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p-ADIC LOCAL COMPACTOIDS

by

W.H. Schikhof

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DEPARTMENT OF MATHEMATICS
CATHOLIC UNIVERSITY
Toernooiveld
6525 ED Nijmegen
The Netherlands
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W.H. Schikhof

ABSTRACT. For a complete local compactoid $A$ in a locally convex space $E$ over a non-archimedean valued field $K$ it is proved that $A = D \oplus B$ where $D$ is a subspace and $B$ is a compactoid. As a corollary Katsaras' Theorem is extended to complete local compactoids.

TERMINOLOGY. Throughout $K$ is a non-archimedean valued field that is complete with respect to the non-trivial valuation $|\cdot|$. A subset $A$ of a $K$-vector space $E$ is absolutely convex if it is a module over the ring $B(0,1) := \{ \lambda \in K : |\lambda| \leq 1 \}$. For a subset $X$ of $E$ we denote by $[X]$ the $K$-vector space generated by $X$, by $\text{co}X$ the smallest absolutely convex subset of $E$ containing $X$. For an absolutely convex set $A \subset E$ we set $A^\circ := A$ if the valuation of $K$ is discrete and $A^\circ := \bigcap \{ \lambda A : \lambda \in K, |\lambda| > 1 \}$ if the valuation of $K$ is dense. $A$ is edged if $A = A^\circ$.

The $K$-Banach space consisting of all sequences $(\xi_1, \xi_2, \ldots)$ in $K$ with $\lim_{n \to \infty} \xi_n = 0$ and with the norm $(\xi_1, \xi_2, \ldots) \mapsto \max_n |\xi_n|$ is denoted $c_0$.

Let $E$ be a locally convex space over $K$. The closure of a set $X \subset E$ is denoted $\overline{X}$. Instead of $\text{co}X$ we write $\text{co}X$. For each continuous seminorm $p$ on $E$, let $E_p$ be the space $E/\text{Ker}p$ with the norm induced by $p$, let $E^\wedge_p$ be its completion. The maps

$$\pi_p : E \to E_p \to E^\wedge_p$$

induce a map

$$E \to \prod_p E^\wedge_p$$

which is, if $E$ is Hausdorff, a linear homeomorphism onto a subspace of the product. An absolutely convex subset $A$ of $E$ is a compactoid if for each zero neighbourhood $U$ in $E$ there exists a finite set $F \subset E$ such that $A \subset U + \text{co}F$. $A$ is a local compactoid in $E$ if for each zero neighbourhood $U$ in $E$ there exists a finite dimensional space $D \subset E$ with $A \subset U + D$.

For terms that are unexplained here we refer to [4].

INTRODUCTION. We quote the following theorem, first proved by Katsaras.
THEOREM ([2],[1]). Let \( A \) be a compactoid in a locally convex space \( E \) over \( K \). Let \( \lambda \in K, \lambda = 1 \) if the valuation of \( K \) is discrete, \( |\lambda| > 1 \) otherwise. Then, for each neighbourhood \( U \) of 0 in \( E \) there exists a finite set \( F \) in \( \lambda A \) such that \( A \subset U + coF \).

The theorem implies that compactoidity of \( A \) is a property of the topological \( B(0,1) \)-module \( A \) and does not depend on the embedding space \( E \).

Surprisingly, Katsaras' Theorem does not extend to local compactoids in general (Example 3.6); we shall prove such a theorem only for complete local compactoids (Theorem 3.4).

Remarks

1. Let \( K \) be spherically (= maximally) complete. Then completeness & local compactoidity is equivalent to \( c \)-compactness ([5], Theorem 11). By using this fact and well-known properties of \( c \)-compact sets one may derive the results of this paper in a much easier way.

2. Because of the previous remark our proofs, although valid for any \( K \), are only of importance if \( K \) is not spherically complete.

§1 LOCAL COMPACTOIDS

Throughout §1 \( E \) is a Hausdorff locally convex space over \( K \). The proofs of the next two Propositions are left to the reader.

PROPOSITION 1.1. Let \( A \) be an absolutely convex subset of \( E \).

(i) If \( A \) is a local compactoid in \( E \) and \( B \subset A \) is absolutely convex then \( B \) is a local compactoid in \( E \).

(ii) If \( A \) is a local compactoid in \( E \) then so is \( \overline{A} \).

(iii) If \( F \) is a Hausdorff locally convex space over \( K \), if \( T : E \to F \) is a continuous linear map and if \( A \) is a local compactoid in \( E \) then \( TA \) is a local compactoid in \( F \).

(iv) \( A \) is a compactoid (in \( E \)) if and only if \( A \) is a bounded local compactoid in \( E \).

PROPOSITION 1.2. Let \( (E_i)_{i \in I} \) be a family of Hausdorff locally convex spaces over \( K \). If, for each \( i \), \( A_i \) is a local compactoid in \( E_i \) then \( \prod_i A_i \) is a local compactoid in \( \prod_i E_i \).

PROPOSITION 1.3. Let \( A \) be a closed local compactoid in a \( K \)-Banach space \( E \). Then \( \overline{A} \) is of countable type and \( A \) is a local compactoid in \( \overline{A} \).

Proof. [3], 6.9 and Theorem 6.7.
**Lemma 1.4.** Let $A$ be a local compactoid in $E$. Then there exists a Hausdorff locally convex space $E_1$ of countable type and a linear homeomorphism of $[A]$ into $E_1$ such that $i(A)$ is a local compactoid in $E_1$.

**Proof.** For each continuous seminorm $p$ the set $\pi_p(A)$ is a local compactoid in $E^\wedge_p$ (Proposition 1.1), hence in a subspace $D_p$ of countable type (Proposition 1.3). By [4], Proposition 4.12 (iii), $E_1 := \prod D_p$ is of countable type. The restriction of the embedding $E \leftarrow \prod E^\wedge_p$ yields a linear homeomorphic embedding $i : [A] \hookrightarrow E_1$. Now $i(A)$ is a subset of $\prod \pi_p(A)$, which is a local compactoid in $E_1$ (Proposition 1.2). Then, $i(A)$ is a local compactoid in $E_1$.

**Corollary 1.5.** If $A$ is a local compactoid in $E$ then $[A]$ is of countable type.

**Proof.** $[A]$ is linearly homeomorphic to a subspace of $E_1$. Now apply [4], Proposition 4.12 (i).

**Proposition 1.6.** Let $E$ be a polar space and let $A$ be a local compactoid in $E$. Then, on $A$, the weak topology $\sigma(E, E')$ and the initial topology coincide. $A$ is complete if and only if $A$ is weakly complete.

**Proof.** The proofs of [4], 5.7-5.11 can easily be modified in such a way that the conclusion of [4], Theorem 5.12 holds for local compactoids, rather than just compactoids.

**Proposition 1.7.** Let $A$ be a local compactoid in $E$. Then, as a topological $B(0,1)$-module, $A$ is isomorphic to a $B(0,1)$-submodule of some power of $K$.

**Proof.** By Lemma 1.4 we may suppose that $E$ is of countable type, hence polar. So, by Proposition 1.6, $A$ is a topological $B(0,1)$-submodule of $(E, \sigma(E, E'))$. The map

$$z \mapsto (f(z))_{f \in E'} \quad (z \in E)$$

is a linear homeomorphism of $(E, \sigma(E, E'))$ into $K^{E'}$. The statements follows.

**§2 Local Compactoids in $K^I$.**

Throughout §2, $E$ is a vector space over $K$ (no topology) and $E^*$ its algebraic dual, with the topology $\sigma(E^*, E)$ of pointwise convergence. Then $E^*$ is Hausdorff, locally convex, complete and of countable
Every absolutely convex subset of $E^*$ is a local compactoid in $E$ as each neighbourhood of 0 in $E^*$ contains a subspace with finite codimension. It is not hard to see that each $\Theta \in (E^*)'$ has the form $f \mapsto f(x)$ ($f \in E^*$) for some $x \in E$, so that we may identify $(E^*)'$ and $E$.

To see the connection with the title of §2 observe that $E$ is the (algebraic) direct sum $\bigoplus_{i \in I} K_i$, where $K_i = K$ for each $i$ and that $E^*$ is linearly homeomorphic to $K^I$.

A subset $X$ of $E$ is $K$-polar if for each $y \in E \setminus X$ there exists an $f \in E^*$ with $|f(X)| < 1$, $|f(y)| > 1$.

For $X \subset E$, $Y \subset E^*$ we set, as usual

$$X^0 := \{f \in E^* : |f(X)| \leq 1\}$$

$$Y^0 := \{x \in E : |Y(x)| \leq 1\}.$$
PROPOSITION 2.3. Let $A$ be a closed absolutely convex subset of $E^*$. Let $D$ be the largest $K$-subspace of $E^*$ that is contained in $A$. Then $D$ is closed. There exists a closed absolutely convex compactoid $B \subset A$ such that $D \cap B = \{0\}$, $D + B = A$, and the canonical map $D \times B \to A$ is a homeomorphism.

Proof.

(i) First assume that $A$ is edged. Then $A = A^{00}$. Trivially, $D$ is closed. $D^0$ has an (algebraic) complement $F$ in $E$. Set

$$B := (F + A^0)^0$$

Then $B$ is closed, edged. Since $F + A^0 \supset A^0$ we have $B \subset A^{00} = A$. Since also $F + A^0 \supset F$ we have $D \cap B \subset D \cap F^0 = D^{00} \cap F^0 = (D^0 + F)^0 = E^0 = \{0\}$. From this it follows, in turn, that $B$ does not contain subspaces except $\{0\}$. By Lemma 2.2, $B$ is a compactoid. Finally we prove that $A \simeq D \times B$.

From $E = F \oplus D^0$ we obtain two standard projections $\pi_1 : E \to F, \pi_2 : E \to D^0$. For each $f \in E^*$ we have $f = f \circ \pi_1 + f \circ \pi_2$. If $f \in A$ then $f \circ \pi_1 \in D^{00}$, so that $f \circ \pi_1 \in A$. Also $f \circ \pi_2 \in F^0$. Then

$$f \circ \pi_2 \in A \cap F^0 = A^{00} \cap F^0 = (A^0 + F)^0 = B.$$  

maps $A$ onto $D \times B$. It follows easily that it is, indeed, a homeomorphism.

(ii) To prove the general case we apply (i) to $A^e$. So $A^e = D \oplus C$ where $D$ is a closed subspace and $C$ is a closed compactoid, both contained in $A^e$. Then $D \subset A$ and $A = D \oplus B$ where $B := A \cap C$, a closed compactoid.

§3 CONCLUSIONS

THEOREM 3.1 (Compare [3], Corollary 6.5). Let $A$ be a complete local compactoid in a Hausdorff locally convex space $E$ over $K$. Then, as a topological $B(0,1)$-module $A$ is a direct sum $D \oplus B$ where $D$ is the largest subspace contained in $A$ and $B$ is some complete compactoid in $A$.

Proof. Immediate from Proposition 1.7 and 2.3.

COROLLARY 3.2. (Compare [3], Lemma 6.3). Let $A$ be a complete local compactoid in a Hausdorff locally convex space over $K$. 

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(i) A does not contain subspaces other than \{0\} then A is a compactoid.

(ii) If \( A \) is unbounded then \( A \) contains a linear space \( \neq \{0\} \).

To prove Theorem 3.4 we need the following lemma.

**Lemma 3.3.** Let \( D \) be a linear subspace of a Hausdorff locally convex space \( E \). Let \( U \) be an absolutely convex zero neighbourhood in \( E \) and let \( D \subset U + Kx \) for some \( x \in E \). Then \( D \subset U + Ka \) for some \( a \in D \).

**Proof.** If \( Kx \subset U \) we may take \( a := 0 \), so assume \( Kx \not\subset U \) i.e. \( p(x) \neq 0 \) where \( p \) is the seminorm associated to \( U \). For each \( \lambda \in K, \lambda \neq 0 \) we have

\[
D = \lambda D \subset \lambda U + Kx
\]

so that for \( d \in D \) and \( n \in \mathbb{N} \) we have a decomposition

\[
d = u_n + \lambda_n x
\]

where \( p(u_n) \leq 1/n \) and \( \lambda_n \in K \). Since also \( p(x) \neq 0 \) it follows easily that \( \lambda := \lim_{n \to \infty} \lambda_n \) exists. Hence, \( u := \lim_{n \to \infty} u_n \) exists and \( p(u) = 0 \). Thus, \( d = u + \lambda x \) i.e.

\[
D \subset \text{Ker}p + Kx
\]

If \( D \subset \text{Ker}p \) we may take again \( a := 0 \). If not then \( x = a + v \) where \( a \in D, v \in \text{Ker}p \). Then \( Kv \in \text{Ker}p \) so that \( D \subset \text{Ker}p + Kv + Ka \subset U + Ka \).

**Theorem 3.4 (Katsaras' Theorem for local compactoids).** Let \( A \) be a complete local compactoid in a Hausdorff locally convex space \( E \) over \( K \). Let \( \lambda \in K, \lambda = 1 \) if the valuation of \( K \) is discrete, \( |\lambda| > 1 \) otherwise. Then for each zero neighbourhood \( U \) in \( E \) there exists a finite dimensional space \( F \subset A \) and finitely many points \( x_1, ..., x_n \in \lambda A \) such that \( A \subset U + F + \text{co}\{x_1, ..., x_n\} \).

**Proof.** We may assume that \( U \) is absolutely convex. Let \( A = D + B \) as in Theorem 3.1. By Katsaras’ Theorem

\[
B \subset U + \text{co}\{x_1, ..., x_n\}
\]
for some \(z_1, ..., z_n \in \lambda B \subseteq \lambda A\). By local compactoidity of \(D\) there exist \(y_1, ..., y_m \in E\) such that

\[D \subseteq U + Ky_1 + ... + Ky_m\]

By repeated application of Lemma 3.3 we can arrange that \(y_1, ..., y_m \in D\). The Theorem follows with \(F := [y_1, ..., y_m]\).

**COROLLARY 3.5.** Let \(A\) be a complete local compactoid in a Hausdorff locally convex space \(E\) over \(K\). Then \(A\) is a local compactoid in \([A]\).

The easy proof is left to the reader.

To see that everything goes wrong if we drop the completeness condition consider the following. (Compare [3], Example 6.4.)

**EXAMPLE 3.6.** There exists a (non-closed) local compactoid \(A\) in \(c_0\) with the following properties.

(i) \(A\) is unbounded.

(ii) \(A\) does not contain linear subspaces other than \(\{0\}\).

(iii) \(A\) is not a local compactoid in \([A]\).

**Proof.** Let \(p \in K, 0 < |p| < 1\). Define

\[z_1 = (p^{-1}, p, 0, 0, ...), \quad z_2 = (p^{-2}, 0, p^2, 0, ...), \quad z_3 = (p^{-3}, 0, 0, p^3, 0, ...), \text{ etc.} \]

and set \(A := co\{z_1, z_2, ...\}\). Then (i),(ii) are clear.

Since

\[\tilde{A} \subseteq K e_1 + \overline{co}\{pe_2, p^2 e_3, ...\}\]

(where \(e_1, e_2, ...\) is the standard base of \(c_0\)), \(A\) is a local compactoid in \(c_0\). To obtain (iii) we prove that there exists no finite dimensional set \(F \subseteq [A]\) with \(A \subseteq U + F\) where \(U = \{x \in c_0 : |x| \leq 1\}\). Suppose such \(F\) does exist. Then we may assume \(F \subseteq A + U, F\) absolutely convex. Suppose \(K a \subseteq F\) for some \(a \neq 0\). Since \(U\) is bounded it is easy to see that then \(K a \subseteq |A|\). But the only subspace \(\neq \{0\}\) of \(\tilde{A}\) is \(K e_1\), so \(a \in K e_1\), which is impossible since \(K e_1 \cap [A] = \{0\}\). Hence, \(F\) contains no subspaces other than \(\{0\}\) so \(F\) is bounded. But then \(A \subseteq U + F\) would be bounded, a contradiction.
REFERENCES


