$p$-ADIC TRIGONOMETRIC POLYNOMIALS

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**INTRODUCTION.** Let $G$ be an abelian group, let $f$ be a bounded complex valued function on $G$ whose translates generate a finite dimensional space. It is well known ([2], 27.7) that $f$ is a linear combination of characters. This conclusion is not valid if the range of $f$ lies in a non-archimedean valued field $K$ rather than $\mathbb{C}$. For example, if $K$ contains the field $\mathbb{Q}_p$ of the $p$-adic numbers and if $G = \mathbb{Z}_p$, the additive group of the $p$-adic integers, it is easily seen that the translates of the function $f : x \mapsto x$ generate a twodimensional space over $K$ whereas $f$ is not a $K$-linear combination of $K$-valued characters (follow the proof of the implication $(\gamma) \Rightarrow (\alpha)$ of Theorem 1.4).

**ABSTRACT.** For an abelian topological group $G$ and an algebraically closed, nontrivially valued, complete field $K$ necessary and sufficient conditions are derived for a representative function $f : G \to K$ to be a finite $K$-linear combination of $K$-valued characters (Theorems 1.4, 2.1, 2.2). Also, a complete description of the set of all representative functions $\mathbb{Z}_p \to K$ is given (Theorem 3.1).

**TERMINOLOGY & STANDARD FACTS.** Throughout this paper $G$ is an additively written abelian topological group, $K$ is an algebraically closed nontrivially valued complete field with valuation $| |$. The set $BC(G \to K)$ consisting of all bounded continuous functions $G \to K$ is a $K$-Banach algebra with respect to pointwise operations and the norm $f \mapsto \|f\|_\infty := \sup \{|f(x)| : x \in G\}$.

A *character* is a nonzero element $\alpha$ of $BC(G \to K)$ for which $\alpha(x+y) = \alpha(x)\alpha(y)$ for all $x, y \in G$. Then $|\alpha(x)| = 1$ for all $x \in G$. Under pointwise multiplication the characters form a group $G_\wedge$. A function $f \in BC(G \to K)$ is a *representative function* (or a *trigonometric polynomial*) if the $K$-linear span $[f_s : s \in G]$ of $\{f_s : s \in G\}$ is finite dimensional. Here, as usual, $f_s(x) := f(s + x)$ for $x \in G$. It is not hard to prove that the collection $\mathfrak{R}(G \to K)$ of all representative functions $G \to K$ is a $K$-subalgebra of $BC(G \to K)$ containing $G_\wedge$. 

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A $G$-module is a Banach space $E$ over $K$ together with a separately continuous structure map $G \times E \to E$

$$(s, x) \mapsto U_s(x) = sx \quad (s \in G, x \in E)$$

such that $s \mapsto U_s$ is a homomorphism of $G$ into the group of invertible (continuous) $K$-linear operators $E \to E$ and such that, for each $x \in G$, $\text{sup} \{ \| U_s x \| : s \in G \}$ is finite. In this paper we shall deal only with finite dimensional $G$-modules.

§1. THE MAIN THEOREM

**Proposition 1.1.** Let $f \in BC(G \to K), f \neq 0$. Then $[f_s : s \in G]$ is onedimensional if and only if $f$ is a multiple of a character.

**Proof.** If $\alpha$ is a character then for each $s \in G$ we have $\alpha_s = \alpha(s) \alpha$ and $[\alpha_s : s \in G]$ is onedimensional. Conversely, suppose $\dim [f_s : s \in G] = 1$. For each $s \in G$ there is a unique $\alpha_s(s) \in K$ for which $f_s = \alpha(s) f_s$. The equality $f_{s+t} = (f_s)_{t}$ yields $\alpha(s) = \alpha(s) \alpha(t)$ for all $s, t \in G$. From $\| f_s \|_\infty = |\alpha(s)| \| f \|_\infty$ we infer that $|\alpha(s)| = 1$ for all $s \in G$. So, $\alpha$ is a character and $f = f(0) \cdot \alpha$.

**Proposition 1.2.** A representative function is uniformly continuous.

**Proof.** Let $f \in \mathcal{R}(G \to K), f \neq 0$ and let $e_1, \ldots, e_n$ be a base of $E := [f_s : s \in G]$. By equivalence of norms (see [3], Theorem 3.15 for the non-archimedean case) there exists a $C > 0$ such that

$$\max_{1 \leq i \leq n} |\lambda_i| \leq C \sum_{i=1}^n \lambda_i e_i \|_{\infty}$$

for all $\lambda_1, \ldots, \lambda_n \in K$. Let $\varepsilon > 0$. There is a neighbourhood $U$ of $0$ in $G$ such that for all $i \in \{1, 2, \ldots, n\}$

$$x \in U \Rightarrow |e_i(x) - e_i(0)| \leq (C n \| f \|_\infty)^{-1} \varepsilon.$$

Now let $s \in G, t \in U$; we shall prove that $|f(s+t) - f(s)| \leq \varepsilon$. There exist $\lambda_1, \ldots, \lambda_n \in K$ (depending on $s$) such that

$$f_s = \sum_{i=1}^n \lambda_i e_i$$

Then

$$f_{s+t} = \sum_{i=1}^n \lambda_i (e_i)_t$$

We see that

$$|f(s+t) - f(s)| = |f_{s+t}(0) - f_s(0)| =$$
\[
\left| \sum_{i=1}^{n} \lambda_i(e_i(t)-e_i(0)) \right| \leq n \max_{1 \leq i \leq n} |\lambda_i||e_i(t)-e_i(0)| \leq
n C\left\| \sum_{i=1}^{n} \lambda_i e_i \right\| (Cn\|f\|_{\infty})^{-1} \varepsilon = \|f\|_{\infty}^{-1} \varepsilon = \varepsilon.
\]

**Proposition 1.3.** Let \( E \) be a \( G \)-module of dimension \( n \in \mathbb{N} \). For each \( m \in \mathbb{N}, 1 \leq m \leq n \), \( E \) has a \( G \)-submodule of dimension \( m \).

**Proof.** By induction on \( m \). To find a onedimensional submodule choose, among all nonzero \( G \)-submodules of \( E \), a \( G \)-submodule \( E_1 \) with minimal dimension. Then \( E_1 \) is simple (i.e., the corresponding representation \( s \mapsto U_s \) is irreducible). As \( K \) is algebraically closed, a standard application of Schur's lemma ([2], 27.9) yields \( \dim E_1 = 1 \). Now let \( m < n \) and let \( E_m \) be an \( m \)-dimensional \( G \)-submodule of \( E \). The quotient \( E/E_m \) is, in an obvious way, a \( G \)-module of dimension \( n - m \geq 1 \). By the first part of the proof it has a onedimensional \( G \)-submodule \( D_1 \). One verifies immediately that \( E_{m+1} := \pi^{-1}(D_1) \), where \( \pi : E \to E/E_m \) is the quotient map, is a \( G \)-submodule of \( E \) whose dimension is \( m+1 \).

We now prove the main theorem. A function \( \mu : G \to K \) is **additive** if \( \mu(s+t) = \mu(s) + \mu(t) \) for all \( s, t \in G \).

**Theorem 1.4.** The following statements on \( G, K \) are equivalent.

\((\alpha)\) Any bounded continuous additive function \( G \to K \) is 0.

\((\beta)\) Each nonzero finite dimensional \( G \)-module over \( K \) is a (direct) sum of onedimensional \( G \)-modules.

\((\gamma)\) Each representative function \( G \to K \) is a finite \( K \)-linear combination of \( K \)-valued characters.

**Proof.** To obtain the implication \((\alpha) \Rightarrow (\beta)\) we shall prove that

\[(*)\]

\[
\begin{cases}
\text{each } n\text{-dimensional } G\text{-module has a base } e_1, \ldots, e_n \\
\text{for which } se_i \in [e_i] \text{ (} s \in G \text{) for each } i \in \{1, \ldots, n\}
\end{cases}
\]

by induction on \( n \). The case \( n = 1 \) is trivial, so suppose \((*)\) is true for some \( n \) and let \( E \) be an \((n+1)\)-dimensional \( G \)-module. According to Proposition 1.3 \( E \) has an \( n \)-dimensional \( G \)-submodule \( D \) which, by the induction hypothesis, has a base \( e_1, \ldots, e_n \) such that \( se_i \in [e_i] \) for all \( s \in G \), all \( i \in \{1, \ldots, n\} \). Choose an \( x \in E \setminus D \); then \( e_1, \ldots, e_n, x \) is base for \( E \). With respect to this base the maps \( U_s (s \in G) \) have the following matrices...
Observe that its entries are continuous functions of $s$ (since each of them has the form $s \mapsto \phi(sy)$ for some $y \in E$, $\phi \in E^*$, the dual space of $E$) and are also bounded by our definition of a $G$-module. Since $U_s$ is invertible we have $\lambda_i(s) \neq 0$ for all $i \in \{1, \ldots, n+1\}$. The equality $U_{s+t} = U_s U_t$ expressed in matrix form yields
\[
\begin{bmatrix}
\lambda_1(s) & \xi_1(s) \\
0 & \\
\lambda_2(s) & \xi_2(s) \\
& \ddots \\
\lambda_n(s) & \xi_n(s) \\
0 & \\
\end{bmatrix}
\begin{bmatrix}
\lambda_1(s+t) \\
0 \\
\lambda_2(s+t) \\
& \ddots \\
\lambda_n(s+t) \\
0 & \\
\end{bmatrix}
= 
\begin{bmatrix}
\lambda_1(s) & \xi_1(s) \\
0 & \\
\lambda_2(s) & \xi_2(s) \\
& \ddots \\
\lambda_n(s) & \xi_n(s) \\
0 & \\
\end{bmatrix}

\begin{bmatrix}
\lambda_1(t) \\
0 \\
\lambda_2(t) \\
& \ddots \\
\lambda_n(t) \\
0 & \\
\end{bmatrix}
\]

(so, each $\lambda_i$ is a character) and

\[
\xi_i(s+t) = \xi_i(s) \lambda_i(t) + \xi_i(t) \lambda_{n+1}(s) \quad (s, t \in G)
\]

for $i \in \{1, \ldots, n\}$. We now complete the proof of (a) $\Rightarrow$ (b) by defining $q_1, \ldots, q_n \in K$ such that for

\[
e_{n+1} := x + \sum_{i=1}^n q_i e_i
\]

we have $s e_{n+1} = \lambda_{n+1}(s) e_{n+1}$ ($s \in G$). That is, we have to choose $q_1, \ldots, q_n$ in such a way that

\[
\xi_i(s) + q_i(\lambda_i(s) - \lambda_{n+1}(s)) = 0 \quad (1 \leq i \leq n, s \in G).
\]

For any $i \in \{1, \ldots, n\}$ we distinguish two cases.

(i) $\lambda_i(t) \neq \lambda_{n+1}(t)$ for some $t \in G$. Then we are forced to choose

\[
q_i := (\lambda_{n+1}(t) - \lambda_i(t))^{-1} \xi_i(t)
\]

Now (**) guarantees that for any $s \in G$

\[
(\lambda_{n+1}(t) - \lambda_i(t))(\xi_i(s) + q_i(\lambda_i(s) - \lambda_{n+1}(s))) = 0
\]

and (***) follows for this $i$.

(ii) $\lambda_i = \lambda_{n+1}$. We shall prove that $\xi_i(s) = 0$ for all $s \in G$ (so that we may choose for $q_i$ and arbitrary element of $K$).

In fact, by (**) we have
\[ \xi_i(s+t) = \lambda_i(s)\xi_i(t) + \xi_i(s)\lambda_i(t) \quad (s, t \in G) \]

After dividing by \( \lambda_i(s+t) = \lambda_i(s)\lambda_i(t) \) we obtain

\[ \mu(s+t) = \mu(s) + \mu(t) \]

where \( \mu := \lambda_i^{-1}\xi_i \) is continuous and bounded. By (a) we have \( \mu=0 \). It follows that \( \xi_i = 0 \).

(\( \beta \Rightarrow \gamma \)). Let \( f \in \mathcal{R}(G \to K) \), \( f \neq 0 \) and let \( E = \{f_s : s \in G\} \). The structure map

\[ (s, g) \mapsto g_s \quad (s \in G, g \in E) \]

makes \( E \) into a finite dimensional \( G \)-module, taking into account that Proposition 1.2 guarantees the continuity of \( s \mapsto g_s \). By (\( \beta \)), \( E \) is the sum of onedimensional \( G \)-modules \([\alpha_1], \ldots, [\alpha_n]\), where Proposition 1.1 tells us that we may assume that \( \alpha_1, \ldots, \alpha_n \) are characters and (\( \gamma \)) follows.

(\( \gamma \Rightarrow (\alpha) \)). Let \( \mu \in BC(G \to K) \) be additive. For each \( s \in G \) we have \( \mu_s = \mu(s) \cdot 1 + \mu \) where 1 is the function with constant value one. So, \([\mu_s : s \in G] = [1, \mu]\) implying that \( \mu \) is a representative function. By (\( \gamma \)) there exist distinct characters \( \alpha_0, \alpha_1, \ldots, \alpha_n \), where \( \alpha_0 \) is the unit character, and \( \lambda_0, \lambda_1, \ldots, \lambda_n \in K \) such that

\[ \mu_s = \sum_{i=0}^{n} \lambda_i \alpha_i \]

The relation \( \mu_s = \mu(s)\alpha_0 + \mu \) yields

\[ \sum_{i=0}^{n} \lambda_i \alpha_i(s)\alpha_i = \mu(s)\alpha_0 + \sum_{i=0}^{n} \lambda_i \alpha_i \quad (s \in G) \]

By linear independence of characters we have equality of the coefficients of \( \alpha_0 \) i.e.

\[ \lambda_0 = \lambda_0 \alpha_0(s) = \mu(s) + \lambda_0 \quad (s \in G) \]

implying \( \mu(s) = 0 \) for all \( s \in G \).

§2. THE MAIN THEOREM FOR VARIOUS GROUND FIELDS

**Theorem 2.1.** If the valuation of \( K \) is archimedean then (\( \alpha \)),(\( \beta \)),(\( \gamma \)) of Theorem 1.4 hold for every topological abelian group \( G \).

**Proof.** Property (\( \alpha \)) of Theorem 1.4 follows from the fact that \( K \) has no bounded additive subgroups other than (0).

Next we turn to the case where the valuation of \( K \) is non-archimedean. First some notations. The residue class field of \( K \) is \( k \). The characteristic of a field \( L \) is \( \text{char} L \). For topological groups \( G_1, G_2 \) the set of all continuous homomorphisms \( G_1 \to G_2 \) is \( \text{Hom}(G_1, G_2) \).
Theorem 2.2. Let the valuation of $K$ be non-archimedean. Then $(\alpha),(\beta),(\gamma)$ of Theorem 1.4 are equivalent to

$$(\delta)' \quad \text{Hom}(G,Q) = (0) \quad \text{(where Q carries the discrete topology)}$$
if $\text{char} K = \text{char} k = 0$,

$$(\delta)'' \quad \text{Hom}(G,Z/pZ) = (0)$$
if $\text{char} K = \text{char} k = p \neq 0$,

$$(\delta)''' \quad \text{Hom}(G,Z_p) = (0)$$
if $\text{char} K = 0$, $\text{char} k = p \neq 0$.

Proof. (a) Assume $\text{char} K = \text{char} k = 0$. We have a natural embedding $Q \to K$ whose image is bounded so $(\alpha)$ of Theorem 1.4 implies $(\delta)'$. To obtain $(\delta)' \Rightarrow (\alpha)$, let $\mu : G \to K$ be a bounded nonzero additive homomorphism; we shall prove that $\text{Hom}(G,Q) \neq (0)$. Let $s \in G$, $\mu(s) \neq 0$ and let

$$\pi : K \to K/(x \in K : |x| < |\mu(s)|) =: H$$

be the canonical quotient map. The discrete group $H$ is torsion free so the formula

$$n \pi(\mu(s)) \mapsto n \quad (n \in \mathbb{Z})$$
defines a homomorphism of the group generated by $\pi(\mu(s))$ into $Q$. By divisibility of $Q$ it can be extended to a homomorphism $\phi : H \to Q$. The map $\phi \circ \pi \circ \mu : G \to Q$ is a continuous homomorphism sending $s$ into 1. Hence $\text{Hom}(G,Q) \neq (0)$.

(b) Assume $\text{char} K = \text{char} k = p \neq 0$. We have a natural embedding $Z/pZ \to K$ so $(\alpha)$ of Theorem 1.4 implies $(\delta)''$. To obtain $(\delta)'' \Rightarrow (\alpha)$, let $\mu : G \to K$ be a bounded nonzero additive homomorphism; we shall prove that $\text{Hom}(G,Z/pZ) \neq (0)$. Define $s, \pi, H$ as in part (a). This time every nonzero element of $H$ has order $p$ so the homomorphism

$$n \pi(\mu(s)) \mapsto n \mod p Z \quad (n \in \mathbb{Z})$$
can be extended to homomorphism $\phi : H \to Z/pZ$. The map $\phi \circ \pi \circ \mu : G \to Z/pZ$ is a continuous homomorphism sending $s$ into $1 \mod p Z$. Hence $\text{Hom}(G,Z/pZ) \neq (0)$.

(c) Assume $\text{char} K = 0$, $\text{char} k = p \neq 0$. Then we may assume $K \simeq \mathbb{Q}_p$. We have a natural embedding $Z_p \to K$ so $(\alpha)$ of Theorem 1.4 implies $(\delta)'''$. To obtain $(\delta)''' \Rightarrow (\alpha)$, let $\mu : G \to K$ be a bounded nonzero additive homomorphism; we shall prove that $\text{Hom}(G,Z_p) \neq (0)$. Now $K$ is, in a natural way, a Banach space over $\mathbb{Q}_p$. Since $\mathbb{Q}_p$ is spherically complete there exists, by the non-archimedean Hahn-Banach Theorem ([3], Theorem 4.15, $(\gamma) \Rightarrow (\alpha)$), a continuous $\mathbb{Q}_p$-linear map $\phi : K \to \mathbb{Q}_p$ that does not vanish on $\mu(G)$. Then $\phi \circ \mu$ is a nonzero bounded continuous homomorphism $G \to \mathbb{Q}_p$. After multiplying it by a suitable element of $\mathbb{Q}_p$ we obtain a nonzero element of $\text{Hom}(G,Z_p)$. 

-- 6 --
Remarks.

1. It is easily seen that \( \text{Hom}(G, \mathbb{Q}) = (0)' \) is equivalent to 'for each open subgroup \( H \) of \( G \) the quotient \( G/H \) is a torsion group'. Similarly, \( \text{Hom}(G, \mathbb{Z}/p\mathbb{Z}) = (0)' \) is equivalent to '\( G \) has no open subgroups of index \( p \)'. Further observe that \( \text{Hom}(G, \mathbb{Z}/p\mathbb{Z}) = (0) \) implies \( \text{Hom}(G, \mathbb{Z}_p) = (0) \).

2. The groups \( \mathbb{Q}_p, \mathbb{Q}_p/\mathbb{Z}_p \) have the property that each nontrivial discrete quotient is an infinite torsion group. Thus, for such groups, each representative function is a linear combination of characters, for any choice of \( K \).

3. In [5] necessary and sufficient conditions are derived on \( G, K \) in order that \( G^*_K \) be an orthonormal set. It is a striking fact that these are the same as in Theorem 2.2 but where \( (5)' \) and \( (5)'' \) are interchanged!

§3. REPRESENTATIVE FUNCTIONS ON \( \mathbb{Z}_p \)

From the previous theory it follows that a representative function \( \mathbb{Z}_p \rightarrow K \) is a linear combination of characters if \( K \) is archimedean and also if \( K \) is non-archimedean and \( \text{char } k = p \). So one may be interested in a description of \( \mathcal{R}(\mathbb{Z}_p \rightarrow K) \) for the remaining case \( \text{char } k = p \). We shall prove the following theorem.

Theorem 3.1. Let \( f : \mathbb{Z}_p \rightarrow K \).

(i) Let \( K \supseteq \mathbb{Q}_p \). Then \( f \in \mathcal{R}(\mathbb{Z}_p \rightarrow K) \) if and only if \( f \) has the form

\[
(*) \quad f = \sum_{i=1}^{n} P_i \alpha_i
\]

where \( n \in \mathbb{N} \), \( P_1, \ldots, P_n \) are polynomial functions, and \( \alpha_1, \ldots, \alpha_n \) are characters.

(ii) Let \( \text{char } K = \text{char } k = p \). Then \( f \in \mathcal{R}(\mathbb{Z}_p \rightarrow K) \) if and only if \( f \) has the form

\[
(**) \quad f = \sum_{i=1}^{n} L_i \alpha_i
\]

where \( n \in \mathbb{N} \), \( L_1, \ldots, L_n \) are locally constant functions and \( \alpha_1, \ldots, \alpha_n \) are characters.

To prove this theorem we need a few lemmas. For technical reasons we shall say that an \( f : \mathbb{Z}_p \rightarrow K \) is a polycharacter if it has the form \( (*) \) if \( K \supseteq \mathbb{Q}_p \) or the form \( (**) \) if \( \text{char } K = \text{char } k = p \). Then Theorem 3.1 reads in short: \( f \in \mathcal{R}(\mathbb{Z}_p \rightarrow K) \Leftrightarrow f \) is a polycharacter.

One half is easy:

Lemma 3.2. Let \( \text{char } k = p \). Each polycharacter \( \mathbb{Z}_p \rightarrow K \) is a representative function.

Proof. If \( K \supseteq \mathbb{Q}_p \) the function \( x \mapsto x \) is an additive homomorphism and therefore is a representative function. For any \( K \), a locally constant function on \( \mathbb{Z}_p \) is constant on cosets of \( p^m \mathbb{Z}_p \) for some \( m \) so its translates generate a space whose dimension is \( \leq p^m \). Now the lemma follows after observing that
\( \mathcal{R}(\mathbb{Z}_p \rightarrow K) \) is a \( K \)-algebra.

For the second half of Theorem 3.1 we introduce the following. A function \( f : \mathbb{N} \rightarrow K \) can be interpolated if there exists a (unique) continuous function \( \tilde{f} : \mathbb{Z}_p \rightarrow K \) whose restriction to \( \mathbb{N} \) is \( f \). We need the following result. (As usual, the symbol \([ \ ]\) indicates the entire part.)

**Lemma 3.3.** Let \( \text{char} \ k = p \neq 0 \).

(i) For \( a \in K, \ a \neq 0 \), the sequence \( n \mapsto a^n \) can be interpolated if and only if \( |a - 1| < 1 \).

(ii) For a continuous function \( f : \mathbb{Z}_p \rightarrow K \) the sequence

\[
    n \mapsto f(0) + f(1) + \ldots + f(n-1)
\]

can be interpolated.

(iii) For each \( m \in \mathbb{N} \) the sequence \( n \mapsto [\frac{n}{p^m}] \), considered as a map \( \mathbb{N} \rightarrow \mathbb{Q}_p \) can be interpolated to a function \( x \mapsto \frac{x}{[p^m]} \) on \( \mathbb{Z}_p \). The function \( x \mapsto x - [\frac{x}{p^m}]p^m \) (\( x \in \mathbb{Z}_p \)) is locally constant.

**Proof.**

(i) See [4], Theorem 32.4.

(ii) See [4], Theorem 34.1 (the assumption \( K \supseteq \mathbb{Q}_p \) is not used in that proof).

(iii) Without trouble one verifies that

\[
    x \mapsto [\frac{x}{p^m}] := a_m + a_{m+1}p + a_{m+2}p^2 + \ldots
\]

where \( x = \sum_{i=0}^{\infty} a_i p^i \) is the standard \( p \)-adic expansion of \( x \), is the required extension.

For the continuous extension \( x \mapsto a^x \) (\( x \in \mathbb{Z}_p \)) of \( n \mapsto a^n \) in Lemma 3.3(i) we shall also write \( a^x \). The continuous extension of \( n \mapsto f(0) + f(1) + \ldots + f(n-1) \) is called the indefinite sum of \( f \), denoted by \( Sf \).

Observe that

\[
    S(f) - Sf = f - f(0)
\]

**Lemma 3.4.** Let \( \text{char} \ k = p \). The indefinite sum of a polycharacter \( \mathbb{Z}_p \rightarrow K \) is again a polycharacter.

**Proof.** We consider two cases.

(i) \( K \supseteq \mathbb{Q}_p \). It is not hard to see that the indefinite sum of a polynomial function is again a polynomial function. By linearity it therefore suffices to prove that for each \( j \in \{0, 1, 2, \ldots\} \) and each \( a \in K \) with \( 0 < |1 - a| < 1 \) the function
where $\omega^j$ is the polynomial $x \mapsto x^j$, is a polycharacter. We shall do this by proving the following statement (*) by induction on $j$.

There is a polynomial function $P_j$ of degree $\leq j$, whose coefficients are rational functions of $a$ and there is a rational function $Q_j$ of $a$ such that for all $n \in \mathbb{N}$ and all $a \in K$ with $0 < |1-a| < 1$

\[ S(\omega^j a^*) = P_j(n)a^n + Q_j(a) \]

For the case $j = 0$ observe that

\[ S(a^*) = a^0 + a^1 + \ldots + a^{n-1} = \frac{1}{a-1}a^n + \frac{1}{1-a} \]

So, (*) holds with $P_0(n) = \frac{1}{a-1}$, $Q_0(a) = \frac{1}{1-a}$.

Now suppose we have (*) for some $j$:

\[ S(\omega^j a^*)(n) = \sum_{i=0}^{n-1} i^j a^i = P_j(n)a^n + Q_j(a) \quad (n \in \mathbb{N}) \]

Then

\[ S(\omega^{j+1} a^*)(n) = \sum_{i=0}^{n-1} i^{j+1} a^i = a \frac{d}{da} \left( \sum_{i=0}^{n-1} i^j a^i \right) \]

\[ = \left( a \frac{d}{da} P_j(n) + n P_j(n) \right)a^n + a \frac{d}{da} Q_j(a) \]

So, if we take $P_{j+1}(n) := a \frac{d}{da} P_j(n) + n P_j(n)$

\[ Q_{j+1}(a) = a \frac{d}{da} Q_j(a) \]

then (*) holds for $j+1$ in place of $j$.

(ii) $\operatorname{char} K = p$. First we prove that $Sf$ is a polycharacter for

\[ f = \xi_p^m \alpha \]

where $m \in \mathbb{N}$, where $\xi_p^m : Z_p$ is the $K$-valued characteristic function of $p^m Z_p$ and where $\alpha$ is a character.

We have for $n \in \mathbb{N}$

\[ (Sf)(n) = \sum_{i=0}^{n-1} \xi_p^m \alpha(i) = \sum_{j=0}^{p^m(n-1)} \alpha(p^m j) \]

If $\alpha(p^m) = 1_K$, the unit element of $K$, we obtain
and we see that $Sf$ is a locally constant function.

If $\alpha(p^n) \neq 1_K$ then $\alpha(x) = a^x \quad (x \in \mathbb{Z}_p)$ where $a \in K$, $0 < |1_K - a| < 1$. We have, for $n \in \mathbb{N}$

$$(Sf)(n) = \frac{\alpha(p^n)^{n+1}}{\alpha(p^n) - 1_K} = \frac{a^{n+1}}{a^n - 1_K}$$

It follows that $Sf$ is a $K$-linear combination of a constant function and the function

$$x \mapsto a^{x-1}p^n = a^{-x} \cdot a^{x-1}p^n$$

which is the product of the character $a^{-x}$ and a locally constant function (Lemma 3.3(iii)). Thus, we may conclude that $S(\xi_{p^n} \alpha)$ is a polycharacter.

By linearity of $S$ and by the remark preceding this lemma the set of all polycharacters $f$ for which $Sf$ is also a polycharacter is a linear space, invariant under translations. A standard reasoning shows that the smallest translation invariant linear space containing all $\xi_{p^n} \alpha$ ($m \in \mathbb{N}$, $\alpha$ character) is the set of all polycharacters which finishes the proof.

**Lemma 3.5.** Let $\text{char} k = p$, let $a \in K$, $|1 - a| < 1$. If $f : \mathbb{Z}_p \rightarrow K$ is a polycharacter and if $g$ is a continuous solution of

$$g(x+1) - ag(x) = f(x) \quad (x \in \mathbb{Z}_p)$$

then $g$ is a polycharacter.

**Proof.** Inductively we arrive easily at

$$g(n) = a^n g(0) + a^{n-1} S(a^{-x}f)(n) \quad (n \in \mathbb{N})$$

By continuity,

$$g = a^n g(0) + a^{n-1} S(a^{-x}f)$$

which is a polycharacter by Lemma 3.4.

Let $L$ denote the operator $BC(\mathbb{Z}_p \rightarrow K) \rightarrow BC(\mathbb{Z}_p \rightarrow K)$ sending $f$ into $f_1$ (recall that $f_1(x) = f(x+1)$).

**Lemma 3.6.** If, for some $a \in K$, the operator $L - af$ is not injective then $|a-1| < 1$.

**Proof.** Let $f \in BC(\mathbb{Z}_p \rightarrow K)$, $f \neq 0$ be such that $Lf - af = 0$. Then $f(x+1) = af(x)$ for all $x \in \mathbb{Z}_p$ so that $f(n) = a^n f(0)$ for all $n \in \mathbb{N}$. We have $f(0) \neq 0$ and, by continuity of $f$, the sequence $x \mapsto a^n$ can be interpolated. By Lemma 3.3(i), $|1 - a| < 1$. 

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Proof of Theorem 3.1. Let $f$ be a representative function, $f \neq 0$; we shall prove that $f$ is a polycharacter. The sequence $f, Lf, L^2f, \ldots$ lies in a finite dimensional space so there is an $n \in \mathbb{N}$ such that $L^n f$ is a $K$-linear combination of $f, Lf, \ldots, L^{n-1} f$. We may choose $n$ minimal. In other words, we have a monic polynomial $P \in K[X]$ with $P(L)(f) = 0$ with minimal degree $n$. As $K$ is algebraically closed $P$ decomposes into linear factors $X - a_1, \ldots, X - a_n$ so we have

$$(L-a_1 I)(L-a_2 I) \ldots (L-a_n I)(f) = 0$$

The operators $L - a_i$ commute and $n$ is minimal so no $L - a_i I$ is injective. By Lemma 3.6, $|a_i - 1| < 1$ for $i \in \{1, \ldots, n\}$.

Lemma 3.5, applied for $a = a_1$, $g = (L-a_2 I) \ldots (L-a_n I)f$ and $f = 0$ yields

$$(L-a_2 I)(L-a_2 I) \ldots (L-a_n I)(f) = g$$

where $g$ is a polycharacter. By repeated application of Lemma 3.5 we can remove all $L - a_i I$ obtaining that $f$ is a polycharacter.

Note. For results on closely related matters see [1].

References


