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A CONNECTION BETWEEN p-ADIC BANACH SPACES AND LOCALLY CONVEX COMPACTOIDS

by

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ABSTRACT. For a vector space $E$ over a non-archimedean valued field $K$ a correspondence $p \mapsto p^0$ is established between seminorms $p$ on $E$ and compactoids $p^0$ in $E^*$. Examination of it yields the solution of two open problems (see §4 and §8) and a reformulation of Serre's renorming problem (see §2). As a by-product results on metrizable compactoids are obtained (see §6).

§0 THE CORRESPONDENCE $p \mapsto p^0$.

Throughout this note $K$ is a non-archimedean valued field, complete with respect to the metric induced by the nontrivial valuation $| |$.

Let $E$ be a $K$-vector space, let $E^*$ be its algebraic dual. A (non-archimedean) seminorm $p$ on $E$ is polar ([3], Definition 3.1), if

$$p = \sup \{ |f| : f \in E^*, |f| \leq p \}$$

Let $P_E$ be the set of all polar seminorms on $E$.

For each $p \in P_E$ we set

$$p^0 = \{ f \in E^* : |f| \leq p \}$$

Then $p^0$ is an absolutely convex, edged ([3],§1b) subset of $E^*$. It is easy to see that $p^0$ is a closed compactoid ([3],§1e) with respect to the topology $\sigma(E^*, E)$, hence complete.
Let \( C_E^* \) be the set of all closed absolutely convex, edged compactoids in \( E^* \) with respect to \( \sigma(E^*, E) \).

**Proposition 0.** The map \( p \mapsto p^0 \) is a bijection of \( P_E \) onto \( C_E^* \). Its inverse assigns to every \( A \in C_E^* \) the seminorm \( p \) given by

\[
p(x) = \sup \{ |f(x)| : f \in A \} \quad (x \in E)
\]

**Proof.** We shall prove surjectivity of \( p \mapsto p^0 \) leaving the (easy) rest of the proof to the reader. So, let \( A \in C_E^* \); we shall prove that \( A = p^0 \) where \( p(x) = \sup \{ |f(x)| : f \in A \} \).

Obviously, \( A \subseteq p^0 \). Now let \( g \in E^* \setminus A \), we prove that \( g \notin p^0 \). The space \( E^* \) is of countable type hence strongly polar ([3], Theorem 4.4). So by [3], Theorem 4.7, there exists a \( \theta \in (E^*, \sigma(E^*, E)^*)' \) such that \( |\theta| \leq 1 \) on \( A \), \( |\theta(g)| > 1 \). But, by [3], lemma 7.1, \( \theta \) has the form \( f \mapsto f(x) \) for some \( x \in E \). Thus, \( |f(x)| \leq 1 \) for \( f \in A \), \( |g(x)| > 1 \) i.e., \( p(x) \leq 1 \) and \( |g(x)| > 1 \) and it follows that \( g \notin p^0 \).

**Remarks.**

1. Let \( K \) be spherically (= maximally) complete. Then each nonarchimedean seminorm \( p \) on \( E \) for which \( p(x) \in \overline{|x|} \) \( (x \in E) \) is polar ([3], Remark following 3.1).

2. Let \( \tau \) be the locally convex topology on \( E \) induced by all nonarchimedean seminorms i.e., \( \tau \) is the strongest among all locally convex topologies on \( E \). It is not hard to see that \( (E, \tau) \) is a complete polar ([3], Definition 3.5) space and that \( (E, \tau) \) and \( (E^*, \sigma(E^*, E)) \) are each others strong dual spaces.
§1 NORMS \( p \) FOR WHICH \( p^0 \) IS \( c' \)-COMPACT

Recall that an absolutely convex subset \( A \) of a locally convex space \( F \) over \( K \) is \( c' \)-compact if for each neighbourhood \( U \) of 0 in \( F \) there exist \( x_1, \ldots, x_n \in A \) (rather than \( x_1, \ldots, x_n \in F \)) such that \( A \subseteq U + \text{co} \{x_1, \ldots, x_n\} \).
(Here \( \text{co} \) indicates the absolutely convex hull)

THEOREM 1.1. For a polar seminorm \( p \) on a \( K \)-vector space \( E \) the following are equivalent.

(a) \( p(x) \in |K| \) for each \( x \in E \). Each onedimensional subspace of \( E \) has a \( p \)-orthocomplement.

(\( \beta \) \( p^0 \) is \( c' \)-compact.

Proof. \((a) \Rightarrow (\beta)\). By [7], Theorem 3.2, it suffices to prove that for each \( \phi \in (E^*, \sigma(E^*, E))' \)

\[
\max \{ |\phi(f)| : f \in p^0 \}
\]

exists. Since \( \phi \) is an evaluation map we therefore have to show that

\[
\max \{ |f(x)| : f \in p^0 \}
\]

exists for each \( x \in E \). This is obviously true if \( p(x) = 0 \). So assume \( p(x) > 0 \). Since \( p(x) \in |K| \) we may assume that \( p(x) = 1 \). For such \( x \) we must prove

\[
\max \{ |f(x)| : f \in p^0 \} = 1
\]

By \((a)\), \( Kx \) has a \( p \)-orthocomplement \( H \). The function

\[
f : \lambda x + h \mapsto \lambda \quad (\lambda \in K, h \in H)
\]
is in $E^*$. We have $|f(x)| = 1$. For $\lambda \in K, h \in H$

$$|f(\lambda x + h)| = |\lambda| = p(\lambda x) \leq \max(p(\lambda x), p(h)) = p(\lambda x + h)$$

so that $f \in p^0$.

$(\beta) \Rightarrow (a)$. Let $x \in E$. The map $f \mapsto |f(x)|$ $(f \in E^*)$

is a continuous seminorm on $(E^*, c(E^*, E))$. By $c'$-compactness its

restriction to $p^0$ has a maximum so there exists a $g \in p^0$ with $|g(x)| = p(x)$

(It follows that $p(x) \in |K|$). We prove that Ker $g$ is a $p$-orthocomplement

of $Kx$. In fact, for $z \in \text{Ker } g$ we have

$$p(x+z) \geq |g(x+z)| = |g(x)| = p(x)$$

Then also

$$p(x+z) \geq p(z)$$

completing the proof of Theorem 1.1.

Note. It is not hard to see that (a) of above is equivalent too.

$(\gamma)$ For each $x \in E$ there exists an $f \in E^*$ with $|f(x)| = p(x)$ and $|f| \leq p$.

For spherically complete $K$ we obtain a simpler form of Theorem 1.1.

COROLLARY 1.2. Let $K$ be spherically complete, let $p$ be a seminorm on

$E$ for which $p(x) \in |K|$ for all $x \in E$.

Then the following are equivalent.

$(a)$ $p(x) \in |K|$ for each $x \in E$.

$(\beta) p^0$ is $c'$-compact.

Proof. By [1], lemma 4.35, each onedimensional subspace has a $p$-ortho-

complement.
§2 APPLICATION: A NEW LIGHT ON SERRE'S RENORMING PROBLEM.

Consider the following two statements (*) and (**).

(*) Let E be a K-vector space and let $|||\ |\ |\ |'$ be a norm on E. Then there exists a norm $|||\ |\ |'$ on E, equivalent to $||\ |\ |$, such that $||x||' \in |K|$ for all $x \in E$.

(**) Let K be spherically complete and let A be a complete absolutely convex compactoid in a Hausdorff locally convex space over K. Then there exist a $\lambda \in K$ with $|\lambda| > 1$ and a $c'$-compact B such that $A \subseteq B \subseteq \lambda A$.

The question as to whether (*) is true or not is known as Serré's renorming problem. See [2] for more details. We are able to reformulate this problem in terms of compactoids:

**PROPOSITION 2.1. The above statements (*) and (**) are equivalent.**

**Proof.** Assume (*). To prove (**) we may assume that A is edged. By [8], Theorem 3, A, as a topological module over $B(0,1) := \{\lambda \in K: |\lambda| \leq 1\}$, is isomorphic to a bounded submodule of $K^I$ for some set I. Let E be the algebraic direct sum $\oplus K_i$ where $K_i = K$ for all $i \in I$.

Then $(E^*, \sigma(E^*, E))$ is in a natural way isomorphic to $K^I$ with the product topology. So we may assume that $A = p^0$ where $p$ is a seminorm on E.

By (*) there exists a seminorm $q$, equivalent to $p$, such that $q(x) \in |K|$ for all $x \in E$. By a suitable scalar multiplication we can arrange that, in addition, $p \leq q \leq |\lambda|p$ for some $\lambda \in K$, $|\lambda| > 1$. Then

$$p^0 \leq q^0 \leq \lambda p^0$$
and $q^0$ is $c'$-compact by Corollary 1.2. This proves (**). Now assume (**).

To prove (*) we may assume (see [2]), that $K$ is spherically complete.

Let $p$ be a norm on $E$. By (***) there is a $c'$-compact $B$ and a $\lambda \in K$, $|\lambda| > 1$
with $p^0 \subset B \subset \lambda p^0$. Then $B = q^0$ for some seminorm $q$ on $E$. We have

$$p \leq q \leq |\lambda| p$$

and $q(x) \in |K|$ for all $x \in E$ by Corollary 1.2.

Note. Serre's renorming problem is still unsettled as far as I know.

§3 NORMS $p$ FOR WHICH $p^0$ IS A KREIN-MILMAN COMPACTOID.

Recall that an absolutely convex subset $A$ of a locally convex space
over $K$ is a $\text{KM-compactoid}$ if it is complete and if $A = \overline{\text{co } X}$ where $X$ is
compact. (Here $\overline{\text{co } X}$ is the closure of $\text{co } X$).

Before stating the theorem we first make some simple observations. Let
$p$ be a norm on $E$. We say that a collection $(e_i)_{i \in I}$ in $E$ is a
$p$-orthonormal base of $E$ if for each $x \in E$ there exist a unique
$(\lambda_i)_{i \in I} \in K^I$ such that $\{i \in I, |\lambda_i| > \varepsilon\}$ is finite for each $\varepsilon > 0$ and

$$x = \sum_{i \in I} \lambda_i e_i$$

$$p(x) = \max_i |\lambda_i|$$

If $(E,p)$ is complete this definition coincides with the usual one.

**Lemma 3.1.** Let $(E,p)$ be a normed space, let $(\hat{E},\hat{p})$ be its completion.
Then $(E,p)$ has a $p$-orthonormal base if and only if $(\hat{E},\hat{p})$ has a $\hat{p}$-orthonormal base.
Proof. It is not hard to see that each p-orthonormal base of \((E, p)\) is also a \(p\)-orthonormal base of \((E^\sim, p^\sim)\). Conversely, let \((e_i)_{i \in I}\) be a \(p\)-orthonormal base of \((E^\sim, p^\sim)\). For each \(i \in I\), choose an \(f_i \in E\) with 
\[ p(e_i - f_i) \leq \frac{1}{2}. \]
By [1], Exercise 5.C, \((f_i)_{i \in I}\) is a \(p\)-orthonormal base of \((E^\sim, p^\sim)\).

Clearly \((f_i)_{i \in I}\) is a \(p\)-orthonormal base of \((E, p)\).

**THEOREM 3.2.** For a polar norm \(p\) on a \(K\)-vector space \(E\) the following are equivalent.

(a) \((E, p)\) has a \(p\)-orthonormal base

(b) \(p^0\) is a KM-compactoid.

**Proof.** (a) \(\Rightarrow\) (b). Let \((e_i)_{i \in I}\) be a \(p\)-orthonormal base of \((E, p)\). The formula

\[ \phi(f) = (f(e_i))_{i \in I} \]

defines a map \(\phi: p^0 \rightarrow B(0,1)^I\). Straightforward verifications show that \(\phi\) is an isomorphism of topological \(B(0,1)^I\)-modules. From [8], Theorem 16 we obtain that \(B(0,1)^I\), hence \(p^0\), is a KM-compactoid.

(b) \(\Rightarrow\) (a). Suppose \(p^0 = \text{co} X\) where \(X\) is a compact subset of \(E^*\).

Let \(C(X^*K)\) be the Banach space of all continuous functions \(X \rightarrow K\), with the supremum norm \(||||_\infty\). Then \(C(X^*K)\) has an orthonormal base. ([1], Theorem 5.22).

The formula

\[ \phi(x)(f) = f(x) \quad (f \in X) \]

defines a \(K\)-linear map \(\phi: E \rightarrow C(X^*K)\). From
\[ \| \phi(x) \|_\infty = \max_{f \in X} |f(x)| = \sup_{f \in X} |f(x)| = \sup_{f \in \mathcal{P}} |f(x)| = p(x) \]

we obtain that \( \phi \) is an isometry \( (E, p) \to (C(X^\mathbb{R}, |x|), \| \cdot \|_\infty) \).

By Grušon's Theorem ([1], 5.9) \( \overline{\phi(E)} \) has an orthonormal base. Then so has \( \phi(E) \) by Lemma 3.1 and has \( E \).

§4 APPLICATION: A COMPLETE \( c^\prime \)-COMPACT SET WHICH IS NOT A KM-COMPACTOID.

We shall give a negative answer to the Problem following Theorem 1.7 of [6].

PROPOSITION 4.1. Let \( K \) be spherically complete, let \( |K| = [0, \infty) \).

Then there exist a locally convex space \( F \) over \( K \) and a complete \( c^\prime \)-compact subset \( A \subset F \) which is not a KM-compactoid.

Proof. Let \( E := l^\infty \) and let \( F := (l^\infty)^* \) (with the topology we agreed upon in §0). Let \( p \) be the standard norm on \( l^\infty \), and set \( A := p^0 \). Since, trivially, \( p(x) \leq |K| \) for all \( x \in l^\infty \), we have that \( p^0 \) is \( c^\prime \)-compact (Corollary 1.2).

However, it is known ([1], Cor. 5.19) that \( l^\infty \) has no orthogonal base so that (Theorem 3.2) \( p^0 \) is not a KM-compactoid.

§5 NORMS \( p \) FOR WHICH \( p^0 \) IS METRIZABLE.

THEOREM 5.1. For a polar seminorm \( p \) on a \( K \)-vector space \( E \) the following are equivalent.

(a) \( (E, p) \) is of countable type ([3], Definition 4.3).

(b) \( p^0 \) is metrizable.
Proof. (a) ⇒ (b). There exist $e_1, e_2, \ldots$ in $E$ with $p(e_i) \leq 1$ for each $i$ such that the $K$-linear span of $e_1, e_2, \ldots$ is $p$-dense in $E$. The formula

$$ \phi(f) = (f(e_1), f(e_2), \ldots) $$

defines a map $\phi : p^0 \to B(0,1)^\mathbb{N}$. Straightforward verifications show that $\phi$ is an isomorphism of topological $B(0,1)$-modules of $p^0$ onto a submodule of $B(0,1)^\mathbb{N}$.

Now $B(0,1)^\mathbb{N}$ is metrizable (the product topology is induced by the metric $(a,b) \mapsto \sup_{i \in \mathbb{N}} |a_i - b_i|2^{-i}$) hence so is $p^0$.

(b) ⇒ (a). Let $\lambda \in K, |\lambda| > 1$. Since $p^0$ is a metrizable compactoid there exist, by [3], Proposition 8.2, $f_1, f_2, \ldots \in \lambda p^0$ with $\lim_{n \to \infty} f_n = 0$ such that

$$ p^0 \subseteq \overline{\co \{f_1, f_2, \ldots\}} \subset \lambda p^0 $$

The map

$$ \phi : x \mapsto (f_1(x), f_2(x), \ldots) \quad (x \in E) $$

is $K$-linear, $\phi(E) \subset c_0$. We have for $x \in E$

$$ ||\phi(x)|| = \sup_{n \in \mathbb{N}} |f_n(x)| = \sup_{n \in \mathbb{N}} \{||g(x)|| : g \in \overline{\co \{f_1, f_2, \ldots\}}\} $$

It follows that

$$ p(x) \leq ||\phi(x)|| \leq |\lambda|p(x) $$

so that $p$ is equivalent to $x \mapsto ||\phi(x)||$, a seminorm of countable type. Hence, $p$ is of countable type.
§6 APPLICATION: DESCRIPTION OF METRIZABLE COMPACTOIDS.

THEOREM 6.1. Let $A$ be an absolutely convex subset of a Hausdorff locally convex space $F$ over $K$. The following are equivalent.

(a) $A$ is a metrizable compactoid.

(b) As a topological $B(0,1)$-module, $A$ is isomorphic to a submodule of $B(0,1)^{\mathbb{N}}$.

(c) As a topological $B(0,1)$-module, $A$ is isomorphic to a submodule of $B(0,1)^{\mathbb{N}}$.

(d) For each $\lambda \in K$, $|\lambda| > 1$ then exist $e_1, e_2, \ldots \in \lambda A$ with $\lim_{n \to \infty} e_n = 0$ and $A \subset \overline{e_1, e_2, \ldots}$.

(e) There exist $e_1, e_2, \ldots \in F$ with $\lim_{n \to \infty} e_n = 0$ and $A \subset \overline{e_1, e_2, \ldots}$.

(f) There exists an ultrametrizable compact $X \subset F$ with $A \subset \overline{X}$.

Proof. (a) $\Rightarrow$ (b). It is not hard to see, by using the absolute convexity of $A$, that $\overline{A}$ is also metrizable. As there is no harm in taking $F$ complete we therefore may assume that $A$ is complete. To prove (b) we also may assume that $A$ is edged. By [8], Theorem 3, $A = B(0,1)^I \subset K^I$ for some set $I$. Like in the proof of Proposition 2.1 we may conclude that $A = p^0$ where $p$ is a polar seminorm on $\oplus_{i \in I} (K_i = K$ for each $i$). Then $p$ is of countable type by Theorem 5.1. From the proof of (a) $\Rightarrow$ (b) of that Theorem we obtain an isomorphism $A = p^0 \cong B(0,1)^{\mathbb{N}}$.

(b) $\Rightarrow$ (c). Choose $\lambda_1, \lambda_2, \ldots \in K$, $|\lambda_1| > |\lambda_2| > \ldots$, $\lim_{n \to \infty} \lambda_n = 0$. The formula

$$\phi((a_i)_{i \in \mathbb{N}}) = (\lambda_1 a_1, \lambda_2 a_2, \ldots) \in c_0$$

defines a $B(0,1)$-module isomorphism of $B(0,1)^{\mathbb{N}}$ onto $C := \overline{\lambda_1 e_1, \lambda_2 e_2, \ldots}$ where $e_1, e_2, \ldots$ are the standard unit vectors in $c_0$. $\phi$ is a homeomorphism.
B(0,1) N → C, and maps A onto a compactoid in c 0.

(γ) ⇒ (δ). See [3], Proposition 8.2.

(δ) ⇒ (ε) is trivial.

(ε) ⇒ (η). [0,e 1,e 2,...] is compact and ultrametrizable.

(η) ⇒ (α). We may assume that F is complete. It suffices to prove the
metrizability of B := co X.

B is a complete, edged compactoid. As before we may assume that B = p 0
for some polar seminorm p on some K-vector space E while B ⊂ E*. The
map $ φ : E → C(X → K) $ defined by

$$ φ(x)(f) = f(x) \quad (f ∈ X) $$

is an isometry $ (E,p) → (C(X → K), || ||_ω) $. 

Now X is ultrametrizable so by [1], Exercise 3.5, C(X → K) is of
countable type. Hence so is p. By Theorem 5.1f B = p 0 is metrizable.

§7 NORMS p FOR WHICH $ (p 0)^1 $ IS OF FINITE TYPE.

Recall that an absolutely convex set A in a locally convex space F over
K is of finite type if for each zero neighbourhood U in F there exists
a finite-dimensional bounded set S ⊂ A such that A ⊂ U + S.

Let us say that a seminorm p on a K-vector space E is of finite type
if Ker p = {x ∈ E : p(x) = 0} has finite codimension.

LEMMA 7.1. Let A be an absolutely convex subset of a locally convex
space F whose topology is generated by a collection of seminorms of
finite type. Then the following are equivalent.

(a) A is a compactoid of finite type.

(b) For each closed linear subspace H of finite codimension there is a
finite dimensional bounded set S ⊂ A with A ⊂ H + S.
Proof. (a) $\Rightarrow$ (b). (Note. This implication holds for any locally convex space $F$.) We may assume $[A] = F$.

$H$ has the form $D^1 := \{ x \in F : f(x) = 0 \text{ for all } f \in D \}$ where $D$ is a finite dimensional subspace of $F'$. Let $f_1, \ldots, f_n$ be a base of $D$. There exist $x_1, \ldots, x_n \in F$ with $f_i(x_j) = \delta_{ij}$ ($i, j \in \{1, \ldots, n\}$). Since $[A] = F$ there exists a $\lambda \in K$, $\lambda \neq 0$ such that $\lambda x_i \in A$ for each $i \in \{1, \ldots, n\}$.

Set

$$U := \bigcap_{i=1}^{n} \{ x \in F : |f_i(x)| \leq |\lambda| \}$$

Then $U$ is a zero neighbourhood in $F$. $A$ is a compactoid of finite type, so there exists a finite dimensional set $S_1 \subseteq A$ with $A \subseteq U + S_1$. Let $x \in U$. Write $x = y + z$ where

$$y := x - \sum_{i=1}^{n} f_i(x) x_i$$

$$z := \sum_{i=1}^{n} f_i(x) x_i$$

Now, since $x \in U$, $|f_i(x)| \leq |\lambda|$ for each $i$ so that $z = \sum_{i=1}^{n} f_i(x) x_i \in A$.

Further, for each $j \in \{1, \ldots, n\}$

$$f_j(y) = f_j(x) - \sum_{i=1}^{n} f_i(x) f_j(x_i) = f_j(x) - f_j(x) = 0$$

and it follows that $y \in D^1 = H$. So $x = y + z \in H + [x_1, \ldots, x_n] \cap A$. We see that

$$A \subseteq U + S_1 \subseteq H + S_2 + S_1$$

where $S_2 := [x_1, \ldots, x_n] \cap A$. Then (b) is proved with $S := S_1 + S_2$.

(b) $\Rightarrow$ (a). Let $U$ be a zero neighbourhood in $F$. Since continuous seminorms are of finite type, $U$ contains a closed subspace $H$ of finite codimension.

By (b) there exists a finite dimensional set $S \subseteq A$ with $S$ bounded and
A \subset H + S. Then A \subset U + S.

From now on we assume that the valuation on \( K \) is dense.

Recall that for an absolutely convex set \( B \) we have \( B^i := \bigcup_{|\lambda|<1} \lambda B \).

**Theorem 7.2.** Let \( p \) be a polar norm on a \( K \)-vector space \( E \). Then the following are equivalent.

(a) For each finite dimensional subspace \( D \) of \( E \) there exists a seminorm \( q \) on \( E \), \( q \) of finite type, \( q \leq p \) and \( q = p \) on \( D \).

(b) \((p^0)^i\) is of finite type.

**Proof.** (a) \(\Rightarrow\) (b). As each continuous seminorm on \( E^* \) is of finite type it suffices to prove, by Lemma 7.1, that for a closed subspace \( H \) of \( E^* \) of finite codimension there exists a finite dimensional set \( S \in (p^0)^i \) such that \((p^0)^i \subset H + S.\)

Now, by (a), there is a seminorm \( q \) of finite type, \( q \leq p \) on \( E \) and \( q = p \) on \( D := H^1. \) Let

\[ S_1 := \{f \in E^*: |f| \leq q\}. \]

We see that \( S_1 \) is finite dimensional and since \( q \leq p \) we have \( S_1 \subset p^0. \)

We now shall prove that \((p^0)^i \subset H + S \) where \( S := (S_1)^i. \)

In fact, let \( f \in (p^0)^i. \) Then there is a \( \lambda \in K, 0 < |\lambda| < 1 \) with \( |f| \leq |\lambda|p. \)

Choose \( \lambda' \in K \) with \( |\lambda| < |\lambda'| < 1. \)

We have \( |f| \leq |\lambda|q \) on \( D \) (since \( p = q \) on \( D \)) so we can extend \( f \) to a \( g \in E^* \) with \( |g| \leq |\lambda'|q \) on \( E. \) (This is because \( q \) is of finite type so that \((E,q)\) is strongly polar.) Now write

\[ f = f - g + g \]

Since \( f = g \) on \( D \) we have \( f - g \in D^1 = H. \)
Also, \(|(\lambda')^{-1}g| \leq q\) so that \((\lambda')^{-1}g \in S_1\) i.e. \(g \in (S_1)^i = S\).

(B) \(\Rightarrow (a)\). By lemma 7.1 there exists a finite dimensional set \(S \subset (p^0)^i\) so that \((p^0)^i = D^i \cap (p^0)^i + S\).

Set \(q(x) := \sup_{h \in S} |h(x)|\). \((x \in E)\).

Then \(q(x) = 0\) for all \(x\) in the space \(S^i\) which has finite codimension.

So \(q\) is of finite type.

Further, for \(x \in E\) we have

\[q(x) = \sup_{h \in S} |h(x)| \leq \sup_{h \in (p^0)^i} |h(x)| = \sup_{h \in p^0} |h(x)| = p(x),\]

so \(q \leq p\). Finally, if \(x \in D\) then

\[p(x) = \sup_{f \in p} |f(x)| = \sup_{f \in (p^0)^i} |f(x)| = \sup_{h \in D^i \cap (p^0)^i} |h(x) + t(x)| + \sup_{t \in S} |t(x)| = q(x).\]

Hence, \(p = q\) on \(D\).

§8 APPLICATION: A COMPLETE COMPACTOID IN \(c_0\) THAT IS NOT OF FINITE TYPE.

If \(K\) is spherically complete each complete absolutely convex compactoid is of finite type (See [4], 2.3).

If \(K\) is not spherically complete the unit ball of \(c_0\) is a complete compactoid for the weak topology but not of finite type (See [5], 1.6). This is a non-metrizable compactoid. A compactoid in \((c_0, ||||)\), not of finite type, is given in [5], 1.4. However this compactoid is not closed. The following example provides an answer to the Problem following 1.5 in [5].
PROPOSITION 8.1. Let $K$ be not spherically complete. Then there exists an absolutely convex complete compactoid in $c_0$ that is not of finite type.

Proof. Let $(K^v, |\ |)$ be the spherical completion of $(K, |\ |)$ in the sense of [1], Theorem 4.49. Let $E$ be a $K$-subspace of $K^v$ of countably infinite dimension and let $p$ be the valuation $|\ |$ restricted to $E$. Then $x, y \in E$, $x \perp y$ in the sense of $p$ implies $x = 0$ or $y = 0$. Obviously, the norm $p$ is of countable type (hence polar) so, by Theorem 5.1, $p^0$ is metrizable and is by Theorem 6.1, isomorphic to a compactoid in $c_0$.

Suppose $p^0$ were of finite type. Then so would $(p^0)^1$ ([5], Proposition 2.4). By Theorem 7.2 we would have a seminorm $q$ on $E$, $q \leq p$, $q$ of finite type, $q(x) = p(x)$ for some $x \in E, x \neq 0$. But then $x \perp \ker q$ in the sense of $p$ (If $q(y) = 0$ then $p(x-y) \geq q(x-y) = q(x) = p(x)$) which is impossible. So, $p^0$ is not of finite type.
REFERENCES


