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# TWO ELEMENTARY PROOFS OF KATSARAS' THEOREM ON P-ADIC COMPACTOIDS

by

S. Caenepeel, W.H. Schikhof

## 0. Introduction

The following 'convexification' of the notion of precompactness plays a central role in p-adic Functional Analysis. Let K be a nonarchimedean nontrivially valued field, and E a locally K-convex space. An absolutely convex subset A of E is called compactoid if for every (absolutely convex) neighbourhood U of 0 in E, there exists a finite subset $S = \{x_1, \ldots, x_n\}$ of E such that $A \subset co(S) + U$. Here co(S) denotes the absolute convex hull of S. Equivalently, we can say : for every absolutely convex neighbourhood U of 0, $\pi_U(A)$ is contained in a finitely generated R-module ; here R is the unit ball in K, and $\pi_U$ is the canonical map $E \to E/U$ in the category of R-modules. A natural question to ask is the following : can we choose S to be subset of A ? Or, equivalently, is $\pi_U(A)$ finitely generated as an R-module ? The answer is affirmative if the valuation of K is discrete, because R is a noetherian ring in that case. If the valuation is dense, then we have an easy counterexample : take $A = \{\lambda \in K : |\lambda| < 1\}$.
It is shown in [3] that, for E a Banach space, one may choose $x_1, \ldots, x_n$ in $\lambda A$, where $\lambda \in K$, $|\lambda| > 1$. For locally convex E it is shown in [1] that it is possible to choose $x_1, \ldots, x_n$ in the K-vector space generated by A, and in [2], [4] that $x_1, \ldots, x_n$ may be chosen in $\lambda A$. Yet, all these proofs are somewhat involved. In this note, both authors present a straightforward and elementary proof. We considered it worth wile to publish our two proofs, since the statement is quite fundamental.

## 1. Proof by the Second Author

1.1. Lemma. Let $A$, $B$ be absolutely convex subsets of a K-vector space $E$. Suppose $A \subset B + co\{x\}$ for some $x \in E$. Let $\lambda \in K$, $0 < |\lambda| < 1$ if the valuation of $K$ is dense, $\lambda = 1$ otherwise. Then there exists an $a \in A$ such that $\lambda A \subset B + co\{a\}$.

Proof. The set $C \subset K$ defined by $C = \{\mu \in K : |\mu| \leq 1, \mu x \in A+B\}$ is absolutely convex. It is not hard to see that there exists a $c \in C$ for which $\lambda C \subset co\{c\} \subset C$. As $c \in C$ there exists an $a \in A$ such that $cx \in a + B$. We claim that $\lambda A \subset B + co\{a\}$. Indeed, if $z \in A$ then $z = b + dx$ for some $b \in B$, $d \in C$ so we have $\lambda z = \lambda b + \lambda dx \in B + co\{cx\} \subset B + co(a + B) \subset B + co\{a\}$. □

1.2. Lemma. Let $E$, $A$, $B$, $\lambda$ be as above. Suppose $A \subset B + co\{x_1, \ldots, x_n\}$ for some $x_1, \ldots, x_n \in E$. Then there exist $a_1, \ldots, a_n \in A$ such that $\lambda A \subset B + co\{a_1, \ldots, a_n\}$.

Proof. Choose $\lambda_1, \ldots, \lambda_n \in K$, $0 < |\lambda_i| < 1$ and $|\Pi_{i=1}^n \lambda_i| \geq |\lambda|$ if the valuation of $K$ is dense, $\lambda_i = 1$ for each $i$ otherwise. By applying Lemma 1.1 with $\lambda_1$ in place of $\lambda$ and $B + co\{x_2, \ldots, x_n\}$ in place of $B$ we find an $a_1 \in A$ such that $\lambda_1 A \subset B + co\{a_1, x_2, \ldots, x_n\}$.
A second application of Lemma 1.1 with $\lambda_1 A$, $\lambda_2$, $B + co\{a_1, x_3, \ldots, x_n\}$ in place of $A$, $\lambda$, $B$ respectively yields an $a_2 \in \lambda_1 A \subset A$ for which $\lambda_1 \lambda_2 A \subset B + co\{a_1, a_2, x_3, \ldots, x_n\}$. Inductively we arrive at points $a_1, \ldots, a_n \in A$ such that $\lambda A \subset \lambda_1 \ldots \lambda_n A \subset B + co\{a_1, \ldots, a_n\}$. □

1.3. Theorem (Katsaras). Let $A$ be an absolutely convex compactoid in a locally convex space over $K$. Let $\lambda \in K$, $|\lambda| > 1$ if the valuation of $K$ is dense, $\lambda = 1$ otherwise. Then for each absolutely convex neighbourhood $U$ of $0$ in $E$ there exist $x_1, \ldots, x_n \in \lambda A$ such that $A \subset U + co\{x_1, \ldots, x_n\}$.

Proof. $\lambda^{-1}U$ is a zero neighbourhood. By definition there exist $y_1, \dots, y_n \in E$ such that $A \subset \lambda^{-1}U + co\{y_1, \dots, y_n\}$. By Lemma 1.2 we can find $a_1, \dots, a_n \in A$ such that $\lambda^{-1}A \subset \lambda^{-1}U + co\{a_1, \dots, a_n\}$, i.e. $A \subset U + co\{x_1, \dots, x_n\}$, where, for each i, $x_i = \lambda a_i \in \lambda A$. □

## 2. Proof by the First Author

In the introduction, we have seen that Theorem 1.3 is trivial if the valuation of K is discrete ; so let us assume from now on that $|K|$ is dense.

2.1. Lemma. Let A be an R-submodule of a finitely generated free R-module, and let $\lambda \in R$ be such that $|\lambda| < 1$. Then we can find $a_1, \dots, a_n \in A$ such that $\lambda A \subset Ra_1 + \dots + Ra_n$.

Proof. $A \subset R^n \subset K^n$. We furnish $K^n$ with the usual supremum norm ; it is well-known (cf. [3]) that every one dimensional subspace of $K^n$ has an orthocomplement. Let us proceed using induction on n. The case n = 1 is trivial. Let $m = \sup\{ \|x\| : x \in A\}$, and choose $a_1 \in A$ such that $\|a_1\| \geq |\lambda'|m$, where $\lambda' \in K$ is such that $|\lambda'|^2 < |\lambda|$. Let $Q : K^n \to Ka_1$ be an orthoprojection, and take $P = I - Q$. Then every $x \in K^n$ may be written under the form $x = \lambda(x)a_1 + Px$, where $\|x\| = \max(|\lambda(x)|\|a_1\|, \|Px\|)$. If $x \in A$, then $|\lambda(x)|\|a_1\| \leq \|x\| \leq m \leq |\lambda'|^{-1}\|a_1\|$, so $|\lambda(x)| \leq |\lambda'|^{-1}$.
Using the induction hypothesis, we find $f_2, \dots, f_n \in PA$ such that $\lambda' PA \subset Rf_2 + \dots + Rf_n$. Lift $f_i$ to an element $a_i \in A$. Then, for $i \geq 2$, we have that $a_i = f_i + \lambda_i a_1$, where $|\lambda_i| \leq |\lambda'|^{-1}$. We now have, for $x \in A$ :
$x = Qx + Px = \lambda(x)a_1 + \Sigma_{i=2}^n \mu_i f_i = (\lambda(x) - \Sigma_{i=2}^n \lambda_i\mu_i)a_1 + \Sigma_{i=2}^n \mu_i a_i$, where $|\lambda(x)|, |\lambda_i|, |\mu_i| \leq |\lambda'|^{-1}$. This implies the result. □

Proof of Theorem 1.3. Write $\mu = \lambda^{-1}$, then $|\mu| < 1$. U is an absolutely convex neighbourhood of 0, so $\pi_{\mu U}(A)$ is a submodule of a finitely generated R-module N. So we have an epimorphism $\phi : R^n \to N$ in the category of

R-modules. By Lemma 2.1, we may find $a_1,\dots, a_n \subset \phi^{-1}(\pi_{\mu U}(A))$ such that $\mu\phi^{-1}(\pi_{\mu U}(A)) \subset Ra_1+\dots+Ra_n$. Choose $u_1,\dots, u_n$ in A such that $\pi_{\mu U}(u_i) = \phi(a_i)$. Then $\mu\pi_{\mu U}(A) \subset R\phi(a_1) + \dots + R\phi(a_n) = R\pi_{\mu U}(u_1) + \dots + R\pi_{\mu U}(u_n)$, hence $\mu A \subset Ru_1 + \dots + Ru_n + \mu U$, and, after multiplication by $\lambda$, $A \subset R\lambda u_1 + \dots R\lambda u_n + U$, and this proves the theorem. □

## References

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