A CONNECTION BETWEEN p-ADIC BANACH SPACES AND LOCALLY CONVEX COMPACTOIDS

by

W.H. SCHIKHOF

Report 8736
December 1987

DEPARTMENT OF MATHEMATICS
CATHOLIC UNIVERSITY
Toernooiveld
6525 ED Nijmegen
The Netherlands
A CONNECTION BETWEEN $p$-ADIC BANACH SPACES AND LOCALLY CONVEX COMPACTOIDS

by

W.H. Schikhof

ABSTRACT. For a vector space $E$ over a non-archimedean valued field $K$ a correspondence $p \leftrightarrow p^0$ is established between seminorms $p$ on $E$ and compactoids $p^0$ in $E^\ast$. Examination of it yields the solution of two open problems (see §4 and §8) and a reformulation of Serre's renorming problem (see §2). As a by-product results on metrizable compactoids are obtained (see §6).

§0 THE CORRESPONDENCE $p \leftrightarrow p^0$.

Throughout this note $K$ is a non-archimedean valued field, complete with respect to the metric induced by the nontrivial valuation $| |$.

Let $E$ be a $K$-vector space, let $E^\ast$ be its algebraic dual. A (non-archimedean) seminorm $p$ on $E$ is polar ([3], Definition 3.1), if

$$p = \sup \{ |f| : f \in E^\ast, |f| \leq p \}$$

Let $P_E$ be the set of all polar seminorms on $E$.

For each $p \in P_E$ we set

$$p^0 = \{ f \in E^\ast : |f| \leq p \}$$

Then $p^0$ is an absolutely convex, edged ([3],§1b) subset of $E^\ast$. It is easy to see that $p^0$ is a closed compactoid ([3],§1e) with respect to the topology $\sigma(E^\ast, E)$, hence complete.
Let $C_E^*$ be the set of all closed absolutely convex, edged compactoids in $E^*$ with respect to $\sigma(E^*,E)$.

**PROPOSITION 0.** The map $p \mapsto p^0$ is a bijection of $P_E$ onto $C_E^*$. Its inverse assigns to every $A \in C_E^*$ the seminorm $p$ given by

$$p(x) = \sup \{|f(x)| : f \in A\} \quad (x \in E)$$

**Proof.** We shall prove surjectivity of $p \mapsto p^0$ leaving the (easy) rest of the proof to the reader. So, let $A \in C_E^*$; we shall prove that $A = p^0$ where $p(x) = \sup \{|f(x)| : f \in A\}$.

Obviously, $A \subseteq p^0$. Now let $g \in E^* \setminus A$, we prove that $g \not\in p^0$. The space $E^*$ is of countable type hence strongly polar ([3], Theorem 4.4). So by [3], Theorem 4.7, there exists a $0 \in (E^*, \sigma(E^*,E))'$ such that $|0| \leq 1$ on $A$, $|0(g)| > 1$. But, by [3], lemma 7.1, $0$ has the form $f \mapsto f(x)$ for some $x \in E$. Thus, $|f(x)| \leq 1$ for $f \in A$, $|g(x)| > 1$ i.e., $p(x) \leq 1$ and $|g(x)| > 1$ and it follows that $g \not\in p^0$.

**Remarks.**

1. Let $K$ be spherically (= maximally) complete. Then each nonarchimedean seminorm $p$ on $E$ for which $p(x) \in \overline{|K|} \ (x \in E)$ is polar ([3], Remark following 3.1).

2. Let $\tau$ be the locally convex topology on $E$ induced by all nonarchimedean seminorms i.e., $\tau$ is the strongest among all locally convex topologies on $E$. It is not hard to see that $(E,\tau)$ is a complete polar ([3], Definition 3.5) space and that $(E,\tau)$ and $(E^*,\sigma(E^*,E))$ are each others strong dual spaces.
§1 NORMS p FOR WHICH p° IS c'-COMPACT

Recall that an absolutely convex subset A of a locally convex space F over K is c'-compact if for each neighborhood U of 0 in F there exist \( x_1, \ldots, x_n \in A \) (rather than \( x_1, \ldots, x_n \in F \)) such that \( A \subseteq U + \text{co} \{x_1, \ldots, x_n\} \). (Here \( \text{co} \) indicates the absolutely convex hull)

THEOREM 1.1. For a polar seminorm p on a K-vector space E the following are equivalent.

(a) \( p(x) \in |K| \) for each \( x \in E \). Each one-dimensional subspace of E has a p-orthocomplement.

(β) \( p^0 \) is c'-compact.

Proof. (a) ⇒ (β). By [7], Theorem 3.2, it suffices to prove that for each \( \langle f \rangle \in (E^*, \sigma(E^*, E))^\prime \)

\[
\max \{ |\phi(f)| : \phi \in p^0 \}
\]

exists. Since \( \phi \) is an evaluation map we therefore have to show that

\[
\max \{ |f(x)| : f \in p^0 \}
\]

exists for each \( x \in E \). This is obviously true if \( p(x) = 0 \). So assume \( p(x) > 0 \). Since \( p(x) \in |K| \) we may assume that \( p(x) = 1 \). For such \( x \) we must prove

\[
\max \{ |f(x)| : f \in p^0 \} = 1
\]

By (a), \( Kx \) has a p-orthocomplement H. The function

\[
f : \lambda x + h \mapsto \lambda \quad (\lambda \in K, h \in H)
\]
is in $E^*$. We have $|f(x)| = 1$. For $\lambda \in K$, $h \in H$

$$|f(\lambda x + h)| = |\lambda| = p(\lambda x) \leq \max(p(\lambda x), p(h)) = p(\lambda x + h)$$

so that $f \in p^0$.

$(\beta) \Rightarrow (\alpha)$. Let $x \in E$. The map $f \mapsto |f(x)|$ ($f \in E^*$) is a continuous seminorm on $(E^*, c(E^*, E))$. By $c'$-compactness its restriction to $p^0$ has a maximum so there exists a $g \in p^0$ with $|g(x)| = p(x)$ (It follows that $p(x) \in |K|$). We prove that $\ker g$ is a $p$-orthocomplement of $Kx$. In fact, for $z \in \ker g$ we have

$$p(x+z) \geq |g(x+z)| = |g(x)| = p(x)$$

Then also

$$p(x+z) \geq p(z)$$

completing the proof of Theorem 1.1.

Note. It is not hard to see that $(\alpha)$ of above is equivalent too.

$(\gamma)$ For each $x \in E$ there exists an $f \in E^*$ with $|f(x)| = p(x)$ and $|f| \leq p$.

For spherically complete $K$ we obtain a simpler form of Theorem 1.1.

COROLLARY 1.2. Let $K$ be spherically complete, let $p$ be a seminorm on $E$ for which $p(x) \in |K|$ for all $x \in E$.

Then the following are equivalent.

$(\alpha) p(x) \in |K|$ for each $x \in E$.

$(\beta) p^0$ is $c'$-compact.

Proof. By [1], lemma 4.35, each onedimensional subspace has a $p$-orthocomplement.
§2 APPLICATION: A NEW LIGHT ON SERRE'S RENORMING PROBLEM.

Consider the following two statements (*) and (**):

(*) Let $E$ be a $K$-vector space and let $||| \cdot |||$ be a norm on $E$. Then there exists a norm $||| \cdot ||'|$ on $E$, equivalent to $||| \cdot |||$, such that $|||x||'| \in |K|$ for all $x \in E$.

(**) Let $K$ be spherically complete and let $A$ be a complete absolutely convex compactoid in a Hausdorff locally convex space over $K$. Then there exist a $\lambda \in K$ with $|\lambda| > 1$ and a $c'$-compact $B$ such that $A \subset B \subset \lambda A$.

The question as to whether (*) is true or not is known as Serre's renorming problem. See [2] for more details. We are able to reformulate this problem in terms of compactoids:

**Proposition 2.1.** The above statements (*) and (**) are equivalent.

**Proof.** Assume (*). To prove (**) we may assume that $A$ is edged. By [8], Theorem 3, $A$, as a topological module over $B(0,1) := \{ \lambda \in K : |\lambda| \leq 1 \}$, is isomorphic to a bounded submodule of $K^I$ for some set $I$. Let $E$ be the algebraic direct sum $\oplus_{i \in I} K_i$ where $K_i = K$ for all $i \in I$.

Then $(E^*, \sigma(E^*, E))$ is in a natural way isomorphic to $K^I$ with the product topology. So we may assume that $A = p^0$ where $p$ is a seminorm on $E$.

By (*) there exists a seminorm $q$, equivalent to $p$, such that $q(x) \in |K|$ for all $x \in E$. By a suitable scalar multiplication we can arrange that, in addition, $p \leq q \leq |\lambda|p$ for some $\lambda \in K$, $|\lambda| > 1$. Then

$$p^0 \leq q^0 \leq \lambda p^0$$
and \( q^0 \) is \( c' \)-compact by Corollary 1.2. This proves (**'). Now assume (**).

To prove (*) we may assume (see [2]), that \( K \) is spherically complete.

Let \( p \) be a norm on \( E \). By (**) there is a \( c' \)-compact \( B \) and a \( \lambda \in K, |\lambda| > 1 \) with \( 0 \in B \subseteq \lambda p^0 \). Then \( B = q^0 \) for some seminorm \( q \) on \( E \). We have

\[
p \leq q \leq |\lambda| p
\]

and \( q(x) \in |K| \) for all \( x \in E \) by Corollary 1.2.

Note. Serre's renorming problem is still unsettled as far as I know.

§3 NORMS \( p \) FOR WHICH \( p^0 \) IS A KREIN-MILMAN COMPACTOID.

Recall that an absolutely convex subset \( A \) of a locally convex space over \( K \) is a \( KM \)-compactoid if it is complete and if \( A = \overline{c^0 X} \) where \( X \) is compact. (Here \( \overline{c^0 X} \) is the closure of \( c^0 X \)).

Before stating the theorem we first make some simple observations. Let \( p \) be a norm on \( E \). We say that a collection \( (e_i) \) in \( E \) is a \( p \)-orthonormal base of \( E \) if for each \( x \in E \) there exist a unique \( \{i \in I, |\lambda_i| \geq \epsilon \} \) is finite for each \( \epsilon > 0 \) and

\[
x = \sum_{i \in I} \lambda_i e_i
\]

\[
p(x) = \max_{i} |\lambda_i|
\]

If \( (E, p) \) is complete this definition coincides with the usual one.

**Lemma 3.1.** Let \( (E, p) \) be a normed space, let \( (\hat{E}, \hat{p}) \) be its completion.

Then \( (E, p) \) has a \( p \)-orthonormal base if and only if \( (\hat{E}, \hat{p}) \) has a

\( \hat{p} \)-orthonormal base.
Proof. It is not hard to see that each \( p \)-orthonormal base of \((E,p)\) is also a \( p \)-orthonormal base of \((E,p')\). Conversely, let \( (e_i)_{i \in I} \) be a \( p \)-orthonormal base of \((E,p')\). For each \( i \in I \), choose an \( f_i \in E \) with \( p(e_i - f_i) \leq \frac{1}{2} \).

By [1], Exercise 5.C, \( (f_i)_{i \in I} \) is a \( p \) orthonormal base of \((E,p')\).

Clearly \( (f_i)_{i \in I} \) is a \( p \)-orthonormal base of \((E,p)\).

**Theorem 3.2.** For a polar norm \( p \) on a \( K \)-vector space \( E \) the following are equivalent.

1. \((E,p)\) has a \( p \)-orthonormal base
2. \( p^0 \) is a KM-compactoid.

**Proof.** (a) \( \Rightarrow \) (b). Let \( (e_i)_{i \in I} \) be a \( p \)-orthonormal base of \((E,p)\). The formula

\[ \phi(f) = (f(e_i))_{i \in I} \]

defines a map \( \phi : p^0 \to B(0,1)^I \). Straightforward verifications show that \( \phi \) is an isomorphism of topological \( B(0,1) \)-modules. From [8], Theorem 16 we obtain that \( B(0,1)^I \), hence \( p^0 \), is a KM-compactoid.

(b) \( \Rightarrow \) (a). Suppose \( p^0 = \text{co} \ X \) where \( X \) is a compact subset of \( E^* \).

Let \( C(X^*K) \) be the Banach space of all continuous functions \( X^* \to K \), with the supremum norm \( || \cdot ||_{\infty} \). Then \( C(X^*K) \) has an orthonormal base.

([1], Theorem 5.22).

The formula

\[ \phi(x)(f) = f(x) \quad (f \in X) \]

defines a \( K \)-linear map \( \phi : E \to C(X^*K) \). From
\[ \| \phi(x) \|_\omega = \max_{f \in X} |f(x)| = \sup_{f \in \mathcal{F}} |f(x)| = \sup_{f \in \mathcal{F}} |f(x)| = p(x) \]

we obtain that \( \phi \) is an isometry \((E, p) \rightarrow (C(X^\omega K), \| \|_\omega)\).

By Gruson's Theorem ([1], 5.9) \( \hat{\phi}(E) \) has an orthonormal base. Then so has \( \hat{\phi}(E) \) by Lemma 3.1 and has \( E \).

§4 APPLICATION: A COMPLETE \( c' \)-COMPACT SET WHICH IS NOT A KM-COMPACTOID.

We shall give a negative answer to the Problem following Theorem 1.7 of [6].

PROPOSITION 4.1. Let \( K \) be spherically complete, let \( |K| = [0, \infty) \). Then there exist a locally convex space \( F \) over \( K \) and a complete \( c' \)-compact subset \( A \subset F \) which is not a KM-compactoid.

Proof. Let \( E := l^\infty \) and let \( F := (l^\infty)^* \) (with the topology we agreed upon in §0). Let \( p \) be the standard norm on \( l^\infty \), and set \( A := p^0 \). Since, trivially, \( p(x) \in |K| \) for all \( x \in l^\infty \), we have that \( p^0 \) is \( c' \)-compact (Corollary 1.2).

However, it is known ([1], Cor. 5.19) that \( l^\infty \) has no orthogonal base so that (Theorem 3.2) \( p^0 \) is not a KM-compactoid.

§5 NORMS \( p \) FOR WHICH \( p^0 \) IS METRIZABLE.

THEOREM 5.1. For a polar seminorm \( p \) on a \( K \)-vector space \( E \) the following are equivalent.

(a) \((E, p)\) is of countable type ([3], Definition 4.3).

(b) \( p^0 \) is metrizable.
Proof. (a) ⇒ (β). There exist $e_1, e_2, \ldots$ in $E$ with $p(e_i) \leq 1$ for each $i$ such that the $K$-linear span of $e_1, e_2, \ldots$ is $p$-dense in $E$. The formula

$$
\phi(f) = (f(e_1), f(e_2), \ldots)
$$

defines a map $\phi : p^0 \to B(0,1)^\mathbb{N}$. Straightforward verifications show that $\phi$ is an isomorphism of topological $B(0,1)$-modules of $p^0$ onto a submodule of $B(0,1)^\mathbb{N}$.

Now $B(0,1)^\mathbb{N}$ is metrizable (the product topology is induced by the metric

$$(a,b) \mapsto \sup_{i \in \mathbb{N}} |a_i - b_i| 2^{-i}$$

hence so is $p^0$.

(β) ⇒ (a). Let $\lambda \in K, |\lambda| > 1$. Since $p^0$ is a metrizable compactoid there exist, by [3], Proposition 8.2, $f_1, f_2, \ldots \in \lambda p^0$ with $\lim_{n \to \infty} f_n = 0$ such that

$$p^0 \subset \overline{\{f_1, f_2, \ldots\}} = \lambda p^0$$

The map

$$\phi : x \mapsto (f_1(x), f_2(x), \ldots) \quad (x \in E)$$

is $K$-linear, $\phi(E) \subset c_0$. We have for $x \in E$

$$||\phi(x)|| = \sup_{n \in \mathbb{N}} |f_n(x)| = \sup \{|g(x)| : g \in \overline{\{f_1, f_2, \ldots\}}\}$$

It follows that

$$p(x) \leq ||\phi(x)|| \leq |\lambda| p(x)$$

so that $p$ is equivalent to $x \mapsto ||\phi(x)||$, a seminorm of countable type. Hence, $p$ is of countable type.
§6 APPLICATION: DESCRIPTION OF METRIZABLE COMPACTOIDS.

THEOREM 6.1. Let $A$ be an absolutely convex subset of a Hausdorff locally convex space $F$ over $K$. The following are equivalent.

(a) $A$ is a metrizable compactoid.

(b) As a topological $B(0,1)$-module, $A$ is isomorphic to a submodule of $B(0,1)^\mathbb{N}$.

(c) As a topological $B(0,1)$-module, $A$ is isomorphic to a compactoid in $C_0$.

(d) For each $\lambda \in K$, $|\lambda| > 1$ then exist $e_1^\lambda, e_2^\lambda, \ldots \in \lambda A$ with $\lim_{n \to \infty} e_n^\lambda = 0$ and $A \subseteq \overline{\text{co}} \{e_1^\lambda, e_2^\lambda, \ldots\}$.

(e) There exist $e_1, e_2, \ldots \in F$ with $\lim_{n \to \infty} e_n = 0$ and $A \subseteq \overline{\text{co}} \{e_1, e_2, \ldots\}$.

(f) There exists an ultrametrizable compact $X \subseteq F$ with $A \subseteq \overline{\text{co}} X$.

Proof. $(a) \Rightarrow (b)$. It is not hard to see, by using the absolute convexity of $A$, that $\overline{A}$ is also metrizable. As there is no harm in taking $F$ complete we therefore may assume that $A$ is complete. To prove $(b)$ we also may assume that $A$ is edged. By [8], Theorem 3, $A \subseteq B(0,1)^I \subseteq K^I$ for some set $I$. Like in the proof of Proposition 2.1 we may conclude that $A = p^0$ where $p$ is a polar seminorm on $\oplus_{i \in I} K_i$ ($K_i = K$ for each $i$). Then $p$ is of countable type by Theorem 5.1. From the proof of $(a) \Rightarrow (b)$ of that Theorem we obtain an isomorphism $A = p^0 \cong B(0,1)^\mathbb{N}$.

$(b) \Rightarrow (c)$. Choose $\lambda_1, \lambda_2, \ldots \in K$, $|\lambda_1| > |\lambda_2| > \ldots$, $\lim_{n \to \infty} \lambda_n = 0$. The formula

$$\phi((a_i)_{i \in \mathbb{N}}) = (\lambda_1 a_1, \lambda_2 a_2, \ldots) \in c_0$$

defines a $B(0,1)$-module isomorphism of $B(0,1)^\mathbb{N}$ onto $C := \overline{\text{co}} \{\lambda_1 e_1, \lambda_2 e_2, \ldots\}$ where $e_1, e_2, \ldots$ are the standard unit vectors in $c_0$. $\phi$ is a homeomorphism.
B(0,1)^\mathbb{N} \to C$, and maps $A$ onto a compactoid in $c_0$.

(γ) ⇒ (δ). See [3], Proposition 8.2.

(δ) ⇒ (ε) is trivial.

(ε) ⇒ (η), $\{0,e_1,e_2,\ldots\}$ is compact and ultrametrizable.

(η) ⇒ (α). We may assume that $F$ is complete. It suffices to prove the metrizability of $B := \overline{co} X$.

$B$ is a complete, edged compactoid. As before we may assume that $B = p^0$ for some polar seminorm $p$ on some $K$-vector space $E$ while $B \subseteq E^*$. The map $\phi : E \to C(X \to K)$ defined by

$$\phi(x) (f) = f(x) \quad (f \in X)$$

is an isometry $(E,p) \to (C(X \to K), ||||_\omega)$.

Now $X$ is ultrametrizable so by [1], Exercise 3.5, $C(X \to K)$ is of countable type. Hence so is $p$. By Theorem 5.1, $B = p^0$ is metrizable.

§7 NORMS $p$ FOR WHICH $(p^0)^i$ IS OF FINITE TYPE.

Recall that an absolutely convex set $A$ in a locally convex space $F$ over $K$ is of finite type if for each zero neighbourhood $U$ in $F$ there exists a finite-dimensional bounded set $S \subset A$ such that $A \subseteq U + S$.

Let us say that a seminorm $p$ on a $K$-vector space $E$ is of finite type if $\ker p = \{x \in E : p(x) = 0\}$ has finite codimension.

LEMMA 7.1. Let $A$ be an absolutely convex subset of a locally convex space $F$ whose topology is generated by a collection of seminorms of finite type. Then the following are equivalent.

(a) $A$ is a compactoid of finite type.

(b) For each closed linear subspace $H$ of finite codimension there is a finite dimensional bounded set $S \subset A$ with $A \subset H + S$. 
Proof. \((a) \Rightarrow (\beta).\) (Note. This implication holds for any locally convex space \(F.\)) We may assume \([A] = F.\)

\(H\) has the form \(D^1 := \{x \in F : f(x) = 0 \text{ for all } f \in D\}\) where \(D\) is a finite dimensional subspace of \(F'.\) Let \(f_1, \ldots, f_n\) be a base of \(D.\) There exist \(x_1, \ldots, x_n \in F\) with \(f_i(x_j) = \delta_{ij}\) \((i, j \in \{1, \ldots, n\}).\) Since \([A] = F\) there exists a \(\lambda \in K, \lambda \neq 0\) such that \(\lambda x_i \in A\) for each \(i \in \{1, \ldots, n\}.\)

Set

\[ U := \bigcap_{i=1}^{n} \{x \in F : |f_i(x)| \leq |\lambda|\} \]

Then \(U\) is a zero neighbourhood in \(F.\) \(A\) is a compactoid of finite type, so there exists a finite dimensional set \(S_1 \subset A\) with \(A \subset U + S_1.\) Let \(x \in U.\) Write \(x = y + z\) where

\[ y := x - \sum_{i=1}^{n} f_i(x) x_i \]
\[ z := \sum_{i=1}^{n} f_i(x) x_i \]

Now, since \(x \in U, |f_i(x)| \leq |\lambda|\) for each \(i\) so that \(z = \sum_{i=1}^{n} f_i(x) x_i \in A.\)

Further, for each \(j \in \{1, \ldots, n\}\)

\[ f_j(y) = f_j(x) - \sum_{i=1}^{n} f_i(x) f_j(x_i) = f_j(x) - f_j(x) = 0 \]

and it follows that \(y \in D^1 = H.\) So \(x = y + z \in H + [x_1, \ldots, x_n] \cap A.\) We see that

\[ A \subset U + S_1 \subset H + S_2 + S_1 \]

where \(S_2 := [x_1, \ldots, x_n] \cap A.\) Then \((\beta)\) is proved with \(S := S_1 + S_2.\)

\((\beta) \Rightarrow (a).\) Let \(U\) be a zero neighbourhood in \(F.\) Since continuous seminorms are of finite type, \(U\) contains a closed subspace \(H\) of finite codimension.

By \((\beta)\) there exists a finite dimensional set \(S \subset A\) with \(S\) bounded and
A \subset H + S. Then A \subset U + S.

From now on we assume that the valuation on K is dense.

Recall that for an absolutely convex set B we have \( B^\dagger := \bigcup |\lambda|<1 \lambda B. \)

**Theorem 7.2.** Let p be a polar norm on a K-vector space E. Then the following are equivalent.

(a) For each finite dimensional subspace D of E there exists a seminorm q on E, q of finite type, q \leq p and q = p on D.

(b) \( (p^0)^1 \) is of finite type.

**Proof.** (a) \(\Rightarrow \) (b). As each continuous seminorm on E* is of finite type it suffices to prove, by Lemma 7.1, that for a closed subspace H of E* of finite codimension there exists a finite dimensional set S \in (p^0)^1 such that \( (p^0)^1 \subset H + S. \)

Now, by (a), there is a seminorm q of finite type, q \leq p on E and q = p on D := H^1. Let

\[ S_1 := \{ f \in E^* : |f| \leq q \}. \]

We see that \( S_1 \) is finite dimensional and since q \leq p we have \( S_1 \subset p^0. \)

We now shall prove that \( (p^0)^1 \subset H + S \) where \( S := (S_1)^1. \)

In fact, let \( f \in (p^0)^1. \) Then there is a \( \lambda \in K, 0 < |\lambda| < 1 \) with \( |f| \leq |\lambda| p. \)

Choose \( \lambda' \in K \) with \( |\lambda| < |\lambda'| < 1. \)

We have \( |f| \leq |\lambda| q \) on D (since p = q on D) so we can extend f to a g \in E* with \( |g| \leq |\lambda'| q \) on E. (This is because q is of finite type so that (E,q) is strongly polar.) Now write

\[ f = f - g + g. \]

Since \( f = g \) on D we have \( f - g \in D^\dagger = H. \)
Also, $| (\lambda ^ t)^{-1} g | \leq q$ so that $(\lambda ^ t)^{-1} g \in S_1$ i.e. $g \in (S_1)^i = S$.

$(\beta) \Rightarrow (a)$. By lemma 7.1 there exists a finite dimensional set $S \subset (p^0)_1$ so that $(p^0)_1 = D^1 \cap (p^0)_1 \cup S$.

Set $q(x) := \sup_{h \in S} |h(x)|$. (x £ E).

Then $q(x) = 0$ for all $x$ in the space $S^1$ which has finite codimension.

So $q$ is of finite type.

Further, for $x \in E$ we have

$$q(x) = \sup_{h \in S} |h(x)| \leq \sup_{h \in (p^0)_1} |h(x)| = \sup_{h \in (p^0)_1} |h(x)| = p(x),$$

so $q \leq p$. Finally, if $x \in D$ then

$$p(x) = \sup_{f \in p^0} |f(x)| = \sup_{f \in (p^0)_1} |f(x)| = \sup_{h \in D^1 \cap (p^0)_1} |h(x) + t(x)|$$

$$= \sup_{t \in S} |t(x)| = q(x).$$

Hence, $p = q$ on $D$.

§8 APPLICATION: A COMPLETE COMPACTOID IN $c_0$ THAT IS NOT OF FINITE TYPE.

If $K$ is spherically complete each complete absolutely convex compactoid is of finite type (See [4], 2.3).

If $K$ is not spherically complete the unit ball of $c_0$ is a complete compactoid for the weak topology but not of finite type (See [5], 1.6). This is a non-metrizable compactoid. A compactoid in $(c_0, || ||)$, not of finite type, is given in [5], 1.4. However this compactoid is not closed. The following example provides an answer to the Problem following 1.5 in [5].
PROPOSITION 8.1. Let $K$ be not spherically complete. Then there exists an absolutely convex complete compactoid in $c_0$ that is not of finite type.

Proof. Let $(K^v, | |)$ be the spherical completion of $(K, | |)$ in the sense of [1], Theorem 4.49. Let $E$ be a $K$-subspace of $K^v$ of countably infinite dimension and let $p$ be the valuation $| |$ restricted to $E$. Then $x, y \in E$, $x \perp y$ in the sense of $p$ implies $x = 0$ or $y = 0$. Obviously, the norm $p$ is of countable type (hence polar) so, by Theorem 5.1, $p^0$ is metrizable and is by Theorem 6.1, isomorphic to a compactoid in $c_0$.

Suppose $p^0$ were of finite type. Then so would $(p^0)^i$ ([5], Proposition 2.4). By Theorem 7.2 we would have a seminorm $q$ on $E$, $q \leq p$, $q$ of finite type, $q(x) = p(x)$ for some $x \in E$, $x \neq 0$. But then $x \perp \text{Ker} q$ in the sense of $p$ (If $q(y) = 0$ then $p(x-y) \geq q(x-y) = q(x) = p(x)$) which is impossible. So, $p^0$ is not of finite type.
REFERENCES


