$p$-ADIC TRIGONOMETRIC POLYNOMIALS

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W.H. SCHIKHOF

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INTRODUCTION. Let $G$ be an abelian group, let $f$ be a bounded complex valued function on $G$ whose translates generate a finite dimensional space. It is well known ([2], 27.7) that $f$ is a linear combination of characters. This conclusion is not valid if the range of $f$ lies in a non-archimedean valued field $K$ rather than $\mathbb{C}$. For example, if $K$ contains the field $\mathbb{Q}_p$ of the $p$-adic numbers and if $G = \mathbb{Z}_p$, the additive group of the $p$-adic integers, it is easily seen that the translates of the function $f : x \mapsto x$ generate a twodimensional space over $K$ whereas $f$ is not a $K$-linear combination of $K$-valued characters (follow the proof of the implication $(\gamma) \Rightarrow (\alpha)$ of Theorem 1.4).

ABSTRACT. For an abelian topological group $G$ and an algebraically closed, nontrivially valued, complete field $K$ necessary and sufficient conditions are derived for a representative function $f : G \to K$ to be a finite $K$-linear combination of $K$-valued characters (Theorems 1.4, 2.1, 2.2). Also, a complete description of the set of all representative functions $\mathbb{Z}_p \to K$ is given (Theorem 3.1).

TERMINOLOGY & STANDARD FACTS. Throughout this paper $G$ is an additively written abelian topological group, $K$ is an algebraically closed nontrivially valued complete field with valuation $| |$. The set $BC(G \to K)$ consisting of all bounded continuous functions $G \to K$ is a $K$-Banach algebra with respect to pointwise operations and the norm $f \mapsto \|f\|_\infty := \sup \{|f(x)| : x \in G\}$.

A character is a nonzero element $\alpha$ of $BC(G \to K)$ for which $\alpha(x+y) = \alpha(x)\alpha(y)$ for all $x, y \in G$. Then $|\alpha(x)| = 1$ for all $x \in G$. Under pointwise multiplication the characters form a group $G^\wedge$. A function $f \in BC(G \to K)$ is a representative function (or a trigonometric polynomial) if the $K$-linear span $[f_s : s \in G]$ of $\{f_s : s \in G\}$ is finite dimensional. Here, as usual, $f_s(x) := f(s+x)$ for $x \in G$. It is not hard to prove that the collection $\mathcal{R}(G \to K)$ of all representative functions $G \to K$ is a $K$-subalgebra of $BC(G \to K)$ containing $G^\wedge$.
A \textit{G-module} is a Banach space $E$ over $K$ together with a separately continuous structure map $G \times E \to E$

$$(s, x) \mapsto U_s(x) = sx \quad (s \in G, x \in E)$$

such that $s \mapsto U_s$ is a homomorphism of $G$ into the group of invertible (continuous) $K$-linear operators $E \to E$ and such that, for each $x \in G$, $\sup\{\|U_s x\| : s \in G\}$ is finite. In this paper we shall deal only with finite dimensional $G$-modules.

§1. THE MAIN THEOREM

\textbf{Proposition 1.1.} Let $f \in BC(G \to K)$, $f \neq 0$. Then $[f_s : s \in G]$ is onedimensional if and only if $f$ is a multiple of a character.

\textbf{Proof.} If $\alpha$ is a character then for each $s \in G$ we have $\alpha_s = \alpha(s) \alpha$ and $[\alpha_s : s \in G]$ is onedimensional. Conversely, suppose $\dim[f_s : s \in G] = 1$. For each $s \in G$ there is a unique $\alpha(s) \in K$ for which $f_s = \alpha(s) f$. The equality $f_{s+t} = (f_s) t$ yields $\alpha(s+t) = \alpha(s) \alpha(t)$ for all $s, t \in G$. From $\|f_s\|_\infty = |\alpha(s)| \|f\|_\infty$ we infer that $|\alpha(s)| = 1$ for all $s \in G$. So, $\alpha$ is a character and $f = f(0) \cdot \alpha$.

\textbf{Proposition 1.2.} A representative function is uniformly continuous.

\textbf{Proof.} Let $f \in \mathcal{B}(G \to K)$, $f \neq 0$ and let $e_1, \ldots, e_n$ be a base of $E := [f_s : s \in G]$. By equivalence of norms (see [3], Theorem 3.15 for the non-archimedean case) there exists a $C > 0$ such that

$$\max_{1 \leq i \leq n} |\lambda_i| \leq C \|\sum_{i=1}^{n} \lambda_i e_i\|_\infty$$

for all $\lambda_1, \ldots, \lambda_n \in K$. Let $\varepsilon > 0$. There is a neighbourhood $U$ of 0 in $G$ such that for all $i \in \{1, 2, \ldots, n\}$

$$x \in U \Rightarrow |e_i(x) - e_i(0)| \leq (C n \|f\|_\infty)^{-1} \varepsilon.$$ 

Now let $s \in G$, $t \in U$; we shall prove that $|f(s + t) - f(s)| \leq \varepsilon$. There exist $\lambda_1, \ldots, \lambda_n \in K$ (depending on $s$) such that

$$f_s = \sum_{i=1}^{n} \lambda_i e_i$$

Then

$$f_{s + t} = \sum_{i=1}^{n} \lambda_i (e_i)_t$$

We see that

$$|f(s + t) - f(s)| = |f_{s + t}(0) - f_s(0)| =$$
\[
\sum_{i=1}^{n} \lambda_i (e_i(t) - e_i(0)) \leq n \max_{1 \leq i \leq n} |\lambda_i| |e_i(t) - e_i(0)| \leq n C \sum_{i=1}^{n} \lambda_i e_i \|C e_i\| \|f\|^{-1} \varepsilon = \|f\| \|f\|^{-1} \varepsilon = \varepsilon.
\]

**Proposition 1.3.** Let \( E \) be a \( G \)-module of dimension \( n \in \mathbb{N} \). For each \( m \in \mathbb{N} \), \( 1 \leq m \leq n \), \( E \) has a \( G \)-submodule of dimension \( m \).

**Proof.** By induction on \( m \). To find a onedimensional submodule choose, among all nonzero \( G \)-submodules of \( E \), a \( G \)-submodule \( E_1 \) with minimal dimension. Then \( E_1 \) is simple (i.e., the corresponding representation \( s \mapsto U_s \) is irreducible). As \( K \) is algebraically closed, a standard application of Schur's lemma ([2], 27.9) yields \( \dim E_1 = 1 \). Now let \( m < n \) and let \( E_m \) be an \( m \)-dimensional \( G \)-submodule of \( E \). The quotient \( E/E_m \) is, in an obvious way, a \( G \)-module of dimension \( n - m \geq 1 \). By the first part of the proof it has a onedimensional \( G \)-submodule \( D_1 \). One verifies immediately that \( E_{m+1} := \pi^{-1}(D_1) \), where \( \pi : E \rightarrow E/E_m \) is the quotient map, is a \( G \)-submodule of \( E \) whose dimension is \( m + 1 \).

We now prove the main theorem. A function \( \mu : G \rightarrow K \) is additive if \( \mu(s+t) = \mu(s) + \mu(t) \) for all \( s, t \in G \).

**Theorem 1.4.** The following statements on \( G, K \) are equivalent.

(a) Any bounded continuous additive function \( G \rightarrow K \) is 0.

(b) Each nonzero finite dimensional \( G \)-module over \( K \) is a (direct) sum of onedimensional \( G \)-modules.

(\gamma) Each representative function \( G \rightarrow K \) is a finite \( K \)-linear combination of \( K \)-valued characters.

**Proof.** To obtain the implication (a) \( \Rightarrow \) (b) we shall prove that

\[
\begin{aligned}
\text{each } n \text{-dimensional } G \text{-module has a base } e_1, \ldots, e_n \\
\text{for which } se_i \in [e_i] (s \in G) \text{ for each } i \in \{1, \ldots, n\}
\end{aligned}
\]

by induction on \( n \). The case \( n = 1 \) is trivial, so suppose (*) is true for some \( n \) and let \( E \) be an \( (n+1) \)-dimensional \( G \)-module. According to Proposition 1.3 \( E \) has an \( n \)-dimensional \( G \)-submodule \( D \) which, by the induction hypothesis, has a base \( e_1, \ldots, e_n \) such that \( se_i \in [e_i] \) for all \( s \in G \), all \( i \in \{1, \ldots, n\} \). Choose an \( x \in E \setminus D \); then \( e_1, \ldots, e_n, x \) is base for \( E \). With respect to this base the maps \( U_s (s \in G) \) have the following matrices.
Observe that its entries are continuous functions of $s$ (since each of them has the form $s \mapsto \phi(sy)$ for some $y \in E$, $\phi \in E^*$, the dual space of $E$) and are also bounded by our definition of a $G$-module. Since $U_s$ is invertible we have $\lambda_i(s) \neq 0$ for all $i \in \{1, \ldots, n+1\}$. The equality $U_{s+t} = U_s U_t$ expressed in matrix form yields

$$h(s+t) = h(s) h(0) (s, t \in G)$$

(so, each $\lambda_i$ is a character) and

$$(**) \quad \xi(s+t) = \lambda_i(s) \xi_i(t) + \xi_i(s) \lambda_{n+1}(t) = \lambda_i(t) \xi_i(s) + \xi_i(t) \lambda_{n+1}(s) \quad (s, t \in G)$$

for $i \in \{1, \ldots, n\}$. We now complete the proof of (a)$\Rightarrow$(b) by defining $q_1, \ldots, q_n \in K$ such that for

$$e_{n+1} := x + \sum_{i=1}^n q_i e_i$$

we have $se_{n+1} = \lambda_{n+1}(s) e_{n+1}$ ($s \in G$). That is, we have to choose $q_1, \ldots, q_n$ in such a way that

$$(***) \quad \xi_i(s) + q_i(\lambda_i(s) - \lambda_{n+1}(s)) = 0 \quad (1 \leq i \leq n, s \in G).$$

For any $i \in \{1, \ldots, n\}$ we distinguish two cases.

(i) $\lambda_i(t) \neq \lambda_{n+1}(t)$ for some $t \in G$. Then we are forced to choose

$$q_i := (\lambda_{n+1}(t) - \lambda_i(t))^{-1} \xi_i(t)$$

Now (**) guarantees that for any $s \in G$

$$(\lambda_{n+1}(t) - \lambda_i(t))(\xi_i(s) + q_i(\lambda_i(s) - \lambda_{n+1}(s))) = 0$$

and (***) follows for this $i$.

(ii) $\lambda_i = \lambda_{n+1}$. We shall prove that $\xi_i(s) = 0$ for all $s \in G$ (so that we may choose for $q_i$ and arbitrary element of $K$).

In fact, by (**) we have
\[ g_i(t) = \lambda_i(t) \lambda_i(t) + \xi_i(t) \lambda_i(t) \quad (s, t \in G) \]

After dividing by \( \lambda_i(s) \lambda_i(t) \) we obtain
\[ \mu(s + t) = \mu(s) + \mu(t) \]

where \( \mu := \lambda_i^{-1} \xi_i \) is continuous and bounded. By \((\alpha)\) we have \( \mu = 0 \). It follows that \( \xi_i = 0 \).

\((\beta) \Rightarrow (\gamma)\). Let \( f \in \mathcal{R}(G \rightarrow K), f \neq 0 \) and let \( E = \{ f_s : s \in G \} \). The structure map
\[(s, g) \mapsto g_s \quad (s \in G, g \in E)\]

makes \( E \) into a finite dimensional \( G \)-module, taking into account that Proposition 1.2 guarantees the continuity of \( s \mapsto g_s \). By \((\beta)\), \( E \) is the sum of onedimensional \( G \)-modules \([a_1], \ldots, [a_n]\), where Proposition 1.1 tells us that we may assume that \( a_1, \ldots, a_n \) are characters and \((\gamma)\) follows.

\((\gamma) \Rightarrow (\alpha)\). Let \( \mu \in BC(G \rightarrow K) \) be additive. For each \( s \in G \) we have \( \mu_s = \mu(s) \cdot 1 + \mu \) where 1 is the function with constant value one. So, \([\mu_s : s \in G] = [1, \mu]\) implying that \( \mu \) is a representative function. By \((\gamma)\) there exist distinct characters \( a_0, a_1, \ldots, a_n \), where \( a_0 \) is the unit character, and \( \lambda_0, \lambda_1, \ldots, \lambda_n \in K \) such that
\[ \mu_s = \sum_{i=0}^{n} \lambda_i a_i \]

The relation \( \mu_s = \mu(s) a_0 + \mu \) yields
\[ \sum_{i=0}^{n} \lambda_i a_i(s) a_i = \mu(s) a_0 + \sum_{i=0}^{n} \lambda_i a_i \quad (s \in G) \]

By linear independence of characters we have equality of the coefficients of \( a_0 \) i.e.
\[ \lambda_0 = \lambda_0 a_0(s) = \mu(s) + \lambda_0 \quad (s \in G) \]

implying \( \mu(s) = 0 \) for all \( s \in G \).

### §2. THE MAIN THEOREM FOR VARIOUS GROUND FIELDS

**Theorem 2.1.** If the valuation of \( K \) is archimedean then \((\alpha),(\beta),(\gamma)\) of Theorem 1.4 hold for every topological abelian group \( G \).

**Proof.** Property \((\alpha)\) of Theorem 1.4 follows from the fact that \( K \) has no bounded additive subgroups other than \((0)\).

Next we turn to the case where the valuation of \( K \) is non-archimedean. First some notations. The residue class field of \( K \) is \( k \). The characteristic of a field \( L \) is \( \text{char} L \). For topological groups \( G_1, G_2 \) the set of all continuous homomorphisms \( G_1 \rightarrow G_2 \) is \( \text{Hom}(G_1, G_2) \).
Theorem 2.2. Let the valuation of $K$ be non-archimedean. Then $(\alpha),(\beta),(\gamma)$ of Theorem 1.4 are equivalent to

\begin{align*}
(\delta)' & \quad \text{Hom}(G,\mathbb{Q}) = (0) \quad \text{(where $\mathbb{Q}$ carries the discrete topology)} \\
& \quad \text{if } \text{char} K = \text{char} k = 0,
(\delta)'' & \quad \text{Hom}(G,\mathbb{Z}/p\mathbb{Z}) = (0) \\
& \quad \text{if } \text{char} K = \text{char} k = p \neq 0,
(\delta)''' & \quad \text{Hom}(G,\mathbb{Z}_p) = (0) \\
& \quad \text{if } \text{char} K = 0, \text{char} k = p \neq 0.
\end{align*}

Proof. (a) Assume char $K$ = char $k$ = 0. We have a natural embedding $\mathbb{Q} \to K$ whose image is bounded so $(\alpha)$ of Theorem 1.4 implies $(\delta)'$. To obtain $(\delta)' \Rightarrow (\alpha)$, let $\mu : G \to K$ be a bounded nonzero additive homomorphism; we shall prove that $\text{Hom}(G,\mathbb{Q}) \neq (0)$. Let $s \in G$, $\mu(s) \neq 0$ and let

$$
\pi : K \to K/(x \in K : |x| < |\mu(s)|) =: H
$$

be the canonical quotient map. The discrete group $H$ is torsion free so the formula

$$
n \pi(\mu(s)) \mapsto n \quad (n \in \mathbb{Z})
$$

defines a homomorphism of the group generated by $\pi(\mu(s))$ into $\mathbb{Q}$. By divisibility of $\mathbb{Q}$ it can be extended to a homomorphism $\phi : H \to \mathbb{Q}$. The map $\phi \circ \pi \circ \mu : G \to \mathbb{Q}$ is a continuous homomorphism sending $s$ into 1. Hence $\text{Hom}(G,\mathbb{Q}) \neq (0)$.

(b) Assume char $K$ = char $k$ = $p \neq 0$. We have a natural embedding $\mathbb{Z}/p\mathbb{Z} \to K$ so $(\alpha)$ of Theorem 1.4 implies $(\delta)''$. To obtain $(\delta)'' \Rightarrow (\alpha)$, let $\mu : G \to K$ be a bounded nonzero additive homomorphism; we shall prove that $\text{Hom}(G,\mathbb{Z}/p\mathbb{Z}) \neq (0)$. Define $s, \pi, H$ as in part (a). This time every nonzero element of $H$ has order $p$ so the homomorphism

$$
n \pi(\mu(s)) \mapsto n \mod p \mathbb{Z} \quad (n \in \mathbb{Z})
$$

can be extended to homomorphism $\phi : H \to \mathbb{Z}/p\mathbb{Z}$. The map $\phi \circ \pi \circ \mu : G \to \mathbb{Z}/p\mathbb{Z}$ is a continuous homomorphism sending $s$ into $1 \mod p \mathbb{Z}$. Hence $\text{Hom}(G,\mathbb{Z}/p\mathbb{Z}) \neq (0)$.

(c) Assume char $K$ = 0, char $k$ = $p \neq 0$. Then we may assume $K \supseteq \mathbb{Q}_p$. We have a natural embedding $\mathbb{Z}_p \to K$ so $(\alpha)$ of Theorem 1.4 implies $(\delta)'''$. To obtain $(\delta)''' \Rightarrow (\alpha)$, let $\mu : G \to K$ be a bounded nonzero additive homomorphism; we shall prove that $\text{Hom}(G,\mathbb{Z}_p) \neq (0)$. Now $K$ is, in a natural way, a Banach space over $\mathbb{Q}_p$. Since $\mathbb{Q}_p$ is spherically complete there exists, by the non-archimedean Hahn-Banach Theorem ([3], Theorem 4.15, $(\gamma) \Rightarrow (\alpha)$), a continuous $\mathbb{Q}_p$-linear map $\phi : K \to \mathbb{Q}_p$ that does not vanish on $\mu(G)$. Then $\phi \circ \mu$ is a nonzero bounded continuous homomorphism $G \to \mathbb{Q}_p$. After multiplying it by a suitable element of $\mathbb{Q}_p$ we obtain a nonzero element of $\text{Hom}(G,\mathbb{Z}_p)$. 

\[\text{-- 6 --}\]
Remarks.

1. It is easily seen that $\text{Hom}(G, \mathbb{Q})=(0)'$ is equivalent to 'for each open subgroup $H$ of $G$ the quotient $G/H$ is a torsion group'. Similarly, $\text{Hom}(G, \mathbb{Z}/p\mathbb{Z})=(0)'$ is equivalent to 'G has no open subgroups of index $p$'. Further observe that $\text{Hom}(G, \mathbb{Z}/p\mathbb{Z})=(0)$ implies $\text{Hom}(G, \mathbb{Z}_p)=(0)$.

2. The groups $\mathbb{Q}_p$, $\mathbb{Q}_p/\mathbb{Z}_p$ have the property that each nontrivial discrete quotient is an infinite torsion group. Thus, for such groups, each representative function is a linear combination of characters, for any choice of $K$.

3. In [5] necessary and sufficient conditions are derived on $G$, $K$ in order that $G^*_K$ be an orthonormal set. It is a striking fact that these are the same as in Theorem 2.2 but where $(5)'$ and $(5)''\prime$ are interchanged!

§3. REPRESENTATIVE FUNCTIONS ON $\mathbb{Z}_p$

From the previous theory it follows that a representative function $\mathbb{Z}_p \rightarrow K$ is a linear combination of characters if $K$ is archimedean and also if $K$ is non-archimedean and char $k\neq p$. So one may be interested in a description of $\mathcal{R}(\mathbb{Z}_p \rightarrow K)$ for the remaining case char $k=p$. We shall prove the following theorem.

Theorem 3.1. Let $f : \mathbb{Z}_p \rightarrow K$.

(i) Let $K \supseteq \mathbb{Q}_p$. Then $f \in \mathcal{R}(\mathbb{Z}_p \rightarrow K)$ if and only if $f$ has the form

\[ f = \sum_{i=1}^{n} P_i \alpha_i \]

where $n \in \mathbb{N}$, $P_1, \ldots, P_n$ are polynomial functions, and $\alpha_1, \ldots, \alpha_n$ are characters.

(ii) Let char $K=\text{char} k=p$. Then $f \in \mathcal{R}(\mathbb{Z}_p \rightarrow K)$ if and only if $f$ has the form

\[ f = \sum_{i=1}^{n} L_i \alpha_i \]

where $n \in \mathbb{N}$, $L_1, \ldots, L_n$ are locally constant functions and $\alpha_1, \ldots, \alpha_n$ are characters.

To prove this theorem we need a few lemmas. For technical reasons we shall say that an $f : \mathbb{Z}_p \rightarrow K$ is a polycharacter if it has the form (*) if $K \supseteq \mathbb{Q}_p$ or the form (**) if char $K=\text{char} k=p$. Then Theorem 3.1 reads in short: $f \in \mathcal{R}(\mathbb{Z}_p \rightarrow K) \iff f$ is a polycharacter.

One half is easy:

Lemma 3.2. Let char $k=p$. Each polycharacter $\mathbb{Z}_p \rightarrow K$ is a representative function.

Proof. If $K \supseteq \mathbb{Q}_p$ the function $x \mapsto x$ is an additive homomorphism and therefore is a representative function. For any $K$, a locally constant function on $\mathbb{Z}_p$ is constant on cosets of $p^m\mathbb{Z}_p$ for some $m$ so its translates generate a space whose dimension is $\leq p^m$. Now the lemma follows after observing that -- 7 --
s(Z_p → K) is a K-algebra.

For the second half of Theorem 3.1 we introduce the following. A function f : N → K can be interpolated if there exists a (unique) continuous function \( f : \mathbb{Z}_p → K \) whose restriction to N is f. We need the following result. (As usual, the symbol \([\cdot]\) indicates the entire part.)

Lemma 3.3. Let char k = p \( \neq 0 \).

(i) For \( a \in K, a \neq 0 \), the sequence \( n \mapsto a^n \) can be interpolated if and only if \(|a - 1| < 1\).

(ii) For a continuous function \( f : \mathbb{Z}_p → K \) the sequence

\[ n \mapsto f(0) + f(1) + ... + f(n-1) \]

can be interpolated.

(iii) For each \( m \in \mathbb{N} \) the sequence \( n \mapsto \frac{n}{p^m} \), considered as a map \( \mathbb{N} → \mathbb{Q}_p \) can be interpolated to a function \( x \mapsto x \cdot \left[ \frac{x}{p^m} \right] \) on \( \mathbb{Z}_p \). The function \( x \mapsto x - \left[ \frac{x}{p^m} \right] p^m \) (\( x \in \mathbb{Z}_p \)) is locally constant.

Proof.

(i) See [4], Theorem 32.4.

(ii) See [4], Theorem 34.1 (the assumption \( K \supseteq \mathbb{Q}_p \) is not used in that proof).

(iii) Without trouble one verifies that

\[ x \mapsto \left[ \frac{x}{p^m} \right] := a_m + a_{m+1}p + a_{m+2}p^2 + ... \]

where \( x = \sum_{i=0}^{\infty} a_i p^i \) is the standard p-adic expansion of \( x \), is the required extension.

For the continuous extension \( x \mapsto a^x \) (\( x \in \mathbb{Z}_p \)) of \( n \mapsto a^n \) in Lemma 3.3(i) we shall also write \( a^* \). The continuous extension of \( n \mapsto f(0) + f(1) + ... + f(n-1) \) is called the indefinite sum of \( f \), denoted by \( Sf \).

Observe that

\[ S(f_i) - Sf = f - f(0) \]

Lemma 3.4. Let char k = p. The indefinite sum of a polycharacter \( \mathbb{Z}_p → K \) is again a polycharacter.

Proof. We consider two cases.

(i) \( K \supseteq \mathbb{Q}_p \). It is not hard to see that the indefinite sum of a polynomial function is again a polynomial function. By linearity it therefore suffices to prove that for each \( j \in \{0, 1, 2, ...\} \) and each \( a \in K \) with \( 0 < |1 - a| < 1 \) the function
where \( \omega^j \) is the polynomial \( x \mapsto x^j \), is a polycharacter. We shall do this by proving the following statement (*) by induction on \( j \).

There is a polynomial function \( P_j \) of degree \( \leq j \), whose coefficients are rational functions of \( a \) and there is a rational function \( Q_j \) of \( a \) such that for all \( n \in \mathbb{N} \) and \( a \in K \) with \( 0 < |1-a| < 1 \)

\[
S(\omega^j a^n) = P_j(n)a^{n} + Q_j(a)
\]

For the case \( j=0 \) observe that

\[
S(a^0)(n) = a^0 + a^1 + \ldots + a^{n-1} = \frac{1}{a-1}a^n + \frac{1}{1-a}
\]

So, (*) holds with \( P_0(n) = \frac{1}{a-1}, \ Q_0(a) = \frac{1}{1-a} \).

Now suppose we have (*) for some \( j \):

\[
S(\omega^{j+1} a^n) = \sum_{i=0}^{n-1} i^j a^i = P_j(n)a^n + Q_j(a) \quad (n \in \mathbb{N})
\]

Then

\[
S(\omega^{j+1} a^n) = \sum_{i=0}^{n-1} i^{j+1} a^i = a \frac{d}{da} \sum_{i=0}^{n-1} i^j a^i = (a \frac{d}{da} P_j(n) + n P_j(n))a^n + a \frac{d}{da} Q_j(a)
\]

So, if we take

\[
P_{j+1}(n) := a \frac{d}{da} P_j(n) + n P_j(n) \\
Q_{j+1}(a) = a \frac{d}{da} Q_j(a)
\]

then (*) holds for \( j+1 \) in place of \( j \).

(ii) \( \text{char } K = p \). First we prove that \( Sf \) is a polycharacter for

\[
f = \xi_{p^mZ_p} \alpha
\]

where \( m \in \mathbb{N} \), where \( \xi_{p^mZ_p} \) is the \( K \)-valued characteristic function of \( p^mZ_p \) and where \( \alpha \) is a character.

We have for \( n \in \mathbb{N} \)

\[
(Sf)(n) = \sum_{i=0}^{n-1} \xi_{p^mZ_p}(i)\alpha(i) = \sum_{j=0}^{[p^m(n-1)]} \alpha(p^m j)
\]

If \( \alpha(p^m) = 1_K \), the unit element of \( K \), we obtain
and we see that \( Sf \) is a locally constant function.

If \( \alpha(p^n) \neq 1 \) then \( \alpha(x) = a^x \) \( (x \in \mathbb{Z}_p) \) where \( a \in K \), \( 0 < |1 - a| < 1 \). We have, for \( n \in \mathbb{N} \)

\[
(Sf)(n) = \frac{\alpha(p^n)|p^n-1|}{\alpha(p^n)-1} = \frac{a^{n-1}p^n-1}{ap^n-1}.
\]

It follows that \( Sf \) is a \( K \)-linear combination of a constant function and the function

\[
x \mapsto a^{x-1}p^n = a^{-x} \cdot a^{x-1}p^n
\]

which is the product of the character \( a^{-x} \) and a locally constant function (Lemma 3.3(iii)). Thus, we may conclude that \( S(\xi_{p^n \mathbb{Z}_p} \alpha) \) is a polycharacter.

By linearity of \( S \) and by the remark preceding this lemma the set of all polycharacters \( f \) for which \( Sf \) is also a polycharacter is a linear space, invariant under translations. A standard reasoning shows that the smallest translation invariant linear space containing all \( \xi_{p^n \mathbb{Z}_p} \alpha \) \( (m \in \mathbb{N}, \alpha \text{ character}) \) is the set of all polycharacters which finishes the proof.

**Lemma 3.5.** Let \( \text{char } k = p \), let \( a \in K \), \( |1 - a| < 1 \). If \( f: \mathbb{Z}_p \to K \) is a polycharacter and if \( g \) is a continuous solution of

\[
g(x+1) - ag(x) = f(x) \quad (x \in \mathbb{Z}_p)
\]

then \( g \) is a polycharacter.

**Proof.** Inductively we arrive easily at

\[
g(n) = a^n g(0) + a^{n-1}S(a^{-x}f)(n) \quad (n \in \mathbb{N})
\]

By continuity,

\[
g = a^n g(0) + a^{n-1}S(a^{-x}f)
\]

which is a polycharacter by Lemma 3.4.

Let \( L \) denote the operator \( BC(\mathbb{Z}_p \to K) \to BC(\mathbb{Z}_p \to K) \) sending \( f \) into \( f_1 \) (recall that \( f_1(x) = f(x+1) \)).

**Lemma 3.6.** If, for some \( a \in K \), the operator \( L - aI \) is not injective then \( |a-1| < 1 \).

**Proof.** Let \( f \in BC(\mathbb{Z}_p \to K), f \neq 0 \) be such that \( Lf - af = 0 \). Then \( f(x+1) = af(x) \) for all \( x \in \mathbb{Z}_p \) so that \( f(n) = a^n f(0) \) for all \( n \in \mathbb{N} \). We have \( f(0) \neq 0 \) and, by continuity of \( f \), the sequence \( x \mapsto a^n \) can be interpolated. By Lemma 3.3(i), \( |1 - a| < 1 \).
Proof of Theorem 3.1. Let \( f \) be a representative function, \( f \neq 0 \); we shall prove that \( f \) is a polycharacter. The sequence \( f, Lf, L^2f, \ldots \) lies in a finite dimensional space so there is an \( n \in \mathbb{N} \) such that \( L^n f \) is a \( K \)-linear combination of \( f, Lf, \ldots, L^{n-1}f \). We may choose \( n \) minimal. In other words, we have a monic polynomial \( P \in K[X] \) with \( P(L)(f) = 0 \) with minimal degree \( n \). As \( K \) is algebraically closed \( P \) decomposes into linear factors \( X-a_1, \ldots, X-a_n \) so we have

\[
(L-a_1I)(L-a_2I) \ldots (L-a_nI)(f) = 0
\]

The operators \( L-a_i \) commute and \( n \) is minimal so no \( L-a_iI \) is injective. By Lemma 3.6, \( \left| a_i - 1 \right| < 1 \) for \( i \in \{1, \ldots, n\} \).

Lemma 3.5, applied for \( a=a_1, g=(L-a_2I) \ldots (L-a_nI)f \) and \( f=0 \) yields

\[
(L-a_2I)(L-a_2I) \ldots (L-a_nI)(f) = g
\]

where \( g \) is a polycharacter. By repeated application of Lemma 3.5 we can remove all \( L-a_i I \) obtaining that \( f \) is a polycharacter.

Note. For results on closely related matters see [1].

References