EPSILON STABILITY OF $p$-ADIC CHARACTERS

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ABSTRACT. The following problem is solved (see Theorem 1.1 below). Let $G$ be an abelian group, written additively, and let $K$ be an algebraically closed nonarchimedean valued field which is complete for the valuation $|\cdot|$. Let $f : G \to T_K := \{\lambda \in K : |\lambda| = 1\}$ be an $\epsilon$-character i.e.

$$|f(x+y) - f(x)f(y)| \leq \epsilon \quad (x, y \in G)$$

where $0 \leq \epsilon < 1$. Does there exist a character $\alpha : G \to T_K$ for which $|\alpha(x) - f(x)| \leq \epsilon$ (for $x \in G$)? Is it unique?

NOTES. The results of this Report will be used in a future paper on $p$-adic almost periodic functions [4]. The expression 'epsilon stability' is taken from [1], p.11, where a closely related problem is discussed.
§1 THE THEOREM

Throughout, let \( G, K, T_K \) be as above. For trivially valued fields \( K \) the above problem has a trivial solution. So from now on we assume that the valuation of \( K \) is non-trivial. The residue class field of \( K \) is \( k \). The characteristic of a field \( L \) is denoted \( \text{char} \ L \).

Let \( 0 < \varepsilon < 1 \). A function \( f : G \rightarrow T_K \) is an \( \varepsilon \)-character if
\[
|f(x+y) - f(x)f(y)| \leq \varepsilon \quad \text{for all } x, y \in G.
\]
As usual, we shall say 'character' instead of '\( 0 \)-character'.

Consider the following statements (E) and (U).

**(E)** For each \( \varepsilon \in [0,1) \) and each \( \varepsilon \)-character \( f : G \rightarrow T_K \) there exists a character \( \alpha : G \rightarrow T_K \) for which \( |f(x) - \alpha(x)| \leq \varepsilon \quad (x \in G) \).

**(U)** For each \( \varepsilon \in [0,1) \) and each \( \varepsilon \)-character \( f : G \rightarrow T_K \) there exists at most one character \( \alpha : G \rightarrow T_K \) for which \( |f(x) - \alpha(x)| \leq \varepsilon \quad (x \in G) \).

The purpose of this note is to prove the following Theorem.

**THEOREM 1.1.**

(i) Let \( \text{char} \ k = 0 \). Then (E) holds for any \( G \), (U) holds if and only if \( G \) is a torsion group.

(ii) Let \( \text{char} \ K = 0 \), \( \text{char} \ k = p \neq 0 \). Then (E) holds if and only if \( G \) has no subgroups of order \( p \), (U) holds if and only if \( G \) has no subgroups of index \( p \).

(iii) Let \( \text{char} \ K = p \neq 0 \). Then (E) holds if and only if \( G \) has no subgroups of order \( p \), (U) holds if and only if no subgroup of \( G/G_p \), where \( G_p := \{ x \in G : p^n x = 0 \text{ for some } n \in \mathbb{N} \} \), has index \( p \).
We have the following trivial corollary.

**COROLLARY 1.2.**

(i) Let char $k = 0$. Then both $(E)$ and $(U)$ hold if and only if $G$ is a torsion group.

(ii) Let char $k = p 
eq 0$. Then both $(E)$ and $(U)$ hold if and only if $G$ has neither subgroups of order $p$, nor subgroups of index $p$.

**REMARK.** The statement 'G has no subgroups of order $p$' is obviously equivalent to 'the map $x \mapsto px$ ($x \in G$) is injective'. It is not hard to see that 'G has no subgroups of index $p$' is equivalent to 'the map $x \mapsto px$ ($x \in G$) is surjective'.

**EXAMPLES.** If $G$ is $p$-free (i.e. if $H_1 \subseteq H_2$ are subgroups then the index $[H_2 : H_1]$, whenever finite, is not divisible by $p$) then $x \mapsto px$ is a bijection. But this conclusion holds also for the additive group of the $p$-adic numbers $\mathbb{Q}_p$. On $\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ the map $x \mapsto px$ is injective but not surjective, on $\mathbb{Q}_p/\mathbb{Z}_p$ the map $x \mapsto px$ is surjective but not injective.

§2 **PRELIMINARIES**

**LEMMA 2.1.** (Elementary properties of $\varepsilon$-characters). Let $0 \leq \varepsilon < 1$.

(i) Let $f : G \to T_K$ be an $\varepsilon$-character. Then

(a) $|f(0) - 1| \leq \varepsilon$.

(b) If $x_1, \ldots, x_n \in G$ then $|f(x_1 + \ldots + x_n) - f(x_1)f(x_2)\ldots f(x_n)| \leq \varepsilon$.

(c) If $g : G \to K$, $|g(x)| \leq \varepsilon$ for all $x \in G$, then $f + g$ is an $\varepsilon$-character.

(ii) $B_1(\varepsilon) := \{\lambda \in K : |1-\lambda| \leq \varepsilon\}$ is a subgroup of $T_K$. Let $\pi : T_K \to T_K/B_1(\varepsilon)$ be the quotient map. Then $f : G \to T_K$ is an $\varepsilon$-character if and only if $\pi \circ f : G \to T_K/B_1(\varepsilon)$ is a homomorphism.

**Proof.** Straightforward.
PROPOSITION 2.2. (Extension of ε-characters) Let $H$ be a subgroup of $G$ and let, for some $\varepsilon \in [0,1)$, $f : H \to T_K$ be an ε-character. Then $f$ can be extended to an ε-character $\tilde{f} : G \to T_K$.

Proof. By lemma 2.1 (i) (c) it suffices to find an ε-character $\tilde{f} : G \to T_K$ for which $|\tilde{f}(h) - f(h)| < \varepsilon$ (for $h \in H$). With the notations as in Lemma 2.1 (ii) we have that $\pi \circ f$ is a homomorphism $H \to T_K/B_1(\varepsilon)$. As $K$ is algebraically closed the group $T_K$, hence $T_K/B_1(\varepsilon)$, is divisible. Therefore, $\pi \circ f$ can be extended to a homomorphism $g : G \to T_K/B_1(\varepsilon)$. Choose any $\rho : T_K/B_1(\varepsilon) \to T_K$ for which $\pi \circ \rho$ is the identity. Then $\tilde{f} := \rho \circ g$ has the required properties.

We quote a lemma needed for the proof of Proposition 3.2.

LEMMA 2.3. Let $\text{char } k = p \neq 0$. Let $a,b \in T_K$ such that $0 < |a-b| < \varepsilon < 1$. Then $|a^p - b^p| < \tau \varepsilon$ where $\tau := \max(\varepsilon, |p|)$. In particular, $|a^p - b^p| < |a-b|$.

Proof. [2], Lemma 32.1.

§3 EXISTENCE

In this section we prove the 'existence part' of Theorem 1.1 i.e. the statements involving (E).

For the case char $k = 0$ we have quite standard methods:

PROPOSITION 3.1. Let char $k = 0$. Then, for any $G$, for each $\varepsilon \in [0,1)$ and each ε-character $f : G \to T_K$ there exists a character $\alpha : G \to T_K$ for which $|f(x) - \alpha(x)| < \varepsilon$ (for $x \in G$).

Proof. We start the construction of $\alpha$ by setting $\alpha(0) := 1$. Then, by Lemma 2.1 (i) (a), $|f - \alpha| < \varepsilon$ on the zero group. Now suppose we have a subgroup $H$ of $G$ and a character $\alpha : H \to T_K$ such that $|f(h) - \alpha(h)| < \varepsilon$ for all $h \in H$. Let $x \in G \setminus H$. We prove that $\alpha$ can be extended to a character $\tilde{\alpha}$ defined on the group $H'$ generated by $H$ and $\{x\}$ such that $|f(h') - \tilde{\alpha}(h')| < \varepsilon$ for all $h' \in H$. (A simple application of Zorn's Lemma then may complete the proof.)
If $H'/H \cong \mathbb{Z}$ we define $\tilde{\alpha}$ by

$$\tilde{\alpha}(nx+h) := f(x)^n\alpha(h) \quad (n \in \mathbb{Z}, h \in H)$$

One proves easily, by using Lemma 2.1(i)(b), that $\tilde{\alpha}$ satisfies the requirements. If $H'/H \cong \mathbb{Z}/q\mathbb{Z}$ for some $q \in \mathbb{N}, q > 1$ we set

$$\tilde{\alpha}(nx+h) := \theta^n\alpha(h) \quad (n \in \{0,1,\ldots,q-1\}, h \in H)$$

where $\theta \in K$ is chosen such that $\theta^q = \alpha(qx)$ (then $\tilde{\alpha}$ is a character) and such that $|\theta-f(x)| \leq \varepsilon$ (then $|\tilde{\alpha}(h')-f(h')| \leq \varepsilon$ for all $h' \in H'$). To see such a $\theta$ exists consider the polynomial $p = x^q-\alpha(qx) \in K[x]$. We have $|p(f(x))| \leq \varepsilon$, $|p'(f(x))| = |q||f(x)^{q-1}| = 1$ (here we use the assumption $\text{char } k = 0$). By Hensel's Lemma there exists a $\theta \in K$ with $p(\theta) = 0$ and $|\theta-f(x)| \leq \varepsilon$.

**REMARKS.**

1. The algebraic closedness of $K$ has not been used in the above proof.
2. Let $B_1(\varepsilon)$, $\pi$ be as in Lemma 2.1. Then any $\rho : T_K/B_1(\varepsilon) \rightarrow T_K$ for which $\pi \circ \rho$ is the identity, is an $\varepsilon$-character of $T_K/B_1(\varepsilon)$. Proposition 3.1 tells that there exists a homomorphism $\tilde{\rho} : T_K/B_1(\varepsilon) \rightarrow T_K$ such that $\pi \circ \tilde{\rho}$ is the identity. As a corollary we obtain that if $\text{char } k = 0$ then $B_1(\varepsilon)$ is a factor in $T_K$.

Next we turn to the case where $\text{char } k = p \neq 0$. To cover also groups like $\mathbb{Q}_p$ we shall use a technique different from the above one.

**PROPOSITION 3.2.** Let $\text{char } k = p \neq 0$. Suppose $x \mapsto px$ is a bijection $G \rightarrow G$.

Then for each $\varepsilon \in [0,1)$ and each $\varepsilon$-character $f : G \rightarrow T_K$ there exists a character $\alpha : G \rightarrow T_K$ such that $|f(x) - \alpha(x)| \leq \varepsilon$ ($x \in G$).
Proof. For each $n \in \mathbb{N}$ let $x \mapsto p^{-n}x \ (x \in G)$ be the inverse of $x \mapsto p^n x$.

For any $n \in \mathbb{N}, \ x \in G$ we have

$$|f(p^{-n}x)^p - f(p^{-n}x)| \leq \varepsilon$$

so that, by Lemma 2.3,

$$|f(p^{-n-1}x)^p p^n - f(p^{-n}x) p^n| \leq \tau n \varepsilon.$$

By completeness of $X$

$$a(x) := \lim_{n \to \infty} f(p^{-n}x)^p$$

exists (uniformly in $x \in G$). For each $n \in \mathbb{N}, \ x \in G$ we have

$$|f(x) - f(p^{-n}x)^p| \leq \varepsilon$$

hence also

$$|f(x) - a(x)| \leq \varepsilon$$

To see that $a$ is a character, let $x, y \in G, \ n \in \mathbb{N}$. We have

$$|f(p^{-n}(x+y)) - f(p^{-n}x)f(p^{-n}y)| \leq \varepsilon$$

Again by Lemma 2.3,

$$|f(p^{-n}(x+y)) p^n - f(p^{-n}x) p^n f(p^{-n}y) p^n| \leq \tau n \varepsilon$$

hence also

$$|a(x+y) - a(x)a(y)| \leq 0.$$

Proposition 3.2 is a stepping stone for
PROPOSITION 3.3. Let \( \text{char } k = p \neq 0 \). Suppose \( x \mapsto px \) is an injection \( G \rightarrow G \). Then the conclusion of Proposition 3.2 holds.

Proof. \( G \) can be embedded into a divisible group \( D \). Set \( G_1 := D/D_p \) (for \( D_p \) see Theorem 1.1(iii)). Then \( G + D + G_1 \) is injective and \( x \mapsto px \) is a bijection \( G_1 \rightarrow G_1 \). We may assume \( G \subseteq G_1 \). By Proposition 2.2 \( f \) can be extended to an \( \varepsilon \)-character \( \tilde{f} \) on \( G_1 \). By Proposition 3.2 there is a character \( \tilde{\alpha} : G_1 \rightarrow T_k \) for which \( |f(x) - \tilde{\alpha}(x)| \leq \varepsilon \) (\( x \in G_1 \)). Set \( \alpha := \tilde{\alpha}|G \).

We now prove the converse to Proposition 3.3.

PROPOSITION 3.4. Let \( \text{char } k = p \neq 0 \). Suppose \( x \mapsto px \) (\( x \in G \)) is not injective. Then there exists an \( \varepsilon \in (0,1) \) and an \( \varepsilon \)-character \( f : G \rightarrow T_k \) such that for every character \( \alpha : G \rightarrow T_k \) there exists an \( x \in G \) with \( |f(x) - \alpha(x)| > \varepsilon \).

Proof. Choose an element \( x \in G \) of order \( p \) and a \( b \in K \) with \( 0 < |1-b| < 1 \).

If \( \text{char } K = 0 \) we assume in addition that \( |1-b| < |1-\theta| \) where \( \theta \) is a primitive \( p \)-th root of unity. Consider the map

\[
g : nx \mapsto b^n \quad (n \in \{0,1,\ldots,p-1\})
\]

defined on the group generated by \( x \). For \( n,m \in \{0,1,\ldots,p-1\} \) we have

\[
|g(nx+mx) - g(nx)g(mx)| = \begin{cases} 
0 & \text{if } n+m < p-1 \\
|b^{n+m-p} - b^{n}b^{m}| & \text{if } n+m \geq p
\end{cases}
\]

so we see that \( g \) is an \( \varepsilon \)-character where \( \varepsilon = |b^{-p+1}| = |b^{p-1}| \).

By Proposition 2.2 \( g \) extends to an \( \varepsilon \)-character \( f : G \rightarrow T_K \). Now let \( \alpha : G \rightarrow T_k \) be a character. If \( \alpha = 1 \) on \( H \) we have \( |f(x) - \alpha(x)| = |b^{-1}| > |b^{P-1}| = \varepsilon \) (Lemma 2.3). Otherwise we have \( \alpha(x) = \theta \) where \( \theta \) is a primitive \( p \)-th root of unity (and \( \text{char } K = 0 \)). Then we have, since \( |1-b| < |1-\theta| \),

\[
|f(x) - \alpha(x)| = |b-\theta| = \max(|b^{-1}|,|1-\theta|) = |1-\theta| > |1-b| > \varepsilon.
\]
§4 UNIQUENESS

In this section we prove the 'uniqueness part' of Theorem 1.1 i.e. the statements involving (U).

PROPOSITION 4.1.

(i) Let \( \text{char } k = 0 \). Then (U) holds if and only if \( G \) is a torsion group.

(ii) Let \( \text{char } K = 0, \text{char } k \neq 0 \). Then (U) holds if and only if \( G \) has no subgroup of index \( p \).

(iii) Let \( \text{char } K = p \neq 0 \). Then (U) holds if and only if each homomorphism \( G \rightarrow \mathbb{Z}_p \) is zero.

Proof. Statement (U) is equivalent to "if \( \alpha, \beta \) are distinct characters then \( \sup \{|\alpha(x) - \beta(x)| : x \in G\} = 1 \)", which is by [3], Proposition 1.1, the same as "the characters form an orthogonal set with respect to the sup norm".

Now apply [3], Theorem 3.1, Theorem 2.2, Theorem 4.3 (where \( G \) has the discrete topology). Proposition 4.1 follows.

To complete the proof of Theorem 1.1 the following lemma remains to be shown.

LEMMA 4.2. Let \( p \) be a prime number. The following are equivalent.

(a) Any homomorphism \( G \rightarrow \mathbb{Z}_p \) is zero.

(b) If \( H \) is a subgroup of \( G/\mathbb{Z}_p \) then \( H \) does not have index \( p \).

The heart of the proof is contained in

LEMMA 4.3. Let \( x \mapsto px \ (x \in G) \) be injective but not surjective. Then there exists a nonzero homomorphism \( G \rightarrow \mathbb{Z}_p \).

Proof. Let \( \pi : G \rightarrow G/pG \) be the quotient map. As \( pG \neq G \) the group \( G/pG \) is in a natural way a nonzero vector space over the field of \( p \) elements.

So there exists an indexed set \( \{e_i\}_{i \in I} \) in \( G \) where \( I \neq \emptyset \) such that \( \{\pi(e_i) : i \in I\} \) is a base of the vector space \( G/pG \). It follows that for
each x ∈ G there exist unique \((\lambda_1^{(1)})_{i \in I}\) where \(\lambda_1^{(1)} \in \{0,1,\ldots,p-1\} \subset \mathbb{N}\) and \(\{i \in I : \lambda_1^{(1)} \neq 0\}\) is finite such that \(x = \sum \lambda_1^{(1)} e_i = px_i\) where \(x_i \in G\). By injectivity of \(x \mapsto px\) also \(x_i\) is unique. By treating \(x_i\) in the same way as we did for \(x\) we find unique \((\lambda_1^{(2)})_{i \in I} \in \{0,1,\ldots,p^2-1\} \subset \mathbb{Z}\) and \(\{i \in I : \lambda_1^{(2)} \neq 0\}\) is finite such that \(x = \sum \lambda_1^{(2)} e_i = p^2x_2\) where \(x_2 \in G\) etc.

Thus, for each \(n \in \mathbb{N}\) there exist unique maps \(\phi_i^{(n)} : G \to \{0,1,\ldots,p^n-1\} \subset \mathbb{Z}\) with \(\{i \in I : \phi_i^{(n)}(x) \neq 0\}\) finite for each \(x \in G\) such that

\[x - \sum \phi_i^{(n)}(x)e_i \in \mathbb{Z}^n G \quad (x \in G)\]

By uniqueness, for each \(i \in I, n \in \mathbb{N}\)

\[\phi_i^{(n+1)}(x) \equiv \phi_i^{(n)}(x) \mod p^n \quad (x \in G)\]

We see that, for any \(j \in I\), the \(p\)-adic limit

\[\phi_j(x) = \lim_{n \to \infty} \phi_j^{(n)}(x) \quad (x \in G)\]

exists and defines a map \(\phi_j : G \to \mathbb{Z}_p\). As \(\phi_j(e_j) = 1\) this map is not zero.

To see that \(\phi_j\) is a homomorphism observe that for each \(n \in \mathbb{N}, x,y \in G\)

\[\sum (\phi_i^{(n)}(x+y) - \phi_i^{(n)}(x) - \phi_i^{(n)}(y))e_i \in \mathbb{Z}^n G\]

By what we have proved above

\[\phi_j^{(n)}(x+y) - \phi_j^{(n)}(x) - \phi_j^{(n)}(y) \equiv 0 \mod p^n\]

i.e.

\[|\phi_j^{(n)}(x+y) - \phi_j^{(n)}(x) - \phi_j^{(n)}(y)|_p \leq p^{-n}\]

which means for \(\phi_j\) that

\[|\phi_j(x+y) - \phi_j(x) - \phi_j(y)|_p \leq p^n\]
Proof of Lemma 4.2. (a) $\Rightarrow$ (b). Suppose we had a subgroup $H$ of $G/G_p$ of index $p$. Then, since $G/G_p$ has no elements of order $p$, the map $x \mapsto px$ is injective but not surjective on $G/G_p$, so Lemma 4.3 gives us a nontrivial homomorphism $\phi : G/G_p \to \mathbb{Z}_p$. But then $G + G/G_p \phi : \mathbb{Z}_p$ is a nontrivial homomorphism $G \to \mathbb{Z}_p$ which conflicts (a). To prove (b) $\Rightarrow$ (a), suppose we had a nontrivial homomorphism $\phi : G \to \mathbb{Z}_p$. Then we may assume $1 \in \text{Im}\phi$. It is easy to see that $H := \phi^{-1}(p\mathbb{Z}_p)$ has index $p$ and contains $G_p$. This violates (a).

PROBLEM. Generalize Theorem 1.1 to topological abelian groups $G$ and continuous ($\epsilon$-)characters.

REFERENCES


