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A CONNECTION BETWEEN p-ADIC BANACH SPACES AND LOCALLY CONVEX COMPACTOIDS

by

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ABSTRACT. For a vector space $E$ over a non-archimedean valued field $K$ a correspondence $p \mapsto p^0$ is established between seminorms $p$ on $E$ and compactoids $p^0$ in $E^*$. Examination of it yields the solution of two open problems (see §4 and §8) and a reformulation of Serre's renorming problem (see §2). As a by-product results on metrizable compactoids are obtained (see §6).

§0 THE CORRESPONDENCE $p \mapsto p^0$.

Throughout this note $K$ is a non-archimedean valued field, complete with respect to the metric induced by the nontrivial valuation $|\cdot|$. Let $E$ be a $K$-vector space, let $E^*$ be its algebraic dual. A (non-archimedean) seminorm $p$ on $E$ is polar ([3], Definition 3.1), if

$$p = \sup \{|f| : f \in E^*, |f| \leq p\}$$

Let $P_E$ be the set of all polar seminorms on $E$. For each $p \in P_E$ we set

$$p^0 = \{f \in E^* : |f| \leq p\}$$

Then $p^0$ is an absolutely convex, edged ([3],§1b) subset of $E^*$. It is easy to see that $p^0$ is a closed compactoid ([3],§1e) with respect to the topology $\sigma(E^*,E)$, hence complete.
Let $C_{E^*}$ be the set of all closed absolutely convex, edged compactoids in $E^*$ with respect to $\sigma(E^*, E)$.

**Proposition 0.** The map $p \mapsto p^0$ is a bijection of $P_E$ onto $C_{E^*}$. Its inverse assigns to every $A \in C_{E^*}$ the seminorm $p$ given by

$$p(x) = \sup \{|f(x)| : f \in A\} \quad (x \in E)$$

**Proof.** We shall prove surjectivity of $p \mapsto p^0$ leaving the (easy) rest of the proof to the reader. So, let $A \in C_{E^*}$; we shall prove that $A = p^0$ where $p(x) = \sup \{|f(x)| : f \in A\}$.

Obviously, $A \subseteq p^0$. Now let $g \in E^* \setminus A$, we prove that $g \notin p^0$. The space $E^*$ is of countable type hence strongly polar ([3], Theorem 4.4). So by ([3], Theorem 4.7, there exists a $\theta \in (E^*, \sigma(E^*, E))'$ such that $|\theta| \leq 1$ on $A$, $|\theta(g)| > 1$. But, by ([3], lemma 7.1, $\theta$ has the form $f \mapsto f(x)$ for some $x \in E$. Thus, $|f(x)| \leq 1$ for $f \in A$, $|g(x)| > 1$ i.e., $p(x) \leq 1$ and $|g(x)| > 1$ and it follows that $g \notin p^0$.

**Remarks.**

1. Let $K$ be spherically (= maximally) complete. Then each nonarchimedean seminorm $p$ on $E$ for which $p(x) \in \overline{|x|}$ ($x \in E$) is polar ([3], Remark following 3.1).

2. Let $\tau$ be the locally convex topology on $E$ induced by all nonarchimedean seminorms i.e., $\tau$ is the strongest among all locally convex topologies on $E$. It is not hard to see that $(E, \tau)$ is a complete polar ([3], Definition 3.5) space and that $(E, \tau)$ and $(E^*, \sigma(E^*, E))$ are each others strong dual spaces.
§1 NORMS p FOR WHICH p° IS c'-COMPACT

Recall that an absolutely convex subset A of a locally convex space F over K is c'-compact if for each neighbourhood U of 0 in F there exist \(x_1, \ldots, x_n \in A\) (rather than \(x_1, \ldots, x_n \in F\)) such that \(A \subseteq U + \operatorname{co} \{x_1, \ldots, x_n\}\).

(Here \(\operatorname{co}\) indicates the absolutely convex hull)

THEOREM 1.1. For a polar seminorm p on a K-vector space E the following are equivalent.

(a) \(p(x) \in |K|\) for each \(x \in E\). Each onedimensional subspace of E has a p-orthocomplement.

(\(\beta\)) \(p^0\) is c'-compact.

Proof. (a) \(\Rightarrow\) (\(\beta\)). By [7], Theorem 3.2, it suffices to prove that for each \(\phi \in (E^*, \sigma(E^*, E))^\prime\)

\[
\max \{|\phi(f)| : f \in p^0\}
\]

exists. Since \(\phi\) is an evaluation map we therefore have to show that

\[
\max \{|f(x)| : f \in p^0\}
\]

events for each \(x \in E\). This is obviously true if \(p(x) = 0\). So assume \(p(x) > 0\). Since \(p(x) \in |K|\) we may assume that \(p(x) = 1\). For such \(x\) we must prove

\[
\max \{|f(x)| : f \in p^0\} = 1
\]

By (a), \(Kx\) has a p-orthocomplement \(H\). The function

\[
f : \lambda x + h \mapsto \lambda \quad (\lambda \in K, h \in H)
\]
is in $E^*$. We have $|f(x)| = 1$. For $\lambda \in K$, $h \in H$

$$|f(\lambda x + h)| = |\lambda| = p(\lambda x) \leq \max(p(\lambda x), p(h)) = p(\lambda x + h)$$

so that $f \in p^0$.

$(f) \Rightarrow (a)$. Let $x \in E$. The map $f \mapsto |f(x)|$ ($f \in E^*$) is a continuous seminorm on $(E^*, C(E^*, E))$. By $c'$-compactness its restriction to $p^0$ has a maximum so there exists a $g \in p^0$ with $|g(x)| = p(x)$ (It follows that $p(x) \leq |K|$). We prove that $\text{Ker} \cdot g$ is a $p$-orthocomplement of $Kx$. In fact, for $z \in \text{Ker} g$ we have

$$p(x+z) \geq |g(x+z)| = |g(x)| = p(x)$$

Then also

$$p(x+z) \geq p(z)$$

completing the proof of Theorem 1.1.

**Note.** It is not hard to see that $(a)$ of above is equivalent too.

(γ) For each $x \in E$ there exists an $f \in E^*$ with $|f(x)| = p(x)$ and $|f| \leq p$.

For spherically complete $K$ we obtain a simpler form of Theorem 1.1.

**COROLLARY 1.2.** Let $K$ be spherically complete, let $p$ be a seminorm on $E$ for which $p(x) \leq |K|$ for all $x \in E$.

Then the following are equivalent.

(a) $p(x) \leq |K|$ for each $x \in E$.

(β) $p^0$ is $c'$-compact.

**Proof.** By [1], lemma 4.35, each onedimensional subspace has a $p$-orthocomplement.
§2 APPLICATION: A NEW LIGHT ON SERRE’S RENORMING PROBLEM.

Consider the following two statements (*) and (**).

(*) Let E be a K-vector space and let \( || \cdot || \) be a norm on E. Then there exists a norm \( || \cdot ||' \) on E, equivalent to \( || \cdot || \), such that
\[
||x||' \leq |x| \quad \text{for all } x \in E.
\]

(**) Let K be spherically complete and let A be a complete absolutely convex compactoid in a Hausdorff locally convex space over K. Then there exist a \( \lambda \in K \) with \( |\lambda| > 1 \) and a \( c' \)-compact B such that
\[ A \subseteq B \subseteq \lambda A. \]

The question as to whether (*) is true or not is known as Serre’s renorming problem. See [2] for more details. We are able to reformulate this problem in terms of compactoids:

**Proposition 2.1.** The above statements (*) and (**) are equivalent.

**Proof.** Assume (*). To prove (**) we may assume that \( \lambda \) is edged. By [8], Theorem 3, A, as a topological module over \( B(0,1) := \{ \lambda \in K : |\lambda| \leq 1 \} \), is isomorphic to a bounded submodule of \( K^I \) for some set I. Let E be the algebraic direct sum \( \oplus K_i \) where \( K_i = K \) for all \( i \in I \).

Then \( (E^*, \sigma(E^*, E)) \) is in a natural way isomorphic to \( K^I \) with the product topology. So we may assume that \( A = p^0 \) where \( p \) is a seminorm on E.

By (*) there exists a seminorm \( q \), equivalent to \( p \), such that \( q(x) \in |K| \) for all \( x \in E \). By a suitable scalar multiplication we can arrange that, in addition, \( p \leq q \leq |\lambda| p \) for some \( \lambda \in K \), \( |\lambda| > 1 \). Then
\[
p^0 \leq q^0 \leq \lambda p^0.
\]
and \( q^0 \) is \( c' \)-compact by Corollary 1.2. This proves (**'). Now assume (**). To prove (*) we may assume (see [2]), that \( K \) is spherically complete. Let \( p \) be a norm on \( E \). By (**) there is a \( c' \)-compact \( B \) and a \( \lambda \in K, |\lambda| > 1 \) with \( p^0 \subset B \subset \lambda p^0 \). Then \( B = q^0 \) for some seminorm \( q \) on \( E \). We have
\[
p \leq q \leq |\lambda| p
\]
and \( q(x) \in |K| \) for all \( x \in E \) by Corollary 1.2.

Note. Serre's renorming problem is still unsettled as far as I know.

§3 NORMS \( p \) FOR WHICH \( p^0 \) IS A KREIN-MILMAN COMPACTOID.

Recall that an absolutely convex subset \( A \) of a locally convex space over \( K \) is a \( KM \)-compactoid if it is complete and if \( A = \text{co} X \) where \( X \) is compact. (Here \( \text{co} X \) is the closure of \( \text{co} X \)).

Before stating the theorem we first make some simple observations. Let \( p \) be a norm on \( E \). We say that a collection \((e_i)_{i \in I}\) in \( E \) is a \( p \)-orthonormal base of \( E \) if for each \( x \in E \) there exist a unique \((\lambda_i)_{i \in I} \subset K^I\) such that \( \{i \in I, |\lambda_i| \geq \epsilon\} \) is finite for each \( \epsilon > 0 \) and
\[
x = \sum_{i \in I} \lambda_i e_i
\]
\[
p(x) = \max_{i} |\lambda_i|
\]
If \((E,p)\) is complete this definition coincides with the usual one.

**Lemma 3.1.** Let \((E,p)\) be a normed space, let \((\hat{E},\hat{p})\) be its completion. Then \((E,p)\) has a \( p \)-orthonormal base if and only if \((\hat{E},\hat{p})\) has a \( \hat{p} \)-orthonormal base.
Proof. It is not hard to see that each $p$-orthonormal base of $(E, p)$ is also a $p$-orthonormal base of $(E', p')$. Conversely, let $(e_i)_{i \in I}$ be a $p$-orthonormal base of $(E', p')$. For each $i \in I$, choose an $f_i \in E$ with $p(e_i - f_i) \leq \frac{1}{2}$.

By [1], Exercise 5.C, $(f_i)_{i \in I}$ is a $p$-orthonormal base of $(E', p')$. Clearly $(f_i)_{i \in I}$ is a $p$-orthonormal base of $(E, p)$.

Theorem 3.2. For a polar norm $p$ on a $K$-vector space $E$ the following are equivalent.

(a) $(E, p)$ has a $p$-orthonormal base

(b) $\mathfrak{p}$ is a KM-compactoid.

Proof. (a) $\Rightarrow$ (b). Let $(e_i)_{i \in I}$ be a $p$-orthonormal base of $(E, p)$. The formula

$$\psi(f) = (f(e_i))_{i \in I}$$

defines a map $\psi : \mathfrak{p} \to B(0, 1)^I$. Straightforward verifications show that $\psi$ is an isomorphism of topological $B(0, 1)$-modules. From [8], Theorem 16 we obtain that $B(0, 1)^I$, hence $\mathfrak{p}$, is a KM-compactoid.

(b) $\Rightarrow$ (a). Suppose $\mathfrak{p} = \text{co} X$ where $X$ is a compact subset of $E^*$. Let $C(X^*K)$ be the Banach space of all continuous functions $X \to K$, with the supremum norm $|| \cdot ||_\infty$. Then $C(X^*K)$ has an orthonormal base. ([1], Theorem 5.22).

The formula

$$\psi(x)(f) = f(x) \quad (f \in X)$$

defines a $K$-linear map $\psi : E \to C(X^*K)$. From
\[ \|f(x)\|_\infty = \max_{f \in \mathcal{F}} |f(x)| = \sup_{f \in \mathcal{F}} |f(x)| = \sup_{f \in \mathcal{P}} |f(x)| = p(x) \]

we obtain that \( \phi \) is an isometry \((E, p) \to (C(X^+, K), \|\|_{\infty})\).

By Gruson's Theorem ([1], 5.9) \( \phi(E) \) has an orthonormal base. Then so has \( \phi(E) \) by Lemma 3.1 and has \( E \).

§4 APPLICATION: A COMPLETE \( c' \)-COMPACT SET WHICH IS NOT A KM-COMPACTOID.

We shall give a negative answer to the Problem following Theorem 1.7 of [6].

PROPOSITION 4.1. Let \( K \) be spherically complete, let \( |K| = [0, \infty) \).

Then there exist a locally convex space \( F \) over \( K \) and a complete \( c' \)-compact subset \( A \subset F \) which is not a KM-compactoid.

Proof. Let \( E := l^\infty \) and let \( F := (l^\infty)^* \) (with the topology we agreed upon in §0). Let \( p \) be the standard norm on \( l^\infty \), and set \( A := p^0 \). Since, trivially, \( p(x) \in |K| \) for all \( x \in l^\infty \), we have that \( p^0 \) is \( c' \)-compact (Corollary 1.2).

However, it is known ([1], Cor. 5.19) that \( l^\infty \) has no orthogonal base so that (Theorem 3.2) \( p^0 \) is not a KM-compactoid.

§5 NORMS \( p \) FOR WHICH \( p^0 \) IS METRIZABLE.

THEOREM 5.1. For a polar seminorm \( p \) on a \( K \)-vector space \( E \) the following are equivalent.

(\( \alpha \))(\( E, p \) is of countable type ([3], Definition 4.3).

(\( \beta \)) \( p^0 \) is metrizable.
Proof. (a) ⇒ (β). There exist $e_1, e_2, \ldots$ in $E$ with $p(e_i) \leq 1$ for each $i$ such that the $K$-linear span of $e_1, e_2, \ldots$ is $p$-dense in $E$. The formula

$$\phi(f) = (f(e_1), f(e_2), \ldots)$$

defines a map $\phi : p^0 \to B(0,1)^N$. Straightforward verifications show that $\phi$ is an isomorphism of topological $B(0,1)$-modules of $p^0$ onto a submodule of $B(0,1)^N$.

Now $B(0,1)^N$ is metrizable (the product topology is induced by the metric

$$(a,b) \mapsto \sup_{i \in \mathbb{N}} |a_i - b_i| 2^{-i}$$

hence so is $p^0$.

(β) ⇒ (a). Let $\lambda \in K$, $|\lambda| > 1$. Since $p^0$ is a metrizable compactoid, there exist, by [3], Proposition 8.2, $f_1, f_2, \ldots \in \lambda p^0$ with $\lim_{n \to \infty} f_n = 0$ such that

$$p^0 \subset \overline{co} \{f_1, f_2, \ldots\} < \lambda p^0$$

The map

$$\phi : x \mapsto (f_1(x), f_2(x), \ldots) \quad (x \in E)$$

is $K$-linear, $\phi(B) \subset c_0$. We have for $x \in E$

$$||\phi(x)|| = \sup_{n \in \mathbb{N}} |f_n(x)| = \sup_{n \in \mathbb{N}} \{ |g(x)| : g \in \overline{co} \{f_1, f_2, \ldots\} \}$$

It follows that

$$p(x) \leq ||\phi(x)|| \leq |\lambda| p(x)$$

so that $p$ is equivalent to $x \mapsto ||\phi(x)||$, a seminorm of countable type. Hence, $p$ is of countable type.
§6 APPLICATION: DESCRIPTION OF METRIZABLE COMPACTOIDS.

THEOREM 6.1. Let $A$ be an absolutely convex subset of a Hausdorff locally convex space $F$ over $K$. The following are equivalent.

(a) $A$ is a metrizable compactoid.

(b) As a topological $B(0,1)$-module, $A$ is isomorphic to a submodule of $B(0,1)^\mathbb{N}$.

(c) As a topological $B(0,1)$-module, $A$ is isomorphic to a compactoid in $C_0$.

(d) For each $\lambda \in K$, $|\lambda| > 1$ there exist $e_1, e_2, \ldots \in A$ with $\lim_{n \to \infty} e_n = 0$ and $A \subseteq \overline{\operatorname{co}} \{e_1, e_2, \ldots\}$.

(e) There exist $e_1, e_2, \ldots \in F$ with $\lim_{n \to \infty} e_n = 0$ and $A \subseteq \overline{\operatorname{co}} \{e_1, e_2, \ldots\}$.

(f) There exists an ultrametrizable compact $X \subseteq F$ with $A \subseteq \overline{\operatorname{co}} X$.

Proof. (a) $\Rightarrow$ (b). It is not hard to see, by using the absolute convexity of $A$, that $\overline{A}$ is also metrizable. As there is no harm in taking $F$ complete we therefore may assume that $A$ is complete. To prove (b) we also may assume that $A$ is edged. By [8], Theorem 3, $A \subseteq B(0,1)^I \subseteq K^I$ for some set $I$. Like in the proof of Proposition 2.1 we may conclude that $A = p^0$ where $p$ is a polar seminorm on $\oplus K_i (K_i = K$ for each $i$). Then $p$ is of countable type by Theorem 5.1. From the proof of (a) $\Rightarrow$ (b) of that Theorem we obtain an isomorphism $A = p^0 \cong B(0,1)^\mathbb{N}$.

(b) $\Rightarrow$ (c). Choose $\lambda_1, \lambda_2, \ldots \in K$, $|\lambda_1| > |\lambda_2| > \ldots$, $\lim_{n \to \infty} \lambda_n = 0$. The formula

$$\phi((a_i)) = (\lambda_1 a_1, \lambda_2 a_2, \ldots) \in c_0$$

defines a $B(0,1)$-module isomorphism of $B(0,1)^\mathbb{N}$ onto $C := \overline{\operatorname{co}} \{\lambda_1 e_1, \lambda_2 e_2, \ldots\}$ where $e_1, e_2, \ldots$ are the standard unit vectors in $c_0$. $\phi$ is a homeomorphism.
B(0,1) → C, and maps A onto a compactoid in c₀.

(γ) ⇒ (δ). See [3], Proposition 8.2.

(δ) ⇒ (ε) is trivial.

(ε) ⇒ (η). {0, e₁, e₂, ...} is compact and ultrametrizable.

(η) ⇒ (α). We may assume that F is complete. It suffices to prove the

metrizability of B := co X.

B is a complete, edged compactoid. As before we may assume that B = p₀

for some polar seminorm p on some K-vector space E while B ⊆ E*. The

map φ : E → C(X + K) defined by

\[ φ(x)(f) = f(x) \quad (f \in X) \]

is an isometry \((E, p) \overset{\phi}{\rightarrow} (C(X + K), \| \cdot \|_\infty)\).

Now X is ultrametrizable so by [1], Exercise 3.5, C(X + K) is of
countable type. Hence so is p. By Theorem 5.1, B = p₀ is metrizable.

§7 NORMS p FOR WHICH \((p₀)^{\mathbb{N}}\) IS OF FINITE TYPE.

Recall that an absolutely convex set A in a locally convex space F over
K is of finite type if for each zero neighbourhood U in F there exists
a finite-dimensional bounded set S ⊆ A such that A ⊆ U + S.

Let us say that a seminorm p on a K-vector space E is of finite type
if \(\text{Ker } p = \{x \in E : p(x) = 0\}\) has finite codimension.

LEMMA 7.1. Let A be an absolutely convex subset of a locally convex
space F whose topology is generated by a collection of seminorms of
finite type. Then the following are equivalent.

(a) A is a compactoid of finite type.

(b) For each closed linear subspace H of finite codimension there is a
finite dimensional bounded set S ⊆ A with A ⊆ H + S.
Proof. \((a) \Rightarrow (\beta)\). (Note. This implication holds for any locally convex space \(F\).) We may assume \([A] = F\).

\(H\) has the form \(D^1 := \{x \in F : f(x) = 0 \text{ for all } f \in D\}\) where \(D\) is a finite dimensional subspace of \(F\). Let \(f_1, \ldots, f_n\) be a base of \(D\). There exist \(x_1, \ldots, x_n \in F\) with \(f_i(x_j) = \delta_{ij} (i, j \in \{1, \ldots, n\})\). Since \([A] = F\) there exists a \(\lambda \in K, \lambda \neq 0\) such that \(\lambda x_i \in A\) for each \(i \in \{1, \ldots, n\}\).

Set

\[
U := \bigcap_{i=1}^{n} \{x \in F : |f_i(x)| \leq |\lambda|\}
\]

Then \(U\) is a zero neighbourhood in \(F\). \(A\) is a compactoid of finite type, so there exists a finite dimensional set \(S_1 \subseteq A\) with \(A \subseteq U + S_1\). Let \(x \in U\). Write \(x = y + z\) where

\[
y := x - \sum_{i=1}^{n} f_i(x) x_i
\]

\[
z := \sum_{i=1}^{n} f_i(x) x_i
\]

Now, since \(x \in U\), \(|f_i(x)| \leq |\lambda|\) for each \(i\) so that \(z = \sum_{i=1}^{n} f_i(x) x_i \in A\).

Further, for each \(j \in \{1, \ldots, n\}\)

\[
f_j(y) = f_j(x) - \sum_{i=1}^{n} f_i(x) f_j(x_i) = f_j(x) - f_j(x) = 0
\]

and it follows that \(y \in D^1 = H\). So \(x = y + z\) \(\in H + [x_1, \ldots, x_n] \cap A\). We see that

\[
A \subseteq U + S_1 \subseteq H + S_2 + S_1
\]

where \(S_2 := [x_1, \ldots, x_n] \cap A\). Then \((\beta)\) is proved with \(S := S_1 + S_2\).

\((\beta) \Rightarrow (a)\). Let \(U\) be a zero neighbourhood in \(F\). Since continuous seminorms are of finite type, \(U\) contains a closed subspace \(H\) of finite codimension.

By \((\beta)\) there exists a finite dimensional set \(S \subseteq A\) with \(S\) bounded and
A ⊆ H + S. Then A ⊆ U + S.

From now on we assume that the valuation on K is dense.

Recall that for an absolutely convex set B we have $B^\perp := \bigcup_{|\lambda| < 1} \lambda B$.

**Theorem 7.2.** Let $p$ be a polar norm on a K-vector space $E$. Then the following are equivalent.

(a) For each finite dimensional subspace $D$ of $E$ there exists a seminorm $q$ on $E$, $q$ of finite type, $q \leq p$ and $q = p$ on $D$.

(b) $(p^0)^\perp$ is of finite type.

**Proof.** (a) $\Rightarrow$ (b). As each continuous seminorm on $E^*$ is of finite type it suffices to prove, by Lemma 7.1, that for a closed subspace $H$ of $E^*$ of finite codimension there exists a finite dimensional set $S \in (p^0)^\perp$ such that $(p^0)^\perp \subseteq H + S$.

Now, by (a), there is a seminorm $q$ of finite type, $q \leq p$ on $E$ and $q = p$ on $D := H^\perp$. Let

$$S_1 := \{ f \in E^* : |f| \leq q \}.$$  

We see that $S_1$ is finite dimensional and since $q \leq p$ we have $S_1 \subseteq p^0$.

We now shall prove that $(p^0)^\perp \subseteq H + S$ where $S := (S_1)^\perp$.

In fact, let $f \in (p^0)^\perp$. Then there is a $\lambda \in K$, $0 < |\lambda| < 1$ with $|f| \leq |\lambda| p$.

Choose $\lambda' \in K$ with $|\lambda| < |\lambda'| < 1$.

We have $|f| \leq |\lambda| q$ on $D$ (since $p = q$ on $D$) so we can extend $f$ to a $g \in E^*$ with $|g| \leq |\lambda'| q$ on $E$. (This is because $q$ is of finite type so that $(E,q)$ is strongly polar.) Now write

$$f = f - g + g$$

Since $f = g$ on $D$ we have $f - g \in D^\perp = H$. 

Also, \( |(\lambda')^{-1}g| \leq q \) so that \( (\lambda')^{-1}g \in S_1 \) i.e. \( g \in (S_1)^i = S \).

\((\beta) \Rightarrow (a)\). By lemma 7.1 there exists a finite dimensional set \( S \subset (p^0)^i \) so that \((p^0)^i = D^\perp \cap (p^0)^i + S \).

Set \( q(x) := \sup_{h \in S} |h(x)| \). (\( x \in E \)).

Then \( q(x) = 0 \) for all \( x \) in the space \( S^\perp \) which has finite codimension.

So \( q \) is of finite type.

Further, for \( x \in E \) we have

\[ q(x) = \sup_{h \in S} |h(x)| \leq \sup_{h \in (p^0)^i} |h(x)| = \sup_{h \in (p^0)^i} |h(x)| = p(x), \]

so \( q \leq p \). Finally, if \( x \in D \) then

\[ p(x) = \sup_{f \in p} |f(x)| = \sup_{f \in (p^0)^i} |f(x)| = \sup_{h \in D^\perp \cap (p^0)^i} |h(x) + t(x)| \]

\[ = \sup_{t \in S} |t(x)| = q(x). \] Hence, \( p = q \) on \( D \).

§8 APPLICATION: A COMPLETE COMPACTOID IN \( c_0 \) THAT IS NOT OF FINITE TYPE.

If \( K \) is spherically complete each complete absolutely convex compactoid is of finite type (See [4], 2.3).

If \( K \) is not spherically complete the unit ball of \( c_0 \) is a complete compactoid for the weak topology but not of finite type (See [5], 1.6).

This is a non-metrizable compactoid. A compactoid in \( (c_0, \| \|) \), not of finite type, is given in [5], 1.4. However this compactoid is not closed. The following example provides an answer to the Problem following 1.5 in [5].
PROPOSITION 8.1. Let $K$ be not spherically complete. Then there exists an absolutely convex complete compactoid in $c_0$ that is not of finite type.

Proof. Let $(K^v, | |)$ be the spherical completion of $(K, | |)$ in the sense of [1], Theorem 4.49. Let $E$ be a $K$-subspace of $K^v$ of countably infinite dimension and let $p$ be the valuation $| |$ restricted to $E$. Then $x, y \in E$, $x \perp y$ in the sense of $p$ implies $x = 0$ or $y = 0$. Obviously, the norm $p$ is of countable type (hence polar) so, by Theorem 5.1, $p^0$ is metrizable and is by Theorem 6.1, isomorphic to a compactoid in $c_0$.

Suppose $p^0$ were of finite type. Then so would $(p^0)^i$ ([5], Proposition 2.4). By Theorem 7.2 we would have a seminorm $q$ on $E$, $q \leq p$, $q$ of finite type, $q(x) = p(x)$ for some $x \in E$, $x \neq 0$. But then $x \perp \text{Ker } q$ in the sense of $p$ (If $q(y) = 0$ then $p(x-y) \geq q(x-y) = q(x) = p(x)$) which is impossible.

So, $p^0$ is not of finite type.
REFERENCES


