_1 \) and \( t \mapsto W_t \) of \( G \) in \( D_2 \) such that

\[
\| V_t d_1 - Q_t d_1 \|_1 \leq c^{-4} \| d_1 \|_2 \quad (t \in G, d_1 \in D_1)
\]

\[
\| W_t d_2 - R_t d_2 \|_1 \leq c^{-4} \| d_2 \|_2 \quad (t \in G, d_2 \in D_2)
\]

Now define

\[
U_t(d_1 + d_2) = V_t d_1 + W_t d_2 \quad (t \in G, d_1 \in D_1, d_2 \in D_2)
\]

Then \( t \mapsto U_t \) is a representation of \( G \) in \( E \). For \( x = d_1 + d_2 \in E \) and \( t \in G \) we have

\[
\| U_t x - A_t x \|_1 = \| V_t d_1 + W_t d_2 - A_t d_1 - A_t d_2 \|_1
\]

\[
\leq \max(\| V_t d_1 - A_1 d_1 \|_1, \| W_t d_2 - A_t d_2 \|_1)
\]

\[
\leq \max(\| V_t d_1 - Q_t d_1 \|_1, \| Q_t d_1 - A_t d_1 \|_1,
\| W_t d_2 - R_t d_2 \|_1, \| R_t d_2 - A_t d_2 \|_1)
\]

\[
\leq \max(c^{-4} \| d_1 \|_2, c^{-3} \| d_1 \|_2, c^{-4} \| d_2 \|_2, c^{-3} \| d_2 \|_2)
\]

\[
\leq c^{-4} \max(\| d_1 \|_2, \| d_2 \|_2) \leq c^{-5} \| d_1 + d_2 \|_2
\]

\[
\leq c^{-5} \| x \|_2.
\]
§13. APPROXIMATION BY REPRESENTATIVE FUNCTIONS

We now shall apply the Focussing Theorem to our almost periodic functions. We shall prove Theorem 13.1 below which will open the way for a more general abelian theory in Chapter III. (As it is convenient for quoting we repeat some of the assumptions made in §9 in the formulation of the Theorem.)

**THEOREM 13.1.** Let $G$ be an abelian group, let $K$ be a nontrivially valued field that is algebraically closed and spherically complete with residue class field $k$. Assume

(i) If the characteristic of $k = 0$ then $G$ is a torsion group.
(ii) If the characteristic of $k = p 
eq 0$ then $x \mapsto px$ is a bijection $G \rightarrow G$.

Then, for each $\varepsilon > 0$ and $f \in \text{AP}(G \rightarrow K)$ there exists a $g \in \mathcal{R}(G \rightarrow K)$ with $\|f - g\|_\infty < \varepsilon$. (In other words, $\mathcal{R}(G \rightarrow K) = \text{AP}(G \rightarrow K)$.)

**Proof.** Let $f \in \text{AP}(G \rightarrow K)$, $\varepsilon > 0$. Assume $f \notin \mathcal{R}(G \rightarrow K)$. Then

$$S := \overline{\text{co } f_G}$$

is infinite dimensional. As $K$ is spherically complete, by [5], Theorem 6.19, this complete compactoid can be decomposed into a $\| \|_\infty$-orthogonal direct sum

$$S = \bigoplus_{n \in \mathbb{N}} S_n$$

where each $S_n$ is a bounded onedimensional absolutely convex subset of $\text{AP}(G \rightarrow K)$ and where $\lim_{n \to \infty} \text{diam } S_n = 0$. (Note. It is right here where we use the assumption of spherical completeness of $K$ in an essential way: for nonspherically complete base fields compactoids can behave very badly, see for example [13], Proposition 8.1.) We may assume that

$$\text{diam } S_1 \geq \text{diam } S_2 \geq \ldots > 0.$$ 

In the rest of the proof we fix an $m \in \mathbb{N}$ such that

$$\text{diam } S_{m+1} < \text{diam } S_m$$

$$\text{diam } S_{m+1} < \frac{1}{2} \varepsilon$$

Let $H$ be the $K$-Banach space generated by $S_1, S_2, \ldots$. Then $H$ is of countable type and shift invariant. Set $D := [S_1 \cup S_2 \cup \ldots \cup S_m]$. One cannot expect $D$ to be shift invariant, so define a map $s \mapsto A_s$ of $G$ into $\mathcal{L}(D)$ as follows.

$$A_s d := Pd_s \quad (d \in D)$$
where $P$ is the obvious projection $H \to D$. Let $q$ be the Minkowski function of $S_1 \oplus \ldots \oplus S_m$ in $D$ (the norm in $D$, associated to $\bigoplus_{i=1}^{m} S_i$).

We claim that $s \mapsto A_s$ is a blurred representation (Definition 9.3) of $G$ in the bi-normed space $(D, \| \|_\infty, \| \|_2)$, where

$$\|d\|_2 := q(d) \text{ diam } S_{m+1} \quad (d \in D)$$

(of course, diam $S_{m+1}$ is meant with respect to $\| \|_\infty$). The proof of this claim runs in several steps.

(i) For each $d \in D$

$$\|d\|_2 \leq \text{diam } S_{m+1} (\text{diam } S_m)^{-1} \|d\|_\infty$$

(so that $(D, \| \|_\infty, \| \|_2)$ is, indeed, a bi-normed space in the sense of §9).

**Proof.** As the valuation of $K$ is dense we only have to prove; If $d \in D$, $\|d\|_\infty < \text{diam } S_m$ then $q(d) \leq 1$. To this end, let $d = d_1 + \ldots + d_m$ where $d_i \in [S_i]$ for each $i \in \{1, \ldots, m\}$. By orthogonality, $\|d_i\|_\infty \leq \|d\|_\infty < \text{diam } S_m \leq \text{diam } S_i$ for each $i$ so that $d_i \in S_i$ for each $i \in \{1, \ldots, m\}$. Thus, $d \in \bigoplus_{i=1}^{m} S_i$, which is, by definition of $q$, contained in the $q$-unit ball of $D$.

(ii) For each $s \in G$, $d \in D$

$$\|A_s d - d_s\|_\infty \leq \|d\|_2.$$

**Proof.** It suffices to prove (scalar multiplication) that

$$d \in \bigoplus_{i=1}^{m} S_i \implies \|A_s d - d_s\|_\infty \leq \text{diam } S_{m+1}.$$

Since $d \in S$ and $S$ is shift invariant

$$d_s = s_1 + s_2 + \ldots \quad (s_i \in S_i \text{ for each } i).$$

So

$$A_s d = P d_s = s_1 + \ldots s_m.$$ 

Hence, $\|A_s d - d_s\|_\infty = \|s_{m+1} + s_{m+2} + \ldots\|_\infty \leq \text{diam } S_{m+1}$.

(iii) Each $A_s$ is a linear isometry $D \to D$ with respect to $\| \|_\infty$ and $\| \|_2$.

**Proof.** Let $d \in D$, $d \neq 0$. From (ii) and (i) we obtain

$$\|A_s d - d_s\|_\infty \leq \|d\|_2 < \|d\|_\infty.$$ 

Hence, $\|A_s d\|_\infty = \|d_s\|_\infty = \|d\|_\infty$, and $A_s$ is an isometry for $\| \|_\infty$. It follows that $|\det A_s| = 1$. Thus, to prove the second part it suffices to show that $A_s$ is a contraction.
for \( \| \cdot \|_2 \), i.e. for the norm \( q \). But this follows easily from the fact that \( A_s \) maps \( \bigoplus_{i=1}^{m} S_i \) into itself (\( S \) is shift invariant and \( P \) maps \( S \) into \( \bigoplus_{i=1}^{m} S_i \)).

(iv) For all \( s, t \in G, d \in D \)

\[
\| A_{s+t}d - A_sA_t d \|_\infty \leq \| d \|_2.
\]

Proof. This follows from

\[
\| A_{s+t}d - A_sA_t d \|_\infty \\
\leq \max(\| A_{s+t}d - d_{s+t} \|_\infty, \| (d_t)_s - (A_s d)_s \|_\infty, \| (A_t d)_s - A_sA_t d \|_\infty)
\]

and

\[
\| A_{s+t}d - d_{s+t} \|_\infty \leq \| d \|_2 \quad \text{(by (ii))},
\]

\[
\| (d_t)_s - (A_s d)_s \|_\infty = \| d_t - A_t d \|_\infty \leq \| d \|_2 \quad \text{(by (i))},
\]

\[
\| (A_t d)_s - A_s(A_t d) \|_\infty \leq \| A_t d \|_2 = \| d \|_2 \quad \text{(by (ii) and (iii))}.
\]

This completes the proof of our claim, so \( s \mapsto A_s \) is a blurred representation of \( G \) in \( (D, \| \cdot \|_\infty, \| \cdot \|_2) \). By the Focussing Theorem 9.4 there exists a representation \( s \mapsto U_s \) of \( G \) in \( D \) such that

\[
(*) \quad \| A_s d - U_s d \|_\infty \leq 2\| d \|_2 \quad (s \in G, \ d \in D)
\]

Now we return to our \( f \) we had chosen in the beginning of our proof. Write

\[
f = \sum_{n=1}^{\infty} f_n \quad (f_n \in S_n \text{ for each } n)
\]

and set \( d := \sum_{n=1}^{\infty} f_n \). Then \( f - d \in \bigoplus_{n>m} S_n \) so

\[
\| f - d \|_\infty \leq \text{diam } S_{m+1} < \frac{1}{2} \varepsilon.
\]

Further, we have \( d \in \bigoplus_{n=1}^{m} S_n \) so that \( q(d) \leq 1 \) i.e. \( \| d \|_2 \leq \text{diam } S_{m+1} \). We find, by using (ii) and (*)

\[
(**) \quad \| d_s - U_s d \|_\infty \leq \max(\| d_s - A_s d \|_\infty, \| A_s d - U_s d \|_\infty)
\]

\[
\leq 2\| d \|_2 \leq 2 \text{ diam } S_{m+1} < \varepsilon.
\]

Now set

\[
g(s) := (U_s d)(0) \quad (s \in G)
\]

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Since $U$ is a finite dimensional representation we have $g \in \mathcal{R}(G \to K)$. From (**) we obtain, for each $s \in G$

$$|d(s) - g(s)| = |d_s(0) - (U_s d)(0)| \leq \|d_s - U_s d\|_\infty < \varepsilon$$

Together with $\|f - d\|_\infty < \frac{1}{2} \varepsilon$ (see above) this implies

$$\|f - g\|_\infty \leq \max(\|f - d\|_\infty, \|d - g\|_\infty) \leq \varepsilon$$

This concludes the proof of Theorem 13.1.
CHAPTER III: ABELIAN THEORY

THROUGHOUT CHAPTER III G IS AN ABELIAN GROUP WITH (K-VALUED) CHARACTER GROUP $G^\lambda_K$.

In this chapter we shall make the theory of invariant means on abelian group into a complete whole. In our set up Theorem 13.1 plays a key role.

§14. THE CASE $\text{char} \ k = 0$.

In §14 the residue class field $k$ of $K$ has characteristic zero.

**PROPOSITION 14.1.** Let $G$ be a torsion group. Then $G^\lambda_K$ is an orthonormal set in $B(G \to K)$. If $K$ is algebraically closed then $\mathcal{R}(G \to K) = [G^\lambda_K]$.

**Proof.** [8], [11] Theorem 1.4 (α) $\implies$ (γ).

**THEOREM 14.2.** Let $G$ be a torsion group, let $K$ be algebraically closed. Then

(i) $G^\lambda_K$ is an orthonormal base of $AP(G \to K)$. More explicitly, for each $f \in AP(G \to K)$ there exists a unique map $\alpha \mapsto \lambda_\alpha$ of $G^\lambda_K$ into $K$ such that $\{\alpha \in G^\lambda_K : |\lambda_\alpha| \geq \varepsilon\}$ is finite for each $\varepsilon > 0$ and

$$f = \sum_{\alpha \in G^\lambda_K} \lambda_\alpha \cdot \alpha \quad (\text{uniformly})$$

$$\|f\|_\infty = \max\{|\lambda_\alpha| : \alpha \in G^\lambda_K\}.$$ 

(ii) The map $\sum \lambda_\alpha \mapsto \lambda_1$ (where $1$ is the unit character) is an invariant mean on $AP(G \to K)$.

**Proof.** We may assume that the valuation of $K$ is nontrivial. Let $L \supseteq K$ be the spherical completion of $K$ in the sense of [6], Theorem 4.49. Then $L$ is algebraically closed ([6], Corollary 4.51), and the conditions of Theorem 13.1 (where the base field is $L$) are satisfied. So we have $AP(G \to L) = \overline{\mathcal{R}(G \to L)}$. Now apply Proposition 14.1 with $L$ in place of $K$ and we find that $G^\lambda_L$ is an orthonormal base of $AP(G \to L)$. We will not offend the reader by proving that the formula

$$M(\sum_{\alpha \in G^\lambda_L} \mu_\alpha \cdot \alpha) = \mu_1 \quad (\mu_\alpha \in L)$$

defines an invariant mean on $AP(G \to L)$. Notice that, if $f \in AP(G \to L)$ has the expansion $\sum_{\alpha \in G^\lambda_L} \mu_\alpha \cdot \alpha$ and $\beta \in G^\lambda_L$, then

$$M(f \beta^{-1}) = \mu_\beta$$

(*)
Now we return to our base field $K$. Let $f \in AP(G \to K)$. Then $f \in AP(G \to L)$ so by the above it has as expansion

$$f = \sum_{\alpha \in \Omega^*_G} \lambda_\alpha \alpha$$

where $\lambda_\alpha \in L$. Now, since $G$ is a torsion group, $\alpha(G)$ consists of roots of unity in $L$. But, as $K$ is algebraically closed, $\alpha(G) \subset K$, so $G^*_L = G^*_K$. To prove that also $\lambda_\alpha \in K$, observe that $f.\alpha^{-1} \in AP(G \to K)$ so that, by Theorem 8.3, $M(f.\alpha^{-1}) \in K$. But, by (*), we have $\lambda_\alpha = M(f.\alpha^{-1})$. Hence $\lambda_\alpha \in K$. We see that $G^*_K$ is an orthonormal base of $AP(G \to K)$. The proof of (ii) is now obvious.

**THEOREM 14.3.** There exists an invariant mean on $AP(G \to K)$ if and only if $G$ is a torsion group.

**Proof.** Let $G$ be not a torsion group. Then we may assume $Z \subset G$. By using the divisibility of the additive group of $Q$ we can find a homomorphism $\phi : G \to Q$ such that $\phi(x) = x$ for all $x \in Z$. As $Q \subset K$ we shall view $\phi$ as a nontrivial additive homomorphism $G \to K$. This $\phi$ is bounded as $\text{char } k = 0$. From $\phi(s + t) = \phi(s) + \phi(t)$ we obtain

$$\phi_x = \phi(s).1 + \phi \in [1, \phi] \quad (s \in G)$$

so that $\phi \in AP(G \to K)$. If $\mu \in AP(G \to K)'$ is shift invariant then from (*) we obtain

$$\mu(\phi) = \mu(\phi_* = \phi(s)\mu(1) + \mu(\phi)$$

i.e.

$$\phi(s)\mu(1) = 0 \quad (s \in G)$$

As $\phi$ is nontrivial, $\mu(1) = 0$. It follows that there does not exist an invariant mean on $AP(G \to K)$.

We now prove the converse (this is Proposition 8.7 (i), but we want also to present a more direct proof). Let $G$ be a torsion group. By Theorem 14.2 there exists an invariant mean on $AP(G \to L)$ where $L$ is an algebraically closed complete field extending $K$. Now apply Theorem 8.3.

We conclude this section by giving a formula expressing $M(f)$ in terms of values of $f$. For a torsion group $G$, let $\mathcal{F}$ be the collection of all finite subgroups of $G$, ordered by inclusion. For any $f : G \to K$

$$H \mapsto \frac{1}{\#H} \sum_{h \in H} f(h) \quad (h \in \mathcal{F})$$
is a net in $K$.

**Theorem 14.4. (Formula for $M(f)$)** Let $G$ be a torsion group, let $f \in \text{AP}(G \to K)$, Let $M$ be the invariant mean on $\text{AP}(G \to K)$. Then

$$M(f) = \lim_{H \in \mathcal{F}} \frac{1}{\#H} \sum_{h \in H} f(h).$$

**Proof.** We may suppose that $K$ is algebraically closed. Let $\varepsilon > 0$. Write $f = \sum_{\alpha \in G_K^*} \lambda_{\alpha} \cdot \alpha$ as in Theorem 14.2, let

$$Y := \{\alpha \in G_K^* : |\lambda_{\alpha}| > \varepsilon\} \cup \{1\}.$$  

Then $Y$ is finite and $\|f - g\|_\infty \leq \varepsilon$ where $g := \sum_{\alpha \in Y} \lambda_{\alpha} \cdot \alpha$. There exists an $H_0 \in \mathcal{F}$ such that

$$(*) \quad \alpha \in Y, \alpha \neq 1 \implies \alpha \text{ is not constant on } H_0$$

We claim that for $H \in \mathcal{F}$, $H \supset H_0$

$$(**) \quad |M(f) - \frac{1}{\#H} \sum_{h \in H} f(h)| \leq \varepsilon$$

In fact, we have by $(*)$

$$\frac{1}{\#H} \sum_{h \in H} \alpha(h) = \begin{cases} 0 & \text{if } \alpha \in Y, \alpha \neq 1 \\ 1 & \text{if } \alpha = 1 \end{cases}$$

Thus,

$$\frac{1}{\#H} \sum_{h \in H} g(h) = \lambda_1 = M(f)$$

But since $\|f - g\|_\infty \leq \varepsilon$ and $|\#H| = 1$

$$|\frac{1}{\#H} \sum_{h \in H} f(h) - \frac{1}{\#H} \sum_{h \in H} g(h)| \leq \varepsilon$$

yielding $(**)$.
§15. THE CASE char \( k = p \neq 0 \)

In §15 the residue class field \( k \) of \( K \) has characteristic \( p \neq 0 \).

\( G \) is called \( p \)-divisible if for each \( x \in G \) there exists an \( y \in G \) with \( py = x \) (i.e. if the map \( x \mapsto px \) is surjective).

In the spirit of §14 we shall prove that there exists an invariant mean on \( AP(G \to K) \) if and only if \( G \) is \( p \)-divisible (Theorem 15.7; compare 14.3) and, that, if \( K \) is algebraically closed and \( G \) is \( p \)-divisible, \( G_K^\wedge \) is an orthonormal base of \( AP(G \to K) \). (Theorem 15.6; compare 14.2). However, the road leading to it is different.

**PROPOSITION 15.1.** Let \( G \) be \( p \)-divisible. Then \( G_K^\wedge \) is an orthonormal set in \( B(G \to K) \). If \( K \) is algebraically closed then \( R(G \to K) = [G_K^\wedge] \).

**Proof.** [8], [11] Theorem 1.4 (\( \alpha \Rightarrow \gamma \)).

**LEMMA 15.2.** Let \( G \) be \( p \)-divisible. Then there exists an abelian group \( G_1 \) such that

(i) \( x \mapsto px \) is a bijection \( G_1 \to G_1 \),

(ii) \( G \) is a quotient of \( G_1 \).

**Proof.** Let \( \Omega := \bigcup_{n \in \mathbb{N}} p^{-n} \mathbb{Z} \), considered as an additive subgroup of \( \mathbb{Q} \). Set

\[
G_1 := \bigoplus_{a \in G} \Omega_a
\]

where \( \Omega_a := \Omega \) for each \( a \in G \). Then \( G_1 \) satisfies (i). For each \( a \in G \) the homomorphism \( \mathbb{Z} \to G \) given by \( n \mapsto na \) \((n \in \mathbb{Z})\) extends, by the \( p \)-divisibility of \( G \), to a homomorphism \( \phi_a : \Omega_a \to G \). The \( \phi_a \) \((a \in G)\) induce naturally a homomorphism \( G_1 \to G \) which is surjective since \( a = \phi_a(1) \) for each \( a \in G \) and (ii) is proved.

**LEMMA 15.3.** Let \( H \) be a subgroup of \( G \). If \( G_K^\wedge \) is an orthonormal base of \( AP(G \to K) \) then \( (G/H)_K^\wedge \) is an orthonormal base of \( AP(G/H \to K) \).

**Proof.** Let \( \pi : G \to G/H \) be the quotient map, let \( f \in AP(G/H \to K) \). Then \( f \circ \pi \in AP(G \to K) \) so it has an expansion

\[
f \circ \pi = \sum_{a \in Y} \lambda_a \cdot \alpha \quad (\lambda_a \in K)
\]

where \( Y \subset G_K^\wedge \), \( Y \) at most countable, \( \lambda_\alpha \neq 0 \) for all \( \alpha \in Y \). For \( s \in H \) we have

\[
f \circ \pi = (f \circ \pi)_s = \sum_{\alpha \in Y} \lambda_\alpha \cdot \alpha(s) \cdot \alpha.
\]
By uniqueness of the expansion we have \( \lambda_\alpha \alpha(s) = \lambda_\alpha \) for all \( \alpha \in Y, \ s \in H \) implying \( \alpha = 1 \) on \( H \). So, for each \( \alpha \in Y \) there is a \( \beta \in (G/H)_K^\alpha \) with \( \alpha = \beta \circ \pi \). In other words, there is a collection \( Z \subset (G/H)_K^\alpha \) for which

\[
f \circ \pi = \sum_{\beta \in Z} \lambda_{\beta \circ \pi} \beta \circ \pi
\]

i.e.

\[
f = \sum_{\beta \in Z} \lambda_{\beta \circ \pi} \beta.
\]

This, together with the obvious fact that \( (G/H)_K^\alpha \) is an orthonormal set, proves Lemma 15.3.

If \( K \) is a subfield of a valued field \( L \) we set

\[
d(x, K) := \inf \{|x - \lambda| : \lambda \in K\} \quad (x \in L)
\]

**Lemma 15.4.** Let \( K \) be algebraically closed and let \( L \supset K \) be a complete valued field extension of \( K \).

Let \( a_0, a_1, \ldots \) be a sequence in \( L \) such that \( |a_0| = 1, \ 0 < d(a_0, K) < 1 \) and \( a_{n+1} = a_n \) \( (n \in \{0, 1, 2, \ldots \}) \). Then

(i) the sequence \( n \mapsto d(a_n, K) \) is increasing,

(ii) \( \lim_{n \to \infty} d(a_n, K) = 1 \)

(iii) for large \( n \), \( d(a_n, K) = d(a_{n+1}, K)^p \).

**Proof.**

(i) Let \( N \in \mathbb{N}_0 \). We have, by the algebraic closedness of \( K \),

\[
d(a_n, K) = d(a_{n+1}, K) = \inf \{|a_{n+1} - \lambda| : \lambda \in K, |\lambda| = 1\} = \inf \{|a_{n+1} - \mu | : \mu \in K, |\mu| = 1\} \leq \inf \{|a_{n+1} - \mu | : \mu \in K, |\mu| = 1\} = d(a_{n+1}, K).
\]

We see that \( n \mapsto d(a_n, K) \) is increasing (and bounded by 1), so that

\[
\rho := \lim_{n \to \infty} d(a_n, K) \leq 1.
\]

Also, \( \rho \geq d(a_0, K) > 0 \).

(ii) To prove that \( \rho = 1 \) we first observe that \( d(a_n, K) < 1 \) for each \( n \) (this is easy to prove). Let \( n \in \mathbb{N} \). Choose a sequence \( \lambda_1, \lambda_2, \ldots \) in \( K \) such that

\[
\lim_{m \to \infty} |a_{n+1} - \lambda_m| = d(a_{n+1}, K)
\]

We may suppose that \( |a_{n+1} - \lambda_m| < 1 \) for all \( m \) (so \( |\lambda_m| = |a_{n+1}| = 1 \)).
By Lemma A.2.5 (see Appendix) we have
\[ d(a_n, K) \leq |a_{n+1}^p - \lambda_n^p| \leq \max(|a_{n+1} - \lambda_n|, |p|)|a_{n+1} - \lambda_n| \]
so by taking \( m \) large and using (i) we find
\[ d(a_n, K) \leq \max(\rho, |p|)\rho \quad (n \in \mathbb{N}) \]
implying \( 0 < \rho \leq \max(\rho, |p|)\rho \) i.e. \( \rho = 1 \).

(iii) By (i) and (ii) there exists an \( m \in \mathbb{N} \) such that \( d(a_n, K) > r^{-\sqrt{|p|}} \) for all \( n \geq m \).
For such \( n \) and any \( \lambda \in K \) with \( |\lambda| = 1 \) we have \( |a_{m+1} - \lambda| > r^{-\sqrt{|p|}} \) so that by Lemma A.2.5
\[ |a_n - \lambda^p| = |a_{n+1}^p - \lambda^p| = |a_{n+1} - \lambda|^p, \]
and \( d(a_n, K) = d(a_{n+1}, K)^p \) follows easily.

The next Proposition is crucial and might be interesting in its own right.

**Proposition 15.5.** Let \( K \) be algebraically closed and let \( L \supset K \) be the spherical completion of \( K \) (in the sense of [6], Theorem 4.49). Let \( T \) be a \( p \)-divisible subgroup of \( \{ \lambda \in L : |\lambda| = 1 \} \) that is also a \( K \)-compactoid. Then \( T \subset K \).

**Proof.** Let \( a_0 \in T, a_0 \notin K \). As the residue class fields of \( L \) and \( K \) are isomorphic we have \( 0 < d(a_0, K) < 1 \). By \( p \)-divisibility we can choose a sequence \( a_0, a_1, \ldots \) in \( T \) with \( a_{n+1}^p = a_n \) for each \( n \). Then \( d(a_n, K) = d(a_{n+1}, K)^p \) for large \( n \) (Lemma 15.4) so, by forgetting a finite initial part of \( a_0, a_1, \ldots \), we obtain a sequence \( b_0, b_1, \ldots \) in \( T \) such that \( \xi := d(b_0, K) \) is strictly between 0 and 1, \( b_{n+1}^p = b_n \) for each \( n \), and \( d(b_n, K) = d(b_{n+1}, K)^p \) for each \( n \). Thus we have
\[ d(b_n, K)^p = \xi \quad \text{for each } n. \]

We shall arrive at a contradiction by proving that the sequence \( b_0, b_1, \ldots \) is \( \xi^{p/(p-1)} \)-orthogonal in the \( K \)-Banach space \( L \). (By [6], Theorem 4.37, \( b_0, b_1, \ldots \) must tend to 0, while \( |b_n| = 1 \) for each \( n \).)
Let \( \lambda_0, \lambda_1, \ldots, \lambda_n \in K \) and consider
\[ x = \lambda_0 b_0 + \lambda_1 b_1 + \ldots + \lambda_n b_n \]
If \( \lambda_0 \neq 0 \) we have
\[ |x| = |\lambda_0 b_0^p + \lambda_1 b_1^{p-1} + \ldots + \lambda_n b_n| = |\lambda_0| \prod_{i=1}^{p^n} |b_n - \omega_i| \]
where, by the algebraic closedness of $K$, the $\omega_i$ are in $K$. We find (and this conclusion is also valid if $\lambda_0 = 0$)

$$|x| \geq |\lambda_0|d(b_n, K)^{p^n} = |\lambda_0|\xi = \xi|\lambda_0 b_0|.$$ 

By the strong triangle inequality we then also have

$$|x| \geq \xi|\lambda_1 b_1 + \ldots \lambda_n b_n|.$$ 

In the above spirit we can treat $y = \lambda_1 b_1 + \ldots + \lambda_n b_n$:

$$|y| = |\lambda_1 b_1^{p^n-1} + \ldots + \lambda_n b_n| \geq |\lambda_1|d(b_n, K)^{p^n-1} = \xi^{1/p}|\lambda_1 b_1|$$

obtaining

$$|x| \geq \xi^{1+\frac{1}{p}}|\lambda_1 b_1|$$

Inductively we find for each $i \in \{0, 1, \ldots, n\}$

$$|x| = |\lambda_0 b_0 + \lambda_1 b_1 + \ldots + \lambda_n b_n| \geq \xi^{1+\frac{1}{p}+\frac{1}{p^2}+\ldots+\frac{1}{p^{n+1}}}|\lambda_i b_i| \geq \xi^{p/(p-1)}|\lambda_i b_i|$$

and the $\xi^{p/(p-1)}$-orthogonality of $b_0, b_1, \ldots$ follows.

After these preparations we come to the point (compare Theorem 14.2).

**THEOREM 15.6.** Let $G$ be a $p$-divisible group, let $K$ be algebraically closed. Then $G_K^\wedge$ is an orthogonal base of $AP(G \to K)$ and the map $\sum_{\alpha \in G_K^\wedge} \lambda_\alpha \cdot \alpha \mapsto \lambda_1$ is an invariant mean on $AP(G \to K)$.

**Proof.** We only need to prove the first statement. We may assume that the valuation is nontrivial. It suffices to consider groups $G$ for which $x \mapsto px$ is bijective (Lemmas 15.2 and 15.3). Let $L$ be the spherical completion of $K$ in the sense of [6], Theorem 4.49. By Theorem 13.1 and Proposition 15.1 $G_L^\wedge$ is an orthonormal base of $AP(G \to L)$. So $f \in AP(G \to K) \subset AP(G \to L)$ has an expansion

$$(*) \quad \tilde{f} = \sum_{\alpha \in Y} \lambda_\alpha \cdot \alpha \quad (\lambda_\alpha \in L)$$

where $Y$ is a subset of $G_L^\wedge$, $Y$ at most countable, $\lambda_\alpha \neq 0$ for all $\alpha \in Y$. Let $\beta \in Y$; we prove that $\beta \in G_K^\wedge$. In fact, let $R : AP(G \to L) \to L$ be the coordinate map

$$\sum_{\alpha \in G_L^\wedge} \mu_\alpha \cdot \alpha \mapsto \mu_\beta.$$
$R$ is continuous, $L$-linear so certainly $K$-linear. It therefore maps the $K$-compactoid $f_G$ onto a $K$-compactoid set in $L$. Since

$$f_s = \sum_{\alpha \in Y} \lambda_\alpha \alpha(s) \alpha \quad (s \in G)$$

we have $R(f_G) = \lambda_\beta \beta(G)$, hence $\beta(G)$ is a $K$-compactoid in $L$, but also a $p$-divisible group. Proposition 15.5 comes to the rescue by telling us that $\beta(G) \subset K$ i.e. $\beta \in G_K^\wedge$. For the proof that also $\lambda_\alpha \in K$ in $(\ast)$ we refer to the proof of Theorem 14.2.

**THEOREM 15.7.** There exists an invariant mean on $AP(G \to K)$ if and only if $G$ is $p$-divisible.

**Proof.** Let $G$ be not $p$-divisible. Then $pG \neq G$ so a standard reasoning yields a surjective homomorphism $G \to C_p$ where $C_p$ is the group of $p$ elements. By Proposition 8.8 there does not exist an invariant mean on $AP(G \to K)$. For the converse, use the previous Theorem and follow the final part of the proof of Theorem 14.3.

**PROBLEM.** Find a formula expressing $M(f)$ in terms of values of $f$ (compare Theorem 14.4).

§16. A FEW CURIOUS PROPERTIES

It is proved in [6], Corollary 8.29, that $SAP(G \to K)$ (see §4) is isomorphic to $C(\bar{G} \to K)$ where $\bar{G}$ is the non-archimedean Bohr compactification of $G$, which result is in perfect harmony to the theory of complex valued almost periodic functions. For $AP(G \to K)$, however, the situation is different.

**THEOREM 16.1.** Let $G$ be divisible. If the characteristic of the residue class field of $K$ is zero, assume in addition that $G$ is a torsion group. Then the norm on $AP(G \to K)$ is multiplicative.

**Proof.** By Theorem 3.3 we may assume that $K$ is algebraically closed. By the Theorems 14.2 and 15.6 and continuity it suffices to prove that $\|fg\|_\infty = \|f\|_\infty \|g\|_\infty$ for $f = \sum_{\alpha \in Y} \lambda_\alpha \cdot \alpha, g = \sum_{\alpha \in Y} \mu_\alpha \cdot \alpha$, where $Y$ is a finite subset of $G_K^\wedge$ and $\lambda_\alpha, \mu_\alpha \in K$. By divisibility of $G$ the group $G_K^\wedge$ is torsion free so there exists an isomorphism $\sigma$ of $\mathbb{Z}^n$ onto the group generated by $Y$. The formula

$$\phi(\sum_{i \in \mathbb{Z}^n} \lambda_i X^i) = \sum_{i \in \mathbb{Z}^n} \lambda_{\sigma(i)} \sigma(i)$$
defines a homomorphism \( \phi \) of \( K[X_1, X_2^{-1}, \ldots, X_n, X_n^{-1}] \) into \( AP(G \rightarrow K) \) whose image contains \( f \) and \( g \). With the Gauss norm on \( K[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}] \) it is easily seen that \( \phi \) is an isometry. It is well known that the Gauss norm is multiplicative. Hence, 
\[
\|fg\|_\infty = \|f\|_\infty \|g\|_\infty.
\]

**Remark.** By Theorem 4.1 \( SAP(G \rightarrow K) \) is generated by idempotents. So, with \( G \) as in Theorem 16.1, we have \( SAP(G \rightarrow K) = K.1 \) Thus, for example, if \( G \) is divisible we have
\[
AP(G \rightarrow Q_p) = SAP(G \rightarrow Q_p) = K.1
\]
\[
AP(G \rightarrow C_p) = [G^\wedge_C_p]_{C_p} \neq [AP(G \rightarrow Q_p)]_{C_p}
\]
(see the second Remark following Theorem 3.3).

The next observation is also worth mentioning as \( \max\{|f(x)| : x \in G\} \) does not always exist for \( f \in AP(G \rightarrow K) \).

**PROPOSITION 16.2** Let \( K \) be algebraically closed with residue class field \( k \). Assume that \( G \) is a torsion group if char \( k = 0 \), \( p \)-divisible if char \( k = p \neq 0 \). Then for each \( AP(G \rightarrow K) \) we have \( \|f\|_\infty \in [K] \).

**Proof.** By Theorems 14.2 and 15.6
\[
\|f\|_\infty = \left\| \sum_{\alpha} \lambda_\alpha \cdot \alpha \right\|_\infty = \max_{\alpha \in G^\wedge_K} |\lambda_\alpha| \in [K].
\]

**Remark.** Do the almost periodic functions separate the points of \( G \)? In general the answer is no (see the above remark). If \( K \) is algebraically closed and has characteristic zero then for any abelian \( G \) the characters, hence the almost periodic functions, separate the points of \( G \). But this conclusion is, in general, false if \( K \) has characteristic \( p \neq 0 \) (if \( G = Q_p/Z_p \), then \( G^\wedge_K \) consists only of the unit character. By Theorem 15.6 we have \( AP(G \rightarrow K) = [G^\wedge_K] = K.1 \)).

§17. ALMOST PERIODIC FUNCTIONS ON \( Q_p \)

As an application of the previous theory we consider the \( K \)-Banach algebra \( CAP(Q_p \rightarrow K) \) consisting of all continuous almost periodic functions \( Q_p \rightarrow K \) (where the additive group \( Q_p \) has the \( p \)-adic topology). We assume that the characteristic of the residue class field \( k \) of \( K \) is \( p \) and that \( K \) is algebraically closed (e.g. \( K = C_p \)).

**THEOREM 17.1** Let \( K \) be as above, let \( f \in CAP(Q_p \rightarrow K) \).
There exists a unique invariant mean $M$ on $\text{CAP}(\mathbb{Q}_p \to K)$.

(ii) The $K$-valued continuous characters form an orthonormal base of $\text{CAP}(\mathbb{Q}_p \to K)$.

(iii) The norm on $\text{CAP}(\mathbb{Q}_p \to K)$ is multiplicative.

(iv) $\|f\|_\infty \in |K|$.

(v) $f$ is uniformly continuous.

(vi) If $a \in \mathbb{Q}_p$, $a \neq 0$ the function $h : x \mapsto f(ax)$ is also almost periodic and $M(h) = M(f)$.

(vii) If $f(\mathbb{Q}_p)$ is compact then $f$ is constant.

(viii) If $f \neq 0$ then $\|f_s - f\|_\infty < \|f\|_\infty$ for all $s \in \mathbb{Q}_p$.

(ix) If $\max\{|f(x)| : x \in \mathbb{Q}_p\}$ exists then $|f|$ is constant.

(x) There exists a sequence $x_1, x_2, \ldots$ in $\mathbb{Q}_p$ with $\lim_{n \to \infty} |x_n| = \infty$ and $\lim_{n \to \infty} |f(x_n)| = \|f\|_\infty$.

(xi) If $X \subset \mathbb{Q}_p$ is compact and $h : X \to K$ is continuous then there is an $f \in \text{CAP}(\mathbb{Q}_p \to K)$ extending $h$. If $\varepsilon > 0$ we may choose $f$ such that

$$\|f\|_\infty \leq (1 + \varepsilon) \max\{|h(x)| : x \in X\}.$$ 

(In other words, the restriction $\text{CAP}(\mathbb{Q}_p \to K) \to C(X \to K)$ is a quotient mapping.)

Proof. Keeping in mind that $\mathbb{Q}_p$ is a divisible group and that $\text{CAP}(\mathbb{Q}_p \to K) \subset \text{AP}(\mathbb{Q}_p \to K)$ is a special $\mathbb{Q}_p$-module in the sense of §7 we have the following.

(i) Theorems 7.6 and 15.7.

(ii) By Theorem 15.6 $f$ has an expansion

$$f = \sum_{\alpha \in Y} \lambda_\alpha \cdot \alpha \quad (\lambda_\alpha \in K)$$

where $Y$ is a subset of $(\mathbb{Q}_p)_K$, $Y$ at most countable, $\lambda_\alpha \neq 0$ for each $\alpha \in Y$. We prove that $\beta \in Y$ implies continuity of $\beta$. The formula

$$\sum \mu_\alpha \cdot \alpha \to \mu_\beta$$

defines a $\phi \in \text{AP}(\mathbb{Q}_p \to K)'$ with $\|\phi\| \leq 1$. By Proposition 5.2

$$s \mapsto \phi(f_s) = \phi(\sum_{\alpha} \lambda_\alpha \alpha(s) \cdot \alpha) = \lambda_\beta \beta(s)$$

is in $(\overline{\mathbb{Q}_p})^*$, hence continuous. As $\lambda_\beta \neq 0$, $\beta$ is continuous.

(iii) Theorem 16.1.

(iv) Proposition 16.2.

(v) Follows from (ii).
(vi) $h$ is bounded and continuous. From $h_s = f_s \circ \tau$ ($s \in \mathbb{Q}_p$), where $\tau$ is the map $x \mapsto ax$, it follows easily that $h \in AP(\mathbb{Q}_p \to K)$. The map $N \in CAP(\mathbb{Q}_p \to K)'$ defined by

$$N(g) = M(g \circ \tau) \quad (g \in CAP(\mathbb{Q}_p \to K))$$

has norm 1, $N(1) = 1$. For each $s \in \mathbb{Q}_p$,

$$N(g_s) = \int g(s + ax)dM(x) = \int g(s + a(x - a^{-1}s))dM(x) = \int g(ax)dM(x)$$

$$= \int (g \circ \tau)dM(x) = N(g).$$

Then, $N$ is an invariant mean on $CAP(\mathbb{Q}_p \to K)$. By uniqueness, $M = N$, and (vi) follows.

(vii) See the Remark following Theorem 16.1.

(viii) By (ii) $f$ has an expansion $\sum \lambda_\alpha \cdot \alpha$ where the $\alpha$'s are continuous characters. Then $f_s = \sum \lambda_\alpha \alpha(s) \cdot \alpha$ so that

(*) \[ \|f_s - f\|_\infty = \max|\lambda_\alpha| \cdot |\alpha(s) - 1| \]

Now for each continuous character $\alpha$ on $\mathbb{Q}_p$, we have $\lim n \to \infty \alpha(p^n s) = \alpha(s) - 1$.

By [9], Theorem 32.2 we have $|\alpha(s) - 1| < 1$, and (viii) follows from (*).

(ix) We may assume $f \neq 0$. Let $|f(a)| = \|f\|_\infty$. Then, for each $x \in \mathbb{Q}_p$,

$$|f(x) - f(a)| = |f_{x-a}(a) - f(a)| \leq \|f_{x-a} - f\|_\infty < \|f\|_\infty = |f(a)|$$

implying $|f(x)| = |f(a)|$.

(x) By (ix) we may suppose that $\max\{|f(x)| : x \in \mathbb{Q}_p\}$ does not exist. Let $x_1, x_2, \ldots$ be a sequence in $\mathbb{Q}_p$ such that $\lim n \to \infty |f(x_n)| = \|f\|_\infty$. If $X \subset \mathbb{Q}_p$ is compact then $\max\{|f(x)| : x \in X\} < \|f\|_\infty$ implying $x_n \notin X$ for large $n$. It follows that $\lim n \to \infty |x_n| = \infty$.

(xi) We may assume $X = \mathbb{Z}_p$ (extend $h$ to a continuous function $h_1$ on some $p^n \mathbb{Z}_p \supset X$, with $h_1(p^n \mathbb{Z}_p) = h(X)$ (for example by [9], Theorem 76.2), extend $x \mapsto h_1(p^n x)$ ($x \in \mathbb{Z}_p$) to an almost periodic function $f_1$ and set $f(x) = f_1(p^n x)$). From [2], Théorème 2 it follows that $h$ has a (nonunique) expansion

$$h = \sum_{n=1}^{\infty} \lambda_n \beta_n$$

where $\beta_1, \beta_2, \ldots$ are distinct locally constant characters on $\mathbb{Z}_p$, where $\lambda_1, \lambda_2, \ldots \in K$ with $\lim n \to \infty \lambda_n = 0$ and $|\lambda_n| \leq (1 + \varepsilon) \max\{|h(x)| : x \in \mathbb{Z}_p\}$ for each $n \in \mathbb{N}$. By the algebraic closedness of $K$ each $\beta_n$ extends to a character $\alpha_n$ of $\mathbb{Q}_p$ (which is automatically continuous). Then $f := \sum_{n=1}^{\infty} \lambda_n \alpha_n$ is the required extension.

**PROBLEM.** (Compare Theorem 17.1 (x)). Let $f \in CAP(\mathbb{Q}_p \to \mathbb{C}_p)$. Let $x_1, x_2, \ldots$ be a sequence in $\mathbb{Q}_p$, $\lim n \to \infty |x_n| = \infty$. Do we have $\lim n \to \infty |f(x_n)| = \|f\|_\infty$?
APPENDIX

Throughout this appendix $K$ is a complete nonarchimedean valued field with a nontrivial valuation $|\cdot|$. We assume throughout that $K$ is algebraically closed, and spherically complete.

§A.1. LINEAR MAPS ON FINITE DIMENSIONAL NORMED SPACES

PROPOSITION A.1.1. Let $(E, \| \cdot \|)$ be an $m$-dimensional normed space over $K$ ($m \in \mathbb{N}$). Let $T \in \mathcal{L}(E)$ and let $\epsilon > 0$. Then there exists an orthogonal basis $e_1, \ldots, e_m$ of $E$ such that

\begin{itemize}
  \item[(i)] $1 + \epsilon \geq \|e_1\| \geq \|e_2\| \geq \ldots \geq \|e_m\| \geq 1,$
  \item[(ii)] the matrix of $T$ with respect to $e_1, \ldots, e_m$ is upper triangular.
\end{itemize}

Proof. By the algebraic closedness of $K$, elementary algebra yields a basis $g_1, \ldots, g_m$ of $E$ such that $T[g_1, \ldots, g_i] \subset [g_1, \ldots, g_i]$ for each $i \in \{1, \ldots, m\}$ i.e., the matrix of $T$ with respect to $g_1, \ldots, g_m$ is upper triangular. As $K$ is spherically complete we can find orthogonal $f_1, f_2, \ldots, f_m$ such that $[f_1, \ldots, f_i] = [g_1, \ldots, g_i]$ for each $i \in \{1, \ldots, m\}$. Obviously, the matrix of $T$ with respect to $f_1, \ldots, f_m$ is upper triangular. Since the valuation of $K$ is dense we can find $\lambda_1, \ldots, \lambda_m \in K$ such that (i) is satisfied where

$$e_i := \lambda_i f_i \quad (1 \leq i \leq m)$$

Then clearly we also have (ii).

PROPOSITION A.1.2. Let $(E, \| \cdot \|)$ be an $m$-dimensional normed space over $K$, let $\epsilon > 0$, let $T \in \mathcal{L}(E)$. Suppose that $e_1, \ldots, e_m$ is an orthogonal base of $E$ such that

$$1 + \epsilon \geq \|e_1\| \geq \|e_2\| \geq \ldots \geq \|e_m\| \geq 1$$

and suppose the matrix $(t_{ij})$ of $T$ with respect to $e_1, \ldots, e_m$ is upper triangular. Let

$$|T| = \max_{i,j} |t_{ij}|$$

$$\|T\| = \sup \left\{ \frac{\|Tx\|}{\|x\|} : x \in E, x \neq 0 \right\}$$

Then

\begin{itemize}
  \item[(i)] $|T| \leq \|T\| \leq (1 + \epsilon)|T|$
  \item[(ii)] If $T$ is invertible,
    $$\frac{\|Tx\|}{\|x\|} \geq (1 + \epsilon)^{-1} |T|^{-m+1} |\det T| \quad (x \in E, x \neq 0)$$
\end{itemize}
Proof. (i) The matrix of $T$ with respect to $e_1, \ldots, e_m$ is

$$
\begin{pmatrix}
t_{11} & t_{12} & \cdots & t_{1m} \\
t_{22} & & & \\
& \ddots & & \\
& & & t_{mm}
\end{pmatrix}
$$

let $i \in \{1, \ldots, m\}$. We have $\|Te_i\| \leq \|T\| \|e_i\|$ and, by orthogonality,

$$
\|Te_i\| = \left\| \sum_{j=1}^{i} t_{ji}e_j \right\| = \max_{j \leq i} |t_{ji}| \|e_j\| \geq (\max_{j \leq i} |t_{ji}|) \|e_i\|.
$$

We find for each $i \in \{1, \ldots, m\}$

$$
\max_{j \leq i} |t_{ji}| \leq \|T\|
$$

implying $|T| \leq \|T\|$. To prove that $\|T\| \leq (1 + \varepsilon)|T|$, let $x = \sum_{i=1}^{m} \xi_i e_i \in E$ where $\xi_i \in K$ for each $i$. We have

$$
\|Tx\| = \left\| \sum_{i=1}^{m} \xi_i Te_i \right\| = \left\| \sum_{i=1}^{m} \sum_{j=1}^{i} \xi_i t_{ji} e_j \right\| \leq |T| \max_{i=1}^{m} \|\xi_i\| \cdot \max_{j=1}^{m} \|e_j\|
$$

$$
\leq (1 + \varepsilon)|T| \max_{i=1}^{m} \|\xi_i\| \leq (1 + \varepsilon)|T| \max_{i=1}^{m} \|\xi_i\| = (1 + \varepsilon)|T||x|.
$$

(ii) The matrix of $T^{-1}$ is also upper triangular so applying (i) to $T^{-1}$ we find

$$
\frac{\|T^{-1}x\|}{\|x\|} \leq (1 + \varepsilon)|T^{-1}| \quad (x \in E, x \neq 0)
$$

i.e.

$$
\frac{\|Ty\|}{\|y\|} \geq (1 + \varepsilon)^{-1}|T^{-1}|^{-1} \quad (y \in E, y \neq 0)
$$

By [4], Theorem 1.13(d) we have $|T^{-1}| \leq |\det T|^{-1}|T|m^{-1}$ and (ii) follows.

PROPOSITION A.1.3. Let $(E, \|\|)$ be an $m$-dimensional normed space over $K$. Let $T \in \mathcal{L}(E)$ with $\|T\| \leq 1$, let $\lambda_1, \ldots, \lambda_m$ be the eigenvalues of $T$ (counted with multiplicity). For any $c \in (0, 1]$, let $G_1, \ldots, G_n$ be the partition of $\{1, \ldots, m\}$ induced by the equivalence relation

$$
i \sim j \text{ if } |\lambda_i - \lambda_j| < c \quad (i, j \in \{1, \ldots, m\})$$

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Set

\[ D_i := \text{Ker} \prod_{j \in G_i} (T - \lambda_j I) \quad (i \in \{1, \ldots, n\}) \]

Then we have:

(i) Each \( D_i \) is invariant under \( T \). The eigenvalues of \( T|D_i \) are precisely \( \{\lambda_j : j \in G_i\} \) (so their distances are \( < c \)).

(ii) \( E = \sum_{i=1}^{n} D_i \).

(iii) \( D_1, \ldots, D_n \) is a \( c^2m^2 \)-orthogonal family of subspaces.

**Proof.** Elementary linear algebra yields (i) and (ii). We shall prove \( c^2m^2 \)-orthogonality of \( D_1, \ldots, D_n \). We may suppose \( n > 1 \). For reasons of symmetry it is enough to prove

\[(*) \quad \|d_1 + d_2 + \ldots + d_n\| \geq c^2m^2\|d_1\| \quad (d_i \in D_i \text{ for each } i)\]

Consider the operator

\[ S := \prod_{i > 1} \prod_{j \in G_i} (T - \lambda_j I) = \prod_{j \in X} (T - \lambda_j I) \]

where \( X := G_2 \cup \ldots \cup G_n \). Then \( S \) is zero on \( D_i \) for each \( i > 1 \) and \( SD_1 \subset D_1, \|S\| \leq 1 \). Hence,

\[(1) \quad \|d_1 + \ldots + d_n\| \geq \|S(d_1 + \ldots + d_n)\| = \|Sd_1\|\]

Let \( \varepsilon > 0 \).

By Proposition A.1.1 there exists an orthogonal base \( e_1, \ldots, e_r \) of \( D_1 \) such that \( 1 + \varepsilon \geq \|e_1\| \geq \ldots \geq \|e_r\| \geq 1 \) and such that the matrix of \( T|D_1 \) with respect to \( e_1, \ldots, e_r \) has the form

\[
\begin{pmatrix}
\lambda_{i_r} & * \\
\vdots & \\
\Theta & \lambda_{i_r}
\end{pmatrix}
\]

where \( \{i_1, \ldots, i_r\} = G_1 \). Proposition A.1.2 (i) yields that the entries of the matrix are all in \( B(0,1) \subset K \). For each \( j \in X \) the matrix of \( (T - \lambda_j I)|D_1 \) has the form

\[
\begin{pmatrix}
\lambda_{i_1} - \lambda_j & * \\
\vdots & \\
\Theta & \lambda_{i_r} - \lambda_j
\end{pmatrix}
\]

and so the matrix of \( S|D_1 \) (with respect to \( e_1, \ldots, e_r \)) has the form

\[
\begin{pmatrix}
\prod_{j \in X}(\lambda_{i_1} - \lambda_j) & * \\
\vdots & \\
\Theta & \prod_{j \in X}(\lambda_{i_r} - \lambda_j)
\end{pmatrix}
\]
We have

\[(2) \quad |\det S|D_1| = \prod_{i \in G_1} \prod_{j \in X} |\lambda_i - \lambda_j| \geq c^{\#G_1 \cdot \#X} \geq c^{\frac{1}{2}m^2}\]

as \(\max\{x(m - x) : x \in \{1, \ldots, m - 1\}\} \leq \frac{1}{2}m^2\)

Now apply Proposition A.1.2 (ii) to \(S|D_1\). We find

\[\|Sd_1\| \geq (1 + \epsilon)^{-1}|S|D_1|^{-1}d_1\| \quad (d_1 \in D_1)\]

Now \(|S|D_1| \leq 1\), so \(|S|D_1|^{\tau+1} \geq 1\). With (2) we obtain

\[\|Sd_1\| \geq (1 + \epsilon)^{-1}c^{\frac{1}{2}m^2}\|d_1\|\]

Combining this with (1) we get

\[\|d_1 + \ldots + d_n\| \geq (1 + \epsilon)^{-1}c^{\frac{1}{2}m^2}\|d_1\|\]

But this inequality holds for each \(\epsilon > 0\) and (*) follows.

**PROPOSITION A.1.4.** Let \(E\) be a finite dimensional space over \(K\), let \(\|\|_1\) and \(\|\|_2\) be norms on \(E\). Then there exists a base of \(E\) that is orthogonal with respect to \(\|\|_1\) and \(\|\|_2\).

Proof. [4], Theorem 1.11 or [1].

§A.2. MATRICES

Throughout §A.2 the characteristic of the residue class field \(k\) of \(K\) is \(p \neq 0\). We fix an \(m \in \mathbb{N}\).

For an \(m \times m\) matrix \(\alpha := (a_{ij})\) with entries in \(K\) we set

\[|\alpha| := \max_{i,j} |a_{i,j}|\]

It is easy to verify that \(\alpha \mapsto |\alpha|\) is an algebra norm on the \(K\)-algebra of all \(m \times m\) matrices with entries in \(K\). We set

\[u := \begin{pmatrix}
1 & \Theta \\
1 & \ddots \\
\Theta & 1
\end{pmatrix}\]
For an $m \times m$-matrix $\alpha$ we define $\rho(\alpha)$ to be the distance of $\alpha$ to $Ku$ i.e.
\[
\rho(\alpha) := \inf \{|\alpha - \lambda u| : \lambda \in K\}
\]
Clearly, $\rho$ is a seminorm on the $K$-vector space of all $m \times m$-matrices, and $\rho(\alpha) \leq |\alpha|$ for all $m \times m$-matrices $\alpha$.

We may express $\rho(\alpha)$ in terms of the $a_{ij}$:

**Lemma A.2.1.** Let $\alpha = (a_{ij})$ be an $m \times m$ matrix. Then
\[
\rho(\alpha) = \max_{i,j} |a_{ii} - a_{jj}| \vee \max_{s \neq t} |a_{st}|.
\]

**Proof.** Denote the right hand expression by $\bar{\rho}(\alpha)$. For each $j \in \{1, \ldots, m\}$ we have
\[
\rho(\alpha) \leq |\alpha - a_{jj}u| = \max_i |a_{ii} - a_{jj}| \vee \max_{s \neq t} |a_{st}|
\]
so that $\rho(\alpha) \leq \bar{\rho}(\alpha)$. To prove the opposite inequality, let $\lambda \in K$. Then
\[
|\alpha - \lambda u| = \max_i |a_{ii} - \lambda| \vee \max_{s \neq t} |a_{st}|.
\]
Since for each $i, j \in \{1, \ldots, m\}$
\[
|a_{ii} - a_{jj}| \leq |a_{ii} - \lambda| \vee |\lambda - a_{jj}|
\]
we have
\[
\bar{\rho}(\alpha) \leq |\alpha - \lambda u|
\]
As $\lambda \in K$ was arbitrary, $\bar{\rho}(\alpha) \leq \rho(\alpha)$.

**Lemma A.2.2.** For $m \times m$ matrices $\alpha = (a_{ij})$, $\beta = (b_{ij})$
\[
\rho(\alpha \beta) \leq |\alpha| \rho(\beta) \vee \rho(\alpha)|\beta|.
\]

**Proof.** Let $\lambda_1, \lambda_2, \ldots$ and $\mu_1, \mu_2, \ldots$ be sequences in $K$ for which
\[
\rho(\alpha) = \lim_{n \to \infty} |\alpha - \lambda_n u|
\]
\[
\rho(\beta) = \lim_{n \to \infty} |\beta - \mu_n u|
\]
Then for all $n \in \mathbb{N}$
\[
|\alpha \beta - \lambda_n \mu_n u| \leq |\alpha \beta - \lambda_n \beta| \vee |\lambda_n \beta - \lambda_n \mu_n u| \\
\leq |\alpha - \lambda_n u| |\beta| \vee |\lambda_n| |\beta - \mu_n u|
\]
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Now we have $|\lambda_n| = |\lambda_n u| \leq |\alpha - \lambda_n u| \vee |\alpha|$ so that $\limsup_{n \to \infty} |\lambda_n| \leq \rho(\alpha) \vee |\alpha| = |\alpha|$ and we find

$$\rho(\alpha \beta) \leq \limsup_{n \to \infty} |\alpha \beta - \lambda_n u| \leq \rho(\alpha) |\beta| \vee |\alpha| \rho(\beta).$$

**DEFINITION A.2.3.** Let $c \in (0,1)$. Let $M_c$ be the set of all upper triangular $m \times m$ matrices $\alpha = (a_{ij})$ for which $|\alpha| \leq 1$ and $\max_{i,j} |a_{ii} - a_{jj}| \leq c$.

It is not hard to see that $M_c$ is a ring. Let $\alpha \in M_c$. From Lemma A.2.2 we obtain $\rho(\alpha^n) \leq \rho(\alpha)$ for each $n \in \mathbb{N}$. In particular

$$n \mapsto \rho(\alpha^n)$$

is decreasing. We shall prove that $\lim_{n \to \infty} \rho(\alpha^n) = 0$ uniformly on $M_c$. More specifically:

**PROPOSITION A.2.4.** Let $k_m$ be a number such that $(p_i^{k_m})$, $(p_2^{k_m}), \ldots, (p_{m-1}^{k_m})$ are all divisible by $p$. Then for each $\alpha \in M_c$, each $n \in \mathbb{N}$

$$\rho(\alpha^{p^{k_m} n}) \leq \tau^n \rho(\alpha)$$

where $\tau := \max(c, |p|)$.

**Proof.** We first prove the statement for $n = 1$ (this is the hard part). Write $\alpha$ as a sum of a diagonal matrix and a nilpotent one:

$$\alpha = \begin{pmatrix} a_{11} & \cdots & * \\ \vdots & \ddots & \vdots \\ \Theta & \cdots & a_{mm} \end{pmatrix} = \begin{pmatrix} a_{11} & \Theta \\ \Theta & a_{mm} \end{pmatrix} + \begin{pmatrix} 0 & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ \Theta & \cdots & 0 \end{pmatrix} = \lambda + \nu$$

An easy computation yields

$$|\lambda \nu - \nu \lambda| = \max_{i \neq j} |a_{ij}| |a_{ii} - a_{jj}| \leq c |\nu|$$

implying

$$(*) \quad |\alpha^{p^{k_m}} - \sum_{i=0}^{p^{k_m}} (p_i^{k_m}) \lambda^{p^{k_m} - i} \nu^i| \leq c |\nu| \leq \tau |\nu|$$

Now, since $\nu^m = 0$ and $(p_1^{k_m}), \ldots, (p_{m-1}^{k_m})$ are divisible by $p$,

$$|\sum_{i=1}^{p^{k_m}} (p_i^{k_m}) \lambda^{p^{k_m} - i} \nu^i| = \sum_{i=1}^{m-1} (p_i^{k_m}) \lambda^{p^{k_m} - i} |\nu^i| \leq |p| \max_{1 \leq i < m} |\nu^i| = \tau |\nu|. $$
Thus, we obtain from (*)

$$|\alpha^{p_{km}} - \lambda^{p_{km}}| \leq \tau|\nu|$$

By Lemma A.2.1 we have $|\nu| = \rho(\nu) \leq \rho(\alpha)$ so that

(***)

$$\rho(\alpha^{p_{km}} - \lambda^{p_{km}}) \leq \tau\rho(\alpha)$$

Next, we estimate $\rho(\lambda^{p_{km}})$. We have

$$\lambda^{p_{km}} = \begin{pmatrix} a_{11}^{p_{km}} & \cdots & \Theta \\ \cdots & \cdots & \cdots \\ \Theta & \cdots & a_{mm}^{p_{km}} \end{pmatrix}$$

Since $|a_{ii} - a_{jj}| \leq c$ we have by Lemma A.2.5 below

$$|a_{ii}^{p_{km}} - a_{jj}^{p_{km}}| \leq \tau|a_{ii} - a_{jj}| \leq \tau\rho(\lambda)$$

So

$$\rho(\lambda^{p_{km}}) \leq \tau\rho(\lambda)$$

Certainly,

$$\rho(\lambda^{p_{km}}) \leq \tau\rho(\lambda) \leq \tau\rho(\alpha)$$

Combined with (***), this implies

$$\rho(\alpha^{p_{km}}) \leq \tau\rho(\alpha)$$

which concludes the proof for $n = 1$. The proof of the induction step $n - 1 \rightarrow n$ is standard. Since $\alpha^{p_{(n-1)km}} \in M_c$ we have by the first part

$$\rho(\alpha^{p_{(n-1)km}}) \leq \tau\rho(\alpha^{p_{(n-1)km}})$$

By the induction hypothesis

$$\rho(\alpha^{p_{(n-1)km}}) \leq \tau^{n-1}\rho(\alpha)$$

yielding

$$\rho(\alpha^{p_{nkm}}) \leq \tau^n\rho(\alpha)$$

**Lemma A.2.5.** Let $a, b \in K$, $|a| = 1$, $|b| = 1$, $|a - b| < 1$. Then $|a^p - b^p| \leq \tau|a - b|$ where $\tau = \max(|a - b|, |p|)$. If, in addition, $|a - b| > \sqrt[p]{|p|}$ then $|a^p - b^p| = |a - b|^p$.

**Proof.** For the first statement, see [9], Lemma 32.1. To prove the second one we may suppose $b = 1$. Set $a := 1 + u$. We have

$$a^p - 1 = u^p + \sum_{i=1}^{p-1} \binom{p}{i} u^i$$

For $i \in \{1, \ldots, p-1\}$

$$|\binom{p}{i} u^i| \leq |p| |u|^i < |u|^{p-1}|u| = |u|^p.$$

It follows that $|a^p - 1| = |u^p| = |a - 1|^p$. 

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REFERENCES