p-ADIC LOCAL COMPACTOIDS

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ABSTRACT. For a complete local compactoid \( A \) in a locally convex space \( E \) over a non-archimedean valued field \( K \) it is proved that \( A = D \oplus B \) where \( D \) is a subspace and \( B \) is a compactoid. As a corollary Katsaras' Theorem is extended to complete local compactoids.

TERMINOLOGY. Throughout \( K \) is a non-archimedean valued field that is complete with respect to the non-trivial valuation \( | \cdot | \). A subset \( A \) of a \( K \)-vector space \( E \) is absolutely convex if it is a module over the ring \( B(0,1) := \{ \lambda \in K : |\lambda| \leq 1 \} \). For a subset \( X \) of \( E \) we denote by \( [X] \) the \( K \)-vector space generated by \( X \), by \( \overline{co}X \) the smallest absolutely convex subset of \( E \) containing \( X \). For an absolutely convex set \( A \subset E \) we set \( A^o := A \) if the valuation of \( K \) is discrete and \( A^e := \bigcap \{ \lambda A : \lambda \in K, |\lambda| > 1 \} \) if the valuation of \( K \) is dense. \( A \) is edged if \( A = A^e \).

The \( K \)-Banach space consisting of all sequences \( (\xi_1, \xi_2, \ldots) \) in \( K \) with \( \lim_{n \to \infty} \xi_n = 0 \) and with the norm \( (\xi_1, \xi_2, \ldots) \mapsto \max_n |\xi_n| \) is denoted \( c_0 \).

Let \( E \) be a locally convex space over \( K \). The closure of a set \( X \subset E \) is denoted \( \overline{X} \). Instead of \( \overline{co}X \) we write \( \overline{\overline{X}} \). For each continuous seminorm \( p \) on \( E \), let \( E_p \) be the space \( E / Ker p \) with the norm induced by \( p \), let \( E_p^\wedge \) be its completion. The maps

\[ \pi_p : E \to E_p \to E_p^\wedge \]

induce a map

\[ E \to \prod_p E_p^\wedge \]

which is, if \( E \) is Hausdorff, a linear homeomorphism onto a subspace of the product. An absolutely convex subset \( A \) of \( E \) is a compactoid if for each zero neighbourhood \( U \) in \( E \) there exists a finite set \( F \subset E \) such that \( A \subset U + \overline{co}F \). \( A \) is a local compactoid in \( E \) if for each zero neighbourhood \( U \) in \( E \) there exists a finite dimensional space \( D \subset E \) with \( A \subset U + D \).

For terms that are unexplained here we refer to [4].

INTRODUCTION. We quote the following theorem, first proved by Katsaras.
THEOREM ([2],[1]). Let $A$ be a compactoid in a locally convex space $E$ over $K$. Let $\lambda \in K, \lambda = 1$ if the valuation of $K$ is discrete, $|\lambda| > 1$ otherwise. Then, for each neighbourhood $U$ of 0 in $E$ there exists a finite set $F$ in $\lambda A$ such that $A \subseteq U + coF$.

The theorem implies that compactoidity of $A$ is a property of the topological $B(0,1)$-module $A$ and does not depend on the embedding space $E$.

Surprisingly, Katsaras' Theorem does not extend to local compactoids in general (Example 3.6); we shall prove such a theorem only for complete local compactoids (Theorem 3.4).

Remarks

1 Let $K$ be spherically (= maximally) complete. Then completeness & local compactoidity is equivalent to c-compactness ([5],Theorem 11). By using this fact and well-known properties of c-compact sets one may derive the results of this paper in a much easier way.

2 Because of the previous remark our proofs, although valid for any $K$, are only of importance if $K$ is not spherically complete.

§1 LOCAL COMPACTOIDS

Throughout §1 $E$ is a Hausdorff locally convex space over $K$. The proofs of the next two Propositions are left to the reader.

PROPOSITION 1.1. Let $A$ be an absolutely convex subset of $E$.

(i) If $A$ is a local compactoid in $E$ and $B \subseteq A$ is absolutely convex then $B$ is a local compactoid in $E$.

(ii) If $A$ is a local compactoid in $E$ then so is $\overline{A}$.

(iii) If $F$ is a Hausdorff locally convex space over $K$, if $T : E \to F$ is a continuous linear map and if $A$ is a local compactoid in $E$ then $TA$ is a local compactoid in $F$.

(iv) $A$ is a compactoid (in $E$) if and only if $A$ is a bounded local compactoid in $E$.

PROPOSITION 1.2. Let $(E_i)_{i \in I}$ be a family of Hausdorff locally convex spaces over $K$. If, for each $i$, $A_i$ is a local compactoid in $E_i$ then $\prod_i A_i$ is a local compactoid in $\prod_i E_i$.

PROPOSITION 1.3. Let $A$ be a closed local compactoid in a $K$-Banach space $E$. Then $\overline{[A]}$ is of countable type and $A$ is a local compactoid in $\overline{[A]}$.

Proof. [3], 6.9 and Theorem 6.7.
LEMMA 1.4. Let $A$ be a local compactoid in $E$. Then there exists a Hausdorff locally convex space $E_1$ of countable type and a linear homeomorphism of $[A]$ into $E_1$ such that $i(A)$ is a local compactoid in $E_1$.

Proof. For each continuous seminorm $p$ the set $\overline{\pi_p(A)}$ is a local compactoid in $E_p^\wedge$ (Proposition 1.1), hence in a subspace $D_p$ of countable type (Proposition 1.3). By [4], Proposition 4.12 (iii), $E_1 := \prod D_p$ is of countable type. The restriction of the embedding $E \hookrightarrow \prod E_p^\wedge$ yields a linear homeomorphic embedding $i : [A] \hookrightarrow E_1$. Now $i(A)$ is a subset of $\prod \overline{\pi_p(A)}$, which is a local compactoid in $E_1$ (Proposition 1.2). Then, $i(A)$ is a local compactoid in $E_1$.

COROLLARY 1.5. If $A$ is a local compactoid in $E$ then $[A]$ is of countable type.

Proof. $[A]$ is linearly homeomorphic to a subspace of $E_1$. Now apply [4], Proposition 4.12 (i).

PROPOSITION 1.6. Let $E$ be a polar space and let $A$ be a local compactoid in $E$. Then, on $A$, the weak topology $\sigma(E, E')$ and the initial topology coincide. $A$ is complete if and only if $A$ is weakly complete.

Proof. The proofs of [4], 5.7-5.11 can easily be modified in such a way that the conclusion of [4], Theorem 5.12 holds for local compactoids, rather than just compactoids.

PROPOSITION 1.7. Let $A$ be a local compactoid in $E$. Then, as a topological $B(0,1)$-module, $A$ is isomorphic to a $B(0,1)$-submodule of some power of $K$.

Proof. By Lemma 1.4 we may suppose that $E$ is of countable type, hence polar. So, by Proposition 1.6, $A$ is a topological $B(0,1)$-submodule of $(E, \sigma(E, E'))$. The map

$$z \mapsto (f(z))_{f \in E'} \quad (z \in E)$$

is a linear homeomorphism of $(E, \sigma(E, E'))$ into $K^{E'}$. The statements follows.

§2 LOCAL COMPACTOIDS IN $K^I$.

Throughout §2, $E$ is a vector space over $K$ (no topology) and $E'$ its algebraic dual, with the topology $\sigma(E', E)$ of pointwise convergence. Then $E'$ is Hausdorff, locally convex, complete and of countable
type. Every absolutely convex subset of $E^*$ is a local compactoid in $E$ as each neighbourhood of 0 in $E^*$ contains a subspace with finite codimension. It is not hard to see that each $\Theta \in (E^*)'$ has the form $f \mapsto f(x)$ $(f \in E^*)$ for some $x \in E$, so that we may identify $(E^*)'$ and $E$.

To see the connection with the title of §2 observe that $E$ is the (algebraic) direct sum $\bigoplus K_i$, where $K_i = K$ for each $i$ and that $E^*$ is linearly homeomorphic to $K^I$.

A subset $X$ of $E$ is $K$-polar if for each $y \in E \setminus X$ there exists an $f \in E^*$ with $|f(X)| \leq 1$, $|f(y)| > 1$.

For $X \subset E$, $Y \subset E^*$ we set, as usual

$$X^0 := \{ f \in E^* : |f(X)| \leq 1 \}$$

$$Y^0 := \{ x \in E : |Y(x)| \leq 1 \}.$$

**Proposition 2.1.** Let $X \subset E$, $Y \subset E^*$.

(i) $X$ is $K$-polar if and only if $X = X^{00}$.

(ii) $Y = Y^{00}$ if and only if $Y$ is closed, (absolutely convex) and edged.

**Proof.** Direct verification yields (i). For (ii) observe that $(E^*)' \cong E$ and that $E^*$ is strongly polar. Now apply [4], Theorem 4.7.

**Remark.** It is easy to see that each linear subspace of $E$ is $K$-polar. If $K$ is spherically complete even each edged subset of $E$ is $K$-polar. However this conclusion is false in general.

**Lemma 2.2.** Let $X \subset E$ be absolutely convex. The following are equivalent.

(a) $X$ is absorbing.

(b) $X^0$ is a compactoid.

(γ) $X^0$ does not contain linear subspaces of $E^*$ other than {0}.

**Proof.** A typical zero neighbourhood in $E^*$ has the form $F^0$ where $F$ is a finite subset of $E$. By (α) we have $\lambda X \supset F$ for some $\lambda \in K$. Then $X^0 \subset \lambda F^0$. It follows that $X^0$ is bounded hence a compactoid (for example from Proposition 1.1(iv)). This proves $(\alpha) \Rightarrow (\beta)$. The implication $(\beta) \Rightarrow (\gamma)$ is easy. To prove $(\gamma) \Rightarrow (\alpha)$, let $f \in E^*$, $f([X]) = \{0\}$. Then $Kf \in X^0$ so that $f = 0$. Then, $[X] = E$ i.e. $X$ is absorbing.

The next Proposition is the heart of this paper.
PROPOSITION 2.3. Let $A$ be a closed absolutely convex subset of $E^*$. Let $D$ be the largest $K$-subspace of $E^*$ that is contained in $A$. Then $D$ is closed. There exists a closed absolutely convex compactoid $B \subset A$ such that $D \cap B = \{0\}$, $D + B = A$, and the canonical map $D \times B \to A$ is a homeomorphism.

Proof.

(i) First assume that $A$ is edged. Then $A = A_0^0$. Trivially, $D$ is closed. $D^0$ has an (algebraic) complement $F$ in $E$. Set

$$B := (F + A^0)^0$$

Then $B$ is closed, edged. Since $F + A^0 \supset A^0$ we have $B \subset A_0^0 = A$. Since also $F + A^0 \supset F$ we have $D \cap B \subset D \cap F^0 = D_0^0 \cap F^0 = (D^0 + F)^0 = E^0 = \{0\}$. From this it follows, in turn, that $B$ does not contain subspaces except $\{0\}$. By Lemma 2.2, $B$ is a compactoid. Finally we prove that $A \simeq D \times B$.

From $E = F \oplus D^0$ we obtain two standard projections $\pi_1 : E \to F, \pi_2 : E \to D^0$. For each $f \in E^*$ we have $f = f \circ \pi_1 + f \circ \pi_2$. If $f \in A$ then $f \circ \pi_1 \in D_0^0$, so that $f \circ \pi_1 \in A$. Also $f \circ \pi_2 \in F^0$. Then $f \circ \pi_2 \in A \cap F^0 = A_0^0 \cap F^0 = (A^0 + F)^0 = B$. Then

$$f \mapsto (f \circ \pi_1, f \circ \pi_2) \quad (f \in A)$$

maps $A$ onto $D \times B$. It follows easily that it is, indeed, a homeomorphism.

(ii) To prove the general case we apply (i) to $A^\circ$. So $A^\circ = D \oplus C$ where $D$ is a closed subspace and $C$ is a closed compactoid, both contained in $A^\circ$. Then $D \subset A$ and $A = D \oplus B$ where $B := A \cap C$, a closed compactoid.

§3 CONCLUSIONS

THEOREM 3.1 (Compare [3], Corollary 6.5). Let $A$ be a complete local compactoid in a Hausdorff locally convex space $E$ over $K$. Then, as a topological $B(0,1)$-module $A$ is a direct sum $D \oplus B$ where $D$ is the largest subspace contained in $A$ and $B$ is some complete compactoid in $A$.

Proof. Immediate from Proposition 1.7 and 2.3.

COROLLARY 3.2. (Compare [3], Lemma 6.3). Let $A$ be a complete local compactoid in a Hausdorff locally convex space over $K$.
(i) A does not contain subspaces other than \{0\} then A is a compactoid.

(ii) If A is unbounded then A contains a linear space \( \neq \{0\} \).

To prove Theorem 3.4 we need the following lemma.

**Lemma 3.3.** Let D be a linear subspace of a Hausdorff locally convex space E. Let U be an absolutely convex zero neighbourhood in E and let \( D \subset U + K z \) for some \( z \in E \). Then \( D \subset U + K a \) for some \( a \in D \).

**Proof.** If \( K z \subset U \) we may take \( a := 0 \), so assume \( K z \not\subset U \) i.e. \( p(z) \neq 0 \) where \( p \) is the seminorm associated to \( U \). For each \( \lambda \in K, \lambda \neq 0 \) we have

\[
D = \lambda D \subset \lambda U + K z
\]

so that for \( d \in D \) and \( n \in \mathbb{N} \) we have a decomposition

\[
d = u_n + \lambda_n z
\]

where \( p(u_n) \leq 1/n \) and \( \lambda_n \in K \). Since also \( p(z) \neq 0 \) it follows easily that \( \lambda := \lim_{n \to \infty} \lambda_n \) exists. Hence, \( u := \lim_{n \to \infty} u_n \) exists and \( p(u) = 0 \). Thus, \( d = u + \lambda z \) i.e.

\[
D \subset K e + K z
\]

If \( D \subset K e \) we may take again \( a := 0 \). If not then \( z = a + v \) where \( a \in D, v \in K e \). Then \( K v \in K e \) so that \( D \subset K e + K v + K a \subset K e + K a \subset U + K a \).

**Theorem 3.4 (Katsaras' Theorem for local compactoids).** Let A be a complete local compactoid in a Hausdorff locally convex space E over K. Let \( \lambda \in K, \lambda = 1 \) if the valuation of K is discrete, \( |\lambda| > 1 \) otherwise. Then for each zero neighbourhood U in E there exists a finite dimensional space \( F \subset A \) and finitely many points \( x_1, ..., x_n \in \lambda A \) such that \( A \subset U + F + \operatorname{co}\{x_1, ..., x_n\} \).

**Proof.** We may assume that U is absolutely convex. Let \( A = D + B \) as in Theorem 3.1. By Katsaras' Theorem

\[
B \subset U + \operatorname{co}\{x_1, ..., x_n\}
\]
for some $x_1, ..., x_n \in \lambda B \subseteq \lambda A$. By local compactoidity of $D$ there exist $y_1, ..., y_m \in E$ such that

$$D \subseteq U + Ky_1 + ... + Ky_m$$

By repeated application of Lemma 3.3 we can arrange that $y_1, ..., y_m \in D$. The Theorem follows with $F := [y_1, ..., y_m]$.

**COROLLARY 3.5.** Let $A$ be a complete local compactoid in a Hausdorff locally convex space $E$ over $K$. Then $A$ is a local compactoid in $[A]$.

The easy proof is left to the reader.

To see that everything goes wrong if we drop the completeness condition consider the following. (Compare [3], Example 6.4.)

**EXAMPLE 3.6.** There exists a (non-closed) local compactoid $A$ in $c_0$ with the following properties.

(i) $A$ is unbounded.

(ii) $A$ does not contain linear subspaces other than $\{0\}$.

(iii) $A$ is not a local compactoid in $[A]$.

**Proof.** Let $p \in K, 0 < |p| < 1$. Define

$$z_1 = (p^{-1}, p, 0, 0, ...)$$

$$z_2 = (p^{-2}, 0, p^3, 0, ...)$$

$$z_3 = (p^{-3}, 0, 0, p^3, 0, ...)$$

etc. and set $A := \mathrm{co}\{z_1, z_2, \ldots\}$. Then (i),(ii) are clear.

Since

$$\overline{A} \subseteq K e_1 + \overline{\mathrm{co}\{p e_2, p^2 e_3, \ldots\}}$$

(where $e_1, e_2, \ldots$ is the standard base of $c_0$), $A$ is a local compactoid in $c_0$. To obtain (iii) we prove that there exists no finite dimensional set $F \subseteq [A]$ with $A \subseteq U + F$ where $U = \{x \in c_0 : ||x|| \leq 1\}$. Suppose such $F$ does exist. Then we may assume $F \subseteq A + U$, $F$ absolutely convex. Suppose $Ka \subseteq F$ for some $a \neq 0$. Since $U$ is bounded it is easy to see that then $Ka \subseteq \overline{A}$. But the only subspace $\neq \{0\}$ of $\overline{A}$ is $Ke_1$, so $a \in Ke_1$, which is impossible since $Ke_1 \cap [A] = \{0\}$. Hence, $F$ contains no subspaces other than $\{0\}$ so $F$ is bounded. But then $A \subseteq U + F$ would be bounded, a contradiction.
REFERENCES


