$p$-ADIC TRIGONOMETRIC POLYNOMIALS

by

W.H. SCHIKHOF

Report 8727
September 1987

DEPARTMENT OF MATHEMATICS
CATHOLIC UNIVERSITY
Toernooiveld
6525 ED Nijmegen
The Netherlands
$p$-ADIC TRIGONOMETRIC POLYNOMIALS

by

W.H. Schikhof

Report 8727
September 1987

DEPARTMENT OF MATHEMATICS
CATHOLIC UNIVERSITY
Toumooiveld
6525 ED Nijmegen
The Netherlands
\textit{p-ADIC TRIGONOMETRIC POLYNOMIALS}

by

W.H. Schikhof

\textbf{INTRODUCTION.} Let \( G \) be an abelian group, let \( f \) be a bounded complex valued function on \( G \) whose translates generate a finite dimensional space. It is well known ([2], 27.7) that \( f \) is a linear combination of characters. This conclusion is not valid if the range of \( f \) lies in a non-archimedean valued field \( K \) rather than \( \mathbb{C} \). For example, if \( K \) contains the field \( \mathbb{Q}_p \) of the \( p \)-adic numbers and if \( G = \mathbb{Z}_p \), the additive group of the \( p \)-adic integers, it is easily seen that the translates of the function \( f : x \mapsto x \) generate a twodimensional space over \( K \) whereas \( f \) is not a \( K \)-linear combination of \( K \)-valued characters (follow the proof of the implication \((\gamma) \Rightarrow (\alpha)\) of Theorem 1.4).

\textbf{ABSTRACT.} For an abelian topological group \( G \) and an algebraically closed, nontrivially valued, complete field \( K \) necessary and sufficient conditions are derived for a representative function \( f : G \rightarrow K \) to be a finite \( K \)-linear combination of \( K \)-valued characters (Theorems 1.4, 2.1, 2.2). Also, a complete description of the set of all representative functions \( \mathbb{Z}_p \rightarrow K \) is given (Theorem 3.1).

\textbf{TERMINOLOGY \& STANDARD FACTS.} Throughout this paper \( G \) is an additively written abelian topological group, \( K \) is an algebraically closed nontrivially valued complete field with valuation \(|
\| \). The set \( BC(G \rightarrow K) \) consisting of all bounded continuous functions \( G \rightarrow K \) is a \( K \)-Banach algebra with respect to pointwise operations and the norm \( f \mapsto \|f\|_\infty := \sup \{|f(x)| : x \in G\} \).

A \textit{character} is a nonzero element \( \alpha \) of \( BC(G \rightarrow K) \) for which \( \alpha(x+y) = \alpha(x)\alpha(y) \) for all \( x, y \in G \). Then \(|\alpha(x)| = 1 \) for all \( x \in G \). Under pointwise multiplication the characters form a group \( G_\hat{K} \). A function \( f \in BC(G \rightarrow K) \) is a \textit{representative function} (or a \textit{trigonometric polynomial}) if the \( K \)-linear span \( \{f_s : s \in G\} \) of \( \{f_s : s \in G\} \) is finite dimensional. Here, as usual, \( f_s(x) := f(s+x) \) for \( x \in G \). It is not hard to prove that the collection \( \mathfrak{R}(G \rightarrow K) \) of all representative functions \( G \rightarrow K \) is a \( K \)-subalgebra of \( BC(G \rightarrow K) \) containing \( G_\hat{K} \).
A $G$–module is a Banach space $E$ over $K$ together with a separately continuous structure map $G \times E \to E$

$$(s, x) \mapsto U_s(x) = sx \quad (s \in G, x \in E)$$

such that $s \mapsto U_s$ is a homomorphism of $G$ into the group of invertible (continuous) $K$-linear operators $E \to E$ and such that, for each $x \in G$, $\sup(\|U_s x\| : s \in G)$ is finite. In this paper we shall deal only with finite dimensional $G$-modules.

§1. THE MAIN THEOREM

**Proposition 1.1.** Let $f \in BC(G \to K)$, $f \neq 0$. Then $[f_s : s \in G]$ is onedimensional if and only if $f$ is a multiple of a character.

**Proof.** If $\alpha$ is a character then for each $s \in G$ we have $\alpha_s = \alpha(s)\alpha$ and $[\alpha_s : s \in G]$ is onedimensional. Conversely, suppose $\dim[f_s : s \in G] = 1$. For each $s \in G$ there is a unique $\alpha(s) \in K$ for which $f_s = \alpha(s)f$. The equality $f_{s+t} = (f_s)_t$ yields $\alpha(s+t) = \alpha(s)\alpha(t)$ for all $s, t \in G$. From $\|f_s\|_\infty = |\alpha(s)|\|f\|_\infty$ we infer that $|\alpha(s)| = 1$ for all $s \in G$. So, $\alpha$ is a character and $f = f(0)\alpha$.

**Proposition 1.2.** A representative function is uniformly continuous.

**Proof.** Let $f \in \mathfrak{S}(G \to K)$, $f \neq 0$ and let $e_1, \ldots, e_n$ be a base of $E := [f_s : s \in G]$. By equivalence of norms (see [3], Theorem 3.15 for the non-archimedean case) there exists a $C > 0$ such that

$$\max_{i \leq n} |\lambda_i| \leq C\|\sum_{i=1}^n \lambda_i e_i\|_\infty$$

for all $\lambda_1, \ldots, \lambda_n \in K$. Let $\varepsilon > 0$. There is a neighbourhood $U$ of $0$ in $G$ such that for all $i \in \{1, 2, \ldots, n\}$

$$x \in U \Rightarrow |e_i(x) - e_i(0)| \leq (C n\|f\|_\infty)^{-1}\varepsilon.$$

Now let $s \in G$, $t \in U$; we shall prove that $\|f(s+t) - f(s)\| \leq \varepsilon$. There exist $\lambda_1, \ldots, \lambda_n \in K$ (depending on $s$) such that

$$f_s = \sum_{i=1}^n \lambda_i e_i$$

Then

$$f_{s+t} = \sum_{i=1}^n \lambda_i (e_i)_t$$

We see that

$$|f(s+t) - f(s)| = |f_{s+t}(0) - f_s(0)| =$$
\[
\left| \sum_{i=1}^{n} \lambda_i (e_i(t) - e_i(0)) \right| \leq n \max_{1 \leq i \leq n} |\lambda_i| |e_i(t) - e_i(0)| \leq \\
nC \left\| \sum_{i=1}^{n} \lambda_i e_i \right\| _{\infty} (Cn \|f\|_{\infty})^{-1} \epsilon = \|f\|_{\infty} \|f\|_{p}^{-1} \epsilon = \epsilon.
\]

**Proposition 1.3.** Let \( E \) be a \( G \)-module of dimension \( n \in \mathbb{N} \). For each \( m \in \mathbb{N}, 1 \leq m \leq n \), \( E \) has a \( G \)-submodule of dimension \( m \).

**Proof.** By induction on \( m \). To find a onedimensional submodule choose, among all nonzero \( G \)-submodules of \( E \), a \( G \)-submodule \( E_1 \) with minimal dimension. Then \( E_1 \) is simple (i.e., the corresponding representation \( s \mapsto U_s \) is irreducible). As \( K \) is algebraically closed, a standard application of Schur's lemma ([2], 27.9) yields \( \dim E_1 = 1 \). Now let \( m < n \) and let \( E_m \) be an \( m \)-dimensional \( G \)-submodule of \( E \). The quotient \( E/E_m \) is, in an obvious way, a \( G \)-module of dimension \( n - m \geq 1 \). By the first part of the proof it has a onedimensional \( G \)-submodule \( D_1 \). One verifies immediately that \( E_{m+1} := \pi^{-1}(D_1) \), where \( \pi : E \to E/E_m \) is the quotient map, is a \( G \)-submodule of \( E \) whose dimension is \( m + 1 \).

We now prove the main theorem. A function \( \mu : G \to K \) is additive if \( \mu(s + t) = \mu(s) + \mu(t) \) for all \( s, t \in G \).

**Theorem 1.4.** The following statements on \( G, K \) are equivalent.

\( \alpha \) Any bounded continuous additive function \( G \to K \) is 0.

\( \beta \) Each nonzero finite dimensional \( G \)-module over \( K \) is a (direct) sum of onedimensional \( G \)-modules.

\( \gamma \) Each representative function \( G \to K \) is a finite \( K \)-linear combination of \( K \)-valued characters.

**Proof.** To obtain the implication \( (\alpha) \Rightarrow (\beta) \) we shall prove that

\[
\left\{ \begin{array}{l}
\text{each } n \text{-dimensional } G \text{-module has a base } e_1, \ldots, e_n \\
\text{for which } se_i \in [e_i] \text{ for each } i \in \{1, \ldots, n\}
\end{array} \right.
\]

by induction on \( n \). The case \( n = 1 \) is trivial, so suppose (*) is true for some \( n \) and let \( E \) be an \( (n + 1) \)-dimensional \( G \)-module. According to Proposition 1.3 \( E \) has an \( n \)-dimensional \( G \)-submodule \( D \) which, by the induction hypothesis, has a base \( e_1, \ldots, e_n \) such that \( se_i \in [e_i] \) for all \( s \in G \), all \( i \in \{1, \ldots, n\} \). Choose an \( x \in E \setminus D \); then \( e_1, \ldots, e_n, x \) is base for \( E \). With respect to this base the maps \( U_s (s \in G) \) have the following matrices
Observe that its entries are continuous functions of $s$ (since each of them has the form $s \mapsto \phi(sy)$ for some $y \in E$, $\phi \in E^*$, the dual space of $E$) and are also bounded by our definition of a $G$-module. Since $U_s$ is invertible we have $\lambda_i(s) \neq 0$ for all $i \in \{1, \ldots, n+1\}$. The equality $U_{s+t} = U_s U_t$ expressed in matrix form yields

$$
\begin{bmatrix}
\lambda_1(s) & \xi_1(s) \\
0 & \lambda_2(s) & \xi_2(s) \\
& \ddots & \ddots \\
& & \lambda_n(s) & \xi_n(s) \\
0 & & & \lambda_{n+1}(s)
\end{bmatrix}
$$

(s, t \in G)

(so, each $\lambda_i$ is a character) and

$$
\xi_i(s+t) = \lambda_i(s)\xi_i(t) + \xi_i(s)\lambda_{n+1}(t) = \lambda_i(t)\xi_i(s) + \xi_i(t)\lambda_{n+1}(s) \\
(s, t \in G)
$$

for $i \in \{1, \ldots, n\}$. We now complete the proof of $(a) \Rightarrow (b)$ by defining $q_1, \ldots, q_n \in K$ such that for

$$
e_{n+1} := x + \sum_{i=1}^n q_i e_i
$$

we have $se_{n+1} = \lambda_{n+1}(s)e_{n+1}$ $(s \in G)$. That is, we have to choose $q_1, \ldots, q_n$ in such a way that

$$
\xi_i(s) + q_i(\lambda_i(s) - \lambda_{n+1}(s)) = 0 \\
(1 \leq i \leq n, s \in G)
$$

For any $i \in \{1, \ldots, n\}$ we distinguish two cases.

(i) $\lambda_i(t) \neq \lambda_{n+1}(t)$ for some $t \in G$. Then we are forced to choose

$$
q_i := (\lambda_{n+1}(t) - \lambda_i(t))^{-1}\xi_i(t)
$$

Now $(**)$ guarantees that for any $s \in G$

$$(\lambda_{n+1}(t) - \lambda_i(t))(\xi_i(s) + q_i(\lambda_i(s) - \lambda_{n+1}(s))) = 0
$$

and $(***)$ follows for this $i$.

(ii) $\lambda_i = \lambda_{n+1}$. We shall prove that $\xi_i(s) = 0$ for all $s \in G$ (so that we may choose for $q_i$ and arbitrary element of $K$).

In fact, by $(**)$ we have
\[ \xi_i(s+t) = \lambda_i(s)\xi_i(t) + \xi_i(s)\lambda_i(t) \quad (s, t \in G) \]

After dividing by \( \lambda_i(s+t) = \lambda_i(s)\lambda_i(t) \) we obtain

\[ \mu(s+t) = \mu(s) + \mu(t) \]

where \( \mu := \lambda_i^{-1}\xi_i \) is continuous and bounded. By \((\alpha)\) we have \( \mu = 0 \). It follows that \( \xi_i = 0 \).

\((\beta)\Rightarrow(\gamma)\). Let \( f \in \mathcal{R}(G \to K) \), \( f \neq 0 \) and let \( E = \{ f_s : s \in G \} \). The structure map

\[ (s,g) \mapsto g_s \quad (s \in G, g \in E) \]

makes \( E \) into a finite dimensional \( G \)-module, taking into account that Proposition 1.2 guarantees the continuity of \( s \mapsto g_s \). By \((\beta)\), \( E \) is the sum of onedimensional \( G \)-modules \([\alpha_1], \ldots, [\alpha_n] \), where Proposition 1.1 tells us that we may assume that \( \alpha_1, \ldots, \alpha_n \) are characters and \((\gamma)\) follows.

\((\gamma)\Rightarrow(\alpha)\). Let \( \mu \in BC(G \to K) \) be additive. For each \( s \in G \) we have \( \mu_s = \mu(s) \cdot 1 + \mu \) where 1 is the function with constant value one. So, \([\mu_s : s \in G] = [1, \mu] \) implying that \( \mu \) is a representative function. By \((\gamma)\) there exist distinct characters \( \alpha_0, \alpha_1, \ldots, \alpha_n \), where \( \alpha_0 \) is the unit character, and \( \lambda_0, \lambda_1, \ldots, \lambda_n \in K \) such that

\[ \mu = \sum_{i=0}^n \lambda_i \alpha_i \]

The relation \( \mu_s = \mu(s)\alpha_0 + \mu \) yields

\[ \sum_{i=0}^n \lambda_i \alpha_i(s)\alpha_i = \mu(s)\alpha_0 + \sum_{i=0}^n \lambda_i \alpha_i \quad (s \in G) \]

By linear independence of characters we have equality of the coefficients of \( \alpha_0 \) i.e.

\[ \lambda_0 = \lambda_0 \alpha_0(s) = \mu(s) + \lambda_0 \quad (s \in G) \]

implying \( \mu(s) = 0 \) for all \( s \in G \).

§2. THE MAIN THEOREM FOR VARIOUS GROUND FIELDS

**Theorem 2.1.** If the valuation of \( K \) is archimedean then \((\alpha), (\beta), (\gamma)\) of Theorem 1.4 hold for every topological abelian group \( G \).

**Proof.** Property \((\alpha)\) of Theorem 1.4 follows from the fact that \( K \) has no bounded additive subgroups other than \((0)\).

Next we turn to the case where the valuation of \( K \) is non-archimedean. First some notations. The residue class field of \( K \) is \( k \). The characteristic of a field \( L \) is \( \text{char} L \). For topological groups \( G_1, G_2 \) the set of all continuous homomorphisms \( G_1 \to G_2 \) is \( \text{Hom}(G_1, G_2) \).
Theorem 2.2. Let the valuation of $K$ be non-archimedean. Then $(\alpha), (\beta), (\gamma)$ of Theorem 1.4 are equivalent to

$(\delta)'$ $\text{Hom}(G, \mathbb{Q}) = (0)$ (where $\mathbb{Q}$ carries the discrete topology)
if $\text{char} K = \text{char} k = 0,$

$(\delta)''$ $\text{Hom}(G, \mathbb{Z}/p\mathbb{Z}) = (0)$
if $\text{char} K = \text{char} k = p \neq 0,$

$(\delta)'''$ $\text{Hom}(G, \mathbb{Z}_p) = (0)$
if $\text{char} K = 0,$ $\text{char} k = p \neq 0.$

Proof. (a) Assume $\text{char} K = \text{char} k = 0.$ We have a natural embedding $\mathbb{Q} \to K$ whose image is bounded so $(\alpha)$ of Theorem 1.4 implies $(\delta)'.$ To obtain $(\delta)' \Rightarrow (\alpha),$ let $\mu : G \to K$ be a bounded nonzero additive homomorphism; we shall prove that $\text{Hom}(G, \mathbb{Q}) \neq (0).$ Let $s \in G,$ $\mu(s) \neq 0$ and let

$$
\pi : K \to K/(x \in K : |x| < |\mu(s)|) = H
$$

be the canonical quotient map. The discrete group $H$ is torsion free so the formula

$$
n \pi(\mu(s)) \mapsto n (n \in \mathbb{Z})
$$

defines a homomorphism of the group generated by $\pi(\mu(s))$ into $\mathbb{Q}.$ By divisibility of $\mathbb{Q}$ it can be extended to a homomorphism $\phi : H \to \mathbb{Q}.$ The map $\phi \circ \pi \circ \mu : G \to \mathbb{Q}$ is a continuous homomorphism sending $s$ into 1. Hence $\text{Hom}(G, \mathbb{Q}) \neq (0)$.

(b) Assume $\text{char} K = \text{char} k = p \neq 0.$ We have a natural embedding $\mathbb{Z}/p\mathbb{Z} \to K$ so $(\alpha)$ of Theorem 1.4 implies $(\delta)'''.$ To obtain $(\delta)'' \Rightarrow (\alpha),$ let $\mu : G \to K$ be a bounded nonzero additive homomorphism; we shall prove that $\text{Hom}(G, \mathbb{Z}/p\mathbb{Z}) \neq (0).$ Define $s, \pi, H$ as in part (a). This time every nonzero element of $H$ has order $p$ so the homomorphism

$$
n \pi(\mu(s)) \mapsto n \mod p \mathbb{Z} (n \in \mathbb{Z})
$$

can be extended to homomorphism $\phi : H \to \mathbb{Z}/p\mathbb{Z}.$ The map $\phi \circ \pi \circ \mu : G \to \mathbb{Z}/p\mathbb{Z}$ is a continuous homomorphism sending $s$ into $1 \mod p \mathbb{Z}.$ Hence $\text{Hom}(G, \mathbb{Z}/p\mathbb{Z}) \neq (0)$.

(c) Assume $\text{char} K = 0,$ $\text{char} k = p \neq 0.$ Then we may assume $K \supseteq \mathbb{Q}_p.$ We have a natural embedding $\mathbb{Z}_p \to K$ so $(\alpha)$ of Theorem 1.4 implies $(\delta)''''.$ To obtain $(\delta)''' \Rightarrow (\alpha),$ let $\mu : G \to K$ be a bounded nonzero additive homomorphism; we shall prove that $\text{Hom}(G, \mathbb{Z}_p) \neq (0).$ Now $K$ is, in a natural way, a Banach space over $\mathbb{Q}_p.$ Since $\mathbb{Q}_p$ is spherically complete there exists, by the non-archimedean Hahn-Banach Theorem ([3], Theorem 4.15, $(\gamma) \Rightarrow (\alpha)),$ a continuous $\mathbb{Q}_p$-linear map $\phi : K \to \mathbb{Q}_p$ that does not vanish on $\mu(G).$ Then $\phi \circ \mu$ is a nonzero bounded continuous homomorphism $G \to \mathbb{Q}_p.$ After multiplying it by a suitable element of $\mathbb{Q}_p$ we obtain a nonzero element of $\text{Hom}(G, \mathbb{Z}_p).$
Remarks.

1. It is easily seen that \( \text{Hom}(G, \mathbb{Q}) = (0) \)' is equivalent to 'for each open subgroup \( H \) of \( G \) the quotient \( G/H \) is a torsion group'. Similarly, \( \text{Hom}(G, \mathbb{Z}/p\mathbb{Z}) = (0) \)' is equivalent to '\( G \) has no open subgroups of index \( p \)'. Further observe that \( \text{Hom}(G, \mathbb{Z}/p\mathbb{Z}) = (0) \) implies \( \text{Hom}(G, \mathbb{Z}_p) = (0) \).

2. The groups \( \mathbb{Q}_p, \mathbb{Q}_p/\mathbb{Z}_p \) have the property that each nontrivial discrete quotient is an infinite torsion group. Thus, for such groups, each representative function is a linear combination of characters, for any choice of \( K \).

3. In [5] necessary and sufficient conditions are derived on \( G, K \) in order that \( G^* \) be an orthonormal set. It is a striking fact that these are the same as in Theorem 2.2 but where \((\delta)'\) and \((\delta)''' \) are interchanged!

§3. REPRESENTATIVE FUNCTIONS ON \( \mathbb{Z}_p \)

From the previous theory it follows that a representative function \( \mathbb{Z}_p \rightarrow K \) is a linear combination of characters if \( K \) is archimedean and also if \( K \) is non-archimedean and \( \text{char } k \neq p \). So one may be interested in a description of \( \mathcal{R}(\mathbb{Z}_p \rightarrow K) \) for the remaining case \( \text{char } k = p \). We shall prove the following theorem.

Theorem 3.1. Let \( f : \mathbb{Z}_p \rightarrow K \).

(i) Let \( K \supset \mathbb{Q}_p \). Then \( f \in \mathcal{R}(\mathbb{Z}_p \rightarrow K) \) if and only if \( f \) has the form

\[
(*) \quad f = \sum_{i=1}^{n} P_i \alpha_i
\]

where \( n \in \mathbb{N} \), \( P_1, \ldots, P_n \) are polynomial functions, and \( \alpha_1, \ldots, \alpha_n \) are characters.

(ii) Let \( \text{char } K = \text{char } k = p \). Then \( f \in \mathcal{R}(\mathbb{Z}_p \rightarrow K) \) if and only if \( f \) has the form

\[
(**) \quad f = \sum_{i=1}^{n} L_i \alpha_i
\]

where \( n \in \mathbb{N} \), \( L_1, \ldots, L_n \) are locally constant functions and \( \alpha_1, \ldots, \alpha_n \) are characters.

To prove this theorem we need a few lemmas. For technical reasons we shall say that an \( f : \mathbb{Z}_p \rightarrow K \) is a polycharacter if it has the form \((*)\) if \( K \supset \mathbb{Q}_p \) or the form \((**)\) if \( \text{char } K = \text{char } k = p \). Then Theorem 3.1 reads in short: \( f \in \mathcal{R}(\mathbb{Z}_p \rightarrow K) \Longleftrightarrow f \) is a polycharacter.

One half is easy:

Lemma 3.2. Let \( \text{char } k = p \). Each polycharacter \( \mathbb{Z}_p \rightarrow K \) is a representative function.

Proof. If \( K \supset \mathbb{Q}_p \) the function \( x \rightarrow x \) is an additive homomorphism and therefore is a representative function. For any \( K \), a locally constant function on \( \mathbb{Z}_p \) is constant on cosets of \( p^m \mathbb{Z}_p \) for some \( m \) so its translates generate a space whose dimension is \( \leq p^m \). Now the lemma follows after observing that
For the second half of Theorem 3.1 we introduce the following. A function $f : \mathbb{N} \to K$ can be interpolated if there exists a (unique) continuous function $\tilde{f} : \mathbb{Z}_p \to K$ whose restriction to $\mathbb{N}$ is $f$. We need the following result. (As usual, the symbol $[ ]$ indicates the entire part.)

**Lemma 3.3.** Let $\text{char } k = p \neq 0$.

(i) For $a \in K$, $a \neq 0$, the sequence $n \mapsto a^n$ can be interpolated if and only if $|a - 1| < 1$.

(ii) For a continuous function $f : \mathbb{Z}_p \to K$ the sequence

$$n \mapsto f(0) + f(1) + \ldots + f(n-1)$$

can be interpolated.

(iii) For each $m \in \mathbb{N}$ the sequence $n \mapsto [\frac{x}{p^m}]$, considered as a map $\mathbb{N} \to \mathbb{Q}_p$ can be interpolated to a function $x \mapsto x - [\frac{x}{p^m}]p^m$ (for $x \in \mathbb{Z}_p$) is locally constant.

**Proof.**

(i) See [4], Theorem 32.4.

(ii) See [4], Theorem 34.1 (the assumption $K \supseteq \mathbb{Q}_p$ is not used in that proof).

(iii) Without trouble one verifies that

$$x \mapsto [\frac{x}{p^m}] := a_m + a_{m+1}p + a_{m+2}p^2 + \ldots$$

where $x = \sum_{i=0}^{\infty} a_i p^i$ is the standard $p$-adic expansion of $x$, is the required extension.

For the continuous extension $x \mapsto a^x$ ($x \in \mathbb{Z}_p$) of $n \mapsto a^n$ in Lemma 3.3(i) we shall also write $a^x$. The continuous extension of $n \mapsto f(0) + f(1) + \ldots + f(n-1)$ is called the *indefinite sum* of $f$, denoted by $Sf$.

Observe that

$$S(f) - Sf = f - f(0)$$

**Lemma 3.4.** Let $\text{char } k = p$. The indefinite sum of a polycharacter $\mathbb{Z}_p \to K$ is again a polycharacter.

**Proof.** We consider two cases.

(i) $K \supseteq \mathbb{Q}_p$. It is not hard to see that the indefinite sum of a polynomial function is again a polynomial function. By linearity it therefore suffices to prove that for each $j \in \{0, 1, 2, \ldots\}$ and each $a \in K$ with $0 < |1-a| < 1$ the function

$$S(f_j) - Sf = f - f(0)$$
\[ S(\omega^i a^*) \]

where \( \omega^i \) is the polynomial \( x \mapsto x^i \), is a polycharacter. We shall do this by proving the following statement (*) by induction on \( j \).

There is a polynomial function \( P_j \) of degree \( \leq j \), whose coefficients are rational functions of \( a \) and there is a rational function \( Q_j \) of \( a \) such that for all \( n \in \mathbb{N} \) and all \( a \in K \) with \( 0 < |1 - a| < 1 \)

\[
S(\omega^i a^*)(n) = P_j(n) a^n + Q_j(a)
\]

For the case \( j = 0 \) observe that

\[
S(a^*)(n) = a^0 + a^1 + \ldots + a^{n-1} = \frac{1}{a-1} a^n + \frac{1}{1-a}
\]

So, (*) holds with \( P_0(n) = \frac{1}{a-1}, Q_0(a) = \frac{1}{1-a} \).

Now suppose we have (*) for some \( j \):

\[
S(\omega^{i+1} a^*)(n) = \sum_{i=0}^{n-1} i i^i a^k = a \frac{d}{da} \sum_{i=0}^{n-1} i i^i
\]

Then

\[
S(\omega^{i+1} a^*)(n) = \sum_{i=0}^{n-1} i i^i a^k = a \frac{d}{da} \sum_{i=0}^{n-1} i i^i
\]

So, if we take

\[
P_{j+1}(n) := a \frac{d}{da} P_j(n) + P_j(n)
\]

\[
Q_{j+1}(a) = a \frac{d}{da} Q_j(a)
\]

then (*) holds for \( j + 1 \) in place of \( j \).

(ii) \( \text{char} K = p \). First we prove that \( Sf \) is a polycharacter for

\[ f = \xi_{p^n Z_p} \alpha \]

where \( m \in \mathbb{N} \), where \( \xi_{p^n Z_p} \) is the \( K \)-valued characteristic function of \( p^n Z_p \) and where \( \alpha \) is a character.

We have for \( n \in \mathbb{N} \)

\[
(Sf)(n) = \sum_{i=0}^{n-1} \xi_{p^n Z_p}(i) \alpha(i) = \sum_{j=0}^{[p^n(n-1)]} \alpha(p^n j)
\]

If \( \alpha(p^n) = 1_K \), the unit element of \( K \), we obtain
\[(Sf)(n) = \left(\frac{n}{p^n} + 1\right) \cdot 1_K\]

and we see that \(Sf\) is a locally constant function.

If \(\alpha(p^n) \neq 1_K\), then \(\alpha(x) = a^x\) \((x \in \mathbb{Z}_p)\) where \(a \in K\), \(0 < |1_K - a| < 1\). We have, for \(n \in \mathbb{N}\)

\[
(Sf)(n) = \frac{\alpha(p^n)^{\frac{n-1}{p^n}}}{\alpha(p^n) - 1_K} = \frac{\frac{n-1}{p^n}}{a^{\frac{n-1}{p^n}} - 1_K}
\]

It follows that \(Sf\) is a \(K\)-linear combination of a constant function and the function

\[x \mapsto a^\frac{x-1}{p^n}\]

which is the product of the character \(a^{-x}\) and a locally constant function (Lemma 3.3(iii)). Thus, we may conclude that \(S(\xi_{p^n\mathbb{Z}_p} a)\) is a polycharacter.

By linearity of \(S\) and by the remark preceding this lemma the set of all polycharacters \(f\) for which \(Sf\) is also a polycharacter is a linear space, invariant under translations. A standard reasoning shows that the smallest translation invariant linear space containing all \(\xi_{p^n\mathbb{Z}_p} a\) \((m \in \mathbb{N}, \alpha \text{ character})\) is the set of all polycharacters which finishes the proof.

**Lemma 3.5.** Let \(\text{char} k = p\), let \(a \in K, |1-a| < 1\). If \(f : \mathbb{Z}_p \to K\) is a polycharacter and if \(g\) is a continuous solution of

\[g(x+1) - ag(x) = f(x) \quad (x \in \mathbb{Z}_p)\]

then \(g\) is a polycharacter.

**Proof.** Inductively we arrive easily at

\[g(n) = a^n g(0) + a^{n-1} S(a^{-x}f)(n) \quad (n \in \mathbb{N})\]

By continuity,

\[g = a^n g(0) + a^{n-1} S(a^{-x}f)\]

which is a polycharacter by Lemma 3.4.

Let \(L\) denote the operator \(BC(\mathbb{Z}_p \to K) \to BC(\mathbb{Z}_p \to K)\) sending \(f\) into \(f_1\) (recall that \(f_1(x) = f(x+1)\)).

**Lemma 3.6.** If, for some \(a \in K\), the operator \(L - af\) is not injective then \(|a-1| < 1\).

**Proof.** Let \(f \in BC(\mathbb{Z}_p \to K), f \neq 0\) be such that \(Lf - af = 0\). Then \(f(x+1) = af(x)\) for all \(x \in \mathbb{Z}_p\) so that \(f(n) = a^n f(0)\) for all \(n \in \mathbb{N}\). We have \(f(0) \neq 0\) and, by continuity of \(f\), the sequence \(x \mapsto a^n\) can be interpolated. By Lemma 3.3(i), \(|1-a| < 1\).
Proof of Theorem 3.1. Let $f$ be a representative function, $f \neq 0$; we shall prove that $f$ is a polycharacter. The sequence $f, Lf, L^2 f, \ldots$ lies in a finite dimensional space so there is an $n \in \mathbb{N}$ such that $L^nf$ is a $K$-linear combination of $f, Lf, \ldots, L^{n-1}f$. We may choose $n$ minimal. In other words, we have a monic polynomial $P \in K[X]$ with $P(L)(f)=0$ with minimal degree $n$. As $K$ is algebraically closed $P$ decomposes into linear factors $X-a_1, \ldots, X-a_n$ so we have

$$(L-a_1 I)(L-a_2 I) \ldots (L-a_n I)(f) = 0$$

The operators $L-a_i$ commute and $n$ is minimal so no $L-a_i I$ is injective. By Lemma 3.6, $|a_i-1|<1$ for $i \in \{1, \ldots, n\}$.

Lemma 3.5, applied for $a=a_1, g=(L-a_2 I), \ldots (L-a_n I)f$ and $f=0$ yields

$$(L-a_2 I)(L-a_2 I) \ldots (L-a_n I)f = g$$

where $g$ is a polycharacter. By repeated application of Lemma 3.5 we can remove all $L-a_i I$ obtaining that $f$ is a polycharacter.

Note. For results on closely related matters see [1].

References


