p-ADIC LOCAL COMPACTOIDS

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ABSTRACT. For a complete local compactoid A in a locally convex space E over a non-archimedean valued field K it is proved that $A = D \oplus B$ where D is a subspace and B is a compactoid. As a corollary Katsaras' Theorem is extended to complete local compactoids.

TERMINOLOGY. Throughout K is a non-archimedean valued field that is complete with respect to the non-trivial valuation $|\ |$. A subset A of a K-vector space E is *absolutely convex* if it is a module over the ring $B(0,1) := \{\lambda \in K : |\lambda| \leq 1\}$. For a subset X of E we denote by $[X]$ the K-vector space generated by X, by coX the smallest absolutely convex subset of E containing X. For an absolutely convex set $A \subset E$ we set $A^e := A$ if the valuation of K is discrete and $A^e := \bigcap\{\lambda A : \lambda \in K, |\lambda| > 1\}$ if the valuation of K is dense. A is *edged* if $A = A^e$.

The K-Banach space consisting of all sequences $(\xi_1, \xi_2, ...)$ in K with $\lim_{n\to\infty} \xi_n = 0$ and with the norm $(\xi_1, \xi_2, ...) \mapsto \max_n |\xi_n|$ is denoted c_0.

Let E be a locally convex space over K. The closure of a set $X \subset E$ is denoted $\overline{X}$. Instead of $\overline{coX}$ we write $\overline{co}X$. For each continuous seminorm p on E, let E_p be the space $E/Kerp$ with the norm induced by p, let $E_p^\wedge$ be its completion. The maps

$$\pi_p : E \to E_p \to E_p^\wedge$$

induce a map

$$E \to \prod_p E_p^\wedge$$

which is, if E is Hausdorff, a linear homeomorphism onto a subspace of the product. An absolutely convex subset A of E is a *compactoid* if for each zero neighbourhood U in E there exists a finite set $F \subset E$ such that $A \subset U + coF$. A is a *local compactoid in* E if for each zero neighbourhood U in E there exists a finite dimensional space $D \subset E$ with $A \subset U + D$.

For terms that are unexplained here we refer to [4].

INTRODUCTION. We quote the following theorem, first proved by Katsaras.

THEOREM ([2],[1]). *Let A be a compactoid in a locally convex space E over K. Let $\lambda \in K, \lambda = 1$ if the valuation of K is discrete, $|\lambda| > 1$ otherwise. Then, for each neighbourhood U of 0 in E there exists a finite set F in λA such that $A \subset U + coF$.*

The theorem implies that compactoidity of A is a property of the topological $B(0,1)$- module A and does not depend on the embedding space E.
Surprisingly, Katsaras' Theorem does not extend to local compactoids in general (Example 3.6); we shall prove such a theorem only for *complete* local compactoids (Theorem 3.4).

Remarks

1 Let K be spherically (= maximally) complete. Then completeness & local compactoidity is equivalent to c-compactness ([5],Theorem 11). By using this fact and well-known properties of c-compact sets one may derive the results of this paper in a much easier way.

2 Because of the previous remark our proofs, although valid for any K, are only of importance if K is *not* spherically complete.

§1 LOCAL COMPACTOIDS

Throughout §1 E is a Hausdorff locally convex space over K. The proofs of the next two Propositions are left to the reader.

PROPOSITION 1.1. *Let A be an absolutely convex subset of E.*

(i) If A is a local compactoid in E and $B \subset A$ is absolutely convex then B is a local compactoid in E.

(ii) If A is a local compactoid in E then so is $\overline{A}$.

(iii) If F is a Hausdorff locally convex space over K, if $T : E \to F$ is a continuous linear map and if A is a local compactoid in E then TA is a local compactoid in F.

(iv) A is a compactoid (in E) if and only if A is a bounded local compactoid in E.

PROPOSITION 1.2. *Let $(E_i)_{i \in I}$ be a family of Hausdorff locally convex spaces over K. If, for each i, A_i is a local compactoid in E_i then $\prod_i A_i$ is a local compactoid in $\prod_i E_i$.*

PROPOSITION 1.3. *Let A be a closed local compactoid in a K-Banach space E. Then $\overline{[A]}$ is of countable type and A is a local compactoid in $\overline{[A]}$.*

Proof. [3], 6.9 and Theorem 6.7.

LEMMA 1.4. *Let A be a local compactoid in E. Then there exists a Hausdorff locally convex space E_1 of countable type and a linear homeomorphism of $[A]$ into E_1 such that $i(A)$ is a local compactoid in E_1.*

Proof. For each continuous seminorm p the set $\overline{\pi_p(A)}$ is a local compactoid in $E_p^\wedge$ (Proposition 1.1), hence in a subspace D_p of countable type (Proposition 1.3). By [4], Proposition 4.12 (iii), $E_1 := \prod_p D_p$ is of countable type. The restriction of the embedding $E \hookrightarrow \prod_p E_p^\wedge$ yields a linear homeomorphic embedding $i : [A] \hookrightarrow E_1$. Now $i(A)$ is a subset of $\prod_p \overline{\pi_p(A)}$, which is a local compactoid in E_1 (Proposition 1.2). Then, $i(A)$ is a local compactoid in E_1.

COROLLARY 1.5. *If A is a local compactoid in E then $[A]$ is of countable type.*

Proof. $[A]$ is linearly homeomorphic to a subspace of E_1. Now apply [4], Proposition 4.12 (i).

PROPOSITION 1.6. *Let E be a polar space and let A be a local compactoid in E. Then, on A, the weak topology $\sigma(E, E')$ and the initial topology coincide. A is complete if and only if A is weakly complete.*

Proof. The proofs of [4], 5.7-5.11 can easily be modified in such a way that the conclusion of [4], Theorem 5.12 holds for local compactoids, rather than just compactoids.

PROPOSITION 1.7. *Let A be a local compactoid in E. Then, as a topological $B(0,1)$-module, A is isomorphic to a $B(0,1)$-submodule of some power of K.*

Proof. By Lemma 1.4 we may suppose that E is of countable type, hence polar. So, by Proposition 1.6, A is a topological $B(0,1)$-submodule of $(E, \sigma(E, E'))$. The map

$$x \mapsto (f(x))_{f \in E'} \qquad (x \in E)$$

is a linear homeomorphism of $(E, \sigma(E, E'))$ into $K^{E'}$. The statements follows.

§2 LOCAL COMPACTOIDS IN K^I.

Throughout §2, E is a vector space over K (no topology) and E^* its algebraic dual, with the topology $\sigma(E^*, E)$ of pointwise convergence. Then E^* is Hausdorff, locally convex, complete and of countable type. Every absolutely convex subset of E^* is a local compactoid in E as each neighbourhood of 0 in E^* contains a subspace with finite codimension. It is not hard to see that each $\Theta \in (E^*)'$ has the form $f \mapsto f(x) \quad (f \in E^*)$ for some $x \in E$, so that we may identify $(E^*)'$ and E.

To see the connection with the title of §2 observe that E is the (algebraic) direct sum $\bigoplus_{i\in I} K_i$, where $K_i = K$ for each i and that E^* is linearly homeomorphic to K^I.

A subset X of E is *K-polar* if for each $y \in E \backslash X$ there exists an $f \in E^*$ with $|f(X)| \leq 1$, $|f(y)| > 1$.

For $X \subset E$, $Y \subset E^*$ we set, as usual

$$X^0 := \{f \in E^* : |f(X)| \leq 1\}$$

$$Y^0 := \{x \in E : |Y(x)| \leq 1\}.$$

PROPOSITION 2.1. *Let $X \subset E$, $Y \subset E^*$.*

(i) X is K-polar if and only if $X = X^{00}$.

(ii) $Y = Y^{00}$ if and only if Y is closed, (absolutely convex) and edged.

Proof. Direct verification yields (i). For (ii) observe that $(E^*)' \cong E$ and that E^* is strongly polar. Now apply [4]. Theorem 4.7.

Remark. It is easy to see that each linear subspace of E is K-polar. If K is spherically complete even each edged subset of E is K-polar. However this conclusion is false in general.

LEMMA 2.2. *Let $X \subset E$ be absolutely convex. The following are equivalent.*

(α) X is absorbing.

(β) X^0 is a compactoid.

(γ) X^0 does not contain linear subspaces of E^* other than $\{0\}$.

Proof. A typical zero neighbourhood in E^* has the form F^0 where F is a finite subset of E. By (α) we have $\lambda X \supset F$ for some $\lambda \in K$. Then $X^0 \subset \lambda F^0$. It follows that X^0 is bounded hence a compactoid (for example from Proposition 1.1.(iv)). This proves $(\alpha) \Rightarrow (\beta)$. The implication $(\beta) \Rightarrow (\gamma)$ is easy. To prove $(\gamma) \Rightarrow (\alpha)$, let $f \in E^*$, $f([X]) = \{0\}$. Then $Kf \in X^0$ so that $f = 0$. Then, $[X] = E$ i.e. X is absorbing.

The next Proposition is the heart of this paper.

PROPOSITION 2.3. *Let A be a closed absolutely convex subset of E^*. Let D be the largest K-subspace of E^* that is contained in A. Then D is closed. There exists a closed absolutely convex compactoid $B \subset A$ such that $D \cap B = \{0\}$, $D + B = A$, and the canonical map $D \times B \to A$ is a homeomorphism.*

Proof.

(i) First assume that A is edged. Then $A = A^{00}$. Trivially, D is closed. D^0 has an (algebraic) complement F in E. Set

$$B := (F + A^0)^0$$

Then B is closed, edged. Since $F + A^0 \supset A^0$ we have $B \subset A^{00} = A$. Since also $F + A^0 \supset F$ we have $D \cap B \subset D \cap F^0 = D^{00} \cap F^0 = (D^0 + F)^0 = E^0 = \{0\}$. From this it follows, in turn, that B does not contain subspaces except $\{0\}$. By Lemma 2.2, B is a compactoid. Finally we prove that $A \simeq D \times B$. From $E = F \oplus D^0$ we obtain two standard projections $\pi_1 : E \to F, \pi_2 : E \to D^0$. For each $f \in E^*$ we have $f = f \circ \pi_1 + f \circ \pi_2$. If $f \in A$ then $f \circ \pi_1 \in D^{00}$, so that $f \circ \pi_1 \in A$. Also $f \circ \pi_2 \in F^0$. Then $f \circ \pi_2 \in A \cap F^0 = A^{00} \cap F^0 = (A^0 + F)^0 = B$. Then

$$f \mapsto (f \circ \pi_1, f \circ \pi_2) \qquad (f \in A)$$

maps A onto $D \times B$. It follows easily that it is, indeed, a homeomorphism.

(ii) To prove the general case we apply (i) to A^e. So $A^e = D \oplus C$ where D is a closed subspace and C is a closed compactoid, both contained in A^e. Then $D \subset A$ and $A = D \oplus B$ where $B := A \cap C$, a closed compactoid.

§3 CONCLUSIONS

THEOREM 3.1 (Compare [3], Corollary 6.5). *Let A be a complete local compactoid in a Hausdorff locally convex space E over K. Then, as a topological $B(0,1)$-module A is a direct sum $D \oplus B$ where D is the largest subspace contained in A and B is some complete compactoid in A.*

Proof. Immediate from Proposition 1.7 and 2.3.

COROLLARY 3.2. (Compare [3], Lemma 6.3). *Let A be a complete local compactoid in a Hausdorff locally convex space over K.*

(i) A does not contain subspaces other than $\{0\}$ then A is a compactoid.

(ii) If A is unbounded then A contains a linear space $\neq \{0\}$.

To prove Theorem 3.4 we need the following lemma.

LEMMA 3.3. *Let D be a linear subspace of a Hausdorff locally convex space E. Let U be an absolutely convex zero neighbourhood in E and let $D \subset U + Kx$ for some $x \in E$. Then $D \subset U + Ka$ for some $a \in D$.*

Proof. If $Kx \subset U$ we may take $a := 0$, so assume $Kx \not\subset U$ i.e. $p(x) \neq 0$ where p is the seminorm associated to U. For each $\lambda \in K, \lambda \neq 0$ we have

$$D = \lambda D \subset \lambda U + Kx$$

so that for $d \in D$ and $n \in \mathbb{N}$ we have a decomposition

$$d = u_n + \lambda_n x$$

where $p(u_n) \leq 1/n$ and $\lambda_n \in K$. Since also $p(x) \neq 0$ it follows easily that $\lambda := \lim_{n \to \inf} \lambda_n$ exists. Hence, $u := \lim_{n \to \inf} u_n$ exists and $p(u) = 0$. Thus, $d = u + \lambda x$ i.e.

$$D \subset Kerp + Kx$$

If $D \subset Kerp$ we may take again $a := 0$. If not then $x = a + v$ where $a \in D, v \in Kerp$. Then $Kv \in Kerp$ so that $D \subset Kerp + Kv + Ka \subset Kerp + Ka \subset U + Ka$.

THEOREM 3.4 (Katsaras' Theorem for local compactoids). *Let A be a complete local compactoid in a Hausdorff locally convex space E over K. Let $\lambda \in K, \lambda = 1$ if the valuation of K is discrete, $|\lambda| > 1$ otherwise. Then for each zero neighbourhood U in E there exists a finite dimensional space $F \subset A$ and finitely many points $x_1, ..., x_n \in \lambda A$ such that $A \subset U + F + co\{x_1, ..., x_n\}$.*

Proof. We may assume that U is absolutely convex. Let $A = D + B$ as in Theorem 3.1. By Katsaras' Theorem

$$B \subset U + co\{x_1, ..., x_n\}$$

for some $x_1, ..., x_n \in \lambda B \subset \lambda A$. By local compactoidity of D there exist $y_1, ..., y_m \in E$ such that

$$D \subset U + Ky_1 + ... + Ky_m$$

By repeated application of Lemma 3.3 we can arrange that $y_1, ..., y_m \in D$. The Theorem follows with $F := [y_1, ..., y_m]$.

COROLLARY 3.5. *Let A be a complete local compactoid in a Hausdorff locally convex space E over K. Then A is a local compactoid in $[A]$.*

The easy proof is left to the reader.

To see that everything goes wrong if we drop the completess condition consider the following. (Compare [3], Example 6.4.)

EXAMPLE 3.6. *There exists a (non-closed) local compactoid A in c_0 with the following properties.*

(i) A is unbounded.

(ii) A does not contain linear subspaces other than $\{0\}$.

(iii) A is not a local compactoid in $[A]$.

Proof. Let $p \in K, 0 < |p| < 1$. Define

$$x_1 = (p^{-1}, p, 0, 0, -)$$
$$x_2 = (p^{-2}, 0, p^2, 0, ...)$$
$$x_3 = (p^{-3}, 0, 0, p^3, 0...)$$

etc. and set $A := co\{x_1, x_2, ...\}$. Then (i),(ii) are clear.

Since

$$\overline{A} \subset Ke_1 + \overline{co}\{pe_2, p^2e_3, ...\}$$

(where $e_1, e_2, ...$ is the standard base of c_0), A is a local compactoid in c_0. To obtain (iii) we prove that there exists no finite dimensional set $F \subset [A]$ with $A \subset U + F$ where $U = \{x \in c_o : ||x|| \leq 1\}$. Suppose such F does exist. Then we may assume $F \subset A + U$, F absolutely convex. Suppose $Ka \subset F$ for some $a \neq 0$. Since U is bounded it is easy to see that then $Ka \subset \overline{A}$. But the only subspace $\neq \{0\}$ of $\overline{A}$ is Ke_1, so $a \in Ke_1$, which is impossible since $Ke_1 \cap [A] = \{0\}$. Hence, F contains no subspaces other than $\{0\}$ so F is bounded. But then $A \subset U + F$ would be bounded, a contradiction.

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