p-ADIC LOCAL COMPACTOIDS

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ABSTRACT. For a complete local compactoid $A$ in a locally convex space $E$ over a non-archimedean valued field $K$ it is proved that $A = D \oplus B$ where $D$ is a subspace and $B$ is a compactoid. As a corollary Katsaras' Theorem is extended to complete local compactoids.

TERMINOLOGY. Throughout $K$ is a non-archimedean valued field that is complete with respect to the non-trivial valuation $|\cdot|$. A subset $A$ of a $K$-vector space $E$ is absolutely convex if it is a module over the ring $B(0,1) := \{ \lambda \in K : |\lambda| \leq 1 \}$. For a subset $X$ of $E$ we denote by $[X]$ the $K$-vector space generated by $X$, by $\text{co}X$ the smallest absolutely convex subset of $E$ containing $X$. For an absolutely convex set $A \subset E$ we set $A^\circ := A$ if the valuation of $K$ is discrete and $A^\circ := \bigcap \{ \lambda A : \lambda \in K, |\lambda| > 1 \}$ if the valuation of $K$ is dense. $A$ is edged if $A = A^\circ$.

The $K$-Banach space consisting of all sequences $(\xi_1, \xi_2, \ldots)$ in $K$ with $\lim_{n \to \infty} \xi_n = 0$ and with the norm $(\xi_1, \xi_2, \ldots) \mapsto \max_n |\xi_n|$ is denoted $c_0$.

Let $E$ be a locally convex space over $K$. The closure of a set $X \subset E$ is denoted $\overline{X}$. Instead of $\text{co}X$ we write $\overline{\text{co}}X$. For each continuous seminorm $p$ on $E$, let $E_p$ be the space $E / \text{Ker} p$ with the norm induced by $p$, let $E_p^\wedge$ be its completion. The maps

$$\pi_p : E \to E_p \to E_p^\wedge$$

induce a map

$$E \to \prod_p E_p^\wedge$$

which is, if $E$ is Hausdorff, a linear homeomorphism onto a subspace of the product. An absolutely convex subset $A$ of $E$ is a compactoid if for each zero neighbourhood $U$ in $E$ there exists a finite set $F \subset E$ such that $A \subset U + \text{co}F$. $A$ is a local compactoid in $E$ if for each zero neighbourhood $U$ in $E$ there exists a finite dimensional space $D \subset E$ with $A \subset U + D$.

For terms that are unexplained here we refer to [4].

INTRODUCTION. We quote the following theorem, first proved by Katsaras.
THEOREM ([2],[1]). Let $A$ be a compactoid in a locally convex space $E$ over $K$. Let $\lambda \in K, \lambda = 1$ if the valuation of $K$ is discrete, $|\lambda| > 1$ otherwise. Then, for each neighbourhood $U$ of $0$ in $E$ there exists a finite set $F$ in $\lambda A$ such that $A \subset U + cOF$.

The theorem implies that compactoidity of $A$ is a property of the topological $B(0,1)$-module $A$ and does not depend on the embedding space $E$.

Surprisingly, Katsaras' Theorem does not extend to local compactoids in general (Example 3.6); we shall prove such a theorem only for complete local compactoids (Theorem 3.4).

Remarks

1. Let $K$ be spherically (= maximally) complete. Then completeness & local compactoidity is equivalent to $c$-compactness ([5],Theorem 11). By using this fact and well-known properties of $c$-compact sets one may derive the results of this paper in a much easier way.

2. Because of the previous remark our proofs, although valid for any $K$, are only of importance if $K$ is not spherically complete.

§1 LOCAL COMPACTOIDS

Throughout §1 $E$ is a Hausdorff locally convex space over $K$. The proofs of the next two Propositions are left to the reader.

PROPOSITION 1.1. Let $A$ be an absolutely convex subset of $E$.

(i) If $A$ is a local compactoid in $E$ and $B \subset A$ is absolutely convex then $B$ is a local compactoid in $E$.

(ii) If $A$ is a local compactoid in $E$ then so is $\overline{A}$.

(iii) If $F$ is a Hausdorff locally convex space over $K$, if $T : E \rightarrow F$ is a continuous linear map and if $A$ is a local compactoid in $E$ then $TA$ is a local compactoid in $F$.

(iv) $A$ is a compactoid (in $E$) if and only if $A$ is a bounded local compactoid in $E$.

PROPOSITION 1.2. Let $(E_i)_{i \in I}$ be a family of Hausdorff locally convex spaces over $K$. If, for each $i$, $A_i$ is a local compactoid in $E_i$ then $\prod_i A_i$ is a local compactoid in $\prod_i E_i$.

PROPOSITION 1.3. Let $A$ be a closed local compactoid in a $K$-Banach space $E$. Then $\overline{A}$ is of countable type and $A$ is a local compactoid in $\overline{A}$.

Proof. [3], 6.9 and Theorem 6.7.
LEMMA 1.4. Let $A$ be a local compactoid in $E$. Then there exists a Hausdorff locally convex space $E_1$ of countable type and a linear homeomorphism of $[A]$ into $E_1$ such that $i(A)$ is a local compactoid in $E_1$.

Proof. For each continuous seminorm $p$ the set $\pi_p(A)$ is a local compactoid in $E_p^\wedge$ (Proposition 1.1), hence in a subspace $D_p$ of countable type (Proposition 1.3). By [4], Proposition 4.12 (iii), $E_1 := \prod D_p$ is of countable type. The restriction of the embedding $E \hookrightarrow \prod E_p^\wedge$ yields a linear homeomorphic embedding $i : [A] \hookrightarrow E_1$. Now $i(A)$ is a subset of $\prod \pi_p(A)$, which is a local compactoid in $E_1$ (Proposition 1.2). Then, $i(A)$ is a local compactoid in $E_1$.

COROLLARY 1.5. If $A$ is a local compactoid in $E$ then $[A]$ is of countable type.

Proof. $[A]$ is linearly homeomorphic to a subspace of $E_1$. Now apply [4], Proposition 4.12 (i).

PROPOSITION 1.6. Let $E$ be a polar space and let $A$ be a local compactoid in $E$. Then, on $A$, the weak topology $\sigma(E, E')$ and the initial topology coincide. $A$ is complete if and only if $A$ is weakly complete.

Proof. The proofs of [4], 5.7-5.11 can easily be modified in such a way that the conclusion of [4], Theorem 5.12 holds for local compactoids, rather than just compactoids.

PROPOSITION 1.7. Let $A$ be a local compactoid in $E$. Then, as a topological $B(0,1)$-module, $A$ is isomorphic to a $B(0,1)$-submodule of some power of $K$.

Proof. By Lemma 1.4 we may suppose that $E$ is of countable type, hence polar. So, by Proposition 1.6, $A$ is a topological $B(0,1)$-submodule of $(E, \sigma(E, E'))$. The map

$$z \mapsto (f(z))_{f \in E'} \quad (z \in E)$$

is a linear homeomorphism of $(E, \sigma(E, E'))$ into $K^{E'}$. The statements follows.

§2 LOCAL COMPACTOIDS IN $K^I$.

Throughout §2, $E$ is a vector space over $K$ (no topology) and $E'$ its algebraic dual, with the topology $\sigma(E', E)$ of pointwise convergence. Then $E^*$ is Hausdorff, locally convex, complete and of countable
type. Every absolutely convex subset of $E^*$ is a local compactoid in $E$ as each neighbourhood of 0 in $E^*$ contains a subspace with finite codimension. It is not hard to see that each $\Theta \in (E^*)'$ has the form $f \mapsto f(x)$ for some $x \in E$, so that we may identify $(E^*)'$ and $E$.

To see the connection with the title of §2 observe that $E$ is the (algebraic) direct sum $\bigoplus_{i \in I} K_i$, where $K_i = K$ for each $i$ and that $E^*$ is linearly homeomorphic to $K^I$.

A subset $X$ of $E$ is $K$-polar if for each $y \in E \setminus X$ there exists an $f \in E^*$ with $|f(x)| \leq 1$, $|f(y)| > 1$.

For $X \subset E$, $Y \subset E^*$ we set, as usual

$$X^0 := \{f \in E^* : |f(X)| \leq 1\}$$

$$Y^0 := \{x \in E : |Y(x)| \leq 1\}.$$

**Proposition 2.1.** Let $X \subset E$, $Y \subset E^*$.

(i) $X$ is $K$-polar if and only if $X = X^{00}$.

(ii) $Y = Y^{00}$ if and only if $Y$ is closed, (absolutely convex) and edged.

**Proof.** Direct verification yields (i). For (ii) observe that $(E^*)' \cong E$ and that $E^*$ is strongly polar. Now apply [4], Theorem 4.7.

**Remark.** It is easy to see that each linear subspace of $E$ is $K$-polar. If $K$ is spherically complete even each edged subset of $E$ is $K$-polar. However this conclusion is false in general.

**Lemma 2.2.** Let $X \subset E$ be absolutely convex. The following are equivalent.

(a) $X$ is absorbing.

(b) $X^0$ is a compactoid.

(c) $X^0$ does not contain linear subspaces of $E^*$ other than $\{0\}$.

**Proof.** A typical zero neighbourhood in $E^*$ has the form $F^0$ where $F$ is a finite subset of $E$. By (a) we have $\lambda X \supset F$ for some $\lambda \in K$. Then $X^0 \subset \lambda F^0$. It follows that $X^0$ is bounded hence a compactoid (for example from Proposition 1.1.(iv)). This proves (a) $\Rightarrow$ (b). The implication (b) $\Rightarrow$ (c) is easy. To prove (c) $\Rightarrow$ (a), let $f \in E^*$, $f([X]) = \{0\}$. Then $Kf \in X^0$ so that $f = 0$. Then, $[X] = E$ i.e. $X$ is absorbing.

The next Proposition is the heart of this paper.
PROPOSITION 2.3. Let $A$ be a closed absolutely convex subset of $E^*$. Let $D$ be the largest $K$-subspace of $E^*$ that is contained in $A$. Then $D$ is closed. There exists a closed absolutely convex compactoid $B \subset A$ such that $D \cap B = \{0\}$, $D + B = A$, and the canonical map $D \times B \to A$ is a homeomorphism.

Proof.

(i) First assume that $A$ is edged. Then $A = A^{00}$. Trivially, $D$ is closed. $D^0$ has an (algebraic) complement $F$ in $E$. Set

$$B := (F + A^0)^0$$

Then $B$ is closed, edged. Since $F + A^0 \supset A^0$ we have $B \subset A^{00} = A$. Since also $F + A^0 \supset F$ we have $D \cap B \subset D \cap F^0 = D^{00} \cap F^0 = (D^0 + F)^0 = E^0 = \{0\}$. From this it follows, in turn, that $B$ does not contain subspaces except $\{0\}$. By Lemma 2.2, $B$ is a compactoid. Finally we prove that $A \simeq D \times B$.

From $E = F \oplus D^0$ we obtain two standard projections $\pi_1 : E \to F, \pi_2 : E \to D^0$. For each $f \in E^*$ we have $f = f \circ \pi_1 + f \circ \pi_2$. If $f \in A$ then $f \circ \pi_1 \in D^{00}$, so that $f \circ \pi_1 \in A$. Also $f \circ \pi_2 \in F^0$. Then $f \circ \pi_2 \in A \cap F^0 = A^{00} \cap F^0 = (A^0 + F)^0 = B$. Then

$$f \mapsto (f \circ \pi_1, f \circ \pi_2) \quad (f \in A)$$

maps $A$ onto $D \times B$. It follows easily that it is, indeed, a homeomorphism.

(ii) To prove the general case we apply (i) to $A^*$. So $A^* = D \oplus C$ where $D$ is a closed subspace and $C$ is a closed compactoid, both contained in $A^*$. Then $D \subset A$ and $A = D \oplus B$ where $B := A \cap C$, a closed compactoid.

§3 CONCLUSIONS

THEOREM 3.1 (Compare [3], Corollary 6.5). Let $A$ be a complete local compactoid in a Hausdorff locally convex space $E$ over $K$. Then, as a topological $B(0,1)$-module $A$ is a direct sum $D \oplus B$ where $D$ is the largest subspace contained in $A$ and $B$ is some complete compactoid in $A$.

Proof. Immediate from Proposition 1.7 and 2.3.

COROLLARY 3.2. (Compare [3], Lemma 6.3). Let $A$ be a complete local compactoid in a Hausdorff locally convex space over $K$. 

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(i) A does not contain subspaces other than \{0\} then A is a compactoid.
(ii) If A is unbounded then A contains a linear space \neq \{0\}.

To prove Theorem 3.4 we need the following lemma.

**Lemma 3.3.** Let D be a linear subspace of a Hausdorff locally convex space E. Let U be an absolutely convex zero neighbourhood in E and let \( D \subset U + Kx \) for some \( x \in E \). Then \( D \subset U + Ka \) for some \( a \in D \).

**Proof.** If \( Kx \subset U \) we may take \( a := 0 \), so assume \( Kx \not\subset U \) i.e. \( p(x) \neq 0 \) where \( p \) is the seminorm associated to \( U \). For each \( \lambda \in K, \lambda \neq 0 \) we have

\[
D = \lambda D \subset \lambda U + Kx
\]

so that for \( d \in D \) and \( n \in \mathbb{N} \) we have a decomposition

\[
d = u_n + \lambda_n x
\]

where \( p(u_n) \leq 1/n \) and \( \lambda_n \in K \). Since also \( p(x) \neq 0 \) it follows easily that \( \lambda := \lim \lambda_n \) exists. Hence, \( u := \lim_{n \to \infty} u_n \) exists and \( p(u) = 0 \). Thus, \( d = u + \lambda z \) i.e.

\[
D \subset Kerp + Kx
\]

If \( D \subset Kerp \) we may take again \( a := 0 \). If not then \( z = a + v \) where \( a \in D, v \in Kerp \). Then \( Kv \in Kerp \) so that \( D \subset Kerp + Kv + Ka \subset U + Ka \).

**Theorem 3.4** (Katsaras' Theorem for local compactoids). Let \( A \) be a complete local compactoid in a Hausdorff locally convex space \( E \) over \( K \). Let \( \lambda \in K, \lambda = 1 \) if the valuation of \( K \) is discrete, \( |\lambda| > 1 \) otherwise. Then for each zero neighbourhood \( U \) in \( E \) there exists a finite dimensional space \( F \subset A \) and finitely many points \( x_1, ..., x_n \in \lambda A \) such that \( A \subset U + F + \text{co}\{x_1, ..., x_n\} \).

**Proof.** We may assume that \( U \) is absolutely convex. Let \( A = D + B \) as in Theorem 3.1. By Katsaras' Theorem

\[
B \subset U + \text{co}\{x_1, ..., x_n\}
\]
for some \( z_1, ..., z_n \in \lambda B \subseteq \lambda A \). By local compactoidity of \( D \) there exist \( y_1, ..., y_m \in E \) such that

\[
D \subseteq U + Ky_1 + ... + Ky_m
\]

By repeated application of Lemma 3.3 we can arrange that \( y_1, ..., y_m \in D \). The Theorem follows with \( F := [y_1, ..., y_m] \).

**COROLLARY 3.5.** Let \( A \) be a complete local compactoid in a Hausdorff locally convex space \( E \) over \( K \). Then \( A \) is a local compactoid in \([A]\).

The easy proof is left to the reader.

To see that everything goes wrong if we drop the completeness condition consider the following. (Compare [3], Example 6.4.)

**EXAMPLE 3.6.** There exists a (non-closed) local compactoid \( A \) in \( c_0 \) with the following properties.

(i) \( A \) is unbounded.

(ii) \( A \) does not contain linear subspaces other than \( \{0\} \).

(iii) \( A \) is not a local compactoid in \([A]\).

**Proof.** Let \( p \in K, 0 < |p| < 1 \). Define

\[
z_1 = (p^{-1}, p, 0, 0, ...)
z_2 = (p^{-2}, 0, p^2, 0, ...)
z_3 = (p^{-3}, 0, 0, p^3, 0, ...)
\]

e tc. and set \( A := co(z_1, z_2, ...) \). Then (i),(ii) are clear.

Since

\[
A \subseteq Ke_1 + \overline{co}(pe_2, p^2e_3, ...)
\]

(where \( e_1, e_2, ... \) is the standard base of \( c_0 \)), \( A \) is a local compactoid in \( c_0 \). To obtain (iii) we prove that there exists no finite dimensional set \( F \subset [A] \) with \( A \subset U + F \) where \( U = \{z \in c_0 : |z| \leq 1\} \). Suppose such \( F \) does exist. Then we may assume \( F \subset A + U \), \( F \) absolutely convex. Suppose \( Ka \subset F \) for some \( a \neq 0 \). Since \( U \) is bounded it is easy to see that then \( Ka \subset \overline{A} \). But the only subspace \( \neq \{0\} \) of \( \overline{A} \) is \( Ke_1 \), so \( a \in Ke_1 \), which is impossible since \( Ke_1 \cap [A] = \{0\} \). Hence, \( F \) contains no subspaces other than \( \{0\} \) so \( F \) is bounded. But then \( A \subset U + F \) would be bounded, a contradiction.
REFERENCES


