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TWO ELEMENTARY PROOFS OF KATSARAS' THEOREM ON P-ADIC
COMPACTOIDS

by

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0. Introduction

The following 'convexification' of the notion of precompactness plays a central role in p-adic Functional Analysis. Let K be a nonarchimedean nontrivially valued field, and E a locally K -convex space. An absolutely convex subset A of E is called compactoid if for every (absolutely convex) neighbourhood U of 0 in E , there exists a finite subset $S = \{x_1, \dots, x_n\}$ of E such that $A \subset \text{co}(S) + U$. Here $\text{co}(S)$ denotes the absolute convex hull of S . Equivalently, we can say : for every absolutely convex neighbourhood U of 0 , $\pi_U(A)$ is contained in a finitely generated R -module ; here R is the unit ball in K , and π_U is the canonical map $E \rightarrow E/U$ in the category of R -modules. A natural question to ask is the following : can we choose S to be subset of A ? Or, equivalently, is $\pi_U(A)$ finitely generated as an R -module ? The answer is affirmative if the valuation of K is discrete, because R is a noetherian ring in that case. If the valuation is dense, then we have an easy counterexample : take $A = \{\lambda \in K : |\lambda| < 1\}$.

It is shown in [3] that, for E a Banach space, one may choose x_1, \dots, x_n in λA , where $\lambda \in K$, $|\lambda| > 1$. For locally convex E it is shown in [1] that it is possible to choose x_1, \dots, x_n in the K -vector space generated by A , and in [2], [4] that x_1, \dots, x_n may be chosen in λA . Yet, all these proofs are somewhat involved. In this note, both authors present a straightforward and elementary proof. We considered it worth while to publish our two proofs, since the statement is quite fundamental.

1. Proof by the Second Author

1.1. Lemma. Let A, B be absolutely convex subsets of a K -vector space E . Suppose $A \subset B + \text{co}\{x\}$ for some $x \in E$. Let $\lambda \in K$, $0 < |\lambda| < 1$ if the valuation of K is dense, $\lambda = 1$ otherwise. Then there exists an $a \in A$ such that $\lambda A \subset B + \text{co}\{a\}$.

Proof. The set $C \subset K$ defined by $C = \{\mu \in K : |\mu| < 1, \mu x \in A+B\}$ is absolutely convex. It is not hard to see that there exists a $c \in C$ for which $\lambda C \subset \text{co}\{c\} \subset C$. As $c \in C$ there exists an $a \in A$ such that $cx \in a + B$. We claim that $\lambda A \subset B + \text{co}\{a\}$. Indeed, if $z \in A$ then $z = b + dx$ for some $b \in B$, $d \in C$ so we have $\lambda z = \lambda b + \lambda dx \in B + \text{co}\{cx\} \subset B + \text{co}\{a + B\} \subset B + \text{co}\{a\}$. \square

1.2. Lemma. Let E, A, B, λ be as above. Suppose $A \subset B + \text{co}\{x_1, \dots, x_n\}$ for some $x_1, \dots, x_n \in E$. Then there exist $a_1, \dots, a_n \in A$ such that $\lambda A \subset B + \text{co}\{a_1, \dots, a_n\}$.

Proof. Choose $\lambda_1, \dots, \lambda_n \in K$, $0 < |\lambda_i| < 1$ and $|\prod_{i=1}^n \lambda_i| > |\lambda|$ if the valuation of K is dense, $\lambda_i = 1$ for each i otherwise. By applying Lemma 1.1 with λ_1 in place of λ and $B + \text{co}\{x_2, \dots, x_n\}$ in place of B we find an $a_1 \in A$ such that $\lambda_1 A \subset B + \text{co}\{a_1, x_2, \dots, x_n\}$.

A second application of Lemma 1.1 with $\lambda_1 A$, λ_2 , $B + \text{co}\{a_1, x_3, \dots, x_n\}$ in place of A , λ , B respectively yields an $a_2 \in \lambda_1 A \subset A$ for which $\lambda_1 \lambda_2 A \subset B + \text{co}\{a_1, a_2, x_3, \dots, x_n\}$. Inductively we arrive at points $a_1, \dots, a_n \in A$ such that $\lambda A \subset \lambda_1 \dots \lambda_n A \subset B + \text{co}\{a_1, \dots, a_n\}$. \square

1.3. Theorem (Katsaras). Let A be an absolutely convex compactoid in a locally convex space over K . Let $\lambda \in K$, $|\lambda| > 1$ if the valuation of K is dense, $\lambda = 1$ otherwise. Then for each absolutely convex neighbourhood U of 0 in E there exist $x_1, \dots, x_n \in \lambda A$ such that $A \subset U + \text{co}\{x_1, \dots, x_n\}$.

Proof. $\lambda^{-1}U$ is a zero neighbourhood. By definition there exist $y_1, \dots, y_n \in E$ such that $A \subset \lambda^{-1}U + \text{co}\{y_1, \dots, y_n\}$. By Lemma 1.2 we can find $a_1, \dots, a_n \in A$ such that $\lambda^{-1}A \subset \lambda^{-1}U + \text{co}\{a_1, \dots, a_n\}$, i.e. $A \subset U + \text{co}\{x_1, \dots, x_n\}$, where, for each i , $x_i = \lambda a_i \in \lambda A$. \square

2. Proof by the First Author

In the introduction, we have seen that Theorem 1.3 is trivial if the valuation of K is discrete ; so let us assume from now on that $|K|$ is dense.

2.1. Lemma. Let A be an R -submodule of a finitely generated free R -module, and let $\lambda \in R$ be such that $|\lambda| < 1$. Then we can find $a_1, \dots, a_n \in A$ such that $\lambda A \subset Ra_1 + \dots + Ra_n$.

Proof. $A \subset R^n \subset K^n$. We furnish K^n with the usual supremum norm ; it is well-known (cf. [3]) that every one dimensional subspace of K^n has an orthocomplement. Let us proceed using induction on n . The case $n = 1$ is trivial. Let $m = \sup\{\|x\| : x \in A\}$, and choose $a_1 \in A$ such that $\|a_1\| > \frac{1}{2}|\lambda'|m$, where $\lambda' \in K$ is such that $|\lambda'|^2 < |\lambda|$. Let $Q : K^n \rightarrow Ka_1$ be an orthoprojection, and take $P = I - Q$. Then every $x \in K^n$ may be written under the form $x = \lambda(x)a_1 + Px$, where $\|x\| = \max(|\lambda(x)|\|a_1\|, \|Px\|)$. If $x \in A$, then $|\lambda(x)|\|a_1\| \leq \|x\| \leq m < |\lambda'|^{-1}\|a_1\|$, so $|\lambda(x)| < |\lambda'|^{-1}$.

Using the induction hypothesis, we find $f_2, \dots, f_n \in PA$ such that $\lambda'PA \subset Rf_2 + \dots + Rf_n$. Lift f_i to an element $a_i \in A$. Then, for $i \geq 2$, we have that $a_i = f_i + \lambda_i a_1$, where $|\lambda_i| < |\lambda'|^{-1}$. We now have, for $x \in A$:
 $x = Qx + Px = \lambda(x)a_1 + \sum_{i=2}^n \mu_i f_i = (\lambda(x) - \sum_{i=2}^n \lambda_i \mu_i) a_1 + \sum_{i=2}^n \mu_i a_i$, where $|\lambda(x)|, |\lambda_i|, |\mu_i| < |\lambda'|^{-1}$. This implies the result. \square

Proof of Theorem 1.3. Write $\mu = \lambda^{-1}$, then $|\mu| < 1$. U is an absolutely convex neighbourhood of 0, so $\pi_{\mu U}(A)$ is a submodule of a finitely generated R -module N . So we have an epimorphism $\phi : R^n \rightarrow N$ in the category of

R -modules. By Lemma 2.1, we may find $a_1, \dots, a_n \in \phi^{-1}(\pi_{\mu U}(A))$ such that $\mu\phi^{-1}(\pi_{\mu U}(A)) \subset Ra_1 + \dots + Ra_n$. Choose u_1, \dots, u_n in A such that $\pi_{\mu U}(u_i) = \phi(a_i)$. Then $\mu\pi_{\mu U}(A) \subset R\phi(a_1) + \dots + R\phi(a_n) = R\pi_{\mu U}(u_1) + \dots + R\pi_{\mu U}(u_n)$, hence $\mu A \subset Ru_1 + \dots + Ru_n + \mu U$, and, after multiplication by λ , $A \subset R\lambda u_1 + \dots + R\lambda u_n + U$, and this proves the theorem. \square

References

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