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WEAK $C^*$-COMPACTNESS IN $p$-ADIC BANACH SPACES

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ABSTRACT. Let \( X \) be a subset of a locally convex space \( E \) of countable type over a nonarchimedean densely valued field \( K \). If, for each continuous linear function \( f : E \to K \), \( \max_{X} |f| \) exists then \( X \) is a compactoid in \( E \).
§ 1. PRELIMINARIES

Throughout, \( K \) is a nonarchimedean nontrivially valued complete field with valuation \(| |\). For fundamentals on Banach spaces and locally convex spaces over \( K \) we refer to [3], [6], [4]. We recall a few definitions and facts and fix some notations. Let \( E \) be a locally convex space over \( K \). \( E' \) is the linear space of all continuous linear functions \( E \to K \). If \( E = (E, || ||) \) is a nonzero Banach space and \( f \in E' \) then
\[
||f|| := \sup \{ |f(x)|/||x|| : x \in E, x \neq 0 \}.
\]
\( E \) is a polar space if there exists a base of continuous seminorms \( p \) for which
\[
p = \sup \{ |f| : f \in E', |f| \leq p \}.
\]

**PROPOSITION 1.1.**

Let \( E \) be a Banach space over \( K \) with a base. Then \( E \) is a polar space.

**Proof.**

By [3], Corollary 3.7 there exists a norm \( || || \) inducing the topology of \( E \) for which \( E \) has an orthogonal base \( \{ e_i : i \in I \} \). It is not hard to see that we even may assume that \( \{ e_i : i \in I \} \) is orthonormal. For each \( i \in I \), let \( f_i \) be the \( i \)-th coordinate function. Let \( x \in E \). Then
\[
x = \sum_{i \in I} f_i(x)e_i \quad \text{and} \quad ||x|| = \max_{i \in I} |f_i(x)|.
\]

For a subset \( X \) of a locally convex space \( E \) over \( K \) we denote by \( \text{co} X \) its absolutely convex hull, by \( [X] \) its \( K \)-linear span, by \( \overline{X} \) its closure. Instead of \( \text{co} X \) we write \( \text{co} X \). For an absolutely convex \( A \subset E \) the formula
\[
p_A(x) = \inf \{ |\lambda| : x \in \lambda A \}
\]
defines, on \( [A] \), the seminorm \( p_A \) associated to \( A \).
PROPOSITION 1.2.

Let $A \subset E$ be absolutely convex. Then

$$\{x \in [A] : p_A(x) < 1\} \subset A \subset \{x \in [A] : p_A(x) \leq 1\}.$$ 

Proof.

Straightforward.

An absolutely convex $A \subset E$ is **edged** if for each $x \in E$ the set

$$\{|\lambda| : \lambda x \in A\}$$

is closed in $|K| := \{|\lambda| : \lambda \in K\}$. $A$ is edged if and only if $A = \{x \in [A] : p_A(x) \leq 1\}$. A subset $X$ of $E$ is a **compactoid** if for each neighbourhood $U$ of $0$ in $E$ there exists a finite set $F \subset E$ such that $X \subset U^\circ \cup F$. For some elementary properties of compactoids, see [2].
§ 2. BANACH SPACES WITH A BASE

In § 2 we assume that the valuation of $K$ is dense.

**Lemma 2.1.**
The closed absolutely convex hull of an orthonormal set in a Banach space over $K$ is edged.

**Proof.**
For an orthonormal set $\{e_i : i \in I\}$, set $A := \overline{\text{co}} \{e_i : i \in I\}$, $D := \overline{\{e_i : i \in I\}}$. Without any trouble one verifies that $D = [A]$, $p_A = \|\|_A$ on $D$ and $A = \{x \in D : \|x\| \leq 1\}$.

**Lemma 2.2.**
Let $E$ be a Banach space of countable type over $K$. Let $A$ be an absolutely convex neighbourhood of $0$ in $E$ and suppose that, for each $f \in E'$, the restriction of $|f|$ to $A$ has a maximum.

(i) $A$ is bounded; $p_A$ is a norm $\|\|$ inducing the topology of $E$.

(ii) Let $\{e_i : i \in I\}$ be a maximal $\|\|_A$-orthonormal set in

$$A_m := \{x \in A : \text{there is an } f \in E' \text{ with } |f(x)| = \max_A |f|\}.$$ 

Then $A = \overline{\text{co}} \{e_i : i \in I\}$.

**Proof.**
(i) $A$ is weakly bounded hence bounded by [4], Corollary 7.7. The interior of $A$ is not empty and $A$ is an additive group so $A$ is open (and closed), and (i) follows.

(ii) $B := \overline{\text{co}} \{e_i : i \in I\}$ is contained in $A$. Suppose $B \neq A$; we shall prove
that \(\{e_i : i \in I\}\) is not maximal yielding a contradiction. The set \(B\) is closed and edged (Lemma 2.1) so by [4], Proposition 4.8 and 3.4, there exists an \(f \in E'\) with \(|f| \leq 1\) on \(B\) and \(|f(y)| > 1\) for some \(y \in A\). Then \(\max_{A} |f| > 1\) and after multiplying \(f\) by a suitable scalar we obtain a \(g \in E'\) for which

\[
|g| < 1 \text{ on } B, \quad \max_{B} |g| = 1.
\]

From

\[
(*) \quad \{x \in E : ||x|| < 1\} \subset A \subset \{x \in E : ||x|| \leq 1\}
\]

(Proposition 1.2) it follows that \(||g|| = 1\). Choose an \(a \in A\) with \(|g(a)| = 1\). We claim that \(\{a\} \cup \{e_i : i \in I\}\) is an orthonormal set in \(A_m\). In fact, we have \(a \in A_m\). By \((*)\), \(||a|| \leq 1\) but also

\[
1 = |g(a)| \leq ||g|| \cdot ||a|| = ||a|| \quad \text{so that } ||a|| = 1.
\]

To prove orthogonality it suffices to show that for a \(K\)-linear combination 

\[
z = \sum_{i \in F} \lambda_i e_i \quad (F \subset I, \text{ } F \text{ finite})
\]

we have

\[
||a-z|| \geq ||a||.
\]

If \(\max |\lambda_i| > 1\) this is an obvious consequence of the strong triangle inequality so assume \(\max |\lambda_i| \leq 1\). Then \(z \in B\) so \(|g(z)| < 1\) and

\[
||a|| = 1 = |g(a)| = |g(a-z)| \leq ||g|| \cdot ||a-z|| = ||a-z||
\]

which finishes the proof.

Remark.

The above proof works for a strongly polar ([4], Definition 3.5) Banach space \(E\), in particular for any Banach space over a spherically complete \(K\).

Two corollaries obtain. The first one we shall need later on. The
second one is rather surprising.

PROPOSITION 2.3.

Let $A$ be an absolutely convex subset of a finite dimensional space $E$ over $K$. The following are equivalent.

(a) For each $f \in E'$, max $|f|$ exists.

(b) For each seminorm $p$ on $E$, max $p$ exists.

(c) There exists a finite set $F \subset A$ with $A = \text{co } F$.

Proof.

$(\gamma) \Rightarrow (\beta)$ is easy, $(\beta) \Rightarrow (\alpha)$ is trivial. To prove $(\alpha) \Rightarrow (\gamma)$ we may assume $[A] = E$. Then $A$ is open (with respect to the unique Banach space topology of $E$). Lemma 2.2 (ii) yields a (finite) set $F$ with $A = \text{co } F = \text{co } F$ (the second equality because each convex set in $E$ is closed).

LEMMA 2.4.

Let $E, A$ be as in Lemma 2.2. Then $E$ is finite dimensional.

Proof.

Suppose $E$ is not finite dimensional. From Lemma 2.2 we infer that $E$ has an orthonormal base $\{e_i : i \in \mathbb{N}\}$ (with respect to the norm $\| \|$) and that $A = \text{co } \{e_i : i \in \mathbb{N}\}$ is the 'closed' unit ball of $(E, \| \|)$. Now choose $\tau_1, \tau_2, \ldots$ in $K$ such that $0 < |\tau_1| < |\tau_2| < \ldots$, $\lim_{n \to \infty} |\tau_n| = 1$ and define $f \in E'$ by the formula $f(\sum_{i \in \mathbb{N}} \lambda_i e_i) = \sum_{i \in \mathbb{N}} \lambda_i \tau_i$ ($\lambda_i \in K$, $\lim_{i \to \infty} \lambda_i = 0$).

We have $\sup_{A} |f| = 1$ but $|f(x)| < 1$ for each $x \in A$, a contradiction.

The proof of the next Lemma is left to the reader.
LEMMA 2.5.

Let $A$ be an absolutely convex subset of a locally convex space $E$ over $K$. Suppose $\max_{A} |f|$ exists for each $f \in E'$.

(i) For each $f \in E'$, $\max_{A} |f|$ exists.

(ii) If $T : E \to F$ is a continuous linear map into a locally convex space $F$ over $K$ then, for each $f \in F'$, $\max_{A} |f|$ exists.

LEMMA 2.6.

Let $E$ be a Banach space over $K$ with a base. Let $A \subset E$ be absolutely convex and suppose that for each $f \in E'$ the restriction of $|f|$ to $A$ has a maximum. Then $A$ is a compactoid.

Proof.

We may assume ([3], Corollary 3.7) that $E$ has an orthogonal base.

Suppose $A$ is not a compactoid. By [3], Theorem 4.38, ($\zeta$) $\Rightarrow$ (a) there exists an orthogonal sequence $e_1, e_2, \ldots$ in $A$ with $\inf ||e_n|| > 0$. Set $D := \{e_1, e_2, \ldots\}$, and choose a linear continuous projection $P : E \to D$ ([3], Corollary 3.18). By Lemma 2.5, for each $f \in D'$ the restriction of $|f|$ to $PA$ has a maximum. Also $PA \supset \text{co} \{e_1, e_2, \ldots\}$ is open in $D$ and $D$ is infinite dimensional. But this is impossible (Lemma 2.4).

We now formulate the main Theorem of this section.

THEOREM 2.7.

Let $E$ be a Banach space with a base over (a densely valued) $K$. For a nonempty subset $X$ of $E$ the following are equivalent.

(a) For each $f \in E'$, $\max_{X} |f|$ exists.

(\beta) For each weakly continuous seminorm $p$, $\max_{X} p$ exists.

(\gamma) For each weak neighbourhood $U$ of $0$ there exists a finite set $P \subset X$...
such that $X \subseteq U + \text{co } F$.

(δ) For each continuous seminorm $p$ on $E$, $\max_X p$ exists.

(ε) For each neighbourhood $U$ of 0 there exists a finite set $F \subseteq X$ such that $X \subseteq U + \text{co } F$.

Proof.

Let (*) be any of the statements (α) - (ε). It is not hard to see that, for a nonempty set $Y \subseteq E$,

(*) holds for $X := Y$ ⇒ (*) holds for $X := \text{co } Y$

Therefore, to prove Theorem 2.7, we may assume that $X$ is absolutely convex. The equivalences (β) ⇒ (γ) and (δ) ⇒ (ε) are proved in [5], Theorem 3.3. Obviously, (δ) ⇒ (β) ⇒ (α). We shall prove (α) ⇒ (β) and (γ) ⇒ (ε). Suppose (α) and let $p$ be a weakly continuous seminorm on $E$. Then $\ker p$ has finite codimension. Let $\pi_p : E \to E/\ker p$ be the quotient map and let $\bar{p}$ be the norm on $E/\ker p$ induced by $p$. For each $f \in (E/\ker p)'$, $\max_{\pi_p(X)} |f|$ exists (Lemma 2.5). Then also (Proposition 2.3)

$$\max_{\pi_p(X)} \bar{p}$$

exists. But $\{p(a) : a \in X\} = \{\bar{p}(x) : x \in \pi_p(X)\}$ and therefore

$$\max_X p$$

exists and we have (β). Now suppose (γ). Lemma 2.6 tells us that $X$ is a compactoid. $E$ is a polar space (Proposition 1.1) so by [4], Theorem 5.12, the weak topology and the norm topology coincide on $X$. Let $U$ be an absolutely convex neighbourhood of 0. There is a weak neighbourhood $V$ of 0 with $V \cap X \subseteq U \cap X$. By (γ) there is a finite set $F \subseteq X$ such that $X \subseteq V + \text{co } F$. Then also $X \subseteq (V \cap X) + \text{co } F \subseteq U + \text{co } F$ and (ε) is proved.
Remark.
In the terminology of [4], Theorem 2.7 entails that weak c'-compactness implies c'-compactness. Compare [1], Proposition 3a, where it is proved that weak c-compactness implies c-compactness.

PROBLEM.
Are (a) - (e) equivalent for a nonempty subset X of a strongly polar ([4], Definition 3.5) Banach space E (in particular, an arbitrary Banach space E over a spherically complete K)? The following example shows that just 'E is a polar space' is not enough.

EXAMPLE.
Let K be not spherically complete, let $A = \{x \in l^\infty : \|x\| \leq 1\}$. Then for each $f \in (l^\infty)'$, $\max_{A} |f|$ exists although A is not a compactoid. [Since $(l^\infty)' \cong c_0$ ([3], Theorem 4.17) an $f \in (l^\infty)'$ has the form

$$x \mapsto \sum_{i \in \mathbb{N}} x_i a_i$$

for some $a \in c_0$.]

§ 3. LOCALLY CONVEX SPACES OF COUNTABLE TYPE

Also in § 3 we assume that the valuation of $K$ is dense. We shall extend the results of § 2 to locally convex spaces of countable type (i.e. for each continuous seminorm $p$ the normed space $E_p := E / \text{Ker } p$ with the norm induced by $p$ is of countable type, see [4], Definition 4.3).

First we extend Lemma 2.6.

Lemma 3.1.

Let $E$ be a locally convex space of countable type over $K$. Let $A \subseteq E$ be absolutely convex and suppose that for each $f \in E'$ the restriction of $|f|$ to $A$ has a maximum. Then $A$ is a compactoid.

Proof.

Let $U$ be a neighbourhood of 0 in $E$. There is a continuous seminorm $p$ such that $\{ x \in E : p(x) \leq 1 \} \subseteq U$. Let $E_p$ be the completion of $E$ (see above). The canonical map $\pi_p : E \to E_p \subseteq E_p$ is continuous. By Lemma 2.5, for each $f \in E_p$ the restriction of $|f|$ to $\pi_p(A)$ has a maximum. Now $E_p$ is of countable type and therefore has a base ([3], Theorem 3.16) so we may apply Lemma 2.6 and conclude that $\pi_p(A)$ is a compactoid in $E_p$, hence in $E$. Since $\pi_p(U)$ is open in $E_p$ there is a finite set $F \subseteq E$ such that $\pi_p(A) \subseteq \pi_p(U) + \text{co } \pi_p F$. We have

$$A \subseteq U + \text{co } F + \text{Ker } p \subseteq U + \text{co } F.$$

Theorem 3.2.

Let $X$ be a nonempty subset of a locally convex space $E$ of countable type over $K$. The statements (a) - (e) of Theorem 2.7 are equivalent.
Proof.

The proof of Theorem 2.7 applies with two modifications in the proof of (γ) ⇒ (e). The compactoidity of X follows from Lemma 3.1 (rather than Lemma 2.6) and the fact that E is a polar space is proved in [4], Theorem 4.4.
§ 4. DISCRETELY VALUED BASE FIELDS

One may wonder what happens to our results if we allow the valuation of $K$ to be discrete. The next two Properties show the deviation from the previous theory.

PROPOSITION 4.1.

Let $A$ be an absolutely convex subset of a locally convex space $E$ over a discretely valued $K$. The following are equivalent.

(a) $A$ is a compactoid.

(b) Each continuous seminorm $E \rightarrow [0, \infty]$ has a maximum on $A$.

Proof.

[5], Remark following Theorem 3.3.

PROPOSITION 4.2.

Let $A, E, K$ be as above. The following are equivalent.

(a) $A$ is a compactoid for the weak topology.

(b) Each weakly continuous seminorm on $E$ has a maximum in $A$.

(γ) For each $f \in E'$, $\max_{A} |f|$ exists.

(δ) $A$ is bounded.

Proof.

For (a) $\iff$ (b) see [5], Remark following Theorem 3.3. (a) $\iff$ (δ) is proved in [4]. (β) $\Rightarrow$ (γ) is obvious. (γ) implies weak boundedness of $A$, hence boundedness ([4], Corollary 7.7), so that we have (γ) $\implies$ (δ).
REFERENCES


