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A CONNECTION BETWEEN p-ADIC BANACH SPACES AND LOCALLY CONVEX COMPACTOIDS

by

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ABSTRACT. For a vector space $E$ over a non-archimedean valued field $K$ a correspondence $p \leftrightarrow p^0$ is established between seminorms $p$ on $E$ and compactoids $p^0$ in $E^*$. Examination of it yields the solution of two open problems (see §4 and §8) and a reformulation of Serre's renorming problem (see §2). As a by-product results on metrizable compactoids are obtained (see §6).

§0 THE CORRESPONDENCE $p \leftrightarrow p^0$.

Throughout this note $K$ is a non-archimedean valued field, complete with respect to the metric induced by the nontrivial valuation $|\cdot|$. Let $E$ be a $K$-vector space, let $E^*$ be its algebraic dual. A (non-archimedean) seminorm $p$ on $E$ is polar ([3], Definition 3.1), if

$$p = \sup \{|f| : f \in E^*, |f| \leq p\}$$

Let $P_E$ be the set of all polar seminorms on $E$. For each $p \in P_E$ we set

$$p^0 = \{f \in E^* : |f| \leq p\}$$

Then $p^0$ is an absolutely convex, edged ([3],§1b) subset of $E^*$. It is easy to see that $p^0$ is a closed compactoid ([3],§1e) with respect to the topology $\sigma(E^*,E)$, hence complete.
Let \( C_\mathcal{E}^* \) be the set of all closed absolutely convex, edged compactoids in \( \mathcal{E}^* \) with respect to \( \sigma(\mathcal{E}^*, \mathcal{E}) \).

**PROPOSITION 0.** The map \( p \mapsto p^0 \) is a bijection of \( P_\mathcal{E} \) onto \( C_\mathcal{E}^* \). Its inverse assigns to every \( A \in C_\mathcal{E}^* \) the seminorm \( p \) given by

\[
p(x) = \sup \{|f(x)| : f \in A\} \quad (x \in \mathcal{E})
\]

**Proof.** We shall prove surjectivity of \( p \mapsto p^0 \) leaving the (easy) rest of the proof to the reader. So, let \( A \in C_\mathcal{E}^* \); we shall prove that \( A = p^0 \) where \( p(x) = \sup \{|f(x)| : f \in A\} \).

Obviously, \( A \subseteq p^0 \). Now let \( g \in \mathcal{E}^* \backslash A \), we prove that \( g \notin p^0 \). The space \( \mathcal{E}^* \) is of countable type hence strongly polar ([3], Theorem 4.4). So by [3], Theorem 4.7, there exists a \( \theta \in (\mathcal{E}^*, \sigma(\mathcal{E}^*, \mathcal{E}))' \) such that \( |\theta| \leq 1 \) on \( A \), \( |\theta(g)| > 1 \). But, by [3], lemma 7.1, \( \theta \) has the form \( f \mapsto f(x) \) for some \( x \in \mathcal{E} \). Thus, \( |f(x)| \leq 1 \) for \( f \in A \), \( |g(x)| > 1 \) i.e., \( p(x) \leq 1 \) and \( |g(x)| > 1 \) and it follows that \( g \notin p^0 \).

**Remarks.**

1. Let \( K \) be spherically (= maximally) complete. Then each nonarchimedean seminorm \( p \) on \( \mathcal{E} \) for which \( p(x) \leq |x| \) (\( x \in \mathcal{E} \)) is polar ([3], Remark following 3.1).

2. Let \( \tau \) be the locally convex topology on \( \mathcal{E} \) induced by all nonarchimedean seminorms i.e., \( \tau \) is the strongest among all locally convex topologies on \( \mathcal{E} \). It is not hard to see that \( (\mathcal{E}, \tau) \) is a complete polar (([3], Definition 3.5) space and that \( (\mathcal{E}, \tau) \) and \( (\mathcal{E}^*, \sigma(\mathcal{E}^*, \mathcal{E})) \) are each others strong dual spaces.
§1 NORMS p FOR WHICH p₀ IS c'-COMPACT

Recall that an absolutely convex subset A of a locally convex space F over \( K \) is c'-compact if for each neighbourhood U of 0 in F there exist \( x_1, \ldots, x_n \in A \) (rather than \( x_1, \ldots, x_n \in F \)) such that \( A \subseteq U + \langle x_1, \ldots, x_n \rangle \). (Here \( \langle \cdot \rangle \) indicates the absolutely convex hull)

**Theorem 1.1.** For a polar seminorm \( p \) on a \( K \)-vector space \( E \) the following are equivalent.

(a) \( p(x) \leq |K| \) for each \( x \in E \). Each one-dimensional subspace of \( E \) has a \( p \)-orthocomplement.

(b) \( p₀ \) is c'-compact.

**Proof.** (a) \( \Rightarrow \) (b). By [7], Theorem 3.2, it suffices to prove that for each \( \langle f \rangle \in (E^*, \sigma(E^*, E))' \)

\[
\max \{ |\phi(f)| : f \in p₀ \}
\]

exists. Since \( \phi \) is an evaluation map we therefore have to show that

\[
\max \{ |f(x)| : f \in p₀ \}
\]

exists for each \( x \in E \). This is obviously true if \( p(x) = 0 \). So assume \( p(x) > 0 \). Since \( p(x) \in |K| \) we may assume that \( p(x) = 1 \). For such \( x \) we must prove

\[
\max \{ |f(x)| : f \in p₀ \} = 1
\]

By (a), \( Kx \) has a \( p \)-orthocomplement \( H \). The function

\[
f : \lambda x + h \mapsto \lambda
\]

(\( \lambda \in K, h \in H \))
is in $E^*$. We have $|f(x)| = 1$. For $\lambda \in K$, $h \in H$

$$|f(\lambda x + h)| = |\lambda| = p(\lambda x) \leq \max(p(\lambda x), p(h)) = p(\lambda x + h)$$

so that $f \in p^0$.

($\beta$) $\Rightarrow$ ($\alpha$). Let $x \in E$. The map $f \mapsto |f(x)|$ is a continuous seminorm on $(E^*, c(E^*, E))$. By c'-compactness its restriction to $p^0$ has a maximum so there exists a $g \in p^0$ with $|g(x)| = p(x)$.

(It follows that $p(x) \in |K|$). We prove that $\text{Ker} \cdot g$ is a $p$-orthocomplement of $Kx$. In fact, for $z \in \text{Ker} \cdot g$ we have

$$p(x+z) \geq |g(x+z)| = |g(x)| = p(x)$$

Then also

$$p(x+z) \geq p(z)$$

completing the proof of Theorem 1.1.

Note. It is not hard to see that ($\alpha$) of above is equivalent too.

($\gamma$) For each $x \in E$ there exists an $f \in E^*$ with $|f(x)| = p(x)$ and $|f| \leq p$.

For spherically complete $K$ we obtain a simpler form of Theorem 1.1.

COROLLARY 1.2. Let $K$ be spherically complete, let $p$ be a seminorm on $E$ for which $p(x) \in |K|$ for all $x \in E$.

Then the following are equivalent.

($\alpha$) $p(x) \in |K|$ for each $x \in E$.

($\beta$) $p^0$ is c'-compact.

Proof. By [1], lemma 4.35, each onedimensional subspace has a p-orthocomplement.
§2 APPLICATION: A NEW LIGHT ON SERRE'S RENORMING PROBLEM.

Consider the following two statements (*) and (**).

(*) Let $E$ be a $K$-vector space and let \(| \cdot \|\) be a norm on $E$. Then there exists a norm \(| \cdot \|'$ on $E$, equivalent to \(| \cdot \|\), such that 
\[ |x|' \leq |K| \text{ for all } x \in E. \]

(**) Let $K$ be spherically complete and let $A$ be a complete absolutely convex compactoid in a Hausdorff locally convex space over $K$. Then there exist a $\lambda \in K$ with $|\lambda| > 1$ and a $c_1$-compact $B$ such that $A \subset B \subset \lambda A$.

The question as to whether (*) is true or not is known as Serre's renorming problem. See [2] for more details. We are able to reformulate this problem in terms of compactoids:

**Proposition 2.1.** The above statements (*) and (**) are equivalent.

**Proof.** Assume (*). To prove (**) we may assume that $A$ is edged. By [8], Theorem 3, $A$, as a topological module over $B(0,1) := \{\lambda \in K : |\lambda| \leq 1\}$, is isomorphic to a bounded submodule of $K^I$ for some set $I$. Let $E$ be the algebraic direct sum $\bigoplus_{i \in I} K_i$ where $K_i = K$ for all $i \in I$.

Then $(E^*, E^* \otimes E)$ is in a natural way isomorphic to $K^I$ with the product topology. So we may assume that $A = p^0$ where $p$ is a seminorm on $E$.

By (*) there exists a seminorm $q$, equivalent to $p$, such that $q(x) \leq |K|$ for all $x \in E$. By a suitable scalar multiplication we can arrange that, in addition, $p \leq q \leq |\lambda| p$ for some $\lambda \in K$, $|\lambda| > 1$. Then 
\[ p^0 \subset q^0 \subset \lambda p^0. \]
and $q^0$ is $c'$-compact by Corollary 1.2. This proves (**). Now assume (**).

To prove (*) we may assume (see [2]), that $K$ is spherically complete.

Let $p$ be a norm on $E$. By (**) there is a $c'$-compact $B$ and a $\lambda \in K$, $|\lambda| > 1$ with $p^0 \subset B \subset \lambda p^0$. Then $B = q^0$ for some seminorm $q$ on $E$. We have

$$p \leq q \leq |\lambda| p$$

and $q(x) \in |K|$ for all $x \in E$ by Corollary 1.2.

Note. Serre's renorming problem is still unsettled as far as I know.

§3 NORMS $p$ FOR WHICH $p^0$ IS A KREIN-MILMAN COMPACTOID.

Recall that an absolutely convex subset $A$ of a locally convex space over $K$ is a $KM$-compactoid if it is complete and if $A = \overline{\text{co}} \ X$ where $X$ is compact. (Here $\overline{\text{co}} \ X$ is the closure of $\text{co} \ X$).

Before stating the theorem we first make some simple observations. Let $p$ be a norm on $E$. We say that a collection $(e_i)_{i \in I}$ in $E$ is a $p$-orthonormal base of $E$ if for each $x \in E$ there exist a unique $(\lambda_i)_{i \in I} \subset K^*$ such that $\{i \in I, |\lambda_i| \geq \varepsilon\}$ is finite for each $\varepsilon > 0$ and

$$x = \sum_{i \in I} \lambda_i e_i$$

$$p(x) = \max_{i} |\lambda_i|$$

If $(E,p)$ is complete this definition coincides with the usual one.

**Lemma 3.1.** Let $(E,p)$ be a normed space, let $(\hat{E},\hat{p})$ be its completion.

Then $(E,p)$ has a $p$-orthonormal base if and only if $(\hat{E},\hat{p})$ has a $\hat{p}$-orthonormal base.
Proof. It is not hard to see that each $p$-orthonormal base of $(E,p)$ is also a $p$-orthonormal base of $(E,p)$. Conversely, let $(e_i)_{i \in I}$ be a $p$-orthonormal base of $(E,p)$. For each $i \in I$, choose an $f_i \in E$ with $p(e_i - f_i) \leq \frac{1}{2}$.

By [1], Exercise 5.C, $(f_i)_{i \in I}$ is a $p$-orthonormal base of $(E,p)$.

Clearly $(f_i)_{i \in I}$ is a $p$-orthonormal base of $(E,p)$.

THEOREM 3.2. For a polar norm $p$ on a $K$-vector space $E$ the following are equivalent.

(a) $(E,p)$ has a $p$-orthonormal base

(b) $p^0$ is a $K$-compactoid.

Proof. (a) $\Rightarrow$ (b). Let $(e_i)_{i \in I}$ be a $p$-orthonormal base of $(E,p)$. The formula

$$\phi(f) = (f(e_i))_{i \in I}$$

defines a map $\phi : p^0 \rightarrow B(0,1)^I$. Straightforward verifications show that $\phi$ is an isomorphism of topological $B(0,1)$-modules. From [8], Theorem 16 we obtain that $B(0,1)^I$, hence $p^0$, is a $K$-compactoid.

(b) $\Rightarrow$ (a). Suppose $p^0 = \text{co} X$ where $X$ is a compact subset of $E^*$. Let $C(X+K)$ be the Banach space of all continuous functions $X + K$, with the supremum norm $\| \|_\infty$. Then $C(X+K)$ has an orthonormal base. ([1], Theorem 5.22).

The formula

$$\phi(x)(f) = f(x) \quad (f \in X)$$

defines a $K$-linear map $\phi : E \rightarrow C(X+K)$. From
\[ \| \phi(x) \|_w = \max \{ f(x) \} = \sup_{f \in \mathcal{X}} \{ f(x) \} = \sup_{f \in \mathcal{P}} \{ f(x) \} = p(x) \]

we obtain that \( \phi \) is an isometry \((E,p) \to (\mathcal{C}(X^K), \| \|_w)\).

By Gruson's Theorem ([1], 5.9) \( \overline{\phi(E)} \) has an orthonormal base. Then so has \( \phi(E) \) by Lemma 3.1 and has \( E \).

§4 APPLICATION: A COMPLETE \( c' \)-COMPACT SET WHICH IS NOT A KM-COMPACTOID.

We shall give a negative answer to the Problem following Theorem 1.7 of [6].

PROPOSITION 4.1. Let \( K \) be spherically complete, let \( |K| = [0,\infty) \).
Then there exist a locally convex space \( F \) over \( K \) and a complete \( c' \)-compact subset \( A \subset F \) which is not a KM-compactoid.

Proof. Let \( E := l^\infty \) and let \( F := (l^\infty)^* \) (with the topology we agreed upon in §0). Let \( p \) be the standard norm on \( l^\infty \), and set \( A := p^0 \). Since, trivially, \( p(x) \in |K| \) for all \( x \in l^\infty \), we have that \( p^0 \) is \( c' \)-compact (Corollary 1.2).

However, it is known ([1], Cor. 5.19) that \( l^\infty \) has no orthogonal base so that (Theorem 3.2) \( p^0 \) is not a KM-compactoid.

§5 NORMS \( p \) FOR WHICH \( p^0 \) IS METRIZABLE.

THEOREM 5.1. For a polar seminorm \( p \) on a \( K \)-vector space \( E \) the following are equivalent.

(a) \((E,p)\) is of countable type ([3], Definition 4.3).

(b) \( p^0 \) is metrizable.
Proof. (a) $\Rightarrow$ (b). There exist $e_1, e_2, \ldots$ in $E$ with $p(e_i) \leq 1$ for each $i$ such that the $K$-linear span of $e_1, e_2, \ldots$ is $p$-dense in $E$. The formula

$$\phi(f) = (f(e_1), f(e_2), \ldots)$$

defines a map $\phi : p^0 \rightarrow B(0,1)^\mathbb{N}$. Straightforward verifications show that $\phi$ is an isomorphism of topological $B(0,1)$-modules of $p^0$ onto a submodule of $B(0,1)^\mathbb{N}$.

Now $B(0,1)^\mathbb{N}$ is metrizable (the product topology is induced by the metric $(a,b) \mapsto \sup \{ |a_i - b_i| 2^{-i} \}$ hence so is $p^0$.

(b) $\Rightarrow$ (a). Let $\lambda \in K$, $|\lambda| > 1$. Since $p^0$ is a metrizable compactoid there exist, by [3], Proposition 8.2, $f_1, f_2, \ldots \in \lambda p^0$ with $\lim_{n \to \infty} f_n = 0$ such that

$$p^0 \subset \sup \{ f_1, f_2, \ldots \} \subset \lambda p^0$$

The map

$$\phi : x \mapsto (f_1(x), f_2(x), \ldots) \quad (x \in E)$$

is $K$-linear, $\phi(E) \subset c_0$. We have for $x \in E$

$$||\phi(x)|| = \sup_{n \in \mathbb{N}} |f_n(x)| = \sup \{ |g(x)| : g \in \sup \{ f_1, f_2, \ldots \} \}$$

It follows that

$$p(x) \leq ||\phi(x)|| \leq |\lambda|p(x)$$

so that $p$ is equivalent to $x \mapsto ||\phi(x)||$, a seminorm of countable type.

Hence, $p$ is of countable type.
§6 APPLICATION: DESCRIPTION OF METRIZABLE COMPACTOIDS.

THEOREM 6.1. Let $A$ be an absolutely convex subset of a Hausdorff locally convex space $F$ over $K$. The following are equivalent.

(a) $A$ is a metrizable compactoid.

(b) As a topological $B(0,1)$-module, $A$ is isomorphic to a submodule of $B(0,1)\mathbb{N}$.

(c) As a topological $B(0,1)$-module, $A$ is isomorphic to a compactoid in $C^0$.

(d) For each $\lambda \in K$, $|\lambda| > 1$ then exist $e_1, e_2, \ldots \in \lambda A$ with $\lim_{n \to \infty} e_n = 0$ and $A \subseteq \overline{\mathbb{e}_1 \oplus \mathbb{e}_2 \oplus \ldots}$.

(e) There exist $e_1, e_2, \ldots \in F$ with $\lim_{n \to \infty} e_n = 0$ and $A \subseteq \overline{\mathbb{e}_1 \oplus \mathbb{e}_2 \oplus \ldots}$.

(f) There exists an ultrametrizable compact $X \subseteq F$ with $A \subseteq \overline{\mathbb{X}}$.

Proof. (a) $\Rightarrow$ (b). It is not hard to see, by using the absolute convexity of $A$, that $\overline{A}$ is also metrizable. As there is no harm in taking $F$ complete we therefore may assume that $A$ is complete. To prove (b) we also may assume that $A$ is edged. By [8], Theorem 3, $A \subseteq B(0,1)^{\mathbb{I}} \subseteq K^{\mathbb{I}}$ for some set $\mathbb{I}$. Like in the proof of Proposition 2.1 we may conclude that $A = p^0$ where $p$ is a polar seminorm on $\oplus_{\mathbb{I}} K (K_i = K$ for each $i$). Then $p$ is of countable type by Theorem 5.1. From the proof of (a) $\Rightarrow$ (b) of that Theorem we obtain an isomorphism $A = p^0 \cong B(0,1)^{\mathbb{N}}$.

(b) $\Rightarrow$ (c). Choose $\lambda_1, \lambda_2, \ldots \in K$, $|\lambda_1| > |\lambda_2| > \ldots$, $\lim_{n \to \infty} \lambda_n = 0$. The formula

$$\phi((a_i)) = (\lambda_1 a_1, \lambda_2 a_2, \ldots) \in C_0$$

for $i \in \mathbb{N}$ defines a $B(0,1)$-module isomorphism of $B(0,1)^{\mathbb{N}}$ onto $C := \overline{\mathbb{e}_1 \oplus \mathbb{e}_2 \oplus \ldots}$ where $e_1, e_2, \ldots$ are the standard unit vectors in $C_0$. $\phi$ is a homeomorphism.
Let $B(0,1)^\mathbb{N} \to \mathcal{C}$, and maps $A$ onto a compactoid in $c_0$.

$(\gamma) \Rightarrow (\delta)$. See [3], Proposition 8.2.

$(\delta) \Rightarrow (\varepsilon)$ is trivial.

$(\varepsilon) \Rightarrow (\eta), \{0,e_1,e_2,\ldots\}$ is compact and ultrametrizable.

$(\eta) \Rightarrow (\alpha)$. We may assume that $F$ is complete. It suffices to prove the

metrizability of $B := \overline{co}X$.

$B$ is a complete, edged compactoid. As before we may assume that $B = p^0$

for some polar seminorm $p$ on some $K$-vector space $E$ while $B \subset E^*$. The

map $\phi : E \to C(X^K)$ defined by

$$\phi(x)(f) = f(x) \quad (f \in X)$$

is an isometry $(E,p) \to (C(X^K), ||| \cdot |||_\omega)$.

Now $X$ is ultrametrizable so by [1], Exercise 3.5, $C(X^K)$ is of

countable type. Hence so is $p$. By Theorem 5.1, $B = p^0$ is metrizable.

§7 NORMS $p$ FOR WHICH $(p_0^1)$ IS OF FINITE TYPE.

Recall that an absolutely convex set $A$ in a locally convex space $F$ over

$K$ is of finite type if for each zero neighbourhood $U$ in $F$ there exists

a finite-dimensional bounded set $S \subset A$ such that $A \subset U + S$.

Let us say that a seminorm $p$ on a $K$-vector space $E$ is of finite type

if $\text{Ker } p = \{x \in E : p(x) = 0\}$ has finite codimension.

**Lemma 7.1.** Let $A$ be an absolutely convex subset of a locally convex

space $F$ whose topology is generated by a collection of seminorms of

finite type. Then the following are equivalent.

$(a)$ $A$ is a compactoid of finite type.

$(b)$ For each closed linear subspace $H$ of finite codimension there is a

finite dimensional bounded set $S \subset A$ with $A \subset H + S$. 


Proof. \((a) \Rightarrow (\beta)\). (Note. This implication holds for any locally convex space \(F\).) We may assume \([A] = F\).

\(H\) has the form \(D' := \{x \in F : f(x) = 0 \text{ for all } f \in D\}\) where \(D\) is a finite dimensional subspace of \(F'\). Let \(f_1, \ldots, f_n\) be a base of \(D\). There exist \(x_1, \ldots, x_n \in F\) with \(f_i(x_j) = \delta_{ij}\) \((i, j \in \{1, \ldots, n\})\). Since \([A] = F\) there exists a \(\lambda \in K, \lambda \neq 0\) such that \(\lambda x_i \in A\) for each \(i \in \{1, \ldots, n\}\).

Set

\[
U := \cap_{i=1}^{n} \{x \in F : |f_i(x)| \leq |\lambda|\}
\]

Then \(U\) is a zero neighbourhood in \(F\). \(A\) is a compactoid of finite type, so there exists a finite dimensional set \(S_1 \subseteq A\) with \(A \subseteq U + S_1\). Let \(x \in U\). Write \(x = y + z\) where

\[
y := x - \xi \sum_{i=1}^{n} f_i(x) x_i
\]

\[
z := \xi \sum_{i=1}^{n} f_i(x) x_i
\]

Now, since \(x \in U\), \(|f_i(x)| \leq |\lambda|\) for each \(i\) so that \(z = \xi \sum_{i=1}^{n} f_i(x) x_i \in A\).

Further, for each \(j \in \{1, \ldots, n\}\)

\[
f_j(y) = f_j(x) - \sum_{i=1}^{n} f_i(x)f_j(x_i) = f_j(x) - f_j(x) = 0
\]

and it follows that \(y \in D' = H\). So \(x = y + z \in H + [x_1, \ldots, x_n] \cap A\). We see that

\[
A \subseteq U + S_1 \subseteq H + S_2 + S_1
\]

where \(S_2 := [x_1, \ldots, x_n] \cap A\). Then \((\beta)\) is proved with \(S := S_1 + S_2\).

\((\beta) \Rightarrow (a)\). Let \(U\) be a zero neighbourhood in \(F\). Since continuous seminorms are of finite type, \(U\) contains a closed subspace \(H\) of finite codimension.

By \((\beta)\) there exists a finite dimensional set \(S \subseteq A\) with \(S\) bounded and
A \subseteq H + S. Then A \subseteq U + S.

From now on we assume that the valuation on K is dense.

Recall that for an absolutely convex set B we have \( B^i := \bigcup_{|\lambda| < 1} \lambda B. \)

**THEOREM 7.2.** Let \( p \) be a polar norm on a K-vector space \( E \). Then the following are equivalent.

(a) For each finite dimensional subspace \( D \) of \( E \) there exists a seminorm \( q \) on \( E \), \( q \) of finite type, \( q \leq p \) and \( q = p \) on \( D \).

(b) \( (p^0)_1 \) is of finite type.

**Proof.** (a) \( \Rightarrow \) (b). As each continuous seminorm on \( E^* \) is of finite type it suffices to prove, by Lemma 7.1, that for a closed subspace \( H \) of \( E^* \) of finite codimension there exists a finite dimensional set \( S \in (p^0)_1 \) such that \( (p^0)_1 \subseteq H + S. \)

Now, by (a), there is a seminorm \( q \) of finite type, \( q \leq p \) on \( E \) and \( q = p \) on \( D := H^1. \) Let

\[
S_1 := \{ f \in E^* : |f| \leq q \}.
\]

We see that \( S_1 \) is finite dimensional and since \( q \leq p \) we have \( S_1 \subseteq p^0. \)

We now shall prove that \( (p^0)_1 \subseteq H + S \) where \( S := (S_1)^\perp. \)

In fact, let \( f \in (p^0)_1 \). Then there is a \( \lambda \in K, 0 < |\lambda| < 1 \) with \( |f| \leq |\lambda| p. \)

Choose \( \lambda' \in K \) with \( |\lambda| < |\lambda'| < 1. \)

We have \( |f| \leq |\lambda| q \) on \( D \) (since \( p = q \) on \( D \)) so we can extend \( f \) to a \( g \in E^* \) with \( |g| \leq |\lambda'| q \) on \( E. \) (This is because \( q \) is of finite type so that \( (E,q) \) is strongly polar.) Now write

\[
f = f - g + g
\]

Since \( f = g \) on \( D \) we have \( f - g \in D^1 = H. \)
Also, \(|(\lambda')^{-1}g| \leq q\) so that \((\lambda')^{-1}g \in S_1\), i.e. \(g \in (S_1)^i = S\).

\((\beta) \Rightarrow (a)\). By lemma 7.1 there exists a finite dimensional set \(S \subset (p^0)^i\) so that \((p^0)^i = D^i \cap (p^0)^i + S\).

Set \(q(x) := \sup_{h \in S} |h(x)|\) \((x \in E)\).

Then \(q(x) = 0\) for all \(x\) in the space \(S^1\) which has finite codimension.

So \(q\) is of finite type.

Further, for \(x \in E\) we have

\[ q(x) = \sup_{h \in S} |h(x)| \leq \sup_{h \in (p^0)^i} |h(x)| = \sup_{h \in (p^0)^i} |h(x)| = p(x), \]

so \(q \leq p\). Finally, if \(x \in D\) then

\[ p(x) = \sup_{f \in p} |f(x)| = \sup_{f \in (p^0)^i} |f(x)| = \sup_{h \in D \cap (p^0)^i} |h(x) + t(x)| \]

\[ = \sup_{t \in S} |t(x)| = q(x). \text{ Hence, } p = q \text{ on } D. \]

§8 APPLICATION: A COMPLETE COMPACTOID IN \(c_0\) THAT IS NOT OF FINITE TYPE.

If \(K\) is spherically complete each complete absolutely convex compactoid is of finite type (See [4], 2.3).

If \(K\) is not spherically complete the unit ball of \(c_0\) is a complete compactoid for the weak topology but not of finite type (See [5], 1.6).

This is a non-metrizable compactoid. A compactoid in \((c_0, \|\|\|\|),\) not of finite type, is given in [5], 1.4. However this compactoid is not closed. The following example provides an answer to the Problem following 1.5 in [5].
PROPOSITION 8.1. Let $K$ be not spherically complete. Then there exists an absolutely convex complete compactoid in $c_0$ that is not of finite type.

Proof. Let $(K^v, | |)$ be the spherical completion of $(K, | |)$ in the sense of [1], Theorem 4.49. Let $E$ be a $K$-subspace of $K^v$ of countably infinite dimension and let $p$ be the valuation $| |$ restricted to $E$. Then $x, y \in E$, $x \perp y$ in the sense of $p$ implies $x = 0$ or $y = 0$. Obviously, the norm $p$ is of countable type (hence polar) so, by Theorem 5.1, $p^0$ is metrizable and is by Theorem 6.1, isomorphic to a compactoid in $c_0$.

Suppose $p^0$ were of finite type. Then so would $(p^0)^1$ ([5], Proposition 2.4).

By Theorem 7.2 we would have a seminorm $q$ on $E$, $q \leq p$, $q$ of finite type, $q(x) = p(x)$ for some $x \in E$, $x \neq 0$. But then $x \perp \text{Ker } q$ in the sense of $p$ (If $q(y) = 0$ then $p(x-y) \geq q(x-y) = q(x) = p(x)$) which is impossible.

So, $p^0$ is not of finite type.
REFERENCES


