EPSILON STABILITY OF p-ADIC CHARACTERS

by

W.H. SCHIKHOF

Report 8726
September 1987

DEPARTMENT OF MATHEMATICS
CATHOLIC UNIVERSITY
Toernooiveld
6525 ED Nijmegen
The Netherlands
EPSILON STABILITY OF p-ADIC CHARACTERS

by

W.H. Schikhof

ABSTRACT. The following problem is solved (see Theorem 1.1 below). Let G be an abelian group, written additively, and let K be an algebraically closed nonarchimedean valued field which is complete for the valuation \(|\cdot\)|. Let \(f : G \to T_K := \{\lambda \in K : |\lambda| = 1\}\) be an \(\varepsilon\)-character i.e.

\[|f(x+y)-f(x)f(y)| \leq \varepsilon \quad (x,y \in G)\]

where \(0 \leq \varepsilon < 1\). Does there exist a character \(\alpha : G \to T_K\) for which

\[|\alpha(x)-f(x)| \leq \varepsilon \quad (x \in G)\]?

Is it unique?

NOTES. The results of this Report will be used in a future paper on p-adic almost periodic functions [4]. The expression 'epsilon stability' is taken from [1], p.11, where a closely related problem is discussed.
§1 THE THEOREM

Throughout, let $G, K, T_K$ be as above. For trivially valued fields $K$ the above problem has a trivial solution. So from now on we assume that the valuation of $K$ is non-trivial. The residue class field of $K$ is $k$. The characteristic of a field $L$ is denoted $\text{char } L$.

Let $0 < \varepsilon < 1$. A function $f : G \to T_K$ is an $\varepsilon$-character if $|f(x+y) - f(x)f(y)| \leq \varepsilon$ for all $x, y \in G$. As usual, we shall say "character" instead of "0-character".

Consider the following statements (E) and (U).

(E) For each $\varepsilon \in [0,1)$ and each $\varepsilon$-character $f : G \to T_K$ there exists a character $\alpha : G \to T_K$ for which $|f(x) - \alpha(x)| \leq \varepsilon$ ($x \in G$).

(U) For each $\varepsilon \in [0,1)$ and each $\varepsilon$-character $f : G \to T_K$ there exists at most one character $\alpha : G \to T_K$ for which $|f(x) - \alpha(x)| \leq \varepsilon$ ($x \in G$).

The purpose of this note is to prove the following Theorem.

**THEOREM 1.1.**

(i) Let $\text{char } k = 0$. Then (E) holds for any $G$, (U) holds if and only if $G$ is a torsion group.

(ii) Let $\text{char } K = 0$, $\text{char } k = p \neq 0$. Then (E) holds if and only if $G$ has no subgroups of order $p$, (U) holds if and only if $G$ has no subgroups of index $p$.

(iii) Let $\text{char } K = p \neq 0$. Then (E) holds if and only if $G$ has no subgroups of order $p$, (U) holds if and only if no subgroup of $G/G_p$, where $G_p := \{x \in G : p^n x = 0 \text{ for some } n \in \mathbb{N}\}$, has index $p$. 
We have the following trivial corollary.

**COROLLARY 1.2.**

(i) Let $\text{char } k = 0$. Then both (E) and (U) hold if and only if $G$ is a **torsion group**.

(ii) Let $\text{char } k = p \neq 0$. Then both (E) and (U) hold if and only if $G$ has neither subgroups of order $p$, nor subgroups of index $p$.

**REMARK.** The statement 'G has no subgroups of order $p$' is obviously equivalent to 'the map $x \mapsto px$ ($x \in G$) is injective'. It is not hard to see that 'G has no subgroups of index $p$' is equivalent to 'the map $x \mapsto px$ ($x \in G$) is surjective'.

**EXAMPLES.** If $G$ is $p$-free (i.e. if $H_1 \subseteq H_2$ are subgroups then the index $[H_2 : H_1]$, whenever finite, is not divisible by $p$) then $x \mapsto px$ is a bijection. But this conclusion holds also for the additive group of the $p$-adic numbers $\mathbb{Q}_p$. On $\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ the map $x \mapsto px$ is injective but not surjective, on $\mathbb{Q}_p/\mathbb{Z}_p$ the map $x \mapsto px$ is surjective but not injective.

§2 PRELIMINARIES

**LEMMA 2.1.** (Elementary properties of $\varepsilon$-characters). Let $0 < \varepsilon < 1$.

(i) Let $f : G \to T_K$ be an $\varepsilon$-character. Then

(a) $|f(0) - 1| \leq \varepsilon$.

(b) If $x_1, \ldots, x_n \in G$ then $|f(x_1 + \ldots + x_n) - f(x_1)f(x_2)\ldots f(x_n)| \leq \varepsilon$.

(c) If $g : G \to K$, $|g(x)| \leq \varepsilon$ for all $x \in G$, then $f+g$ is an $\varepsilon$-character.

(ii) $B_1(\varepsilon) := \{\lambda \in K : |1-\lambda| < \varepsilon\}$ is a subgroup of $T_K$. Let $\pi : T_K \to T_K/B_1(\varepsilon)$ be the quotient map. Then $f : G \to T_K$ is an $\varepsilon$-character if and only if

$\pi f : G \to T_K/B_1(\varepsilon)$ is a homomorphism.

**Proof.** Straightforward.
PROPOSITION 2.2. (Extension of ε-characters) Let \( H \) be a subgroup of \( G \) and let, for some \( \varepsilon \in [0,1) \), \( f : H \rightarrow T_K \) be an \( \varepsilon \)-character. Then \( f \) can be extended to an \( \varepsilon \)-character \( \tilde{f} : G \rightarrow T_K \).

Proof. By lemma 2.1 (i) (c) it suffices to find an \( \varepsilon \)-character \( \tilde{f} : G \rightarrow T_K \) for which \( |\tilde{f}(h) - f(h)| < \varepsilon \) \( (h \in H) \). With the notations as in Lemma 2.1 (ii) we have that \( \pi \circ f \) is a homomorphism \( H \rightarrow T_K/B_1(\varepsilon) \). As \( K \) is algebraically closed the group \( T_K \) hence \( T_K/B_1(\varepsilon) \), is divisible. Therefore, \( \pi \circ f \) can be extended to a homomorphism \( g : G \rightarrow T_K/B_1(\varepsilon) \). Choose any \( \rho : T_K/B_1(\varepsilon) \rightarrow T_K \) for which \( \pi \circ \rho \) is the identity. Then \( \tilde{f} := \rho \circ g \) has the required properties.

We quote a lemma needed for the proof of Proposition 3.2.

LEMMA 2.3. Let \( \text{char } k = p \neq 0 \). Let \( a, b \in T_K \) such that \( 0 < |a-b| < \varepsilon < 1 \). Then \( |a^p-b^p| < \tau \varepsilon \) where \( \tau := \max(\varepsilon, |p|) \). In particular, \( |a^p-b^p| < |a-b| \).

Proof. [2], Lemma 32.1.

§3 EXISTENCE

In this section we prove the 'existence part' of Theorem 1.1 i.e. the statements involving \((E)\).

For the case \( \text{char } k = 0 \) we have quite standard methods:

PROPOSITION 3.1. Let \( \text{char } k = 0 \). Then, for any \( G \), for each \( \varepsilon \in [0,1) \) and each \( \varepsilon \)-character \( f : G \rightarrow T_K \) there exists a character \( \alpha : G \rightarrow T_K \) for which \( |f(x) - \alpha(x)| < \varepsilon \) \( (x \in G) \).

Proof. We start the construction of \( \alpha \) by setting \( \alpha(0) := 1 \). Then, by Lemma 2.1 (i) (a), \( |f-\alpha| \leq \varepsilon \) on the zero group. Now suppose we have a subgroup \( H \) of \( G \) and a character \( \alpha : H \rightarrow T_K \) such that \( |f(h) - \alpha(h)| \leq \varepsilon \) for all \( h \in H \). Let \( x \in G \setminus H \). We prove that \( \alpha \) can be extended to a character \( \tilde{\alpha} \) defined on the group \( H' \) generated by \( H \) and \( \{x\} \) such that \( |f(h') - \tilde{\alpha}(h')| \leq \varepsilon \) for all \( h' \in H \). (A simple application of Zorn's Lemma then may complete the proof.)
If $H'/H \cong \mathbb{Z}$ we define $\tilde{\alpha}$ by

$$\tilde{\alpha}(nx+h) := f(x)^n \alpha(h) \quad (n \in \mathbb{Z}, h \in H)$$

One proves easily, by using Lemma 2.1(i)(b), that $\tilde{\alpha}$ satisfies the requirements. If $H'/H \cong \mathbb{Z}/q\mathbb{Z}$ for some $q \in \mathbb{N}, q > 1$ we set

$$\tilde{\alpha}(nx+h) := \theta^n \alpha(h) \quad (n \in \{0,1, \ldots, q-1\}, h \in H)$$

where $\theta \in K$ is chosen such that $\theta^q = \alpha(qx)$ (then $\tilde{\alpha}$ is a character) and such that $|\theta-f(x)| \leq \varepsilon$ (then $|\tilde{\alpha}(h')-f(h')| \leq \varepsilon$ for all $h' \in H'$). To see such a $\theta$ exists consider the polynomial $p = x^q - \alpha(qx) \in K[X]$. We have $|p(f(x))| \leq \varepsilon, |p'(f(x))| = |q||f(x)|^{q-1} = 1$ (here we use the assumption $\text{char } k = 0$). By Hensel's Lemma there exists a $\theta \in K$ with $p(\theta) = 0$ and $|\theta-f(x)| \leq \varepsilon$.

REMARKS.

1. The algebraic closedness of $K$ has not been used in the above proof.
2. Let $B_1(\varepsilon), \pi$ be as in Lemma 2.1. Then any $\rho : T_K/B_1(\varepsilon) \to T_K$ for which $\pi \circ \rho$ is the identity, is an $\varepsilon$-character of $T_K/B_1(\varepsilon)$. Proposition 3.1 tells that there exists a homomorphism $\tilde{\rho} : T_K/B_1(\varepsilon) \to T_K$ such that $\pi \circ \tilde{\rho}$ is the identity. As a corollary we obtain that if $\text{char } k = 0$ then $B_1(\varepsilon)$ is a factor in $T_K$.

Next we turn to the case where $\text{char } k = p \neq 0$. To cover also groups like $\mathbb{Q}_p$ we shall use a technique different from the above one.

PROPOSITION 3.2. Let $\text{char } k = p \neq 0$. Suppose $x \mapsto px$ is a bijection $G \to G$. Then for each $\varepsilon \in (0,1)$ and each $\varepsilon$-character $f : G \to T_K$ there exists a character $\alpha : G \to T_K$ such that $|f(x)-\alpha(x)| \leq \varepsilon$ ($x \in G$).
Proof. For each $n \in \mathbb{N}$ let $x \mapsto p^{-n}x$ ($x \in G$) be the inverse of $x \mapsto p^n x$.

For any $n \in \mathbb{N}$, $x \in G$ we have

$$|f(p^{-n-1}x)^p - f(p^{-n}x)| \leq \varepsilon$$

so that, by Lemma 2.3,

$$|f(p^{-n-1}x)^{p^{n+1}} - f(p^{-n}x)^p| \leq n \varepsilon.$$ 

By completeness of $K$

$$\alpha(x) := \lim_{n \to \infty} f(p^{-n}x)^p$$

exists (uniformly in $x \in G$). For each $n \in \mathbb{N}$, $x \in G$ we have

$$|f(x) - f(p^{-n}x)^p| \leq \varepsilon$$

giving also

$$|f(x) - \alpha(x)| \leq \varepsilon.$$

To see that $\alpha$ is a character, let $x, y \in G$, $n \in \mathbb{N}$. We have

$$|f(p^{-n}(x+y)) - f(p^{-n}x)f(p^{-n}y)| \leq \varepsilon$$

Again by Lemma 2.3,

$$|f(p^{-n}(x+y))^p - f(p^{-n}x)^p f(p^{-n}y)^p| \leq n \varepsilon$$

hence also

$$|\alpha(x+y) - \alpha(x)\alpha(y)| \leq 0.$$

Proposition 3.2 is a stepping stone for
PROPOSITION 3.3. Let char \( k = p \neq 0 \). Suppose \( x \mapsto px \) is an injection \( G \rightarrow G \). Then the conclusion of Proposition 3.2 holds.

Proof. \( G \) can be embedded into a divisible group \( D \). Set \( G_1 := D/D_p \) (for \( D_p \) see Theorem 1.1(iii)). Then \( G + D + G_1 \) is injective and \( x \mapsto px \) is a bijection \( G_1 \rightarrow G_1 \). We may assume \( G \subseteq G_1 \). By Proposition 2.2 \( f \) can be extended to an \( \epsilon \)-character \( \tilde{f} \) on \( G_1 \). By Proposition 3.2 there is a character \( \tilde{\alpha} : G_1 \rightarrow \mathbb{T}_k \) for which \( |\tilde{f}(x) - \tilde{\alpha}(x)| \leq \epsilon \) (\( x \in G_1 \)). Set \( \alpha := \tilde{\alpha}|G \).

We now prove the converse to Proposition 3.3.

PROPOSITION 3.4. Let char \( k = p \neq 0 \). Suppose \( x \mapsto px \) \( (x \in G) \) is not injective. Then there exists an \( \epsilon \in (0,1) \) and an \( \epsilon \)-character \( f : G + T_K \) such that for every character \( \alpha : G + T_K \) there exists an \( x \in G \) with \( |f(x) - \alpha(x)| > \epsilon \).

Proof. Choose an element \( x \in G \) of order \( p \) and a \( b \in K \) with \( 0 < |1-b| < 1 \).

If char \( K = 0 \) we assume in addition that \( |1-b| < |1-\theta| \) where \( \theta \) is a primitive \( p^{\text{th}} \) root of unity. Consider the map

\[
g : nx \mapsto b^n \quad (n \in \{0,1,\ldots,p-1\})
\]

defined on the group generated by \( x \). For \( n,m \in \{0,1,\ldots,p-1\} \) we have

\[
|g(nx+mx) - g(nx)g(mx)| = \begin{cases} 0 & \text{if } n+m \leq p-1 \\ |b^{n+m-p}b^mb^m| & \text{if } n+m > p \end{cases}
\]

so we see that \( g \) is an \( \epsilon \)-character where \( \epsilon = |b^{-p-1}| = |b^{p-1}| \).

By Proposition 2.2 \( g \) extends to an \( \epsilon \)-character \( f : G + T_K \). Now let \( \alpha : G + T_K \) be a character. If \( \alpha = 1 \) on \( H \) we have \( |f(x)-\alpha(x)| = |b^{-1}| > |b^{p-1}| = \epsilon \) (Lemma 2.3). Otherwise we have \( \alpha(x) = \theta \) where \( \theta \) is a primitive \( p^{\text{th}} \) root of unity (and char \( K = 0 \)). Then we have, since \( |1-b| < |1-\theta| \),

\[
|f(x)-\alpha(x)| = |b-\theta| = \max(|b^{-1}|,|1-\theta|) = |1-\theta| > |1-b| > \epsilon.
\]
§4 UNIQUENESS

In this section we prove the 'uniqueness part' of Theorem 1.1 i.e. the statements involving \((U)\).

PROPOSITION 4.1.

(i) Let \(\text{char } k = 0\). Then \((U)\) holds if and only if \(G\) is a torsion group.

(ii) Let \(\text{char } K = 0\), char \(k \neq p \neq 0\). Then \((U)\) holds if and only if \(G\) has no subgroup of index \(p\).

(iii) Let \(\text{char } K = p \neq 0\). Then \((U)\) holds if and only if each homomorphism \(G \rightarrow \mathbb{Z}/p\) is zero.

Proof. Statement \((U)\) is equivalent to "if \(\alpha, \beta\) are distinct characters then \(\sup\{|\alpha(x) - \beta(x)| : x \in G\} = 1\)"; which is by [3], Proposition 1.1, the same as "the characters form an orthogonal set with respect to the sup norm".

Now apply [3], Theorem 3.1, Theorem 2.2, Theorem 4.3 (where \(G\) has the discrete topology). Proposition 4.1 follows.

To complete the proof of Theorem 1.1 the following lemma remains to be shown.

LEMMA 4.2. Let \(p\) be a prime number. The following are equivalent:

(a) Any homomorphism \(G \rightarrow \mathbb{Z}/p\) is zero.

(b) If \(H\) is a subgroup of \(G/G_p\) then \(H\) does not have index \(p\).

The heart of the proof is contained in

LEMMA 4.3. Let \(x \mapsto px\) \((x \in G)\) be injective but not surjective. Then there exists a nonzero homomorphism \(G \rightarrow \mathbb{Z}/p\).

Proof. Let \(\pi : G \rightarrow G/pG\) be the quotient map. As \(pG \neq G\) the group \(G/pG\) is in a natural way a nonzero vector space over the field of \(p\) elements.

So there exists an indexed set \((e_i)_{i \in I}\) in \(G\) where \(I \neq \emptyset\) such that \(\{\pi(e_i) : i \in I\}\) is a base of the vector space \(G/pG\). It follows that for
each $x \in G$ there exist unique $(\lambda^{(1)}_i)_{i \in I}$ where $\lambda^{(1)}_i \in \{0,1,\ldots,p-1\} \subset \mathbb{Z}$
and \{i \in I : \lambda^{(1)}_i \neq 0\} is finite such that $x = \sum \lambda^{(1)}_i e_i = px_i$ where $x_i \in G$.
By injectivity of $x \rightarrow px$ also $x_i$ is unique. By treating $x_i$ in the same way as we did for $x$ we find unique $(\lambda^{(2)}_i)_{i \in I} \in \{0,1,\ldots,p^2-1\} \subset \mathbb{Z}$,
\{i \in I : \lambda^{(2)}_i \neq 0\} is finite such that $x = \sum \lambda^{(2)}_i e_i = p^2 x_2$ where $x_2 \in G$ etc.

Thus, for each $n \in \mathbb{N}$ there exist unique maps $\phi^{(n)}_i : G \rightarrow \{0,1,\ldots,p^n-1\} \subset \mathbb{Z}$
with \{i \in I : \phi^{(n)}_i(x) \neq 0\} finite for each $x \in G$ such that

$$x = \sum \phi^{(n)}_i (x) e_i \in p^n G \quad (x \in G)$$

By uniqueness, for each $i \in I$, $n \in \mathbb{N}$

$$\phi^{(n+1)}_i (x) \equiv \phi^{(n)}_i (x) \mod p^n \quad (x \in G)$$

We see that, for any $j \in I$, the p-adic limit

$$\phi_j(x) = \lim_{n \to \infty} \phi^{(n)}_j (x) \quad (x \in G)$$

exists and defines a map $\phi_j : G \rightarrow \mathbb{Z}_p$. As $\phi_j(e_j) = 1$ this map is not zero.

To see that $\phi_j$ is a homomorphism observe that for each $n \in \mathbb{N}$, $x,y \in G$

$$\sum_{i} (\phi^{(n)}_i (x+y) - \phi^{(n)}_i (x) - \phi^{(n)}_i (y)) e_i \in p^n G$$

By what we have proved above

$$\phi^{(n)}_j (x+y) - \phi^{(n)}_j (x) - \phi^{(n)}_j (y) \equiv 0 \mod p^n$$

i.e.

$$|\phi^{(n)}_j (x+y) - \phi^{(n)}_j (x) - \phi^{(n)}_j (y)|_p \leq p^{-n}$$

which means for $\phi_j$ that

$$|\phi_j (x+y) - \phi_j (x) - \phi_j (y)| \leq 0$$
Proof of Lemma 4.2. (a) \rightarrow (b). Suppose we had a subgroup \( H \) of \( G/G_p \) of index \( p \). Then, since \( G/G_p \) has no elements of order \( p \), the map \( x \mapsto px \) is injective but not surjective on \( G/G_p \), so Lemma 4.3 gives us a nontrivial homomorphism \( \phi : G/G_p \rightarrow \mathbb{Z}_p \). But then \( G + G/G_p \rightarrow \mathbb{Z}_p \) is a nontrivial homomorphism \( G \rightarrow \mathbb{Z}_p \) which conflicts (a). To prove (b) \rightarrow (a), suppose we had a nontrivial homomorphism \( \phi : G \rightarrow \mathbb{Z}_p \). Then we may assume \( 1 \in \text{Im} \phi \).

It is easy to see that \( H := \phi^{-1}(p\mathbb{Z}_p) \) has index \( p \) and contains \( g_p \). This violates (a).

PROBLEM. Generalize Theorem 1.1 to topological abelian groups \( G \) and continuous (\( \epsilon \)-)characters.

REFERENCES


