A CONNECTION BETWEEN p-ADIC BANACH SPACES AND LOCALLY CONVEX COMPACTOIDS

by

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ABSTRACT. For a vector space $E$ over a non-archimedean valued field $K$ a
 correspondence $p \leftrightarrow p^0$ is established between seminorms $p$ on $E$ and
 compactoids $p^0$ in $E^*$. Examination of it yields the solution of two open problems (see §4 and
 §8) and a reformulation of Serre's renorming problem (see §2). As a
 by-product results on metrizable compactoids are obtained (see §6).

§0 THE correspondENcE $p \leftrightarrow p^0$.

Throughout this note $K$ is a non-archimedean valued field, complete with
 respect to the metric induced by the nontrivial valuation $| |$.

Let $E$ be a $K$-vector space, let $E^*$ be its algebraic dual. A (non-archimedean)
 seminorm $p$ on $E$ is polar ([3], Definition 3.1), if

$$p = \sup \{ |f| : f \in E^*, |f| \leq p \}$$

Let $P_E$ be the set of all polar seminorms on $E$.

For each $p \in P_E$ we set

$$p^0 = \{ f \in E^* : |f| \leq p \}$$

Then $p^0$ is an absolutely convex, edged ([3],§1b) subset of $E^*$. It is
easy to see that $p^0$ is a closed compactoid ([3],§1e) with respect to the
topology $\sigma(E^*,E)$, hence complete.
Let \( C_E^* \) be the set of all closed absolutely convex, edged compactoids in \( E^* \) with respect to \( \sigma(E^*, E) \).

**Proposition 0.** The map \( p \mapsto p^0 \) is a bijection of \( P_E \) onto \( C_E^* \). Its inverse assigns to every \( A \in C_E^* \) the seminorm \( p \) given by

\[
p(x) = \sup \{|f(x)| : f \in A\} \quad (x \in E)
\]

**Proof.** We shall prove surjectivity of \( p \mapsto p^0 \) leaving the (easy) rest of the proof to the reader. So, let \( A \in C_E^* \); we shall prove that \( A = p^0 \) where \( p(x) = \sup \{|f(x)| : f \in A\} \).

Obviously, \( A \subseteq p^0 \). Now let \( g \in E^* \setminus A \), we prove that \( g \notin p^0 \). The space \( E^* \) is of countable type hence strongly polar ([3], Theorem 4.4). So by [3], Theorem 4.7, there exists a \( \theta \in (E^*, \sigma(E^*, E))' \) such that \( |\theta| \leq 1 \) on \( A \), \( |\theta(g)| > 1 \). But, by [3], lemma 7.1, \( \theta \) has the form \( f \mapsto f(x) \) for some \( x \in E \). Thus, \( |f(x)| \leq 1 \) for \( f \in A \), \( |g(x)| > 1 \) i.e., \( p(x) \leq 1 \) and \( |g(x)| > 1 \) and it follows that \( g \notin p^0 \).

**Remarks.**

1. Let \( K \) be spherically (= maximally) complete. Then each nonarchimedean seminorm \( p \) on \( E \) for which \( p(x) \in \overline{\mathbb{R}} \) \( (x \in E) \) is polar ([3], Remark following 3.1).

2. Let \( \tau \) be the locally convex topology on \( E \) induced by all nonarchimedean seminorms i.e., \( \tau \) is the strongest among all locally convex topologies on \( E \). It is not hard to see that \( (E, \tau) \) is a complete polar ([3], Definition 3.5) space and that \( (E, \tau) \) and \( (E^*, \sigma(E^*, E)) \) are each others strong dual spaces.
§1 NORMS $p$ FOR WHICH $p^0$ IS $c^1$-COMPACT

Recall that an absolutely convex subset $A$ of a locally convex space $F$ over $K$ is $c^1$-compact if for each neighbourhood $U$ of $0$ in $F$ there exist $x_1, \ldots, x_n \in A$ (rather than $x_1, \ldots, x_n \in F$) such that $A \subset U + \operatorname{co} \{x_1, \ldots, x_n\}$.
(Here $\operatorname{co}$ indicates the absolutely convex hull)

THEOREM 1.1. For a polar seminorm $p$ on a $K$-vector space $E$ the following are equivalent.

(a) $p(x) \in |K|$ for each $x \in E$. Each onedimensional subspace of $E$ has a $p$-orthocomplement.

(β) $p^0$ is $c^1$-compact.

Proof. (a) ⇒ (β). By [7], Theorem 3.2, it suffices to prove that for each $\langle f \rangle \in (E^*, \sigma(E^*, E))^*$

$$\max \{|\phi(f)| : f \in p^0\}$$

exists. Since $\phi$ is an evaluation map we therefore have to show that

$$\max \{|f(x)| : f \in p^0\}$$

exists for each $x \in E$. This is obviously true if $p(x) = 0$. So assume $p(x) > 0$. Since $p(x) \in |K|$ we may assume that $p(x) = 1$. For such $x$ we must prove

$$\max \{|f(x)| : f \in p^0\} = 1$$

By (a), $Kx$ has a $p$-orthocomplement $H$. The function

$$f : \lambda x + h \mapsto \lambda \quad (\lambda \in K, \ h \in H)$$
is in $E^*$. We have $|f(x)| = 1$. For $\lambda \in K$, $h \in H$

$$|f(\lambda x + h)| = |\lambda| = p(\lambda x) \leq \max(p(\lambda x), p(h)) = p(\lambda x + h)$$

so that $f \in p^0$.

$(\beta) \Rightarrow (\alpha)$. Let $x \in E$. The map $f \mapsto |f(x)|$ ($f \in E^*$)

is a continuous seminorm on $(E^*, C(E^*,E))$. By $c'$-compactness its

restriction to $p^0$ has a maximum so there exists a $g \in p^0$ with $|g(x)| = p(x)$

(It follows that $p(x) \in |K|$). We prove that $\text{Ker} \cdot g$ is a $p$-orthocomplement

of $Kx$. In fact, for $z \in \text{Ker} g$ we have

$$p(x+z) \geq |g(x+z)| = |g(x)| = p(x)$$

Then also

$$p(x+z) \geq p(z)$$

completing the proof of Theorem 1.1.

Note. It is not hard to see that $(\alpha)$ of above is equivalent too.

$(\gamma)$ For each $x \in E$ there exists an $f \in E^*$ with $|f(x)| = p(x)$ and $|f| \leq p$.

For spherically complete $K$ we obtain a simpler form of Theorem 1.1.

**COROLLARY 1.2.** Let $K$ be spherically complete, let $p$ be a seminorm on

$E$ for which $p(x) \in |K|$ for all $x \in E$.

Then the following are equivalent.

$(\alpha) p(x) \in |K|$ for each $x \in E$.

$(\beta) p^0$ is $c'$-compact.

**Proof.** By [1], lemma 4.35, each onedimensional subspace has a $p$-orthocomplement.
§2 APPLICATION: A NEW LIGHT ON SERRE’S RENORMING PROBLEM.

Consider the following two statements (*) and (**).

(*) Let $E$ be a $K$-vector space and let $||\cdot||$ be a norm on $E$. Then there exists a norm $||\cdot||'$ on $E$, equivalent to $||\cdot||$, such that $||x||' \leq |K|$ for all $x \in E$.

(**) Let $K$ be spherically complete and let $A$ be a complete absolutely convex compactoid in a Hausdorff locally convex space over $K$. Then there exist a $\lambda \in K$ with $|\lambda| > 1$ and a $c'$-compact $B$ such that $A \subseteq B \subseteq \lambda A$.

The question as to whether (*) is true or not is known as Serre’s renorming problem. See [2] for more details. We are able to reformulate this problem in terms of compactoids:

**Proposition 2.1.** The above statements (*) and (**) are equivalent.

**Proof.** Assume (*). To prove (**) we may assume that $A$ is edged. By [8], Theorem 3, $A$, as a topological module over $B(0,1) := \{\lambda \in K: |\lambda| \leq 1\}$, is isomorphic to a bounded submodule of $K^I$ for some set $I$. Let $E$ be the algebraic direct sum $\oplus_{i \in I} K_i$ where $K_i = K$ for all $i \in I$.

Then $(E^*, \sigma(E^*, E))$ is in a natural way isomorphic to $K^I$ with the product topology. So we may assume that $A = p^0$ where $p$ is a seminorm on $E$.

By (*) there exists a seminorm $q$, equivalent to $p$, such that $q(x) \leq |K|$ for all $x \in E$. By a suitable scalar multiplication we can arrange that, in addition, $p \leq q \leq |\lambda|p$ for some $\lambda \in K$, $|\lambda| > 1$. Then

$$ p^0 \leq q^0 \leq \lambda p^0 $$
and \(q^0\) is \(c'\)-compact by Corollary 1.2. This proves \((**')\). Now assume \((**)\).

To prove \((*)\) we may assume (see [2]), that \(K\) is spherically complete.

Let \(p\) be a norm on \(E\). By \((**)\) there is a \(c'\)-compact \(B\) and a \(\lambda \in K, |\lambda| > 1\) with \(p^0 \subset B \subset \lambda p^0\). Then \(B = q^0\) for some seminorm \(q\) on \(E\). We have

\[ p \leq q \leq |\lambda|p \]

and \(q(x) \in |K|\) for all \(x \in E\) by Corollary 1.2.

Note. Serre's renorming problem is still unsettled as far as I know.

§3 Norms \(p\) for Which \(p^0\) is a Krein-Milman Compactoid.

Recall that an absolutely convex subset \(A\) of a locally convex space over \(K\) is a \(KM\)-compactoid if it is complete and if \(A = \overline{\text{co} X}\) where \(X\) is compact. (Here \(\text{co} X\) is the closure of \(\text{co} X\)).

Before stating the theorem we first make some simple observations. Let \(p\) be a norm on \(E\). We say that a collection \((e_i)\) in \(E\) is a \(p\)-orthonormal base of \(E\) if for each \(x \in E\) there exist a unique \((\lambda_i)_{i \in I} \subset K^I\) such that \(\{i \in I, |\lambda_i| \geq \varepsilon\}\) is finite for each \(\varepsilon > 0\) and

\[ x = \sum_{i \in I} \lambda_i e_i \]

\[ p(x) = \max_{i} |\lambda_i| \]

If \((E, p)\) is complete this definition coincides with the usual one.

**Lemma 3.1.** Let \((E, p)\) be a normed space, let \((\hat{E}, \hat{p})\) be its completion. Then \((E, p)\) has a \(p\)-orthonormal base if and only if \((\hat{E}, \hat{p})\) has a \(\hat{p}\)-orthonormal base.
Proof. It is not hard to see that each $p$-orthonormal base of $(E, p)$ is also a $p$-orthonormal base of $(E, p)$. Conversely, let $(e_i)_{i \in I}$ be a $p$-orthonormal base of $(E, p)$. For each $i \in I$, choose an $f_i \in E$ with $p(e_i - f_i) \leq \frac{1}{2}$.

By [1], Exercise 5.C, $(f_i)_{i \in I}$ is a $p$-orthonormal base of $(E, p)$.

Clearly $(f_i)_{i \in I}$ is a $p$-orthonormal base of $(E, p)$.

**Theorem 3.2.** For a polar norm $p$ on a $K$-vector space $E$ the following are equivalent.

(a) $(E, p)$ has a $p$-orthonormal base

(b) $p^0$ is a KM-compactoid.

**Proof.** (a) $\Rightarrow$ (b). Let $(e_i)_{i \in I}$ be a $p$-orthonormal base of $(E, p)$. The formula

$$
\phi(f) = (f(e_i))_{i \in I}
$$

defines a map $\phi : p^0 \rightarrow B(0,1)^I$. Straightforward verifications show that $\phi$ is an isomorphism of topological $B(0,1)$-modules. From [8], Theorem 16 we obtain that $B(0,1)^I$, hence $p^0$, is a KM-compactoid.

(b) $\Rightarrow$ (a). Suppose $p^0 = \text{co} \, X$ where $X$ is a compact subset of $E^*$. Let $C(X*K)$ be the Banach space of all continuous functions $X \times K$, with the supremum norm $|| ||_\infty$. Then $C(X*K)$ has an orthonormal base. ([1], Theorem 5.22).

The formula

$$
\phi(x)(f) = f(x) \quad (f \in X)
$$

defines a $K$-linear map $\phi : E \rightarrow C(X*K)$. From
\[ ||\phi(x)||_\infty = \max_{f \in \mathcal{F}} |f(x)| = \sup_{f \in \mathcal{F}} |f(x)| = \sup_{f \in \mathcal{F} \cup \{0\}} |f(x)| = p(x) \]

we obtain that \( \phi \) is an isometry \((E, p) \to (C(X^\omega K), ||||_\infty)\).

By Gruson's Theorem ([1], 5.9) \( \hat{\phi}(E) \) has an orthonormal base. Then so has \( \hat{\phi}(E) \) by Lemma 3.1 and has \( E \).

§4 APPLICATION: A COMPLETE \( c' \)-COMPACT SET WHICH IS NOT A KM-COMPACTOID.

We shall give a negative answer to the Problem following Theorem 1.7 of [6].

**Proposition 4.1.** Let \( K \) be spherically complete, let \( |K| = [0, \omega) \).

Then there exist a locally convex space \( F \) over \( K \) and a complete \( c' \)-compact subset \( A \subset F \) which is not a KM-compactoid.

**Proof.** Let \( E := l^\omega \) and let \( F := (l^\omega)^* \) (with the topology we agreed upon in §0). Let \( p \) be the standard norm on \( l^\omega \), and set \( A := p^0 \). Since, trivially, \( p(x) \in |K| \) for all \( x \in l^\omega \), we have that \( p^0 \) is \( c' \)-compact (Corollary 1.2).

However, it is known ([1], Cor. 5.19) that \( l^\omega \) has no orthogonal base so that (Theorem 3.2) \( p^0 \) is not a KM-compactoid.

§5 NORMS \( p \) FOR WHICH \( p^0 \) IS METRIZABLE.

**Theorem 5.1.** For a polar seminorm \( p \) on a \( K \)-vector space \( E \) the following are equivalent.

(a) \((E, p)\) is of countable type ([3], Definition 4.3).

(b) \( p^0 \) is metrizable.
Proof. (a) ⇒ (b). There exist \( e_1, e_2, \ldots \) in \( E \) with \( p(e_i) \leq 1 \) for each \( i \) such that the \( K \)-linear span of \( e_1, e_2, \ldots \) is \( p \)-dense in \( E \). The formula

\[
\phi(f) = (f(e_1), f(e_2), \ldots)
\]

defines a map \( \phi : p^0 \rightarrow B(0,1)^{\mathbb{N}} \). Straightforward verifications show that \( \phi \) is an isomorphism of topological \( B(0,1) \)-modules of \( p^0 \) onto a submodule of \( B(0,1)^{\mathbb{N}} \).

Now \( B(0,1)^{\mathbb{N}} \) is metrizable (the product topology is induced by the metric \( (a,b) \mapsto \sup_{i \in \mathbb{N}} |a_i - b_i| 2^{-i} \)) hence so is \( p^0 \).

(b) ⇒ (a). Let \( \lambda \in K, |\lambda| > 1 \). Since \( p^0 \) is a metrizable compactoid there exist, by [3], Proposition 8.2, \( f_1, f_2, \ldots \in \lambda p^0 \) with \( \lim_{n \to \infty} f_n = 0 \) such that

\[
p^0 \subseteq \overline{\text{co}} \{ f_1, f_2, \ldots \} \subseteq \lambda p^0
\]

The map

\[
\phi : x \mapsto (f_1(x), f_2(x), \ldots) \quad (x \in E)
\]

is \( K \)-linear, \( \phi(E) \subseteq c_0 \). We have for \( x \in E \)

\[
||\phi(x)|| = \sup_{n \in \mathbb{N}} |f_n(x)| = \sup \{ |g(x)| : g \in \overline{\text{co}} \{ f_1, f_2, \ldots \} \}
\]

It follows that

\[
p(x) \leq ||\phi(x)|| \leq |\lambda| p(x)
\]

so that \( p \) is equivalent to \( x \mapsto ||\phi(x)|| \), a seminorm of countable type. Hence, \( p \) is of countable type.
§6 APPLICATION: DESCRIPTION OF METRIZABLE COMPACTOIDS.

THEOREM 6.1. Let \( A \) be an absolutely convex subset of a Hausdorff locally convex space \( F \) over \( K \). The following are equivalent.

(a) \( A \) is a metrizable compactoid.

(b) As a topological \( B(0,1) \)-module, \( A \) is isomorphic to a submodule of \( B(0,1)^\mathbb{N} \).

(c) As a topological \( B(0,1) \)-module, \( A \) is isomorphic to a compactoid in \( c_0 \).

(d) For each \( \lambda \in K \), \( |\lambda| > 1 \) then exist \( e_1, e_2, \ldots \in \lambda A \) with \( \lim_{n \to \infty} e_n = 0 \) and \( A \subset \text{co} \{e_1, e_2, \ldots\} \).

(e) There exist \( e_1, e_2, \ldots \in F \) with \( \lim_{n \to \infty} e_n = 0 \) and \( A \subset \text{co} \{e_1, e_2, \ldots\} \).

(n) There exists an ultrametrizable compact \( X \subset F \) with \( A \subset \text{co} X \).

Proof. (a) \( \Rightarrow \) (b). It is not hard to see, by using the absolute convexity of \( A \), that \( \overline{A} \) is also metrizable. As there is no harm in taking \( F \) complete we therefore may assume that \( A \) is complete. To prove (b) we also may assume that \( A \) is edged. By [8], Theorem 3, \( A \subset B(0,1)^I \subset K^I \) for some set \( I \). Like in the proof of Proposition 2.1 we may conclude that \( A = p^0 \) where \( p \) is a polar seminorm on \( \oplus K \) \((K_i = K \) for each \( i \)). Then \( p \) is of countable type by Theorem 5.1. From the proof of (a) \( \Rightarrow \) (b) of that theorem we obtain an isomorphism \( A = p^0 \cong B(0,1)^\mathbb{N} \).

(b) \( \Rightarrow \) (c). Choose \( \lambda_1, \lambda_2, \ldots \in K \), \( |\lambda_1| > |\lambda_2| > \ldots \), \( \lim_{n \to \infty} \lambda_n = 0 \). The formula

\[
\phi((a_i)_{i \in \mathbb{N}}) = (\lambda_1 a_1, \lambda_2 a_2, \ldots) \in c_0
\]

defines a \( B(0,1) \)-module isomorphism of \( B(0,1)^\mathbb{N} \) onto \( C := \text{co} \{\lambda_1 e_1, \lambda_2 e_2, \ldots\} \) where \( e_1, e_2, \ldots \) are the standard unit vectors in \( c_0 \). \( \phi \) is a homeomorphism.
B(0,1) \mathbb{N} \to C$, and maps $A$ onto a compactoid in $c_0$.

(γ) ⇒ (δ). See [3], Proposition 8.2.

(δ) ⇒ (ε) is trivial.

(ε) ⇒ (η), $\{0,e_1,e_2,\ldots\}$ is compact and ultrametrizable.

(η) ⇒ (α). We may assume that $F$ is complete. It suffices to prove the

metrizability of $B := \text{co } X$.

$B$ is a complete, edged compactoid. As before we may assume that $B = p^0$

for some polar seminorm $p$ on some $K$-vector space $E$ while $B \subset E^*$. The

map $\phi : E \to C(X; K)$ defined by

$$
\phi(x)(f) = f(x) \quad (f \in X)
$$

is an isometry $(E, p) \to (C(X; K), \| \cdot \|_\infty)$.

Now $X$ is ultrametrizable so by [1], Exercise 3.5, $C(X; K)$ is of

countable type. Hence so is $p$. By Theorem 5.1, $B = p^0$ is metrizable.

§7 NORMS $\varpi$ FOR WHICH $(\varpi^0)^1$ IS OF FINITE TYPE.

Recall that an absolutely convex set $A$ in a locally convex space $F$ over

$K$ is of finite type if for each zero neighbourhood $U$ in $F$ there exists

a finite-dimensional bounded set $S \subset A$ such that $A \subset U + S$.

Let us say that a seminorm $p$ on a $K$-vector space $E$ is of finite type

if $\ker p = \{x \in E : p(x) = 0\}$ has finite codimension.

LEMMA 7.1. Let $A$ be an absolutely convex subset of a locally convex

space $F$ whose topology is generated by a collection of seminorms of

finite type. Then the following are equivalent.

(a) $A$ is a compactoid of finite type.

(b) For each closed linear subspace $H$ of finite codimension there is a

finite dimensional bounded set $S \subset A$ with $A \subset H + S$. 
Proof. (a) \Rightarrow (\beta). (Note. This implication holds for any locally convex space \( F \).) We may assume \( \{A\} = F \).

\( H \) has the form \( D^1 := \{ x \in F : f(x) = 0 \text{ for all } f \in D \} \) where \( D \) is a finite dimensional subspace of \( F' \). Let \( f_1, \ldots, f_n \) be a base of \( D \). There exist \( x_1, \ldots, x_n \in F \) with \( f_i(x_j) = \delta_{ij} \) \((i, j \in \{1, \ldots, n\})\). Since \( \{A\} = F \) there exists a \( \lambda \in K, \lambda \neq 0 \) such that \( \lambda x_i \in A \) for each \( i \in \{1, \ldots, n\} \).

Set

\[
U := \bigcap_{i=1}^{n} \{ x \in F : |f_i(x)| \leq |\lambda| \}
\]

Then \( U \) is a zero neighbourhood in \( F \). \( A \) is a compactoid of finite type, so there exists a finite dimensional set \( S_1 \subseteq A \) with \( A \subseteq U + S_1 \). Let \( x \in U \). Write \( x = y + z \) where

\[
y := x - \sum_{i=1}^{n} f_i(x) x_i
\]

\[
z := \sum_{i=1}^{n} f_i(x) x_i
\]

Now, since \( x \in U \), \( |f_i(x)| \leq |\lambda| \) for each \( i \) so that \( z = \sum_{i=1}^{n} f_i(x) x_i \in A \).

Further, for each \( j \in \{1, \ldots, n\} \)

\[
f_j(y) = f_j(x) - \sum_{i=1}^{n} f_i(x) f_j(x_i) = f_j(x) - f_j(x) = 0
\]

and it follows that \( y \in D^1 = H \). So \( x = y + z \in H + [x_1, \ldots, x_n] \cap A \). We see that

\[
A \subseteq U + S_1 \subseteq H + S_2 + S_1
\]

where \( S_2 := [x_1, \ldots, x_n] \cap A \). Then \( (\beta) \) is proved with \( S := S_1 + S_2 \).

\( (\beta) \Rightarrow (a) \). Let \( U \) be a zero neighbourhood in \( F \). Since continuous seminorms are of finite type, \( U \) contains a closed subspace \( H \) of finite codimension.

By \( (\beta) \) there exists a finite dimensional set \( S \subseteq A \) with \( S \) bounded and
A ⊆ H + S. Then A ⊆ U + S.

From now on we assume that the valuation on K is dense.

Recall that for an absolutely convex set B we have \( B^i := \bigcup_{|\lambda| < 1} \lambda B \).

**THEOREM 7.2.** Let \( p \) be a polar norm on a K-vector space \( E \). Then the following are equivalent.

(a) For each finite dimensional subspace \( D \) of \( E \) there exists a seminorm \( q \) on \( E \), \( q \) of finite type, \( q \leq p \) and \( q = p \) on \( D \).

(b) \((p^0)^i\) is of finite type.

**Proof.** (a) ⇒ (b). As each continuous seminorm on \( E^* \) is of finite type it suffices to prove, by Lemma 7.1, that for a closed subspace \( H \) of \( E^* \) of finite codimension there exists a finite dimensional set \( S \in (p^0)^i \) such that \((p^0)^i \subseteq H + S\).

Now, by (a), there is a seminorm \( q \) of finite type, \( q \leq p \) on \( E \) and \( q = p \) on \( D := H^1 \). Let

\[
S_1 := \{ f \in E^* : |f| \leq q \}.
\]

We see that \( S_1 \) is finite dimensional and since \( q \leq p \) we have \( S_1 \subseteq p^0 \).

We now shall prove that \((p^0)^i \subseteq H + S\) where \( S := (S_1)^i \).

In fact, let \( f \in (p^0)^i \). Then there is a \( \lambda \in K \), \( 0 < |\lambda| < 1 \) with \( |\hat{f}| \leq |\lambda| p \).

Choose \( \lambda' \in K \) with \( |\lambda| < |\lambda'| < 1 \).

We have \( |f| \leq |\lambda| q \) on \( D \) (since \( p = q \) on \( D \)) so we can extend \( f \) to a \( g \in E^* \) with \( |g| \leq |\lambda'| q \) on \( E \). (This is because \( q \) is of finite type so that \((E, q)\) is strongly polar.) Now write

\[
f = f - g + g
\]

Since \( f = g \) on \( D \) we have \( f - g \in D^1 = H \).
Also, \(|(\lambda')^{-1}g| \leq q\) so that \((\lambda')^{-1}g \in S\) i.e. \(g \in (S_1)^i = S\).

\((\beta) \Rightarrow (a).\) By lemma 7.1 there exists a finite dimensional set \(S \subset (p_0)^i\) so that \((p_0)^i = D^i \cap (p_0)^i + S.\)

Set \(q(x) := \sup_{h \in S} |h(x)|.\) (x \(\in E).\)

Then \(q(x) = 0\) for all \(x\) in the space \(S^i\) which has finite codimension.

So \(q\) is of finite type.

Further, for \(x \in E\) we have

\[q(x) = \sup_{h \in S} |h(x)| \leq \sup_{h \in (p_0)^i} |h(x)| = \sup_{h \in p_0} |h(x)| = p(x),\]

so \(q \leq p.\) Finally, if \(x \in D\) then

\[p(x) = \sup_{f \in p} |f(x)| = \sup_{f \in (p_0)^i} |f(x)| = \sup_{h \in D^i \cap (p_0)^i} |h(x) + t(x)|\]

\[= \sup_{t \in S} |t(x)| = q(x).\] Hence, \(p = q\) on \(D.\)

§8 APPLICATION: A COMPLETE COMPACTOID IN \(C_0\) THAT IS NOT OF FINITE TYPE.

If \(K\) is spherically complete each complete absolutely convex compactoid is of finite type (See [4], 2.3).

If \(K\) is not spherically complete the unit ball of \(C_0\) is a complete compactoid for the weak topology but not of finite type (See [5], 1.6).

This is a non-metrizable compactoid. A compactoid in \((C_0, |||||)\), not of finite type, is given in [5], 1.4. However this compactoid is not closed. The following example provides an answer to the Problem following 1.5 in [5].
PROPOSITION 8.1. Let \( K \) be not spherically complete. Then there exists an absolutely convex complete compactoid in \( c_0 \) that is not of finite type.

**Proof.** Let \((K^v, | |)\) be the spherical completion of \((K, | |)\) in the sense of [1], Theorem 4.49. Let \( E \) be a \( K \)-subspace of \( K^v \) of countably infinite dimension and let \( p \) be the valuation \(| |\) restricted to \( E \). Then \( x, y \in E \), \( x \perp y \) in the sense of \( p \) implies \( x = 0 \) or \( y = 0 \). Obviously, the norm \( p \) is of countable type (hence polar) so, by Theorem 5.1, \( p^0 \) is metrizable and is by Theorem 6.1, isomorphic to a compactoid in \( c_0 \).

Suppose \( p^0 \) were of finite type. Then so would \((p^0)^i\) ([5], Proposition 2.4). By Theorem 7.2 we would have a seminorm \( q \) on \( E \), \( q \leq p \), \( q \) of finite type, \( q(x) = p(x) \) for some \( x \in E \), \( x \neq 0 \). But then \( x \perp \text{Ker} \ q \) in the sense of \( p \) (If \( q(y) = 0 \) then \( p(x-y) \geq q(x-y) = q(x) = p(x) \)) which is impossible. So, \( p^0 \) is not of finite type.
REFERENCES


