A CONNECTION BETWEEN p-ADIC BANACH SPACES AND LOCALLY CONVEX COMPACTOIDS

by

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ABSTRACT. For a vector space $E$ over a non-archimedean valued field $K$ a correspondence $p \mapsto p^0$ is established between seminorms $p$ on $E$ and compactoids $p^0$ in $E^*$. Examination of it yields the solution of two open problems (see §4 and §8) and a reformulation of Serre's renorming problem (see §2). As a by-product results on metrizable compactoids are obtained (see §6).

§0 THE CORRESPONDENCE $p \mapsto p^0$.

Throughout this note $K$ is a non-archimedean valued field, complete with respect to the metric induced by the nontrivial valuation $|.|$. Let $E$ be a $K$-vector space, let $E^*$ be its algebraic dual. A (non-archimedean) seminorm $p$ on $E$ is polar ([3], Definition 3.1), if

$$p = \sup \{|f| : f \in E^*, |f| \leq p\}$$

Let $P_E$ be the set of all polar seminorms on $E$. For each $p \in P_E$ we set

$$p^0 = \{f \in E^* : |f| \leq p\}$$

Then $p^0$ is an absolutely convex, edged ([3],§1b) subset of $E^*$. It is easy to see that $p^0$ is a closed compactoid ([3],§1e) with respect to the topology $\sigma(E^*,E)$, hence complete.
Let $\mathcal{C}_E^*$ be the set of all closed absolutely convex, edged compactoids in $E^*$ with respect to $\sigma(E^*,E)$.

**Proposition 0.** The map $p \mapsto p^0$ is a bijection of $P_E$ onto $\mathcal{C}_E^*$. Its inverse assigns to every $A \in \mathcal{C}_E^*$ the seminorm $p$ given by

$$p(x) = \sup \{|f(x)| : f \in A\} \quad (x \in E)$$

**Proof.** We shall prove surjectivity of $p \mapsto p^0$ leaving the (easy) rest of the proof to the reader. So, let $A \in \mathcal{C}_E^*$; we shall prove that $A = p^0$ where $p(x) = \sup \{|f(x)| : f \in A\}$.

Obviously, $A \subset p$. Now let $g \in E^* \setminus A$, we prove that $g \not\in p^0$. The space $E^*$ is of countable type hence strongly polar ([3], Theorem 4.4). So by [3], Theorem 4.7, there exists a $\theta \in (E^*,\sigma(E^*,E))^\prime$ such that $|\theta| \leq 1$ on $A$, $|\theta(g)| > 1$. But, by [3], lemma 7.1, $\theta$ has the form $f \mapsto f(x)$ for some $x \in E$. Thus, $|f(x)| \leq 1$ for $f \in A$, $|g(x)| > 1$ i.e., $p(x) \leq 1$ and $|g(x)| > 1$ and it follows that $g \not\in p^0$.

**Remarks.**

1. Let $K$ be spherically (= maximally) complete. Then each nonarchimedean seminorm $p$ on $E$ for which $p(x) \in |K|$ ($x \in E$) is polar ([3], Remark following 3.1).

2. Let $\tau$ be the locally convex topology on $E$ induced by all nonarchimedean seminorms i.e., $\tau$ is the strongest among all locally convex topologies on $E$. It is not hard to see that $(E,\tau)$ is a complete polar ([3], Definition 3.5) space and that $(E,\tau)$ and $(E^*,\sigma(E^*,E))$ are each others strong dual spaces.
§1 NORMS \( p \) FOR WHICH \( p^0 \) IS \( c' \)-COMPACT

Recall that an absolutely convex subset \( A \) of a locally convex space \( F \) over \( K \) is \( c' \)-compact if for each neighbourhood \( U \) of \( 0 \) in \( F \) there exist \( x_1, \ldots, x_n \in A \) (rather than \( x_1, \ldots, x_n \in F \)) such that \( A \subseteq U + \operatorname{co} \{ x_1, \ldots, x_n \} \). (Here \( \operatorname{co} \) indicates the absolutely convex hull)

**Theorem 1.1.** For a polar seminorm \( p \) on a \( K \)-vector space \( E \) the following are equivalent.

(a) \( p(x) \in |K| \) for each \( x \in E \). Each one-dimensional subspace of \( E \) has a \( p \)-orthocomplement.

(b) \( p^0 \) is \( c' \)-compact.

**Proof.** (a) \( \Rightarrow \) (b). By [7], Theorem 3.2, it suffices to prove that for each \( \phi \in (E^*, \sigma(E^*, E))' \)

\[
\max \{|\phi(f)| : f \in p^0\}
\]

exists. Since \( \phi \) is an evaluation map we therefore have to show that

\[
\max \{|f(x)| : f \in p^0\}
\]

exists for each \( x \in E \). This is obviously true if \( p(x) = 0 \). So assume \( p(x) > 0 \). Since \( p(x) \in |K| \) we may assume that \( p(x) = 1 \). For such \( x \) we must prove

\[
\max \{|f(x)| : f \in p^0\} = 1
\]

By (a), \( Kx \) has a \( p \)-orthocomplement \( H \). The function

\[
f : \lambda x + h \mapsto \lambda \quad (\lambda \in K, h \in H)
\]
is in $E^*$. We have $|f(x)| = 1$. For $\lambda \in K$, $h \in H$

$$|f(\lambda x + h)| = |\lambda| = p(\lambda x) \leq \max(p(\lambda x), p(h)) = p(\lambda x + h)$$

so that $f \in p^0$.

($\beta$) $\Rightarrow$ ($\alpha$). Let $x \in E$. The map $f \mapsto |f(x)|$ ($f \in E^*$)
is a continuous seminorm on $(E^*, c(E^*, E))$. By $c'$-compactness its
restriction to $p^0$ has a maximum so there exists a $g \in p^0$ with $|g(x)| = p(x)$
(It follows that $p(x) \in |K|$). We prove that $\ker g$ is a $p$-orthocomplement
of $Kx$. In fact, for $z \in \ker g$ we have

$$p(x+z) \geq |g(x+z)| = |g(x)| = p(x)$$

Then also

$$p(x+z) \geq p(z)$$

completing the proof of Theorem 1.1.

Note. It is not hard to see that ($\alpha$) of above is equivalent too.

($\gamma$) For each $x \in E$ there exists an $f \in E^*$ with $|f(x)| = p(x)$ and $|f| \leq p$.

For spherically complete $K$ we obtain a simpler form of Theorem 1.1.

**Corollary 1.2.** Let $K$ be spherically complete, let $p$ be a seminorm on
$E$ for which $p(x) \in |K|$ for all $x \in E$.

Then the following are equivalent.

($\alpha$) $p(x) \in |K|$ for each $x \in E$.

($\beta$) $p^0$ is $c'$-compact.

**Proof.** By [1], lemma 4.35, each onedimensional subspace has a $p$-orthocomplement.
§2 APPLICATION: A NEW LIGHT ON SERRE'S RENORMING PROBLEM.

Consider the following two statements (*) and (**).

(*) Let $E$ be a $K$-vector space and let $|| \cdot ||$ be a norm on $E$. Then there exists a norm $|| \cdot ||'$ on $E$, equivalent to $|| \cdot ||$, such that $||x||' \leq |x|$ for all $x \in E$.

(**) Let $K$ be spherically complete and let $A$ be a complete absolutely convex compactoid in a Hausdorff locally convex space over $K$. Then there exist a $\lambda \in K$ with $|\lambda| > 1$ and a $c'$-compact $B$ such that $A \subset B \subset \lambda A$.

The question as to whether (*) is true or not is known as Serre's renorming problem. See [2] for more details. We are able to reformulate this problem in terms of compactoids:

PROPOSITION 2.1. The above statements (*) and (**) are equivalent.

Proof. Assume (*). To prove (**) we may assume that $A$ is edged. By [8], Theorem 3, $A$, as a topological module over $B(0,1) := \{ \lambda \in K : |\lambda| \leq 1 \}$, is isomorphic to a bounded submodule of $K^I$ for some set $I$. Let $E$ be the algebraic direct sum $\bigoplus_{i \in I} K_i$ where $K_i = K$ for all $i \in I$.

Then $(E^*, \sigma(E^*, E))$ is in a natural way isomorphic to $K^I$ with the product topology. So we may assume that $A = p^0$ where $p$ is a seminorm on $E$.

By (*) there exists a seminorm $q$, equivalent to $p$, such that $q(x) \leq |x|$ for all $x \in E$. By a suitable scalar multiplication we can arrange that, in addition, $p \leq q \leq |\lambda|p$ for some $\lambda \in K$, $|\lambda| > 1$. Then

$$ p^0 \leq q^0 \leq \lambda p^0 $$
and \( q^0 \) is \( c' \)-compact by Corollary 1.2. This proves (**'). Now assume (**).

To prove (*) we may assume (see [2]), that \( K \) is spherically complete.

Let \( p \) be a norm on \( E \). By (**) there is a \( c' \)-compact \( B \) and a \( \lambda \in K, |\lambda| > 1 \) with \( p^0 \subset B \subset \lambda p^0 \). Then \( B = q^0 \) for some seminorm \( q \) on \( E \). We have

\[
p \leq q \leq |\lambda|p\]

and \( q(x) \in |K| \) for all \( x \in E \) by Corollary 1.2.

**Note.** Serre's renorming problem is still unsettled as far as I know.

### §3 NORMS \( p \) FOR WHICH \( p^0 \) IS A KREIN-MILMAN COMPACTOID.

Recall that an absolutely convex subset \( A \) of a locally convex space over \( K \) is a **KM-compactoid** if it is complete and if \( A = \overline{\text{co} \, X} \) where \( X \) is compact. (Here \( \overline{\text{co} \, X} \) is the closure of \( \text{co} \, X \)).

Before stating the theorem we first make some simple observations. Let \( p \) be a norm on \( E \). We say that a collection \( (e_i) \) in \( E \) is a \( p \)-orthonormal base of \( E \) if for each \( x \in E \) there exist a unique \( (\lambda_i)_{i \in I} \subset K^* \) such that \( \{i \in I, |\lambda_i| \geq \varepsilon \} \) is finite for each \( \varepsilon > 0 \) and

\[
x = \sum_{i \in I} \lambda_i e_i
\]

\[
p(x) = \max_{i} |\lambda_i|
\]

If \( (E,p) \) is complete this definition coincides with the usual one.

**Lemma 3.1.** Let \( (E,p) \) be a normed space, let \( (E^\sim, p^\sim) \) be its completion. Then \( (E,p) \) has a \( p \)-orthonormal base if and only if \( (E^\sim, p^\sim) \) has a \( p^\sim \)-orthonormal base.
Proof. It is not hard to see that each p-orthonormal base of \((E, p)\) is also a \(p\)-orthonormal base of \((E, p)\). Conversely, let \((e_i)_{i \in I}\) be a \(p\)-orthonormal base of \((E, p)\). For each \(i \in I\), choose an \(f_i \in E\) with \(p(e_i - f_i) \leq \frac{1}{2}\).

By [1], Exercise 5.C, \((f_i)_{i \in I}\) is a \(p\)-orthonormal base of \((E, p)\).

Clearly \((f_i)_{i \in I}\) is a \(p\)-orthonormal base of \((E, p)\).

**Theorem 3.2.** For a polar norm \(p\) on a \(K\)-vector space \(E\) the following are equivalent.

(a) \((E, p)\) has a \(p\)-orthonormal base

(b) \(p^0\) is a KM-compactoid.

**Proof.** (a) \(\Rightarrow\) (b). Let \((e_i)_{i \in I}\) be a \(p\)-orthonormal base of \((E, p)\). The formula

\[
\phi(f) = (f(e_i))_{i \in I}
\]

defines a map \(\phi : p^0 \to B(0,1)^I\). Straightforward verifications show that \(\phi\) is an isomorphism of topological \(B(0,1)\)-modules. From [8], Theorem 16 we obtain that \(B(0,1)^I\), hence \(p^0\), is a KM-compactoid.

(b) \(\Rightarrow\) (a). Suppose \(p^0 = \text{co } X\) where \(X\) is a compact subset of \(E^*\).

Let \(C(X^*K)\) be the Banach space of all continuous functions \(X^* \to K\), with the supremum norm \(|| \cdot \||_\infty\). Then \(C(X^*K)\) has an orthonormal base. ([1], Theorem 5.22).

The formula

\[
\phi(x)(f) = f(x) \quad (f \in X)
\]

defines a \(K\)-linear map \(\phi : E \to C(X^*K)\). From
\[ |\phi(x)|_w = \max |f(x)| = \sup_{f \in X} |f(x)| = \sup_{f \in \mathcal{C}(X,K)} |f(x)| = p(x) \]

we obtain that \( \phi \) is an isometry \( (E,p) \rightarrow (\mathcal{C}(X,K), |\cdot|_w) \).

By Gruson's Theorem ([1], 5.9) \( \phi(E) \) has an orthonormal base. Then so has \( \phi(E) \) by Lemma 3.1 and has \( E \).

\section*{§4 APPLICATION: A COMPLETE \( c' \)-COMPACT SET WHICH IS NOT A KM-COMPACTOID.}

We shall give a negative answer to the Problem following Theorem 1.7 of [6].

\textbf{PROPOSITION 4.1.} Let \( K \) be spherically complete, let \( |K| = [0,\infty) \).

Then there exist a locally convex space \( F \) over \( K \) and a complete \( c' \)-compact subset \( A \subset F \) which is not a KM-compactoid.

\textbf{Proof.} Let \( E := l^\infty \) and let \( F := (l^\infty)^* \) (with the topology we agreed upon in §0). Let \( p \) be the standard norm on \( l^\infty \), and set \( A := p^0 \). Since, trivially, \( p(x) \in |K| \) for all \( x \in l^\infty \), we have that \( p^0 \) is \( c' \)-compact (Corollary 1.2).

However, it is known ([1], Cor. 5.19) that \( l^\infty \) has no orthogonal base so that (Theorem 3.2) \( p^0 \) is not a KM-compactoid.

\section*{§5 NORMS \( p \) FOR WHICH \( p^0 \) IS METRIZABLE.}

\textbf{THEOREM 5.1.} For a polar seminorm \( p \) on a \( K \)-vector space \( E \) the following are equivalent.

\( (a) \) \( (E,p) \) is of countable type ([3], Definition 4.3).

\( (b) \) \( p^0 \) is metrizable.
Proof. (a) ⇒ (b). There exist $e_1, e_2, \ldots$ in $E$ with $p(e_i) \leq 1$ for each $i$ such that the $K$-linear span of $e_1, e_2, \ldots$ is $p$-dense in $E$. The formula

$$\phi(f) = (f(e_1), f(e_2), \ldots)$$

defines a map $\phi : p^0 \to B(0,1)^\mathbb{N}$. Straightforward verifications show that $\phi$ is an isomorphism of topological $B(0,1)$-modules of $p^0$ onto a submodule of $B(0,1)^\mathbb{N}$.

Now $B(0,1)^\mathbb{N}$ is metrizable (the product topology is induced by the metric

$$(a,b) \mapsto \sup_{i \in \mathbb{N}} |a_i - b_i|2^{-i}$$

hence so is $p^0$.

(b) ⇒ (a). Let $\lambda \in K, |\lambda| > 1$. Since $p^0$ is a metrizable compactoid, there exist, by [3], Proposition 8.2, $f_1, f_2, \ldots \in \lambda p^0$ with $\lim_{n \to \infty} f_n = 0$ such that

$$p^0 \subset \text{co} \{f_1, f_2, \ldots\} \subset \lambda p^0$$

The map

$$\phi : x \mapsto (f_1(x), f_2(x), \ldots) \quad (x \in E)$$

is $K$-linear, $\phi(E) \subset c_0$. We have for $x \in E$

$$||\phi(x)|| = \sup_{n \in \mathbb{N}} |f_n(x)| = \sup \{ |g(x)| : g \in \text{co} \{f_1, f_2, \ldots\} \}$$

It follows that

$$p(x) \leq ||\phi(x)|| \leq |\lambda|p(x)$$

so that $p$ is equivalent to $x \mapsto ||\phi(x)||$, a seminorm of countable type. Hence, $p$ is of countable type.
§6 APPLICATION: DESCRIPTION OF METRIZABLE COMPACTOIDS.

THEOREM 6.1. Let $A$ be an absolutely convex subset of a Hausdorff locally convex space $F$ over $K$. The following are equivalent.

(a) $A$ is a metrizable compactoid.

(b) As a topological $B(0,1)$-module, $A$ is isomorphic to a submodule of $B(0,1)^\mathbb{N}$.

(c) As a topological $B(0,1)$-module, $A$ is isomorphic to a compactoid in $C_0^*$.

(d) For each $\lambda \in K$, $|\lambda| > 1$ then exist $e_1, e_2, \ldots \in \lambda A$ with $\lim_{n \to \infty} e_n = 0$ and $A \subseteq \overline{\{e_1, e_2, \ldots\}}$.

(e) There exist $e_1, e_2, \ldots \in F$ with $\lim_{n \to \infty} e_n = 0$ and $A \subseteq \overline{\{e_1, e_2, \ldots\}}$.

(f) There exists an ultrametrizable compact $X \subseteq F$ with $A \subseteq \overline{X}$.

Proof. (a) $\Rightarrow$ (b). It is not hard to see, by using the absolute convexity of $A$, that $\overline{A}$ is also metrizable. As there is no harm in taking $F$ complete we therefore may assume that $A$ is complete. To prove (b) we also may assume that $A$ is edged. By [8], Theorem 3, $A \subseteq B(0,1)^I \subset K^I$ for some set $I$. Like in the proof of Proposition 2.1 we may conclude that $A = \overline{p^0}$ where $p$ is a polar seminorm on $\oplus K_i$ ($K_i = K$ for each $i$). Then $p$ is of countable type by Theorem 5.1. From the proof of (a) $\Rightarrow$ (b) of that Theorem we obtain an isomorphism $A = \overline{p^0} \cong B(0,1)^\mathbb{N}$.

(b) $\Rightarrow$ (c). Choose $\lambda_1, \lambda_2, \ldots \in K$, $|\lambda_1| > |\lambda_2| > \ldots$, $\lim_{n \to \infty} \lambda_n = 0$. The formula

$$\phi((a_i)_{i \in \mathbb{N}}) = (\lambda_1 a_1, \lambda_2 a_2, \ldots) \in C_0$$

defines a $B(0,1)$-module isomorphism of $B(0,1)^\mathbb{N}$ onto $C := \overline{\{e_1, e_2, \ldots\}}$ where $e_1, e_2, \ldots$ are the standard unit vectors in $C_0$. $\phi$ is a homeomorphism.
$\mathbb{N} \to C$, and maps $A$ onto a compactoid in $c_0$.

(γ) $\Rightarrow$ (δ). See [3], Proposition 8.2.

(δ) $\Rightarrow$ (ε) is trivial.

(ε) $\Rightarrow$ (η), $\{0, e_1, e_2, \ldots\}$ is compact and ultrametrizable.

(η) $\Rightarrow$ (α). We may assume that $F$ is complete. It suffices to prove the

metrizability of $B := \operatorname{co} X$.

$B$ is a complete, edged compactoid. As before we may assume that $B = p^0$

for some polar seminorm $p$ on some $K$-vector space $E$ while $B \subseteq E^*$. The

map $\phi : E \to C(X \otimes K)$ defined by

$$
\phi(x)(f) = f(x) \quad (f \in X)
$$

is an isometry $(E, p) \to (C(X \otimes K), \| \cdot \|_\alpha)$.

Now $X$ is ultrametrizable so by [1], Exercise 3.5, $C(X \otimes K)$ is of

countable type. Hence so is $p$. By Theorem 5.1f $B = p^0$ is metrizable.

§7 NORMS $p$ FOR WHICH $(p^0)_1$ IS OF FINITE TYPE.

Recall that an absolutely convex set $A$ in a locally convex space $F$ over

$K$ is of finite type if for each zero neighbourhood $U$ in $F$ there exists

a finite-dimensional bounded set $S \subseteq A$ such that $A \subseteq U + S$.

Let us say that a seminorm $p$ on a $K$-vector space $E$ is of finite type

if $\operatorname{Ker} p = \{x \in E : p(x) = 0\}$ has finite codimension.

LEMMA 7.1. Let $A$ be an absolutely convex subset of a locally convex

space $F$ whose topology is generated by a collection of seminorms of

finite type. Then the following are equivalent.

(a) $A$ is a compactoid of finite type.

(b) For each closed linear subspace $H$ of finite codimension there is a

finite dimensional bounded set $S \subseteq A$ with $A \subseteq H + S$. 


Proof. \((a) \Rightarrow (\beta)\). (Note. This implication holds for any locally convex space \(F\).) We may assume \([A] = F\).

\(H\) has the form \(D^1 := \{x \in F : f(x) = 0 \text{ for all } f \in D\}\) where \(D\) is a finite dimensional subspace of \(F'\). Let \(f_1, \ldots, f_n\) be a base of \(D\). There exist \(x_1, \ldots, x_n \in F\) with \(f_i(x_j) = \delta_{ij}\) \((i, j \in \{1, \ldots, n\})\). Since \([A] = F\) there exists a \(\lambda \in K, \lambda \neq 0\) such that \(\lambda x_1 \in A\) for each \(i \in \{1, \ldots, n\}\).

Set

\[ U := \bigcap_{i=1}^n \{x \in F : |f_i(x)| \leq |\lambda|\} \]

Then \(U\) is a zero neighbourhood in \(F\). \(A\) is a compactoid of finite type, so there exists a finite dimensional set \(S^1 \subseteq A\) with \(A \subseteq U + S^1\). Let \(x \in U\). Write \(x = y + z\) where

\[ y := x - \sum_{i=1}^n f_i(x_i) x_i \]
\[ z := \sum_{i=1}^n f_i(x_i) x_i \]

Now, since \(x \in U\), \(|f_i(x_i)| \leq |\lambda|\) for each \(i\) so that \(z = \sum_{i=1}^n f_i(x_i) x_i \in A\).

Further, for each \(j \in \{1, \ldots, n\}\)

\[ f_j(y) = f_j(x) - \sum_{i=1}^n f_i(x_i)f_j(x_i) = f_j(x) - f_j(x) = 0 \]

and it follows that \(y \in D^1 = H\). So \(x = y + z\) \(\in H + [x_1, \ldots, x_n] \cap A\). We see that

\[ A \subseteq U + S^1 \subseteq H + S^2 + S^1 \]

where \(S_2 := [x_1, \ldots, x_n] \cap A\). Then \((\beta)\) is proved with \(S := S^1 + S^2\).

\((\beta) \Rightarrow (a)\). Let \(U\) be a zero neighbourhood in \(F\). Since continuous seminorms are of finite type, \(U\) contains a closed subspace \(H\) of finite codimension.

By \((\beta)\) there exists a finite dimensional set \(S \subseteq A\) with \(S\) bounded and
A ⊆ H + S. Then A ⊆ U + S.

From now on we assume that the valuation on K is dense.

Recall that for an absolutely convex set B we have $B^i := \bigcup_{|\lambda| < 1} \lambda B$.

**THEOREM 7.2.** Let $p$ be a polar norm on a K-vector space E. Then the following are equivalent.

(a) For each finite dimensional subspace $D$ of $E$ there exists a seminorm $q$ on $E$, $q$ of finite type, $q \leq p$ and $q = p$ on $D$.

(b) $(p^0)^i$ is of finite type.

**Proof.** (a) $\Rightarrow$ (b). As each continuous seminorm on $E^*$ is of finite type it suffices to prove, by Lemma 7.1, that for a closed subspace $H$ of $E^*$ of finite codimension there exists a finite dimensional set $S \in (p^0)^i$ such that $(p^0)^i \subseteq H + S$.

Now, by (a), there is a seminorm $q$ of finite type, $q \leq p$ on $E$ and $q = p$ on $D := H^1$. Let

$$S_1 := \{f \in E^*: |f| \leq q\}.$$

We see that $S_1$ is finite dimensional and since $q \leq p$ we have $S_1 \subseteq p^0$.

We now shall prove that $(p^0)^i \subseteq H + S$ where $S := (S_1)^i$.

In fact, let $f \in (p^0)^i$. Then there is a $\lambda \in K$, $0 < |\lambda| < 1$ with $|f| \leq |\lambda|p$.

Choose $\lambda' \in K$ with $|\lambda| < |\lambda'| < 1$.

We have $|f| \leq |\lambda|q$ on $D$ (since $p = q$ on $D$) so we can extend $f$ to a $g \in E^*$ with $|g| \leq |\lambda'|q$ on $E$. (This is because $q$ is of finite type so that $(E,q)$ is strongly polar.) Now write

$$f = f - g + g$$

Since $f = g$ on $D$ we have $f - g \in D^1 = H$. 
Also, $|({\lambda}')^{-1}g| \leq q$ so that $({\lambda}')^{-1}g \in S_1$ i.e. $g \in (S_1)^i = S$.

$(\beta) \Rightarrow (a)$. By lemma 7.1 there exists a finite dimensional set $S \subset (p_0)^i$ so that $(p_0)^i = D^i \cap (p_0)^i + S$.

Set $q(x) := \sup_{h \in S} |h(x)|$. $(x \in E)$.

Then $q(x) = 0$ for all $x$ in the space $S^1$ which has finite codimension.

So $q$ is of finite type.

Further, for $x \in E$ we have

$q(x) = \sup_{h \in S} |h(x)| \leq \sup_{h \in (p_0)^i} |h(x)| = \sup_{h \in (p_0)^i} |h(x)| = p(x)$,

so $q \leq p$. Finally, if $x \in D$ then

$p(x) = \sup_{f \in p} |f(x)| = \sup_{f \in (p_0)^i} |f(x)| = \sup_{h \in (p_0)^i} |h(x) + t(x)|$

$= \sup_{t \in S} |t(x)| = q(x)$. Hence, $p = q$ on $D$.

§8 APPLICATION: A COMPLETE COMPACTOID IN $c_0$ THAT IS NOT OF FINITE TYPE.

If $K$ is spherically complete each complete absolutely convex compactoid is of finite type (See [4], 2.3).

If $K$ is not spherically complete the unit ball of $c_0$ is a complete compactoid for the weak topology but not of finite type (See [5], 1.6).

This is a non-metrizable compactoid. A compactoid in $(c_0, ||||)$, not of finite type, is given in [5], 1.4. However this compactoid is not closed. The following example provides an answer to the Problem following 1.5 in [5].
PROPOSITION 8.1. Let $K$ be not spherically complete. Then there exists an absolutely convex complete compactoid in $c_0$ that is not of finite type.

Proof. Let $(K^v, | |)$ be the spherical completion of $(K, | |)$ in the sense of [1], Theorem 4.49. Let $E$ be a $K$-subspace of $K^v$ of countably infinite dimension and let $p$ be the valuation $| |$ restricted to $E$. Then $x, y \in E$, $x \perp y$ in the sense of $p$ implies $x = 0$ or $y = 0$. Obviously, the norm $p$ is of countable type (hence polar) so, by Theorem 5.1, $p^0$ is metrizable and is by Theorem 6.1, isomorphic to a compactoid in $c_0$.

Suppose $p^0$ were of finite type. Then so would $(p^0)^1$ ([5], Proposition 2.4). By Theorem 7.2 we would have a seminorm $q$ on $E$, $q \leq p$, $q$ of finite type, $q(x) = p(x)$ for some $x \in E$, $x \neq 0$. But then $x \perp \text{Ker } q$ in the sense of $p$ (If $q(y) = 0$ then $p(x-y) \geq q(x-y) = q(x) = p(x)$) which is impossible. So, $p^0$ is not of finite type.
REFERENCES


