A CONNECTION BETWEEN p-ADIC BANACH SPACES AND LOCALLY CONVEX COMPACTOIDS

by

W.H. SCHIKHOF

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DEPARTMENT OF MATHEMATICS
CATHOLIC UNIVERSITY
Toernooiveld
6525 ED Nijmegen
The Netherlands

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ABSTRACT. For a vector space E over a non-archimedean valued field K a correspondence $p \mapsto p^0$ is established between seminorms p on E and compactoids p^0 in E^*.
Examination of it yields the solution of two open problems (see §4 and §8) and a reformulation of Serre's renorming problem (see §2). As a by-product results on metrizable compactoids are obtained (see §6).

§0 THE CORRESPONDENCE $p \mapsto p^0$.

Throughout this note K is a non-archimedean valued field, complete with respect to the metric induced by the nontrivial valuation $| \ |$.
Let E be a K-vector space, let E^* be its algebraic dual. A (non-archimedean) seminorm p on E is <u>polar</u> ([3], Definition 3.1), if

$$p = \sup \{|f| : f \in E^*, |f| \le p\}$$

Let $\mathcal{P}_E$ be the set of all polar seminorms on E.
For each $p \in \mathcal{P}_E$ we set

$$p^0 = \{f \in E^* : |f| \le p\}$$

Then p^0 is an absolutely convex, edged ([3],§1b) subset of E^*. It is easy to see that p^0 is a closed compactoid ([3],§1e) with respect to the topology $\sigma(E^*,E)$, hence complete.

Let C_E* be the set of all closed absolutely convex, edged compactoids in E^* with respect to $\sigma(E^*,E)$.

PROPOSITION 0. <u>The map</u> $p \mapsto p^0$ <u>is a bijection of</u> P_E <u>onto</u> C_E*. <u>Its inverse assigns to every</u> $A \in C_E*$ <u>the seminorm</u> p <u>given by</u>

$$p(x) = \sup \{|f(x)| : f \in A\} \qquad (x \in E)$$

<u>Proof</u>. We shall prove surjectivity of $p \mapsto p^0$ leaving the (easy) rest of the proof to the reader. So, let $A \in C_E*$; we shall prove that $A = p^0$ where $p(x) = \sup \{|f(x)| : f \in A\}$.
Obviously, $A \subset p^0$. Now let $g \in E^* \setminus A$, we prove that $g \notin p^0$. The space E^* is of countable type hence strongly polar ([3],Theorem 4.4). So by [3], Theorem 4.7, there exists a $\theta \in (E^*, \sigma(E^*,E))'$ such that $|\theta| \le 1$ on A, $|\theta(g)| > 1$. But, by [3], lemma 7.1, θ has the form $f \mapsto f(x)$ for some $x \in E$. Thus, $|f(x)| \le 1$ for $f \in A$, $|g(x)| > 1$ i.e., $p(x) \le 1$ and $|g(x)| > 1$ and it follows that $g \notin p^0$.

<u>Remarks</u>.

1. Let K be spherically (= maximally) complete. Then each nonarchimedean seminorm p on E for which $p(x) \in \overline{|K|}$ $(x \in E)$ is polar ([3], Remark following 3.1).

2. Let τ be the locally convex topology on E induced by all nonarchimedean seminorms i.e., τ is the strongest among all locally convex topologies on E. It is not hard to see that (E,τ) is a complete polar ([3],Definition 3.5) space and that (E,τ) and $(E^*,\sigma(E^*,E))$ are each others strong dual spaces.

§1 NORMS p FOR WHICH p^0 IS c'-COMPACT

Recall that an absolutely convex subset A of a locally convex space F over K is c'-compact if for each neighbourhood U of 0 in F there exist $x_1,\dots,x_n \in A$ (rather than $x_1,\dots,x_n \in F$) such that $A \subset U + \text{co}\,\{x_1,\dots,x_n\}$. (Here co indicates the absolutely convex hull)

THEOREM 1.1. For a polar seminorm p on a K-vector space E the following are equivalent.

(α) $p(x) \in |K|$ for each $x \in E$. Each onedimensional subspace of E has a p-orthocomplement.

(β) p^0 is c'-compact.

Proof. (α) $\Rightarrow$ (β). By [7], Theorem 3.2, it suffices to prove that for each $\phi \in (E^*, \sigma(E^*,E))'$

$$\max\,\{|\phi(f)| : f \in p^0\}$$

exists. Since ϕ is an evaluation map we therefore have to show that

$$\max\,\{|f(x)| : f \in p^0\}$$

exists for each $x \in E$. This is obviously true if $p(x) = 0$. So assume $p(x) > 0$. Since $p(x) \in |K|$ we may assume that $p(x) = 1$. For such x we must prove

$$\max\,\{|f(x)| : f \in p^0\} = 1$$

By (α), Kx has a p-orthocomplement H. The function

$$f : \lambda x + h \mapsto \lambda \qquad (\lambda \in K,\ h \in H)$$

is in E^*. We have $|f(x)| = 1$. For $\lambda \in K$, $h \in H$

$|f(\lambda x + h)| = |\lambda| = p(\lambda x) \leq \max(p(\lambda x), p(h)) = p(\lambda x + h)$

so that $f \in p^0$.

(β) $\Rightarrow$ (α). Let $x \in E$. The map $f \mapsto |f(x)|$ $(f \in E^*)$

is a continuous seminorm on $(E^*, \sigma(E^*,E))$. By c'-compactness its restriction to p^0 has a maximum so there exists a $g \in p^0$ with $|g(x)| = p(x)$ (It follows that $p(x) \in |K|$). We prove that Ker g is a p-orthocomplement of Kx. In fact, for $z \in$ Ker g we have

$$p(x+z) \geq |g(x+z)| = |g(x)| = p(x)$$

Then also

$$p(x+z) \geq p(z)$$

completing the proof of Theorem 1.1.

<u>Note</u>. It is not hard to see that (α) of above is equivalent too.

(γ) <u>For each</u> $x \in E$ <u>there exists an</u> $f \in E^*$ <u>with</u> $|f(x)| = p(x)$ <u>and</u> $|f| \leq p$.

For spherically complete K we obtain a simpler form of Theorem 1.1.

COROLLARY 1.2. <u>Let</u> K <u>be spherically complete</u>, <u>let</u> p <u>be a seminorm on</u> E <u>for which</u> $p(x) \in \overline{|K|}$ <u>for all</u> $x \in E$.

<u>Then the following are equivalent</u>.

(α) $p(x) \in |K|$ <u>for each</u> $x \in E$.

(β) p^0 <u>is</u> c'-<u>compact</u>.

<u>Proof</u>. By [1], lemma 4.35, each onedimensional subspace has a p-orthocomplement.

§2 APPLICATION: A NEW LIGHT ON SERRE'S RENORMING PROBLEM.

Consider the following two statements (*) and (**).

(*) Let E be a K-vector space and let $\| \;\|$ be a norm on E. Then there exists a norm $\| \;\|'$ on E, equivalent to $\| \;\|$, such that $\|x\|' \in |K|$ for all $x \in E$.

(**) Let K be spherically complete and let A be a complete absolutely convex compactoid in a Hausdorff locally convex space over K. Then there exist a $\lambda \in K$ with $|\lambda| > 1$ and a c'-compact B such that $A \subset B \subset \lambda A$.

The question as to whether (*) is true or not is known as Serre's renorming problem. See [2] for more details. We are able to reformulate this problem in terms of compactoids:

PROPOSITION 2.1. <u>The above statements</u> (*) <u>and</u> (**) <u>are equivalent</u>.

<u>Proof</u>. Assume (*). To prove (**) we may assume that A is edged. By [8], Theorem 3, A, as a topological module over $B(0,1) := \{\lambda \in K: |\lambda| \leq 1\}$, is isomorphic to a bounded submodule of K^I for some set I. Let E be the algebraic direct sum $\bigoplus_{i\in I} K_i$ where $K_i = K$ for all $i \in I$.
Then $(E^*, \sigma(E^*,E))$ is in a natural way isomorphic to K^I with the product topology. So we may assume that $A = p^0$ where p is a seminorm on E.
By (*) there exists a seminorm q, equivalent to p, such that $q(x) \in |K|$ for all $x \in E$. By a suitable scalar multiplication we can arrange that, in addition, $p \leq q \leq |\lambda|p$ for some $\lambda \in K$, $|\lambda| > 1$. Then

$$p^0 \subset q^0 \subset \lambda p^0$$

and q^0 is c'-compact by Corollary 1.2. This proves (**). Now assume (**). To prove (*) we may assume (see [2]), that K is spherically complete. Let p be a norm on E. By (**) there is a c'-compact B and a $\lambda \in K$, $|\lambda| > 1$ with $p^0 \subset B \subset \lambda p^0$. Then $B = q^0$ for some seminorm q on E. We have

$$p \leq q \leq |\lambda| p$$

and $q(x) \in |K|$ for all $x \in E$ by Corollary 1.2.

Note. Serre's renorming problem is still unsettled as far as I know.

§3 NORMS p FOR WHICH p^0 IS A KREIN-MILMAN COMPACTOID.

Recall that an absolutely convex subset A of a locally convex space over K is a KM-compactoid if it is complete and if $A = \overline{co}\, X$ where X is compact. (Here $\overline{co}\, X$ is the closure of co X).

Before stating the theorem we first make some simple observations. Let p be a norm on E. We say that a collection $(e_i)_{i \in I}$ in E is a p-orthonormal base of E if for each $x \in E$ there exist a unique $(\lambda_i)_{i \in I} \subset K^I$ such that $\{i \in I, |\lambda_i| \geq \varepsilon\}$ is finite for each $\varepsilon > 0$ and

$$x = \sum_{i \in I} \lambda_i e_i$$

$$p(x) = \max_i |\lambda_i|$$

If (E,p) is complete this definition coincides with the usual one.

LEMMA 3.1. *Let (E,p) be a normed space, let $(\bar{E}, \bar{p})$ be its completion. Then (E,p) has a p-orthonormal base if and only if $(\hat{E}, \hat{p})$ has a $\hat{p}$-orthonormal base.*

Proof. It is not hard to see that each p-orthonormal base of (E,p) is also a $\bar{p}$-orthonormal base of $(\hat{E},\hat{p})$. Conversely, let $(e_i)_{i\in I}$ be a $\hat{p}$-orthonormal base of $(\bar{E},\bar{p})$. For each $i \in I$, choose an $f_i \in E$ with $\bar{p}(e_i - f_i) \le \frac{1}{2}$.
By [1], Exercise 5.C, $(f_i)_{i\in I}$ is a $\hat{p}$ orthonormal base of $(\hat{E},\bar{p})$.
Clearly $(f_i)_{i\in I}$ is a p-orthonormal base of (E,p).

THEOREM 3.2. For a polar norm p on a K-vector space E the following are equivalent.

(α) (E,p) has a p-orthonormal base

(β) p^0 is a KM-compactoid.

Proof. (α) ⇒ (β). Let $(e_i)_{i\in I}$ be a p-orthonormal base of (E,p). The formula

$$\phi(f) = (f(e_i))_{i\in I}$$

defines a map $\phi : p^0 \to B(0,1)^I$. Straightforward verifications show that ϕ is an isomorphism of topological B(0,1)-modules. From [8], Theorem 16 we obtain that $B(0,1)^I$, hence p^0, is a KM-compactoid.
(β) ⇒ (α). Suppose $p^0 = \overline{co}\, X$ where X is a compact subset of E^*.
Let $C(X\to K)$ be the Banach space of all continuous functions $X \to K$, with the supremum norm $\| \|_\infty$. Then $C(X\to K)$ has an orthonormal base. ([1], Theorem 5.22).
The formula

$$\phi(x)\ (f) = f(x) \qquad\qquad (f \in X)$$

defines a K-linear map $\phi : E \to C(X\to K)$. From

$$||\phi(x)||_\infty = \max_{f\in X} |f(x)| = \sup_{f\in coX} |f(x)| = \sup_{f\in p^0} |f(x)| = p(x)$$

we obtain that ϕ is an isometry $(E,p) \to (C(X\to K), || \ ||_\infty)$.

By Gruson's Theorem ([1], 5.9) $\overline{\phi(E)}$ has an orthonormal base. Then so has $\phi(E)$ by Lemma 3.1 and has E.

§4 *APPLICATION: A COMPLETE c'-COMPACT SET WHICH IS NOT A KM-COMPACTOID.*

We shall give a negative answer to the Problem following Theorem 1.7 of [6].

PROPOSITION 4.1. Let K be spherically complete, let $|K| = [0,\infty)$. Then there exist a locally convex space F over K and a complete c'-compact subset $A \subset F$ which is not a KM-compactoid.

Proof. Let $E := l^\infty$ and let $F := (l^\infty)^*$ (with the topology we agreed upon in §0). Let p be the standard norm on l^∞, and set $A := p^0$. Since, trivially, $p(x) \in |K|$ for all $x \in l^\infty$, we have that p^0 is c'-compact (Corollary 1.2).

However, it is known ([1], Cor. 5.19) that l^∞ has no orthogonal base so that (Theorem 3.2) p^0 is not a KM-compactoid.

§5 NORMS p FOR WHICH p^0 IS METRIZABLE.

THEOREM 5.1. For a polar seminorm p on a K-vector space E the following are equivalent.

(α) (E,p) is of countable type ([3], Definition 4.3).

(β) p^0 is metrizable.

Proof. $(\alpha) \Rightarrow (\beta)$. There exist $e_1, e_2, \ldots$ in E with $p(e_i) \le 1$ for each i such that the K-linear span of $e_1, e_2, \ldots$ is p-dense in E. The formula

$$\phi(f) = (f(e_1), f(e_2), \ldots)$$

defines a map $\phi : p^0 \to B(0,1)^{\mathbb{N}}$. Straightforward verifications show that ϕ is an isomorphism of topological $B(0,1)$-modules of p^0 onto a submodule of $B(0,1)^{\mathbb{N}}$.

Now $B(0,1)^{\mathbb{N}}$ is metrizable (the product topology is induced by the metric $(a,b) \mapsto \sup_{i\in\mathbb{N}} |a_i - b_i| 2^{-i}$) hence so is p^0.

$(\beta) \Rightarrow (\alpha)$. Let $\lambda \in K$, $|\lambda| > 1$. Since p^0 is a metrizable compactoid there exist, by [3], Proposition 8.2, $f_1, f_2, \ldots \in \lambda p^0$ with $\lim_{n\to\infty} f_n = 0$ such that

$$p^0 \subset \overline{co}\,\{f_1, f_2, \ldots\} \subset \lambda p^0$$

The map

$$\phi : x \mapsto (f_1(x), f_2(x), \ldots) \qquad (x \in E)$$

is K-linear, $\phi(E) \subset c_0$. We have for $x \in E$

$$\|\phi(x)\| = \sup_{n\in\mathbb{N}} |f_n(x)| = \sup\{|g(x)| : g \in \overline{co}\,\{f_1 f_2, \ldots\}\}$$

It follows that

$$p(x) \le \|\phi(x)\| \le |\lambda| p(x)$$

so that p is equivalent to $x \mapsto \|\phi(x)\|$, a seminorm of countable type. Hence, p is of countable type.

§6 APPLICATION: DESCRIPTION OF METRIZABLE COMPACTOIDS.

THEOREM 6.1. *Let A be an absolutely convex subset of a Hausdorff locally convex space F over K. The following are equivalent.*

(α) *A is a metrizable compactoid.*

(β) *As a topological* $B(0,1)$*-module, A is isomorphic to a submodule of* $B(0,1)^{\mathbb{N}}$.

(γ) *As a topological* $B(0,1)$*-module, A is isomorphic to a compactoid in* c_0.

(δ) *For each* $\lambda \in K$, $|\lambda| > 1$ *then exist* $e_1, e_2, \ldots \in \lambda A$ *with* $\lim_{n\to\infty} e_n = 0$ *and* $A \subset \overline{co}\,\{e_1, e_2, \ldots\}$.

(ε) *There exist* $e_1, e_2, \ldots \in F$ *with* $\lim_{n\to\infty} e_n = 0$ *and* $A \subset \overline{co}\,\{e_1, e_2, \ldots\}$.

(η) *There exists an ultrametrizable compact* $X \subset F$ *with* $A \subset \overline{co}\, X$.

Proof. (α) $\Rightarrow$ (β). It is not hard to see, by using the absolute convexity of A, that $\overline{A}$ is also metrizable. As there is no harm in taking F complete we therefore may assume that A is complete. To prove (β) we also may assume that A is edged. By [8], Theorem 3, $A \subset B(0,1)^I \subset K^I$ for some set I. Like in the proof of Proposition 2.1 we may conclude that $A = p^0$ where p is a polar seminorm on $\bigoplus_{i\in I} K_i$ ($K_i = K$ for each i). Then p is of countable type by Theorem 5.1. From the proof of (α) $\Rightarrow$ (β) of that Theorem we obtain an isomorphism $A = p^0 \hookrightarrow B(0,1)^{\mathbb{N}}$.

(β) $\Rightarrow$ (γ). Choose $\lambda_1, \lambda_2, \ldots \in K$, $|\lambda_1| > |\lambda_2| > \ldots$, $\lim_{n\to\infty} \lambda_n = 0$. The formula

$$\phi((a_i)_{i\in\mathbb{N}}) = (\lambda_1 a_1 \lambda_2 a_2, \ldots) \in c_0$$

defines a $B(0,1)$-module isomorphism of $B(0,1)^{\mathbb{N}}$ onto $C := \overline{co}\,\{\lambda_1 e_1, \lambda_2 e_2, \ldots\}$ where $e_1, e_2, \ldots$ are the standard unit vectors in c_0. ϕ is a homeomorphism

$B(0,1)^{\mathbb{N}} \to C$, and maps A onto a compactoid in c_0.

$(\gamma) \Rightarrow (\delta)$. See [3], Proposition 8.2.

$(\delta) \Rightarrow (\varepsilon)$ is trivial.

$(\varepsilon) \Rightarrow (\eta)$. $\{0, e_1, e_2, \ldots\}$ is compact and ultrametrizable.

$(\eta) \Rightarrow (\alpha)$. We may assume that F is complete. It suffices to prove the metrizability of $B := \overline{\text{co}}\ X$.

B is a complete, edged compactoid. As before we may assume that $B = p^0$ for some polar seminorm p en some K-vector space E while $B \subset E^*$. The map $\phi : E \to C(X \to K)$ defined by

$$\phi(x)\ (f) = f(x) \qquad (f \in X)$$

is an isometry $(E,p) \to (C(X \to K), \|\ \|_\infty)$.

Now X is ultrametrizable so by [1], Exercise 3.5, $C(X \to K)$ is of countable type. Hence so is p. By Theorem 5.1, $B = p^0$ is metrizable.

§7 NORMS p FOR WHICH $(p^0)^i$ IS OF FINITE TYPE.

Recall that an absolutely convex set A in a locally convex space F over K is <u>of finite type</u> if for each zero neighbourhood U in F there exists a finite-dimensional bounded set $S \subset A$ such that $A \subset U + S$.

Let us say that a seminorm p on a K-vector space E is <u>of finite type</u> if $\text{Ker}\ p = \{x \in E : p(x) = 0\}$ has finite codimension.

LEMMA 7.1. *Let A be an absolutely convex subset of a locally convex space F whose topology is generated by a collection of seminorms of finite type. Then the following are equivalent.*

(α) *A is a compactoid of finite type.*

(β) *For each closed linear subspace H of finite codimension there is a finite dimensional bounded set $S \subset A$ with $A \subset H + S$.*

Proof. $(\alpha) \Rightarrow (\beta)$. (Note. This implication holds for any locally convex space F.) We may assume $[\![A]\!] = F$.

H has the form $D^{\perp} := \{x \in F : f(x) = 0 \text{ for all } f \in D\}$ where D is a finite dimensional subspace of F'. Let $f_1, \ldots, f_n$ be a base of D. There exist $x_1, \ldots, x_n \in F$ with $f_i(x_j) = \delta_{ij}$ $(i,j \in \{1,\ldots,n\})$. Since $[\![A]\!] = F$ there exists a $\lambda \in K$, $\lambda \neq 0$ such that $\lambda x_i \in A$ for each $i \in \{1,\ldots,n\}$. Set

$$U := \bigcap_{i=1}^{n} \{x \in F : |f_i(x)| \leq |\lambda|\}$$

Then U is a zero neighbourhood in F. A is a compactoid of finite type, so there exists a finite dimensional set $S_1 \subset A$ with $A \subset U + S_1$. Let $x \in U$. Write $x = y + z$ where

$$y := x - \sum_{i=1}^{n} f_i(x)\, x_i$$

$$z := \sum_{i=1}^{n} f_i(x)\, x_i$$

Now, since $x \in U$, $|f_i(x)| \leq |\lambda|$ for each i so that $z = \sum_{i=1}^{n} f_i(x)\, x_i \in A$.

Further, for each $j \in \{1,\ldots,n\}$

$$f_j(y) = f_j(x) - \sum_{i=1}^{n} f_i(x) f_j(x_i) = f_j(x) - f_j(x) = 0$$

and it follows that $y \in D^{\perp} = H$. So $x = y + z$ $\in H + [\![x_1,\ldots,x_n]\!] \cap A$. We see that

$$A \subset U + S_1 \subset H + S_2 + S_1$$

where $S_2 := [\![x_1,\ldots,x_n]\!] \cap A$. Then (β) is proved with $S := S_1 + S_2$.

$(\beta) \Rightarrow (\alpha)$. Let U be a zero neighbourhood in F. Since continuous seminorms are of finite type, U contains a closed subspace H of finite codimension. By (β) there exists a finite dimensional set $S \subset A$ with S bounded and

$A \subset H + S$. Then $A \subset U + S$.

From now on we assume that the valuation on K is dense.
Recall that for an absolutely convex set B we have $B^i := \bigcup_{|\lambda|<1} \lambda B$.

THEOREM 7.2. Let p be a polar norm on a K-vector space E. Then the following are equivalent.

(α) For each finite dimensional subspace D of E there exists a seminorm q on E, q of finite type, $q \leq p$ and $q = p$ on D.

(β) $(p^0)^i$ is of finite type.

Proof. (α) ⇒ (β). As each continuous seminorm on E^* is of finite type it suffices to prove, by Lemma 7.1, that for a closed subspace H of E^* of finite codimension there exists a finite dimensional set $S \in (p^0)^i$ such that $(p^0)^i \subset H + S$.

Now, by (α), there is a seminorm q of finite type, $q \leq p$ on E and $q = p$ on $D := H^{\perp}$. Let

$$S_1 := \{f \in E^* : |f| \leq q\}.$$

We see that S_1 is finite dimensional and since $q \leq p$ we have $S_1 \subset p^0$.
We now shall prove that $(p^0)^i \subset H + S$ where $S := (S_1)^i$.
In fact, let $f \in (p^0)^i$. Then there is a $\lambda \in K$, $0 < |\lambda| < 1$ with $|f| \leq |\lambda| p$.
Choose $\lambda' \in K$ with $|\lambda| < |\lambda'| < 1$.
We have $|f| \leq |\lambda|\, q$ on D (since $p = q$ on D) so we can extend f to a $g \in E^*$ with $|g| \leq |\lambda'|\, q$ on E. (This is because q is of finite type so that (E,q) is strongly polar.) Now write

$$f = f - g + g$$

Since $f = g$ on D we have $f - g \in D^{\perp} = H$.

Also, $|(\lambda')^{-1}g| \le q$ so that $(\lambda')^{-1}g \in S_1$ i.e. $g \in (S_1)^i = S$.

$(\beta) \Rightarrow (\alpha)$. By lemma 7.1 there exists a finite dimensional set $S \subset (p^0)^i$ so that $(p^0)^i = D^{\perp} \cap (p^0)^i + S$.

Set $q(x) := \sup_{h\in S} |h(x)|$. $\qquad (x \in E)$.

Then $q(x) = 0$ for all x in the space $S^{\perp}$ which has finite codimension. So q is of finite type.

Further, for $x \in E$ we have

$$q(x) = \sup_{h\in S}|h(x)| \le \sup_{h\in (p^0)^i} |h(x)| = \sup_{h\in p^0} |h(x)| = p(x),$$

so $q \le p$. Finally, if $x \in D$ then

$$p(x) = \sup_{f\in p^0} |f(x)| = \sup_{f\in (p^0)^i} |f(x)| = \sup_{\substack{h\in D^{\perp}\cap (p^0)^i \\ t\in S}} |h(x) + t(x)|$$

$$= \sup_{t\in S}|t(x)| = q(x).$$ Hence, $p = q$ on D.

§8 APPLICATION: A COMPLETE COMPACTOID IN c_0 THAT IS NOT OF FINITE TYPE.

If K is spherically complete each complete absolutely convex compactoid is of finite type (See [4], 2.3).

If K is not spherically complete the unit ball of c_0 is a complete compactoid for the weak topology but not of finite type (See [5], 1.6). This is a non-metrizable compactoid. A compactoid in $(c_0, \|\ \|)$, not of finite type, is given in [5], 1.4. However this compactoid is not closed. The following example provides an answer to the Problem following 1.5 in [5].

PROPOSITION 8.1. *Let K be not spherically complete. Then there exists an absolutely convex complete compactoid in c_0 that is not of finite type.*

Proof. Let $(K^\vee, |\ |)$ be the spherical completion of $(K, |\ |)$ in the sense of [1], Theorem 4.49. Let E be a K-subspace of $K^\vee$ of countably infinite dimension and let p be the valuation $|\ |$ restricted to E. Then $x, y \in E$, $x \perp y$ in the sense of p implies $x = 0$ or $y = 0$. Obviously, the norm p is of countable type (hence polar) so, by Theorem 5.1, p^0 is metrizable and is by Theorem 6.1, isomorphic to a compactoid in c_0.
Suppose p^0 were of finite type. Then so would $(p^0)^i$ ([5], Proposition 2.4). By Theorem 7.2 we would have a seminorm q on E, $q \le p$, q of finite type, $q(x) = p(x)$ for some $x \in E$, $x \neq 0$. But then $x \perp \mathrm{Ker}\, q$ in the sense of p (If $q(y) = 0$ then $p(x-y) \ge q(x-y) = q(x) = p(x)$) which is impossible.
So, p^0 is not of finite type.

REFERENCES

[1] A.C.M. van Rooij: Non-Archimedean Functional Analysis. Marcel Dekker, New York (1978).

[2] A.C.M. van Rooij: Notes on p-adic Banach spaces. Report 7633, Mathematisch Instituut, Katholieke Universiteit, Nijmegen, 1-62 (1976).

[3] W.H. Schikhof: Locally convex spaces over nonspherically complete valued fields. Groupe d'étude d'analyse ultramétrique 12 no. 24, 1-33 (1984/85).

[4] W.H. Schikhof: Some properties of c-compact sets in p-adic spaces. Report 8632, Department of Mathematics, Catholic University, Nijmegen (1986), 1-12.

[5] W.H. Schikhof: Ultrametric compactoids of finite type. Report 8634, Department of Mathematics, Catholic University, Nijmegen (1986), 1-19.

[6] W.H. Schikhof: The closed convex hull of a compact set in a non-Archimedean locally convex space. Report 8646, Department of Mathematics, Catholic University, Nijmegen (1986), 1-11.

[7] W.H. Schikhof: Weak c'-compactness in p-adic Banach spaces. Report 8648, Department of Mathematics, Catholic University, Nijmegen (1986), 1-13.

[8] W.H. Schikhof: Compact-like sets in non-archimedean functional analysis. Proceedings of the Conference on p-adic analysis, Hengelhoef (1986).