p-ADIC TRIGONOMETRIC POLYNOMIALS

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W.H. SCHIKHOF

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INTRODUCTION. Let G be an abelian group, let f be a bounded complex valued function on G whose translates generate a finite dimensional space. It is well known ([2], 27.7) that f is a linear combination of characters. This conclusion is not valid if the range of f lies in a non-archimedean valued field K rather than C. For example, if K contains the field $\mathbb{Q}_p$ of the p-adic numbers and if $G=\mathbb{Z}_p$, the additive group of the p-adic integers, it is easily seen that the translates of the function $f : x \mapsto x$ generate a twodimensional space over K whereas f is *not* a K-linear combination of K-valued characters (follow the proof of the implication $(\gamma) \Rightarrow (\alpha)$ of Theorem 1.4).

ABSTRACT. For an abelian topological group G and an algebraically closed, nontrivially valued, complete field K necessary and sufficient conditions are derived for a representative function $f : G \rightarrow K$ to be a finite K-linear combination of K-valued characters (Theorems 1.4, 2.1, 2.2). Also, a complete description of the set of all representative functions $\mathbb{Z}_p \rightarrow K$ is given (Theorem 3.1).

TERMINOLOGY & STANDARD FACTS. Throughout this paper G is an additively written abelian topological group, K is an algebraically closed nontrivially valued complete field with valuation $|\ |$. The set $BC(G \rightarrow K)$ consisting of all bounded continuous functions $G \rightarrow K$ is a K-Banach algebra with respect to pointwise operations and the norm $f \mapsto \|f\|_\infty := \sup\{|f(x)| : x \in G\}$.

A *character* is a nonzero element α of $BC(G \rightarrow K)$ for which $\alpha(x+y)=\alpha(x)\alpha(y)$ for all $x,y \in G$. Then $|\alpha(x)|=1$ for all $x \in G$. Under pointwise multiplication the characters form a group $G_K^\wedge$. A function $f \in BC(G \rightarrow K)$ is a *representative function* (or a *trigonometric polynomial*) if the K-linear span $[f_s : s \in G]$ of $\{f_s : s \in G\}$ is finite dimensional. Here, as usual, $f_s(x) := f(s+x)$ for $x \in G$. It is not hard to prove that the collection $\mathcal{R}(G \rightarrow K)$ of all representative functions $G \rightarrow K$ is a K-subalgebra of $BC(G \rightarrow K)$ containing $G_K^\wedge$.

A G*-module* is a Banach space E over K together with a separately continuous structure map $G \times E \to E$

$$(s, x) \mapsto U_s(x) = sx \qquad (s \in G, x \in E)$$

such that $s \mapsto U_s$ is a homomorphism of G into the group of invertible (continuous) K-linear operators $E \to E$ and such that, for each $x \in G$, $\sup\{\|U_s x\| : s \in G\}$ is finite. In this paper we shall deal only with finite dimensional G-modules.

§1. THE MAIN THEOREM

Proposition 1.1. *Let $f \in BC(G \to K)$, $f \neq 0$. Then $[f_s : s \in G]$ is onedimensional if and only if f is a multiple of a character.*

Proof. If α is a character then for each $s \in G$ we have $\alpha_s = \alpha(s)\alpha$ and $[\alpha_s : s \in G]$ is onedimensional. Conversely, suppose $\dim[f_s : s \in G] = 1$. For each $s \in G$ there is a unique $\alpha(s) \in K$ for which $f_s = \alpha(s) f$. The equality $f_{s+t} = (f_s)_t$ yields $\alpha(s+t) = \alpha(s)\alpha(t)$ for all $s, t \in G$. From $\|f_s\|_\infty = |\alpha(s)| \|f\|_\infty$ we infer that $|\alpha(s)| = 1$ for all $s \in G$. So, α is a character and $f = f(0) \cdot \alpha$.

Proposition 1.2. *A representative function is uniformly continuous.*

Proof. Let $f \in \mathcal{R}(G \to K)$, $f \neq 0$ and let $e_1, \ldots, e_n$ be a base of $E := [f_s : s \in G]$. By equivalence of norms (see [3], Theorem 3.15 for the non-archimedean case) there exists a $C > 0$ such that

$$\max_{1 \leqslant i \leqslant n} |\lambda_i| \leqslant C \|\sum_{i=1}^{n} \lambda_i e_i\|_\infty$$

for all $\lambda_1, \ldots, \lambda_n \in K$. Let $\varepsilon > 0$. There is a neighbourhood U of 0 in G such that for all $i \in \{1, 2, \ldots, n\}$

$$x \in U \Rightarrow |e_i(x) - e_i(0)| \leqslant (Cn\|f\|_\infty)^{-1} \varepsilon.$$

Now let $s \in G$, $t \in U$; we shall prove that $|f(s+t) - f(s)| \leqslant \varepsilon$. There exist $\lambda_1, \ldots, \lambda_n \in K$ (depending on s) such that

$$f_s = \sum_{i=1}^{n} \lambda_i e_i$$

Then

$$f_{s+t} = \sum_{i=1}^{n} \lambda_i (e_i)_t$$

We see that

$$|f(s+t) - f(s)| = |f_{s+t}(0) - f_s(0)| =$$

$$|\sum_{i=1}^{n} \lambda_i (e_i(t) - e_i(0))| \leqslant n \max_{1 \leqslant i \leqslant n} |\lambda_i||e_i(t) - e_i(0)| \leqslant$$

$$n\, C \|\sum_{i=1}^{n} \lambda_i e_i\|_\infty \cdot (Cn\|f\|_\infty)^{-1}\varepsilon = \|f_s\|_\infty \|f\|_\infty^{-1} \varepsilon = \varepsilon.$$

Proposition 1.3. *Let E be a G-module of dimension $n \in \mathbb{N}$. For each $m \in \mathbb{N}$, $1 \leqslant m \leqslant n$, E has a G-submodule of dimension m.*

Proof. By induction on m. To find a onedimensional submodule choose, among all nonzero G-submodules of E, a G-submodule E_1 with minimal dimension. Then E_1 is simple (i.e., the corresponding representation $s \mapsto U_s$ is irreducible). As K is algebraically closed, a standard application of Schur's lemma ([2], 27.9) yields $\dim E_1 = 1$. Now let $m < n$ and let E_m be an m-dimensional G-submodule of E. The quotient E/E_m is, in an obvious way, a G-module of dimension $n-m \geqslant 1$. By the first part of the proof it has a onedimensional G-submodule D_1. One verifies immediately that $E_{m+1} := \pi^{-1}(D_1)$, where $\pi : E \to E/E_m$ is the quotient map, is a G-submodule of E whose dimension is $m+1$.

We now prove the main theorem. A function $\mu : G \to K$ is *additive* if $\mu(s+t) = \mu(s) + \mu(t)$ for all $s, t \in G$.

Theorem 1.4. *The following statements on G, K are equivalent.*

(α) *Any bounded continuous additive function $G \to K$ is 0.*

(β) *Each nonzero finite dimensional G-module over K is a (direct) sum of onedimensional G-modules.*

(γ) *Each representative function $G \to K$ is a finite K-linear combination of K-valued characters.*

Proof. To obtain the implication (α)$\Rightarrow$(β) we shall prove that

$$(*) \qquad \begin{cases} \text{each } n\text{-dimensional } G\text{-module has a base } e_1, \ldots, e_n \\ \text{for which } se_i \in [e_i] \ (s \in G) \text{ for each } i \in \{1, \ldots, n\} \end{cases}$$

by induction on n. The case $n=1$ is trivial, so suppose (*) is true for some n and let E be an $(n+1)$-dimensional G-module. According to Proposition 1.3 E has an n-dimensional G-submodule D which, by the induction hypothesis, has a base $e_1, \ldots, e_n$ such that $se_i \in [e_i]$ for all $s \in G$, all $i \in \{1, \ldots, n\}$. Choose an $x \in E \setminus D$; then $e_1, \ldots, e_n$, x is base for E. With respect to this base the maps $U_s (s \in G)$ have the following matrices

$$\begin{bmatrix} \lambda_1(s) & & & & \xi_1(s) \\ & & 0 & & \\ & \lambda_2(s) & & & \xi_2(s) \\ & & & & \cdot \\ & & & & \cdot \\ & & & & \cdot \\ & & & \lambda_n(s) & \xi_n(s) \\ & 0 & & & \\ & & & & \lambda_{n+1}(s) \end{bmatrix}$$

Observe that its entries are continuous functions of s (since each of them has the form $s \mapsto \phi(sy)$ for some $y \in E$, $\phi \in E^*$, the dual space of E) and are also bounded by our definition of a G-module. Since U_s is invertible we have $\lambda_i(s) \neq 0$ for all $i \in \{1,\dots,n+1\}$. The equality $U_{s+t} = U_s U_t$ espressed in matrix form yields

$$\lambda_i(s+t) = \lambda_i(s)\lambda_i(t) \qquad (s, t \in G)$$

(so, each λ_i is a character) and

(**) $$\xi_i(s+t) = \lambda_i(s)\xi_i(t) + \xi_i(s)\lambda_{n+1}(t) = \lambda_i(t)\xi_i(s) + \xi_i(t)\lambda_{n+1}(s) \quad (s, t \in G)$$

for $i \in \{1,\dots,n\}$. We now complete the proof of $(\alpha) \Rightarrow (\beta)$ by defining $q_1,\dots,q_n \in K$ such that for

$$e_{n+1} := x + \sum_{i=1}^{n} q_i e_i$$

we have $se_{n+1} = \lambda_{n+1}(s)e_{n+1}$ $(s \in G)$. That is, we have to choose $q_1,\dots,q_n$ in such a way that

(***) $$\xi_i(s) + q_i(\lambda_i(s) - \lambda_{n+1}(s)) = 0 \quad (1 \leq i \leq n, s \in G).$$

For any $i \in \{1,\dots,n\}$ we distinguish two cases.

(i) $\lambda_i(t) \neq \lambda_{n+1}(t)$ for some $t \in G$. Then we are forced to choose

$$q_i := (\lambda_{n+1}(t) - \lambda_i(t))^{-1}\xi_i(t)$$

Now (**) guarantees that for any $s \in G$

$$(\lambda_{n+1}(t) - \lambda_i(t))(\xi_i(s) + q_i(\lambda_i(s) - \lambda_{n+1}(s))) = 0$$

and (***) follows for this i.

(ii) $\lambda_i = \lambda_{n+1}$. We shall prove that $\xi_i(s) = 0$ for all $s \in G$ (so that we may choose for q_i and arbitrary element of K).

In fact, by (**) we have

$$\xi_i(s+t) = \lambda_i(s)\xi_i(t)+\xi_i(s)\lambda_i(t) \quad (s,t\in G)$$

After dividing by $\lambda_i(s+t)=\lambda_i(s)\lambda_i(t)$ we obtain

$$\mu(s+t) = \mu(s)+\mu(t)$$

where $\mu := \lambda_i^{-1}\xi_i$ is continuous and bounded. By (α) we have $\mu=0$. It follows that $\xi_i=0$.
(β)$\Rightarrow$(γ). Let $f\in\mathcal{R}(G\to K)$, $f\neq 0$ and let $E=[f_s : s\in G]$. The structure map

$$(s,g)\mapsto g_s \quad (s\in G, g\in E)$$

makes E into a finite diemensional G-module, taking into account that Proposition 1.2 guarantees the continuity of $s\mapsto g_s$. By (β), E is the sum of onedimensional G-modules $[\alpha_1],\ldots,[\alpha_n]$, where Proposition 1.1 tells us that we may assume that $\alpha_1,\ldots,\alpha_n$ are characters and (γ) follows.

(γ)$\Rightarrow$(α). Let $\mu\in BC(G\to K)$ be additive. For each $s\in G$ we have $\mu_s=\mu(s)\cdot 1+\mu$ where 1 is the function with constant value one. So, $[\mu_s : s\in G] = [1,\mu]$ implying that μ is a representative function. By (γ) there exist distinct characters $\alpha_0,\alpha_1,\ldots,\alpha_n$, where α_0 is the unit character, and $\lambda_0,\lambda_1,\ldots,\lambda_n\in K$ such that

$$\mu = \sum_{i=0}^{n}\lambda_i\alpha_i$$

The relation $\mu_s=\mu(s)\alpha_0+\mu$ yields

$$\sum_{i=0}^{n}\lambda_i\alpha_i(s)\alpha_i = \mu(s)\alpha_0 + \sum_{i=0}^{n}\lambda_i\alpha_i \quad (s\in G)$$

By linear independence of characters we have equality of the coefficients of α_0 i.e.

$$\lambda_0 = \lambda_0\alpha_0(s) = \mu(s)+\lambda_0 \quad (s\in G)$$

implying $\mu(s)=0$ for all $s\in G$.

§2. THE MAIN THEOREM FOR VARIOUS GROUND FIELDS

Theorem 2.1. *If the valuation of K is archimedean then* (α),(β),(γ) *of Theorem 1.4 hold for every topological abelian group G.*
Proof. Property (α) of Theorem 1.4 follows from the fact that K has no bounded additive subgroups other than (0).

Next we turn to the case where the valuation of K is non-archimedean. First some notations. The residue class field of K is k. The characteristic of a field L is char$L$. For topological groups G_1, G_2 the set of all continuous homomorphisms $G_1\to G_2$ is Hom(G_1,G_2).

Theorem 2.2. *Let the valuation of K be non-archimedean. Then* (α),(β),(γ) *of Theorem 1.4 are equivalent to*

(δ)′ Hom(G,$\mathbb{Q}$)=(0) (*where* $\mathbb{Q}$ *carries the discrete topology*)
if char$K$ = char$k$ = 0,

(δ)″ Hom(G,$\mathbb{Z}/p\mathbb{Z}$) = (0)
if char$K$ = char$k$ = $p \neq 0$,

(δ)‴ Hom(G,$\mathbb{Z}_p$) = (0)
if char$K$ = 0, char$k$ = $p \neq 0$.

Proof. (a) Assume char$K$ = char$k$ = 0. We have a natural embedding $\mathbb{Q} \to K$ whose image is bounded so (α) of Theorem 1.4 implies (δ)′. To obtain (δ)′$\Rightarrow$(α), let $\mu : G \to K$ be a bounded nonzero additive homomorphism; we shall prove that Hom(G,$\mathbb{Q}$)$\neq$(0). Let $s \in G$, $\mu(s) \neq 0$ and let

$$\pi : K \to K/\{x \in K : |x| < |\mu(s)|\} =: H$$

be the canonical quotient map. The discrete group H is torsion free so the formula

$$n\,\pi(\mu(s)) \mapsto n \qquad (n \in \mathbb{Z})$$

defines a homomorphism of the group generated by $\pi(\mu(s))$ into $\mathbb{Q}$. By divisibility of $\mathbb{Q}$ it can be extended to a homomorphism $\phi : H \to \mathbb{Q}$. The map $\phi \circ \pi \circ \mu : G \to \mathbb{Q}$ is a continuous homomorphism sending s into 1. Hence Hom(G,$\mathbb{Q}$)$\neq$(0).

(b) Assume char$K$ = char$k$ = $p \neq 0$. We have a natural embedding $\mathbb{Z}/p\mathbb{Z} \to K$ so (α) of Theorem 1.4 implies (δ)″. To obtain (δ)″$\Rightarrow$(α), let $\mu : G \to K$ be a bounded nonzero additive homomorphism; we shall prove that Hom(G,$\mathbb{Z}/p\mathbb{Z}$)$\neq$(0). Define s,π,H as in part (a). This time every nonzero element of H has order p so the homomorphism

$$n\,\pi(\mu(s)) \mapsto n \bmod p\,\mathbb{Z} \qquad (n \in \mathbb{Z})$$

can be extended to homomorphism $\phi : H \to \mathbb{Z}/p\mathbb{Z}$. The map $\phi \circ \pi \circ \mu : G \to \mathbb{Z}/p\mathbb{Z}$ is a continuous homomorphism sending s into 1 mod p $\mathbb{Z}$. Hence Hom(G,$\mathbb{Z}/p\mathbb{Z}$)$\neq$(0).

(c) Assume char$K$=0, char$k$=$p \neq 0$. Then we may assume $K \supset \mathbb{Q}_p$. We have a natural embedding $\mathbb{Z}_p \to K$ so (α) of Theorem 1.4 implies (δ)‴. To obtain (δ)‴$\Rightarrow$(α), let $\mu : G \to K$ be a bounded nonzero additive homomorphism; we shall prove that Hom(G,$\mathbb{Z}_p$)$\neq$(0). Now K is, in a natural way, a Banach space over $\mathbb{Q}_p$. Since $\mathbb{Q}_p$ is spherically complete there exists, by the non-archimedean Hahn-Banach Theorem ([3], Theorem 4.15, (γ)$\Rightarrow$(α)), a continuous $\mathbb{Q}_p$-linear map $\phi : K \to \mathbb{Q}_p$ that does not vanish on $\mu(G)$. Then $\phi \circ \mu$ is a nonzero bounded continuous homomorphism $G \to \mathbb{Q}_p$. After multiplying it by a suitable element of $\mathbb{Q}_p$ we obtain a nonzero element of Hom(G,$\mathbb{Z}_p$).

Remarks.

1. It is easily seen that 'Hom(G,Q)=(0)' is equivalent to 'for each open subgroup H of G the quotient G/H is a torsion group'. Similarly, 'Hom(G,$\mathbb{Z}/p\mathbb{Z}$)=(0)' is equivalent to 'G has no open subgroups of index p'. Further observe that Hom(G,$\mathbb{Z}/p\mathbb{Z}$)=(0) implies Hom(G,$\mathbb{Z}_p$)=(0).
2. The groups $\mathbb{Q}_p$, $\mathbb{Q}_p/\mathbb{Z}_p$ have the property that each nontrivial discrete quotient is an infinite torsion group. Thus, for such groups, each representative function is a linear combination of characters, for any choice of K.
3. In [5] necessary and sufficient conditions are derived on G, K in order that $G_K^{\wedge}$ be an orthonormal set. It is a striking fact that these are the same as in Theorem 2.2 but where $(\delta)''$ and $(\delta)'''$ are interchanged!

§3. REPRESENTATIVE FUNCTIONS ON $\mathbb{Z}_p$

From the previous theory it follows that a representative function $\mathbb{Z}_p \to K$ is a linear combination of characters if K is archimedean and also if K is non-archimedean and char$k \neq p$. So one may be interested in a description of $\mathcal{R}(\mathbb{Z}_p \to K)$ for the remaining case char$k = p$. We shall prove the following theorem.

Theorem 3.1. *Let* $f : \mathbb{Z}_p \to K$.

(i) *Let* $K \supset \mathbb{Q}_p$. *Then* $f \in \mathcal{R}(\mathbb{Z}_p \to K)$ *if and only if* f *has the form*

$$(*) \qquad f = \sum_{i=1}^{n} P_i \alpha_i$$

where $n \in \mathbb{N}$, $P_1,\dots,P_n$ *are polynomial functions, and* $\alpha_1,\dots,\alpha_n$ *are characters.*

(ii) *Let* char$K$=char$k=p$. *Then* $f \in \mathcal{R}(\mathbb{Z}_p \to K)$ *if and only if* f *has the form*

$$(**) \qquad f = \sum_{i=1}^{n} L_i \alpha_i$$

where $n \in \mathbb{N}$, $L_1,\dots,L_n$ *are locally constant functions and* $\alpha_1,\dots,\alpha_n$ *are characters.*

To prove this theorem we need a few lemmas. For technical reasons we shall say that an $f : \mathbb{Z}_p \to K$ is a *polycharacter* if it has the form (*) if $K \supset \mathbb{Q}_p$ or the form (**) if char$K$=char$k=p$. Then Theorem 3.1 reads in short: $f \in \mathcal{R}(\mathbb{Z}_p \to K) \Leftrightarrow f$ is a polycharacter.
One half is easy:

Lemma 3.2. *Let* char$k=p$. *Each polycharacter* $\mathbb{Z}_p \to K$ *is a representative function.*

Proof. If $K \supset \mathbb{Q}_p$ the function $x \mapsto x$ is an additive homomorphism and therefore is a representative function. For any K, a locally constant function on $\mathbb{Z}_p$ is constant on cosets of $p^m\mathbb{Z}_p$ for some m so its translates generate a space whose dimension is $\leq p^m$. Now the lemma follows after observing that

$\mathcal{R}(\mathbb{Z}_p \rightarrow K)$ is a K-algebra.

For the second half of Theorem 3.1 we introduce the following. A function $f : \mathbb{N} \rightarrow K$ *can be interpolated* if there exists a (unique) continuous function $\bar{f} : \mathbb{Z}_p \rightarrow K$ whose restriction to $\mathbb{N}$ is f. We need the following result. (As usual, the symbol [ ] indicates the entire part.)

Lemma 3.3. *Let char* $k = p \neq 0$.

(i) *For* $a \in K$, $a \neq 0$, *the sequence* $n \mapsto a^n$ *can be interpolated if and only if* $|a-1|<1$.

(ii) *For a continuous function* $f : \mathbb{Z}_p \rightarrow K$ *the sequence*

$$n \mapsto f(0) + f(1) + \ldots + f(n-1)$$

can be interpolated.

(iii) *For each* $m \in \mathbb{N}$ *the sequence* $n \mapsto [\frac{n}{p^m}]$, *considered as a map* $\mathbb{N} \rightarrow \mathbb{Q}_p$ *can be interpolated to a function* $x \mapsto [\frac{x}{p^m}]$ *on* $\mathbb{Z}_p$. *The function* $x \mapsto x - [\frac{x}{p^m}]p^m$ $(x \in \mathbb{Z}_p)$ *is locally constant.*

Proof.

(i) See [4], Theorem 32.4.

(ii) See [4], Theorem 34.1 (the assumption $K \supset \mathbb{Q}_p$ is not used in that proof).

(iii) Without trouble one verifies that

$$x \mapsto [\frac{x}{p^m}] := a_m + a_{m+1}p + a_{m+2}p^2 + \ldots$$

where $x = \sum_{i=0}^{\infty} a_i p^i$ is the standard p-adic expansion of x, is the required extension.

For the continuous extension $x \mapsto a^x$ $(x \in \mathbb{Z}_p)$ of $n \mapsto a^n$ in Lemma 3.3(i) we shall also write a^{∞}. The continuous extension of $n \mapsto f(0)+f(1)+\ldots+f(n-1)$ is called *the indefinite sum of* f, denoted by Sf. Observe that

$$S(f_1) - Sf = f - f(0)$$

Lemma 3.4. *Let* char $k = p$. *The indefinite sum of a polycharacter* $\mathbb{Z}_p \rightarrow K$ *is again a polycharacter.*

Proof. We consider two cases.

(i) $K \supset \mathbb{Q}_p$. It is not hard to see that the indefinite sum of a polynomial function is again a polynomial function. By linearity it therefore suffices to prove that for each $j \in \{0,1,2,\ldots\}$ and each $a \in K$ with $0<|1-a|<1$ the function

$$S(\varkappa^j a^x),$$

where $\varkappa^j$ is the polynomial $x \mapsto x^j$, is a polycharacter. We shall do this by proving the following statement (*) by induction on j.

(*) There is a polynomial function P_j of degree $\leq j$, whose coefficients are rational functions of a and there is a rational function Q_j of a such that for all $n \in \mathbb{N}$ and all $a \in K$ with $0<|1-a|<1$

$$S(\varkappa^j a^x)(n) = P_j(n)a^n + Q_j(a)$$

For the case $j=0$ observe that

$$S(a^x)(n) = a^0+a^1+\ldots+a^{n-1} = \frac{1}{a-1}a^n + \frac{1}{1-a}$$

So, (*) holds with $P_0(n) = \frac{1}{a-1}$, $Q_0(a) = \frac{1}{1-a}$.

Now suppose we have (*) for some j:

$$S(\varkappa^j a^x)(n) = \sum_{i=0}^{n-1} i^j a^i = P_j(n)a^n + Q_j(a) \quad (n \in \mathbb{N})$$

Then

$$S(\varkappa^{j+1} a^x)(n) = \sum_{i=0}^{n-1} i^{j+1} a^k = a\frac{d}{da}\sum_{i=0}^{n-1} i^j a^i$$

$$= (a\frac{d}{da}P_j(n)+n\,P_j(n))a^n + a\frac{d}{da}Q_j(a)$$

So, if we take

$$P_{j+1}(n) := a\frac{d}{da}P_j(n) + n\,P_j(n)$$

$$Q_{j+1}(a) = a\frac{d}{da}Q_j(a)$$

then (*) holds for $j+1$ in place of j.

(ii) $\operatorname{char} K = p$. First we prove that Sf is a polycharacter for

$$f = \xi_{p^m \mathbb{Z}_p}\alpha$$

where $m \in \mathbb{N}$, where $\xi_{p^m\mathbb{Z}_p}$ is the K-valued characteristic function of $p^m\mathbb{Z}_p$ and where α is a character. We have for $n \in \mathbb{N}$

$$(Sf)(n) = \sum_{i=0}^{n-1} \xi_{p^m\mathbb{Z}_p}(i)\alpha(i) = \sum_{j=0}^{[p^{-m}(n-1)]} \alpha(p^m j)$$

If $\alpha(p^m) = 1_K$, the unit element of K, we obtain

$$(Sf)(n) = ([\frac{n-1}{p^m}]+1)\cdot 1_K$$

and we see that Sf is a locally constant function.

If $\alpha(p^m) \neq 1_K$ then $\alpha(x) = a^x \quad (x \in \mathrm{Z}_p)$ where $a \in K$, $0<|1_K-a|<1$. We have, for $n \in \mathrm{N}$

$$(Sf)(n) = \frac{\alpha(p^m)^{[\frac{n-1}{p^m}]+1} - 1_K}{\alpha(p^m)-1_K} = \frac{a^{[\frac{n-1}{p^m}]p^m} - 1_K}{a^{p^m}-1_K}$$

It follows that Sf is a K-linear combination of a constant function and the function

$$x \mapsto a^{[\frac{x-1}{p^m}]p^m} = a^{-x}\cdot a \cdot a^{x-1-[\frac{x-1}{p^m}]p^m}$$

which is the product of the character a^{-x} and a locally constant function (Lemma 3.3(iii)). Thus, we may conclude that $S(\xi_{p^m\mathrm{Z}_p}\alpha)$ is a polycharacter.

By linearity of S and by the remark preceding this lemma the set of all polycharacters f for which Sf is also a polycharacter is a linear space, invariant under translations. A standard reasoning shows that the smallest translation invariant linear space containing all $\xi_{p^m\mathrm{Z}_p}\alpha$ ($m \in \mathrm{N}$, α character) is the set of all polycharacters which finishes the proof.

Lemma 3.5. *Let* char$k=p$, *let* $a \in K$, $|1-a|<1$. *If* $f : \mathrm{Z}_p \to K$ *is a polycharacter and if* g *is a continuous solution of*

$$g(x+1) - ag(x) = f(x) \quad (x \in \mathrm{Z}_p)$$

then g *is a polycharacter.*

Proof. Inductively we arrive easily at

$$g(n) = a^n g(0) + a^{n-1}S(a^{-x}f)(n) \quad (n \in \mathrm{N})$$

By continuity,

$$g = a^x g(0) + a^{x-1}S(a^{-x}f)$$

which is a polycharacter by Lemma 3.4.

Let L denote the operator $BC(\mathrm{Z}_p \to K) \to BC(\mathrm{Z}_p \to K)$ sending f into f_1 (recall that $f_1(x) = f(x+1)$).

Lemma 3.6. *If, for some* $a \in K$, *the operator* $L-aI$ *is not injective then* $|a-1|<1$.

Proof. Let $f \in BC(\mathrm{Z}_p \to K)$, $f \neq 0$ be such that $Lf-af=0$. Then $f(x+1)=af(x)$ for all $x \in \mathrm{Z}_p$ so that $f(n)=a^n f(0)$ for all $n \in \mathrm{N}$. We have $f(0)\neq 0$ and, by continuity of f, the sequence $x \mapsto a^n$ can be interpolated. By Lemma 3.3(i), $|1-a|<1$.

Proof of Theorem 3.1. Let f be a representative function, $f \neq 0$; we shall prove that f is a polycharacter. The sequence $f, Lf, L^2f, \ldots$ lies in a finite dimensional space so there is an $n \in \mathbb{N}$ such that $L^n f$ is a K-linear combination of $f, Lf, \ldots, L^{n-1}f$. We may choose n minimal. In other words, we have a monic polynomial $P \in K[X]$ with $P(L)(f)=0$ with minimal degree n. As K is algebraically closed P decomposes into linear factors $X-a_1, \ldots, X-a_n$ so we have

$$(L-a_1I)(L-a_2I) \ldots (L-a_nI)(f) = 0$$

The operators $L-a_i$ commute and n is minimal so no $L-a_iI$ is injective. By Lemma 3.6, $|a_i-1|<1$ for $i \in \{1, \ldots, n\}$.

Lemma 3.5, applied for $a=a_1$, $g=(L-a_2I), \ldots (L-a_nI)f$ and $f=0$ yields

$$(L-a_2I)(L-a_2I) \ldots (L-a_nI)(f) = g$$

where g is a polycharacter. By repeated application of Lemma 3.5 we can remove all $L-a_iI$ obtaining that f is a polycharacter.

Note. For results on closely related matters see [1].

References

[1] A.M.M. GOMMERS. Non-archimedean harmonic analysis on groups without Haar measure. Thesis. Nijmegen (1979).

[2] E. HEWITT and K.A. ROSS. Abstract Harmonic Analysis II. Springer-Verlag, Berlin etc. (1970).

[3] A.C.M. VAN ROOIJ. Non-Archimedean Functional Analysis. Marcel Dekker, New York (1978).

[4] W.H. SCHIKHOF. Ultrametric Calculus. Cambridge University Press (1984).

[5] W.H. SCHIKHOF. Orthogonality of p-adic characters. Proc. Kon. Ned. Akad. Wet. A 89, 337-344 (1986).