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A CONNECTION BETWEEN p-ADIC BANACH SPACES AND LOCALLY CONVEX
COMPACTOIDS

by

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ABSTRACT. For a vector space $E$ over a non-archimedean valued field $K$ a
 correspondence $p \mapsto p^0$ is established between seminorms $p$ on $E$ and
 compactoids $p^0$ in $E^*$. Examination of it yields the solution of two open
 problems (see §4 and §8) and a reformulation of Serre's renorming problem
 (see §2). As a by-product results on metrizable compactoids are obtained (see §6).

§0 THE CORRESPONDENCE $p \mapsto p^0$.

Throughout this note $K$ is a non-archimedean valued field, complete with
respect to the metric induced by the nontrivial valuation $|\cdot|$. Let $E$ be a
K-vector space, let $E^*$ be its algebraic dual. A (non-archimedean) seminorm $p$ on $E$ is polar ([3], Definition 3.1), if

$$p = \sup \{ |f| : f \in E^*, |f| \leq p \}$$

Let $P_E$ be the set of all polar seminorms on $E$. For each $p \in P_E$ we set

$$p^0 = \{ f \in E^* : |f| \leq p \}$$

Then $p^0$ is an absolutely convex, edged ([3], §1b) subset of $E^*$. It is easy to see that $p^0$ is a closed compactoid ([3], §1e) with respect to the topology $\sigma(E^*, E)$, hence complete.
Let $C_E^*$ be the set of all closed absolutely convex, edged compactoids in $E^*$ with respect to $\sigma(E^*,E)$.

**PROPOSITION 0.** The map $p \mapsto p^0$ is a bijection of $P_E$ onto $C_E^*$. Its inverse assigns to every $A \in C_E^*$ the seminorm $p$ given by

$$p(x) = \sup \{|f(x)| : f \in A\} \quad (x \in E)$$

**Proof.** We shall prove surjectivity of $p \mapsto p^0$, leaving the (easy) rest of the proof to the reader. So, let $A \in C_E^*$; we shall prove that $A = p^0$ where $p(x) = \sup \{|f(x)| : f \in A\}$.

Obviously, $A \subset p^0$. Now let $g \in E^* \setminus A$, we prove that $g \not\in p^0$. The space $E^*$ is of countable type; hence strongly polar ([3], Theorem 4.4). So by [3], Theorem 4.7, there exists a $0 \in (E^*, \sigma(E^*,E))'$ such that $|0| \leq 1$ on $A$, $|0(g)| > 1$. But, by [3], lemma 7.1, $\theta$ has the form $f \mapsto f(x)$ for some $x \in E$. Thus, $|f(x)| \leq 1$ for $f \in A$, $|g(x)| > 1$ i.e., $p(x) \leq 1$ and $|g(x)| > 1$ and it follows that $g \not\in p^0$.

**Remarks.**

1. Let $K$ be spherically (= maximally) complete. Then each nonarchimedean seminorm $p$ on $E$ for which $p(x) \in \bar{K}$ ($x \in E$) is polar ([3], Remark following 3.1).

2. Let $\tau$ be the locally convex topology on $E$ induced by all nonarchimedean seminorms i.e., $\tau$ is the strongest among all locally convex topologies on $E$. It is not hard to see that $(E,\tau)$ is a complete polar ([3], Definition 3.5) space and that $(E,\tau)$ and $(E^*, \sigma(E^*,E))$ are each others strong dual spaces.
§1 NORMS \( p \) FOR WHICH \( p^0 \) IS c'-COMPACT

Recall that an absolutely convex subset \( A \) of a locally convex space \( F \) over \( K \) is c'-compact if for each neighbourhood \( U \) of 0 in \( F \) there exist \( x_1, \ldots, x_n \in A \) (rather than \( x_1, \ldots, x_n \in F \)) such that \( A \subseteq U + \text{co} \{x_1, \ldots, x_n\} \).
(Here co indicates the absolutely convex hull)

**THEOREM 1.1.** For a polar seminorm \( p \) on a \( K \)-vector space \( E \) the following are equivalent.

(a) \( p(x) \in |K| \) for each \( x \in E \). Each onedimensional subspace of \( E \) has a \( p \)-orthocomplement.

(\( \beta \)) \( p^0 \) is c'-compact.

**Proof.** (a) \( \Rightarrow \) (\( \beta \)). By [7], Theorem 3.2, it suffices to prove that for each \( \phi \in (E^*, \sigma(E^*,E))^\prime \)

\[
\max \{ |\phi(f)| : f \in p^0 \}
\]

exists. Since \( \phi \) is an evaluation map we therefore have to show that

\[
\max \{ |f(x)| : f \in p^0 \}
\]

exists for each \( x \in E \). This is obviously true if \( p(x) = 0 \). So assume \( p(x) > 0 \). Since \( p(x) \in |K| \) we may assume that \( p(x) = 1 \). For such \( x \) we must prove

\[
\max \{ |f(x)| : f \in p^0 \} = 1
\]

By (a), \( Kx \) has a \( p \)-orthocomplement \( H \). The function

\[
f : \lambda x + h \mapsto \lambda \quad (\lambda \in K, \ h \in H)
\]
is in $E^*$. We have $|f(x)| = 1$. For $\lambda \in K, h \in H$

$$|f(\lambda x + h)| = |\lambda| = p(\lambda x) \leq \max(p(\lambda x), p(h)) = p(\lambda x + h)$$

so that $f \in p_0^1$.

$(\beta) \Rightarrow (\alpha)$. Let $x \in E$. The map $f \mapsto |f(x)|$ ($f \in E^*$) is a continuous seminorm on $(E^*, \sigma(E^*, E))$. By $c'$-compactness its restriction to $p_0^1$ has a maximum so there exists a $g \in p_0^1$ with $|g(x)| = p(x)$ (It follows that $p(x) \in |K|$). We prove that $\ker g$ is a $p$-orthocomplement of $Kx$. In fact, for $z \in \ker g$ we have

$$p(x+z) \leq |g(x+z)| = |g(x)| = p(x)$$

Then also

$$p(x+z) \leq p(z)$$

completing the proof of Theorem 1.1.

Note. It is not hard to see that (a) of above is equivalent too.

$(\gamma)$ For each $x \in E$ there exists an $f \in E^*$ with $|f(x)| = p(x)$ and $|f| \leq p$.

For spherically complete $K$ we obtain a simpler form of Theorem 1.1.

**COROLLARY 1.2.** Let $K$ be spherically complete, let $p$ be a seminorm on $E$ for which $p(x) \in |K|$ for all $x \in E$.

Then the following are equivalent.

$(\alpha) p(x) \in |K|$ for each $x \in E$.

$(\beta) p_0^1$ is $c'$-compact.

**Proof.** By [1], lemma 4.35, each one-dimensional subspace has a $p$-orthocomplement.
§2 APPLICATION: A NEW LIGHT ON SERRE'S RENORMING PROBLEM.

Consider the following two statements (*) and (**).

(*) Let E be a K-vector space and let $|||\ |||$ be a norm on E. Then there exists a norm $||| \ |||'$ on E, equivalent to $||| \ |||$, such that $|||x|||' \leq |K|$ for all $x \in E$.

(**) Let K be spherically complete and let A be a complete absolutely convex compactoid in a Hausdorff locally convex space over K. Then there exist a $\lambda \in K$ with $|\lambda| > 1$ and a $c'$-compact B such that $A \subseteq B \subseteq \lambda A$.

The question as to whether (*) is true or not is known as Serre's renorming problem. See [2] for more details. We are able to reformulate this problem in terms of compactoids:

PROPOSITION 2.1. The above statements (*) and (**) are equivalent.

Proof. Assume (*). To prove (**) we may assume that A is edged. By [8], Theorem 3, A, as a topological module over $B(0,1) := \{ \lambda \in K : |\lambda| \leq 1 \}$, is isomorphic to a bounded submodule of $K^I$ for some set I. Let E be the algebraic direct sum $\oplus K_i$ where $K_i = K$ for all $i \in I$. Then $(E^*, \sigma(E^*, E))$ is in a natural way isomorphic to $K^I$ with the product topology. So we may assume that $A = p^0$ where p is a seminorm on E.

By (*) there exists a seminorm $q$, equivalent to $p$, such that $q(x) \in |K|$ for all $x \in E$. By a suitable scalar multiplication we can arrange that, in addition, $p \leq q \leq |\lambda|p$ for some $x \in K$, $|\lambda| > 1$. Then

$p^0 \leq q^0 \leq \lambda p^0$
and $q^0$ is $c'$-compact by Corollary 1.2. This proves (**). Now assume (**).

To prove (*) we may assume (see [2]), that $K$ is spherically complete.

Let $p$ be a norm on $E$. By (**) there is a $c'$-compact $B$ and a $\lambda \in K$, $|\lambda| > 1$
with $p^0 \subset B \subset \lambda p^0$. Then $B = q^0$ for some seminorm $q$ on $E$. We have

$$p \leq q \leq |\lambda|p$$

and $q(x) \in |K|$ for all $x \in E$ by Corollary 1.2.

Note. Serre's renorming problem is still unsettled as far as I know.

§3 NORMS $p$ FOR WHICH $p^0$ IS A KREIN-MILMAN COMPACTOID.

Recall that an absolutely convex subset $A$ of a locally convex space
over $K$ is a KM-compactoid if it is complete and if $A = \overline{\text{co} X}$ where $X$ is
compact. (Here $\overline{\text{co} X}$ is the closure of $\text{co} X$).

Before stating the theorem we first make some simple observations. Let

$p$ be a norm on $E$. We say that a collection $(e_i)_{i \in I}$ in $E$ is a

$p$-orthonormal base of $E$ if for each $x \in E$ there exist a unique

$(\lambda_i)_{i \in I} \subset K^I$ such that $\{i \in I, |\lambda_i| > \epsilon\}$ is finite for each $\epsilon > 0$ and

$$x = \sum_{i \in I} \lambda_i e_i$$

$$p(x) = \max_{i} |\lambda_i|$$

If $(E,p)$ is complete this definition coincides with the usual one.

**Lemma 3.1.** Let $(E,p)$ be a normed space, let $(\hat{E},\hat{p})$ be its completion.

Then $(E,p)$ has a $p$-orthonormal base if and only if $(\hat{E},\hat{p})$ has a

$\hat{p}$-orthonormal base.
Proof. It is not hard to see that each \( p \)-orthonormal base of \((E,p)\) is also a \( \hat{p} \)-orthonormal base of \((\hat{E},\hat{p})\). Conversely, let \( (e_i)_{i \in I} \) be a \( \hat{p} \)-orthonormal base of \((\hat{E},\hat{p})\). For each \( i \in I \), choose an \( f_i \in E \) with 
\[
p(e_i - f_i) \leq \frac{1}{2}.
\]
By [1], Exercise 5.C, \( (f_i)_{i \in I} \) is a \( \hat{p} \)-orthonormal base of \((\hat{E},\hat{p})\).

Clearly \( (f_i)_{i \in I} \) is a \( p \)-orthonormal base of \((E,p)\).

**THEOREM 3.2.** For a polar norm \( p \) on a \( K \)-vector space \( E \) the following are equivalent.

(a) \((E,p)\) has a \( p \)-orthonormal base

(3) \( p^0 \) is a KM-compactoid.

Proof. (a) \( \Rightarrow \) (3). Let \( (e_i)_{i \in I} \) be a \( p \)-orthonormal base of \((E,p)\). The formula 
\[
\phi(f) = (f(e_i))_{i \in I}
\]
defines a map \( \phi : p^0 \rightarrow B(0,1)^I \). Straightforward verifications show that \( \phi \) is an isomorphism of topological \( B(0,1)^I \)-modules. From [8], Theorem 16 we obtain that \( B(0,1)^I \), hence \( p^0 \), is a KM-compactoid.

(3) \( \Rightarrow \) (a). Suppose \( p^0 = \text{co} X \) where \( X \) is a compact subset of \( E^* \).

Let \( C(X*K) \) be the Banach space of all continuous functions \( X + K \), with the supremum norm \( || ||_\infty \). Then \( C(X*K) \) has an orthonormal base.

([1], Theorem 5.22).

The formula 
\[
\phi(x) (f) = f(x) \quad (f \in X)
\]
defines a \( K \)-linear map \( \phi : E \rightarrow C(X*K) \). From
\[ \|\phi(x)\|_w = \max_{f \in X} |f(x)| = \sup_{f \in \mathcal{F}} |f(x)| = p(x) \]

we obtain that \( \phi \) is an isometry \((E, p) \to (C(X^K), \| \|_w)\).

By Gruson's Theorem ([1], 5.9) \( \overline{\phi(E)} \) has an orthonormal base. Then so has \( \overline{\phi(E)} \) by Lemma 3.1 and has \( E \).

§4 APPLICATION: A COMPLETE \( c' \)-COMPACT SET WHICH IS NOT A KM-COMPACTOID.

We shall give a negative answer to the Problem following Theorem 1.7 of [6].

PROPOSITION 4.1. Let \( K \) be spherically complete, let \( |K| = \mathbb{R}_+ \).

Then there exist a locally convex space \( F \) over \( K \) and a complete \( c' \)-compact subset \( A \subset F \) which is not a KM-compactoid.

Proof. Let \( E := l^\infty \) and let \( F := (l^\infty)^* \) (with the topology we agreed upon in §0). Let \( p \) be the standard norm on \( l^\infty \), and set \( A := \overline{p^0} \). Since, trivially, \( p(x) \in |K| \) for all \( x \in l^\infty \), we have that \( \overline{p^0} \) is \( c' \)-compact (Corollary 1.2).

However, it is known ([1], Cor. 5.19) that \( l^\infty \) has no orthogonal base so that (Theorem 3.2) \( \overline{p^0} \) is not a KM-compactoid.

§5 NORMS \( p \) FOR WHICH \( p^0 \) IS METRIZABLE.

THEOREM 5.1. For a polar seminorm \( p \) on a \( K \)-vector space \( E \) the following are equivalent.

(α) \( (E, p) \) is of countable type ([3], Definition 4.3).

(β) \( p^0 \) is metrizable.
Proof. (a) ⇒ (B). There exist \( e_1, e_2, \ldots \) in \( E \) with \( p(e_i) \leq 1 \) for each \( i \) such that the \( K \)-linear span of \( e_1, e_2, \ldots \) is \( p \)-dense in \( E \). The formula

\[ \phi(f) = (f(e_1), f(e_2), \ldots) \]

defines a map \( \phi : p^0 \to B(0,1)^\mathbb{N} \). Straightforward verifications show that \( \phi \) is an isomorphism of topological \( B(0,1) \)-modules of \( p^0 \) onto a submodule of \( B(0,1)^\mathbb{N} \).

Now \( B(0,1)^\mathbb{N} \) is metrizable (the product topology is induced by the metric

\[ ((a,b)) \mapsto \sup_{i \in \mathbb{N}} |a_i - b_i|2^{-i} \]

hence so is \( p^0 \).

(B) ⇒ (a). Let \( \lambda \in K, |\lambda| > 1 \). Since \( p^0 \) is a metrizable compactoid there exist, by [3], Proposition 8.2, \( f_1, f_2, \ldots \in \lambda p^0 \) with \( \lim_{n \to \infty} f_n = 0 \) such that

\[ p^0 \leq \sup \{ f_1, f_2, \ldots \} < \lambda p^0 \]

The map

\[ \phi : x \mapsto (f_1(x), f_2(x), \ldots) \quad \text{(x \in E)} \]

is \( K \)-linear, \( \phi(E) \subset c_0 \). We have for \( x \in E \)

\[ ||\phi(x)|| = \sup_{n \in \mathbb{N}} |f_n(x)| = \sup \{ |g(x)| : g \in \sup \{ f_1, f_2, \ldots \} \} \]

It follows that

\[ p(x) \leq ||\phi(x)|| \leq |\lambda| p(x) \]

so that \( p \) is equivalent to \( x \mapsto ||\phi(x)|| \), a seminorm of countable type.

Hence, \( p \) is of countable type.
§6 APPLICATION: DESCRIPTION OF METRIZABLE COMPACTOIDS.

THEOREM 6.1. Let $A$ be an absolutely convex subset of a Hausdorff locally convex space $F$ over $K$. The following are equivalent.

(a) $A$ is a metrizable compactoid.

(b) As a topological $B(0,1)$-module, $A$ is isomorphic to a submodule of $B(0,1)^\mathbb{N}$.

(c) As a topological $B(0,1)$-module, $A$ is isomorphic to a compactoid in $C_0$.

(d) For each $\lambda \in K$, $|\lambda| > 1$ then exist $e_1, e_2, \ldots \in \lambda A$ with $\lim_{n \to \infty} e_n = 0$ and $A \subseteq \operatorname{co} \{e_1, e_2, \ldots\}$.

(e) There exist $e_1, e_2, \ldots \in F$ with $\lim_{n \to \infty} e_n = 0$ and $A \subseteq \operatorname{co} \{e_1, e_2, \ldots\}$.

(f) There exists an ultrametrizable compact $X \subseteq F$ with $A \subseteq \operatorname{co} X$.

Proof. (a) $\Rightarrow$ (b). It is not hard to see, by using the absolute convexity of $A$, that $\overline{A}$ is also metrizable. As there is no harm in taking $F$ complete we therefore may assume that $A$ is complete. To prove (b) we also may assume that $A$ is edged. By [8], Theorem 3, $A = B(0,1)^I \subseteq K^I$ for some set $I$. Like in the proof of Proposition 2.1 we may conclude that $A = \overline{p^0}$ where $p$ is a polar seminorm on $\oplus_{i \in I} K_i$ ($K_i = K$ for each $i$). Then $p$ is of countable type by Theorem 5.1. From the proof of (a) $\Rightarrow$ (b) of that Theorem we obtain an isomorphism $A = p^0 \cong B(0,1)^\mathbb{N}$.

(b) $\Rightarrow$ (c). Choose $\lambda_1, \lambda_2, \ldots \in K$, $|\lambda_1| > |\lambda_2| > \ldots$, $\lim_{n \to \infty} \lambda_n = 0$. The formula

$$\phi(\mathbf{a}) = (\lambda_1 a_1, \lambda_2 a_2, \ldots) \in c_0$$

$$\phi(\mathbf{a}) = (\lambda_1 a_1, \lambda_2 a_2, \ldots) \in c_0$$

defines a $B(0,1)$-module isomorphism of $B(0,1)^\mathbb{N}$ onto $C := \overline{\operatorname{co} \{ \lambda_1 e_1, \lambda_2 e_2, \ldots \}}$ where $e_1, e_2, \ldots$ are the standard unit vectors in $c_0$. $\phi$ is a homeomorphism
\[ B(0,1)^\mathbb{N} \rightarrow C, \text{ and maps } \text{A onto a compactoid in } C_0. \]

(y) \Rightarrow (\delta). See [3], Proposition 8.2.

(\delta) \Rightarrow (\varepsilon) \text{ is trivial.}

(\varepsilon) \Rightarrow (\eta), \{0, e_1, e_2, \ldots\} \text{ is compact and ultrametrizable.}

(\eta) \Rightarrow (\alpha). We may assume that \( F \) is complete. It suffices to prove the

metrizability of \( B := \text{co} X. \)

\( B \) is a complete, edged compactoid. As before we may assume that \( B = p^0 \)

for some polar seminorm \( p \) en some K-vector space \( E \) while \( B \subset E^*. \) The

map \( \phi : E \rightarrow C(X \times K) \) defined by

\[ \phi(x)(f) = f(x) \quad (f \in X) \]

is an isometry \( (E, p) \rightarrow (C(X \times K), ||| \cdot |||_\infty). \)

Now \( X \) is ultrametrizable so by [1], Exercise 3.5, \( C(X \times K) \) is of

countable type. Hence so is \( p. \) By Theorem 5.1, \( B = p^0 \) is metrizable.

\section{Norms \( p \) For Which \( (p^0)^* \) is of Finite Type.}

Recall that an absolutely convex set \( A \) in a locally convex space \( F \) over

\( K \) is of finite type if for each zero neighbourhood \( U \) in \( F \) there exists

a finite-dimensional bounded set \( S \subset A \) such that \( A \subset U + S. \)

Let us say that a seminorm \( p \) on a K-vector space \( E \) is of finite type

if \( \ker p = \{x \in E : p(x) = 0\} \) has finite codimension.

\textbf{Lemma 7.1.} Let \( A \) be an absolutely convex subset of a locally convex

space \( F \) whose topology is generated by a collection of seminorms of

finite type. Then the following are equivalent.

(\alpha) \( A \) is a compactoid of finite type.

(\beta) For each closed linear subspace \( H \) of finite codimension there is a

finite dimensional bounded set \( S \subset A \) with \( A \subset H + S. \)
Proof. (a) ⇒ (β). (Note. This implication holds for any locally convex space F.) We may assume \( \{A\} = F \).

H has the form \( D^\perp := \{ x \in F : f(x) = 0 \text{ for all } f \in D \} \) where D is a finite dimensional subspace of \( F' \). Let \( f_1, \ldots, f_n \) be a base of D. There exist \( x_1, \ldots, x_n \in F \) with \( f_i(x_j) = \delta_{ij} \) (\( i, j \in \{1, \ldots, n\} \)). Since \( \{A\} = F \), there exists a \( \lambda \in K, \lambda \neq 0 \) such that \( \lambda x_i \in A \) for each \( i \in \{1, \ldots, n\} \).

Set

\[
U := \bigcap_{i=1}^n \{ x \in F : |f_i(x)| \leq |\lambda| \}
\]

Then U is a zero neighbourhood in \( F \). A is a compactoid of finite type, so there exists a finite dimensional set \( S_1 \subseteq A \) with \( A \subseteq U + S_1 \). Let \( x \in U \). Write \( x = y + z \) where

\[
y := x - \sum_{i=1}^n f_i(x) x_i
\]

\[
z := \sum_{i=1}^n f_i(x) x_i
\]

Now, since \( x \in U \), \( |f_i(x)| \leq |\lambda| \) for each \( i \) so that \( z = \sum_{i=1}^n f_i(x) x_i \in A \).

Further, for each \( j \in \{1, \ldots, n\} \)

\[
f_j(y) = f_j(x) - \sum_{i=1}^n f_i(x)f_j(x_i) = f_j(x) - f_j(x) = 0
\]

and it follows that \( y \in D^\perp = H \). So \( x = y + z \)

\( \in H + [x_1, \ldots, x_n] \cap A \). We see that

\[
A \subseteq U + S_1 \subseteq H + S_2 + S_1
\]

where \( S_2 := [x_1, \ldots, x_n] \cap A \). Then (β) is proved with \( S := S_1 + S_2 \).

(β) ⇒ (a). Let \( U \) be a zero neighbourhood in \( F \). Since continuous seminorms are of finite type, \( U \) contains a closed subspace \( H \) of finite codimension.

By (β) there exists a finite dimensional set \( S \subseteq A \) with \( S \) bounded and
A \subset H + S. Then A \subset U + S.

From now on we assume that the valuation on K is dense.

Recall that for an absolutely convex set B we have \( B^i := \bigcup_{|\lambda| < 1} \lambda B. \)

**Theorem 7.2.** Let \( p \) be a polar norm on a K-vector space \( E \). Then the following are equivalent.

(a) For each finite dimensional subspace \( D \) of \( E \) there exists a seminorm \( q \) on \( E \), \( q \) of finite type, \( q \leq p \) and \( q = p \) on \( D \).

(b) \( (p^0)^{1\ast} \) is of finite type.

**Proof.** (a) \( \Rightarrow \) (b). As each continuous seminorm on \( E^* \) is of finite type it suffices to prove, by Lemma 7.1, that for a closed subspace \( H \) of \( E^* \) of finite codimension there exists a finite dimensional set \( S \in (p^0)^{1\ast} \) such that \( (p^0)^{1\ast} \subset H + S. \)

Now, by (a), there is a seminorm \( q \) of finite type, \( q \leq p \) on \( E \) and \( q = p \) on \( D := \overline{H}. \) Let

\[
S_1 := \{ f \in E^* : |f| \leq q \}.
\]

We see that \( S_1 \) is finite dimensional and since \( q \leq p \) we have \( S_1 \subset p^0. \)

We now shall prove that \( (p^0)^{1\ast} \subset H + S \) where \( S := (S_1)^{1\ast}. \)

In fact, let \( f \in (p^0)^{1\ast}. \) Then there is a \( \lambda \in K \), \( 0 < |\lambda| < 1 \) with \( |f| \leq |\lambda| p. \)

Choose \( \lambda' \in K \) with \( |\lambda| < |\lambda'| < 1. \)

We have \( |f| \leq |\lambda| q \) on \( D \) (since \( p = q \) on \( D \)) so we can extend \( f \) to a \( g \in E^* \) with \( |g| \leq |\lambda'| q \) on \( E. \) (This is because \( q \) is of finite type so that \( (E,q) \) is strongly polar.) Now write

\[
f = f - g + g \]

Since \( f = g \) on \( D \) we have \( f - g \in D^\perp = H. \)
Also, $|\lambda'^{-1}g| \leq q$ so that $(\lambda'^{-1}g) \in S_1$ i.e. $g \in (S_1)^i = S$.

$(B) \Rightarrow (a)$. By lemma 7.1 there exists a finite dimensional set $S \subset (p^0)^i$ so that $(p^0)^i = D^\perp \cap (p^0)^i + S$.

Set $q(x) := \sup_{h \in S} |h(x)|$. (x $\in E$).

Then $q(x) = 0$ for all $x$ in the space $S^\perp$ which has finite codimension.

So $q$ is of finite type.

Further, for $x \in E$ we have

$q(x) = \sup_{h \in S} |h(x)| \leq \sup_{h \in (p^0)^i} |h(x)| = \sup_{h \in (p^0)^i} |h(x)| = p(x)$,

so $q \leq p$. Finally, if $x \in D$ then

$p(x) = \sup_{f \in p} |f(x)| = \sup_{f \in (p^0)^i} |f(x)| = \sup_{h \in D^\perp \cap (p^0)^i} |h(x) + t(x)|$

$= \sup_{t \in S} |t(x)| = q(x)$. Hence, $p = q$ on $D$.

§8 APPLICATION: A COMPLETE COMPACTOID IN $c_0$ THAT IS NOT OF FINITE TYPE.

If $K$ is spherically complete each complete absolutely convex compactoid is of finite type (See [4], 2.3).

If $K$ is not spherically complete the unit ball of $c_0$ is a complete compactoid for the weak topology but not of finite type (See [5], 1.6).

This is a non-metrizable compactoid. A compactoid in $(c_0, ||||)$, not of finite type, is given in [5], 1.4. However this compactoid is not closed. The following example provides an answer to the Problem following 1.5 in [5].
PROPOSITION 8.1. Let K be not spherically complete. Then there exists an absolutely convex complete compactoid in $c_0$ that is not of finite type.

Proof. Let $(K^V, | |)$ be the spherical completion of $(K, | |)$ in the sense of [1], Theorem 4.49. Let $E$ be a $K$-subspace of $K^V$ of countably infinite dimension and let $p$ be the valuation $| |$ restricted to $E$. Then $x, y \in E, x \perp y$ in the sense of $p$ implies $x = 0$ or $y = 0$. Obviously, the norm $p$ is of countable type (hence polar) so, by Theorem 5.1, $p^0$ is metrizable and is by Theorem 6.1, isomorphic to a compactoid in $c_0$.

Suppose $p^0$ were of finite type. Then so would $(p^0)^i$ ([5], Proposition 2.4).

By Theorem 7.2 we would have a seminorm $q$ on $E$, $q \leq p$, $q$ of finite type, $q(x) = p(x)$ for some $x \in E, x \neq 0$. But then $x \perp \text{Ker } q$ in the sense of $p$ (If $q(y) = 0$ then $p(x-y) \geq q(x-y) = q(x) = p(x)$) which is impossible.

So, $p^0$ is not of finite type.
REFERENCES


