EPSILON STABILITY OF p-ADIC CHARACTERS

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Report 8726
September 1987

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ABSTRACT. The following problem is solved (see Theorem 1.1 below). Let G be an abelian group, written additively, and let K be an algebraically closed nonarchimedean valued field which is complete for the valuation \(|\cdot|\). Let \(f : G \rightarrow T_K := \{\lambda \in K : |\lambda| = 1\}\) be an \(\varepsilon\)-character i.e.

\[|f(x+y) - f(x)f(y)| \leq \varepsilon \quad (x, y \in G)\]

where \(0 \leq \varepsilon < 1\). Does there exist a character \(\alpha : G \rightarrow T_K\) for which \(|\alpha(x) - f(x)| \leq \varepsilon\) \((x \in G)\)? Is it unique?

NOTES. The results of this Report will be used in a future paper on p-adic almost periodic functions [4]. The expression 'epsilon stability' is taken from [1], p.11, where a closely related problem is discussed.
§1 THE THEOREM

Throughout, let $G, K, T_K$ be as above. For trivially valued fields $K$ the above problem has a trivial solution. So from now on we assume that the valuation of $K$ is non-trivial. The residue class field of $K$ is $k$. The characteristic of a field $L$ is denoted $\text{char} L$.

Let $0 < \varepsilon < 1$. A function $f : G \rightarrow T_K$ is an $\varepsilon$-character if $|f(x+y)-f(x)f(y)| \leq \varepsilon$ for all $x, y \in G$. As usual, we shall say 'character' instead of '0-character'.

Consider the following statements (E) and (U).

\begin{align*}
\text{(E)} & \quad \text{For each } \varepsilon \in (0,1) \text{ and each } \varepsilon \text{-character } f : G \rightarrow T_K \text{ there exists a character } \alpha : G \rightarrow T_K \text{ for which } |f(x)-\alpha(x)| \leq \varepsilon \quad (x \in G). \\
\text{(U)} & \quad \text{For each } \varepsilon \in (0,1) \text{ and each } \varepsilon \text{-character } f : G \rightarrow T_K \text{ there exists at most one character } \alpha : G \rightarrow T_K \text{ for which } |f(x)-\alpha(x)| \leq \varepsilon \quad (x \in G).
\end{align*}

The purpose of this note is to prove the following Theorem.

**THEOREM 1.1.**

(i) Let $\text{char } k = 0$. Then (E) holds for any $G$, (U) holds if and only if $G$ is a torsion group.

(ii) Let $\text{char } K = 0$, $\text{char } k = p \neq 0$. Then (E) holds if and only if $G$ has no subgroups of order $p$, (U) holds if and only if $G$ has no subgroups of index $p$.

(iii) Let $\text{char } K = p \neq 0$. Then (E) holds if and only if $G$ has no subgroups of order $p$, (U) holds if and only if no subgroup of $G/G_p$, where $G_p := \{ x \in G : p^n x = 0 \text{ for some } n \in \mathbb{N} \}$, has index $p$. 

We have the following trivial corollary.

COROLLARY 1.2.

(i) Let \( \text{char } k = 0 \). Then both \((E)\) and \((U)\) hold if and only if \( G \) is a torsion group.

(ii) Let \( \text{char } k = p \neq 0 \). Then both \((E)\) and \((U)\) hold if and only if \( G \) has neither subgroups of order \( p \), nor subgroups of index \( p \).

REMARK. The statement '\( G \) has no subgroups of order \( p \)' is obviously equivalent to 'the map \( x \mapsto px \ (x \in G) \) is injective'. It is not hard to see that '\( G \) has no subgroups of index \( p \)' is equivalent to 'the map \( x \mapsto px \ (x \in G) \) is surjective'.

EXAMPLES. If \( G \) is \( p \)-free (i.e. if \( H_1 \subset H_2 \) are subgroups then the index \( [H_2 : H_1] \), whenever finite, is not divisible by \( p \)) then \( x \mapsto px \) is a bijection. But this conclusion holds also for the additive group of the \( p \)-adic numbers \( \mathbb{Q}_p \). On \( \mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x|_p \leq 1\} \) the map \( x \mapsto px \) is injective but not surjective, on \( \mathbb{Q}_p / \mathbb{Z}_p \) the map \( x \mapsto px \) is surjective but not injective.

§2 PRELIMINARIES

LEMMA 2.1. (Elementary properties of \( \varepsilon \)-characters). Let \( 0 \leq \varepsilon < 1 \).

(i) Let \( f : G \rightarrow T_K \) be an \( \varepsilon \)-character. Then

(a) \(|f(0)-1| \leq \varepsilon \).

(b) If \( x_1, \ldots, x_n \in G \) then \(|f(x_1 + \cdots + x_n) - f(x_1)f(x_2) \cdots f(x_n)| \leq \varepsilon \).

(c) If \( g : G \rightarrow K \), \(|g(x)| \leq \varepsilon \) for all \( x \in G \), then \( f+g \) is an \( \varepsilon \)-character.

(ii) \( B_1(\varepsilon) := \{\lambda \in K : |1-\lambda| \leq \varepsilon\} \) is a subgroup of \( T_K \). Let \( \pi : T_K \rightarrow T_K / B_1(\varepsilon) \) be the quotient map. Then \( f : G \rightarrow T_K \) is an \( \varepsilon \)-character if and only if \( \pi \circ f : G \rightarrow T_K / B_1(\varepsilon) \) is a homomorphism.

Proof. Straightforward.
PROPOSITION 2.2. (Extension of ε-characters) Let $H$ be a subgroup of $G$ and let, for some $ε \in (0,1)$, $f : H \to T_K$ be an ε-character. Then $f$ can be extended to an ε-character $\tilde{f} : G \to T_K$.

Proof. By Lemma 2.1 (i) (c) it suffices to find an ε-character $\tilde{f} : G \to T_K$ for which $|\tilde{f}(h) - f(h)| < ε$ $(h \in H)$. With the notations as in Lemma 2.1 (ii) we have that $π \circ f$ is a homomorphism $H \to T_K/B_1(ε)$. As $K$ is algebraically closed the group $T_K$, hence $T_K/B(ε)$, is divisible. Therefore, $π \circ f$ can be extended to a homomorphism $g : G \to T_K/B_1(ε)$. Choose any $ρ : T_K/B_1(ε) \to T_K$ for which $π \circ ρ$ is the identity. Then $\tilde{f} := ρ \circ g$ has the required properties.

We quote a lemma needed for the proof of Proposition 3.2.

LEMMA 2.3. Let $char k = p \neq 0$. Let $a, b \in T_K$ such that $0 < |a-b| < ε < 1$. Then $|a^p - b^p| < τε$ where $τ := \max(ε, |p|)$. In particular, $|a^p - b^p| < |a-b|$.

Proof. [2], Lemma 32.1.

§3 EXISTENCE

In this section we prove the 'existence part' of Theorem 1.1 i.e. the statements involving $(E)$.

For the case $char k = 0$ we have quite standard methods:

PROPOSITION 3.1. Let $char k = 0$. Then, for any $G$, for each $ε \in (0,1)$ and each ε-character $f : G \to T_K$ there exists a character $α : G \to T_K$ for which $|f(x) - α(x)| < ε$ $(x \in G)$.

Proof. We start the construction of $α$ by setting $α(0) := 1$. Then, by Lemma 2.1 (i) (a), $|f-α| < ε$ on the zero group. Now suppose we have a subgroup $H$ of $G$ and a character $α : H \to T_K$ such that $|f(h) - α(h)| < ε$ for all $h \in H$. Let $x \in G \setminus H$. We prove that $α$ can be extended to a character $\tilde{α}$ defined on the group $H'$ generated by $H$ and $\{x\}$ such that $|f(h') - \tilde{α}(h')| < ε$ for all $h' \in H$. (A simple application of Zorn's Lemma then may complete the proof.)
If $H'/H \cong \mathbb{Z}$ we define $\tilde{\alpha}$ by

$$\tilde{\alpha}(nx+h) := f(x)^n a(h) \quad (n \in \mathbb{Z}, h \in H)$$

One proves easily, by using Lemma 2.1(i)(b), that $\tilde{\alpha}$ satisfies the requirements. If $H'/H \cong \mathbb{Z}/q\mathbb{Z}$ for some $q \in \mathbb{N}, q > 1$ we set

$$\tilde{\alpha}(nx+h) := \Theta^n a(h) \quad (n \in \{0,1, \ldots , q-1\}, h \in H)$$

where $\Theta \in K$ is chosen such that $\Theta^q = a(qx)$ (then $\tilde{\alpha}$ is a character) and such that $|\Theta-f(x)| \leq \epsilon$ (then $|\tilde{\alpha}(h') - f(h')| \leq \epsilon$ for all $h' \in H'$). To see such a $\Theta$ exists consider the polynomial $p = x^q - a(qx) \in K[x]$. We have $|p(f(x))| \leq \epsilon$, $|p'(f(x))| = |q||f(x)|^{q-1} = 1$ (here we use the assumption char $k = 0$). By Hensel's Lemma there exists a $\Theta \in K$ with $P(\Theta) = 0$ and $|\Theta-f(x)| \leq \epsilon$.

REMARKS.

1. The algebraic closedness of $K$ has not been used in the above proof.

2. Let $B_1(\varepsilon), \pi$ be as in Lemma 2.1. Then any $\rho : T_K/B_1(\varepsilon) \to T_K$ for which $\pi \circ \rho$ is the identity, is an $\varepsilon$-character of $T_K/B_1(\varepsilon)$. Proposition 3.1 tells that there exists a homomorphism $\tilde{\rho} : T_K/B_1(\varepsilon) \to T_K$ such that $\pi \circ \tilde{\rho}$ is the identity. As a corollary we obtain that if char $k = 0$ then $B_1(\varepsilon)$ is a factor in $T_K$.

Next we turn to the case where char $k = p \neq 0$. To cover also groups like $\mathbb{Q}_p$ we shall use a technique different from the above one.

PROPOSITION 3.2. Let char $k = p \neq 0$. Suppose $x \mapsto px$ is a bijection $G \to G$.

Then for each $\varepsilon \in (0,1)$ and each $\varepsilon$-character $f : G \to T_K$ there exists a character $\alpha : G \to T_K$ such that $|f(x) - \alpha(x)| \leq \varepsilon$ $(x \in G)$.
Proof. For each \( n \in \mathbb{N} \) let \( x \mapsto p^{-n}x \) \((x \in G)\) be the inverse of \( x \mapsto p^n x \).

For any \( n \in \mathbb{N}, x \in G \) we have

\[
|f(p^{-n-1}x)^p - f(p^{-n}x)| \leq \varepsilon
\]

so that, by Lemma 2.3,

\[
|f(p^{-n-1}x)^{p^{n+1}} - f(p^{-n}x)^p| \leq \tau^n \varepsilon.
\]

By completeness of \( K \)

\[
a(x) := \lim_{n \to \infty} f(p^{-n}x)^p^n
\]

exists (uniformly in \( x \in G \)). For each \( n \in \mathbb{N}, x \in G \) we have

\[
|f(x) - f(p^{-n}x)^p^n| \leq \varepsilon
\]

hence also

\[
|f(x) - a(x)| \leq \varepsilon
\]

To see that \( a \) is a character, let \( x, y \in G, n \in \mathbb{N} \). We have

\[
|f(p^{-n}(x+y)) - f(p^{-n}x)f(p^{-n}y)| \leq \varepsilon
\]

Again by Lemma 2.3,

\[
|f(p^{-n}(x+y))^p^n - f(p^{-n}x)^p f(p^{-n}y)^p| \leq \tau^n \varepsilon
\]

hence also

\[
|a(x+y) - a(x)a(y)| \leq 0.
\]

Proposition 3.2 is a stepping stone for
PROPOSITION 3.3. Let char k = p ≠ 0. Suppose x ↦ px is an injection G → G. Then the conclusion of Proposition 3.2 holds.

Proof. G can be embedded into a divisible group D. Set G₁ := D/Dₚ (for D see Theorem 1.1(iii)). Then G + D + G₁ is injective and x ↦ px is a bijection G₁ → G₁. We may assume G ⊂ G₁. By Proposition 2.2 f can be extended to an ε-character f on G₁. By Proposition 3.2 there is a character α : G₁ → Tₖ for which |f(x) - α(x)| ≤ ε (x ∈ G₁). Set α := α|G.

We now prove the converse to Proposition 3.3.

PROPOSITION 3.4. Let char k = p ≠ 0. Suppose x ↦ px (x ∈ G) is not injective. Then there exists an ε ∈ (0,1) and an ε-character f : G → Tₖ such that for every character α : G → Tₖ there exists an x ∈ G with |f(x) - α(x)| > ε.

Proof. Choose an element x ∈ G of order p and a b ∈ K with 0 < |1-b| < 1. If char K = 0 we assume in addition that |1-b| < |1-θ| where θ is a primitive p-th root of unity. Consider the map
g : nx ↦ bⁿ (n ∈ {0,1,...,p-1})
defined on the group generated by x. For n,m ∈ {0,1,...,p-1} we have

|g(nx+mx) - g(nx)g(mx)| = \begin{cases} 0 & \text{if } n+m ≤ p-1 \\
|b^{n+m-p}b^n b^m| & \text{if } n+m > p \\
\end{cases}

so we see that g is an ε-character where ε = |b⁻ᵖ⁻¹| = |b⁻¹|.

By Proposition 2.2 g extends to an ε-character f : G → Tₖ. Now let α : G → Tₖ be a character. If α = 1 on H we have |f(x) - α(x)| = |b⁻¹| > |b⁻¹| = ε (Lemma 2.3). Otherwise we have α(x) = θ where θ is a primitive p-th root of unity (and char K = 0). Then we have, since |1-b| < |1-θ|,

|f(x) - α(x)| = |b-θ| = \text{max}(|b⁻¹|,|1-θ|) = |1-θ| > |1-b| > ε.
§4 UNIQUENESS

In this section we prove the 'uniqueness part' of Theorem 1.1 i.e. the statements involving (U).

PROPOSITION 4.1.

(i) Let \( \text{char } k = 0 \). Then (U) holds if and only if \( G \) is a torsion group.

(ii) Let \( \text{char } K = 0, \text{char } k \neq p \neq 0 \). Then (U) holds if and only if \( G \) has no subgroup of index \( p \).

(iii) Let \( \text{char } K = p \neq 0 \). Then (U) holds if and only if each homomorphism \( G \to \mathbb{Z}_p \) is zero.

Proof. Statement (U) is equivalent to "if \( \alpha, \beta \) are distinct characters then \( \sup \{ |\alpha(x) - \beta(x)| : x \in G \} = 1 \)", which is by [3], Proposition 1.1, the same as "the characters form an orthogonal set with respect to the sup norm". Now apply [3], Theorem 3.1, Theorem 2.2, Theorem 4.3 (where \( G \) has the discrete topology). Proposition 4.1 follows.

To complete the proof of Theorem 1.1 the following lemma remains to be shown.

LEMMA 4.2. Let \( p \) be a prime number. The following are equivalent.

(a) Any homomorphism \( G \to \mathbb{Z}_p \) is zero.

(b) If \( H \) is a subgroup of \( G/\mathbb{Z}_p \) then \( H \) does not have index \( p \).

The heart of the proof is contained in

LEMMA 4.3. Let \( x \mapsto px \ (x \in G) \) be injective but not surjective. Then there exists a nonzero homomorphism \( G \to \mathbb{Z}_p \).

Proof. Let \( \pi : G \to G/pG \) be the quotient map. As \( pG \neq G \) the group \( G/pG \) is in a natural way a nonzero vector space over the field of \( p \) elements. So there exists an indexed set \( \{ e_i \}_{i \in I} \) in \( G \) where \( I \neq \emptyset \) such that \( \{ \pi(e_i) : i \in I \} \) is a base of the vector space \( G/pG \). It follows that for
each \( x \in G \) there exist unique \( (\lambda^{(1)}_i)_{i \in I} \), where \( \lambda^{(1)}_i \in \{0, 1, \ldots, p-1\} \subseteq \mathbb{N} \) and \( \{i \in I : \lambda^{(1)}_i \neq 0\} \) is finite such that \( x - \sum \lambda^{(1)}_i x_i = p x_1 \) where \( x_1 \in G \).

By injectivity of \( x \mapsto px \) also \( x_1 \) is unique. By treating \( x_1 \) in the same way as we did for \( x \) we find unique \( (\lambda^{(2)}_i)_{i \in I} \) \( \in \{0, 1, \ldots, p^2 - 1\} \subseteq \mathbb{Z} \),

\( \{i \in I : \lambda^{(2)}_i \neq 0\} \) is finite such that \( x - \sum \lambda^{(2)}_i x_i = p^2 x_2 \) where \( x_2 \in G \) etc.

Thus, for each \( n \in \mathbb{N} \) there exist unique maps \( \phi^{(n)}_i : G \to \{0, 1, \ldots, p^n - 1\} \subseteq \mathbb{Z} \) with \( \{i \in I : \phi^{(n)}_i(x) \neq 0\} \) finite for each \( x \in G \) such that

\[
x - \sum \phi^{(n)}_i(x) e_i \in p^n G \quad (x \in G)
\]

By uniqueness, for each \( i \in I \), \( n \in \mathbb{N} \)

\[
\phi^{(n+1)}_i(x) \equiv \phi^{(n)}_i(x) \mod p^n \quad (x \in G)
\]

We see that, for any \( j \in I \), the \( p \)-adic limit

\[
\phi_j(x) = \lim_{n \to \infty} \phi^{(n)}_j(x) \quad (x \in G)
\]

exists and defines a map \( \phi_j : G \to \mathbb{Z}_p \). As \( \phi_j(e_j) = 1 \) this map is not zero.

To see that \( \phi_j \) is a homomorphism observe that for each \( n \in \mathbb{N}, x, y \in G \)

\[
\sum \phi^{(n)}_i(x + y) - \phi^{(n)}_i(x) - \phi^{(n)}_i(y)) e_i \in p^n G
\]

By what we have proved above

\[
\phi^{(n)}_j(x + y) - \phi^{(n)}_j(x) - \phi^{(n)}_j(y) \equiv 0 \mod p^n
\]

i.e.

\[
|\phi^{(n)}_j(x + y) - \phi^{(n)}_j(x) - \phi^{(n)}_j(y)|_p \leq p^{-n}
\]

which means for \( \phi_j \) that

\[
|\phi_j(x + y) - \phi_j(x) - \phi_j(y)| \leq 0
\]
Proof of Lemma 4.2. (a) $\Rightarrow$ (b). Suppose we had a subgroup H of $G/G_p$ of index p. Then, since $G/G_p$ has no elements of order p, the map $x \mapsto px$ is injective but not surjective on $G/G_p$, so Lemma 4.3 gives us a nontrivial homomorphism $\phi : G/G_p \to \mathbb{Z}_p$. But then $G + G/G_p \phi : \mathbb{Z}_p$ is a nontrivial homomorphism $G \to \mathbb{Z}_p$ which conflicts (a). To prove (b) $\Rightarrow$ (a), suppose we had a nontrivial homomorphism $\phi : G \to \mathbb{Z}_p$. Then we may assume $1 \in \text{Im} \phi$. It is easy to see that $H := \phi^{-1}(p\mathbb{Z}_p)$ has index p and contains $G_p$. This violates (a).

PROBLEM. Generalize Theorem 1.1 to topological abelian groups G and continuous $(\varepsilon,\delta)$-characters.

REFERENCES


