WEAK C'-COMPACTNESS IN p-ADIC BANACH SPACES

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ABSTRACT. Let X be a subset of a locally convex space E of countable type over a nonarchimedean densely valued field K. If, for each continuous linear function $f : E \to K$, $\max_X |f|$ exists then X is a compactoid in E.

§ 1. PRELIMINARIES

Throughout, K is a nonarchimedean nontrivially valued complete field with valuation $|\ |$. For fundamentals on Banach spaces and locally convex spaces over K we refer to [3], [6], [4]. We recall a few definitions and facts and fix some notations. Let E be a locally convex space over K. E' is the linear space of all continuous linear functions $E \to K$. If $E = (E, \|\ \|)$ is a nonzero Banach space and $f \in E'$ then $\|f\| := \sup\{|f(x)|/\|x\| : x \in E\ ,\ x \neq 0\}$. E is a <u>polar space</u> if there exists a base of continuous seminorms p for which $p = \sup\{|f| : f \in E'\ ,\ |f| \leq p\}$.

PROPOSITION 1.1.

<u>Let E be a Banach space over K with a base. Then E is a polar space.</u>

<u>Proof.</u>

By [3], Corollary 3.7 there exists a norm $\|\ \|$ inducing the topology of E for which E has an orthogonal base $\{e_i : i \in I\}$. It is not hard to see that we even may assume that $\{e_i : i \in I\}$ is orthonormal. For each $i \in I$, let f_i be the i-th coordinate function. Let $x \in E$. Then $x = \sum_{i \in I} f_i(x)e_i$ and $\|x\| = \max_{i \in I} |f_i(x)|$.

For a subset X of a locally convex space E over K we denote by co X its absolutely convex hull, by [X] its K-linear span, by $\overline{X}$ its closure. Instead of $\overline{\text{co } X}$ we write $\overline{\text{co}}\ X$. For an absolutely convex $A \subset E$ the formula

$$p_A(x) = \inf\{|\lambda| : x \in \lambda A\}$$

defines, on [A], the <u>seminorm</u> p_A <u>associated to</u> A.

PROPOSITION 1.2.

Let $A \subset E$ *be absolutely convex. Then*

$$\{x \in [A] : p_A(x) < 1\} \subset A \subset \{x \in [A] : p_A(x) \leq 1\}.$$

Proof.

Straightforward.

An absolutely convex $A \subset E$ is *edged* if for each $x \in E$ the set $\{|\lambda| : \lambda x \in A\}$ is closed in $|K| := \{|\lambda| : \lambda \in K\}$. A is edged if and only if $A = \{x \in [A] : p_A(x) \leq 1\}$. A subset X of E is a *compactoid* if for each neighbourhood U of 0 in E there exists a finite set $F \subset E$ such that $X \subset U + \mathrm{co}\, F$. For some elementary properties of compactoids, see [2].

§ 2. BANACH SPACES WITH A BASE

In § 2 we assume that the valuation of K is dense.

LEMMA 2.1.

The closed absolutely convex hull of an orthonormal set in a Banach space over K is edged.

Proof.

For an orthonormal set $\{e_i : i \in I\}$, set $A := \overline{co}\,\{e_i : i \in I\}$, $D := \overline{[e_i : i \in I]}$. Without any trouble one verifies that $D = [A]$, $p_A = \|\ \ \|$ on D and $A = \{x \in D : \|x\| \leq 1\}$.

LEMMA 2.2.

Let E be a Banach space of countable type over K. Let A be an absolutely convex neighbourhood of 0 in E and suppose that, for each $f \in E'$, the restriction of $|f|$ to A has a maximum.

(i) A is bounded; p_A is a norm $\|\ \ \|$ inducing the topology of E.

(ii) Let $\{e_i : i \in I\}$ be a maximal $\|\ \ \|$-orthonormal set in

$$A_m := \{x \in A : \text{there is an } f \in E' \text{ with } |f(x)| = \max_A |f|\}.$$

Then $A = \overline{co}\,\{e_i : i \in I\}$.

Proof.

(i) A is weakly bounded hence bounded by [4], Corollary 7.7. The interior of A is not empty and A is an additive group so A is open (and closed) and (i) follows.

(ii) $B := \overline{co}\,\{e_i : i \in I\}$ is contained in A. Suppose $B \neq A$; we shall prove that $\{e_i : i \in I\}$ is not maximal yielding a contradiction. The set B is closed and edged (Lemma 2.1) so by [4], Proposition 4.8 and 3.4, there exists an $f \in E'$ with $|f| \le 1$ on B and $|f(y)| > 1$ for some $y \in A$. Then $\max_A |f| > 1$ and after multiplying f by a suitable scalar we obtain a $g \in E'$ for which

$$|g| < 1 \text{ on } B \ , \ \max_B |g| = 1.$$

From

$$(^*) \qquad \{x \in E : \|x\| < 1\} \subset A \subset \{x \in E : \|x\| \le 1\}$$

(Proposition 1.2) it follows that $\|g\| = 1$. Choose an $a \in A$ with $|g(a)| = 1$. We claim that $\{a\} \cup \{e_i : i \in I\}$ is an orthonormal set in A_m. In fact, we have $a \in A_m$. By $(^*)$, $\|a\| \le 1$ but also $1 = |g(a)| \le \|g\| \ \|a\| = \|a\|$ so that $\|a\| = 1$. To prove orthogonality it suffices to show that for a K-linear combination $z = \sum_{i \in F} \lambda_i e_i$ ($F \subset I$, F finite) we have

$$\|a-z\| \ge \|a\|.$$

If $\max |\lambda_i| > 1$ this is an obvious consequence of the strong triangle inequality so assume $\max |\lambda_i| \le 1$. Then $z \in B$ so $|g(z)| < 1$ and

$$\|a\| = 1 = |g(a)| = |g(a-z)| \le \|g\| \ \|a-z\| = \|a-z\|$$

which finishes the proof.

Remark.

The above proof works for a strongly polar ([4], Definition 3.5) Banach space E, in particular for any Banach space over a spherically complete K.

Two corollaries obtain. The first one we shall need later on. The second one is rather surprising.

PROPOSITION 2.3.

Let A be an absolutely convex subset of a finite dimensional space E over K. The following are equivalent.

(α) *For each* $f \in E'$, $\max_A |f|$ *exists.*

(β) *For each seminorm* p *on* E, $\max_A p$ *exists.*

(γ) *There exists a finite set* $F \subset A$ *with* $A = \text{co } F$.

Proof.

(γ) ⇒ (β) is easy, (β) ⇒ (α) is trivial. To prove (α) ⇒ (γ) we may assume [A] = E. Then A is open (with respect to the unique Banach space topology of E). Lemma 2.2 (ii) yields a (finite) set F with $A = \overline{\text{co}}\, F = \text{co } F$ (the second equality because each convex set in E is closed).

LEMMA 2.4.

Let E,A be as in Lemma 2.2. Then E is finite dimensional.

Proof.

Suppose E is not finite dimensional. From Lemma 2.2 we infer that E has an orthonormal base $\{e_i : i \in \mathbb{N}\}$ (with respect to the norm $\| \;\; \|$) and that $A = \overline{\text{co}}\, \{e_i : i \in \mathbb{N}\}$ is the 'closed' unit ball of $(E, \| \;\; \|)$. Now choose $\tau_1, \tau_2, \ldots$ in K such that $0 < |\tau_1| < |\tau_2| < \ldots$, $\lim_{n\to\infty} |\tau_n| = 1$ and define $f \in E'$ by the formula

$$f(\sum_{i \in \mathbb{N}} \lambda_i e_i) = \sum_{i \in \mathbb{N}} \lambda_i \tau_i \qquad (\lambda_i \in K, \lim_{i\to\infty} \lambda_i = 0)$$

We have $\sup_A |f| = 1$ but $|f(x)| < 1$ for each $x \in A$, a contradiction.

The proof of the next Lemma is left to the reader.

LEMMA 2.5.

Let A be an absolutely convex subset of a locally convex space E over K. Suppose $\max_A |f|$ exists for each $f \in E'$.

(i) *For each $f \in E'$, $\max_{\bar{A}} |f|$ exists.*

(ii) *If $T : E \to F$ is a continuous linear map into a locally convex space F over K then, for each $f \in F'$, $\max_{TA} |f|$ exists.*

LEMMA 2.6.

Let E be a Banach space over K with a base. Let $A \subset E$ be absolutely convex and suppose that for each $f \in E'$ the restriction of $|f|$ to A has a maximum. Then A is a compactoid.

Proof.

We may assume ([3], Corollary 3.7) that E has an orthogonal base. Suppose A is not a compactoid. By [3], Theorem 4.38, $(\zeta) \Rightarrow (\alpha)$ there exists an orthogonal sequence $e_1, e_2, \ldots$ in A with $\inf_n \|e_n\| > 0$. Set $D := \overline{[e_1, e_2, \ldots]}$, and choose a linear continuous projection $P : E \to D$ ([3], Corollary 3.18). By Lemma 2.5, for each $f \in D'$ the restriction of $|f|$ to $\overline{PA}$ has a maximum. Also $\overline{PA} \supset \overline{co}\,\{e_1, e_2, \ldots\}$ is open in D and D is infinite dimensional. But this is impossible (Lemma 2.4).

We now formulate the main Theorem of this section.

THEOREM 2.7.

Let E be a Banach space with a base over (a densely valued) K. For a nonempty subset X of E the following are equivalent.

(α) *For each $f \in E'$, $\max_X |f|$ exists.*

(β) *For each weakly continuous seminorm p, $\max_X p$ exists.*

(γ) *For each weak neighbourhood U of 0 there exists a finite set $F \subset X$*

such that $X \subset U+\text{co } F$.

(δ) *For each continuous seminorm* p *on* E, $\max_X p$ *exists.*

(ε) *For each neighbourhood* U *of* 0 *there exists a finite set* $F \subset X$ *such that* $X \subset U+\text{co } F$.

Proof.

Let (*) be any of the statements (α) – (ε). It is not hard to see that, for a nonempty set $Y \subset E$,

$$(*) \text{ holds for } X := Y \Leftrightarrow (*) \text{ holds for } X := \text{co } Y$$

Therefore, to prove Theorem 2.7, we may assume that X is absolutely convex. The equivalences (β) ⇔ (γ) and (δ) ⇔ (ε) are proved in [5], Theorem 3.3. Obviously, (δ) ⇒ (β) ⇒ (α). We shall prove (α) ⇒ (β) and (γ) ⇒ (ε). Suppose (α) and let p be a weakly continuous seminorm on E. Then $\text{Ker } p$ has finite codimension. Let $\pi_p : E \to E/\text{Ker } p$ be the quotient map and let $\overline{p}$ be the norm on $E/\text{Ker } p$ induced by p. For each $f \in (E/\text{Ker } p)'$, $\max_{\pi_p(X)} |f|$ exists (Lemma 2.5). Then also (Proposition 2.3)

$$\max_{\pi_p(X)} \overline{p}$$

exists. But $\{p(a) : a \in X\} = \{\overline{p}(x) : x \in \pi_p(X)\}$ and therefore

$$\max_X p$$

exists and we have (β). Now suppose (γ). Lemma 2.6 tells us that X is a compactoid. E is a polar space (Proposition 1.1) so by [4], Theorem 5.12, the weak topology and the norm topology coincide on X. Let U be an absolutely convex neighbourhood of 0. There is a weak neighbourhood V of 0 with $V \cap X \subset U \cap X$. By (γ) there is a finite set $F \subset X$ such that $X \subset V+\text{co } F$. Then also $X \subset (V \cap X)+\text{co } F \subset U+\text{co } F$ and (ε) is proved.

Remark.

In the terminology of [4], Theorem 2.7 entails that weak c'-compactness implies c'-compactness. Compare [1], Proposition 3a, where it is proved that weak c-compactness implies c-compactness.

PROBLEM.

Are (α) – (ε) equivalent for a nonempty subset X of a strongly polar ([4], Definition 3.5) Banach space E (in particular, an arbitrary Banach space E over a spherically complete K) ? The following example shows that just 'E is a polar space' is not enough.

EXAMPLE.

Let K be not spherically complete, let $A = \{x \in l^\infty : \|x\| \leq 1\}$. Then for each $f \in (l^\infty)'$, $\max_A |f|$ exists although A is not a compactoid. [Since $(l^\infty)' \simeq c_0$ ([3], Theorem 4.17) an $f \in (l^\infty)'$ has the form

$$x \mapsto \sum_{i \in \mathbb{N}} x_i a_i \qquad (x \in l^\infty)$$

for some $a \in c_0$.]

§ 3. LOCALLY CONVEX SPACES OF COUNTABLE TYPE

Also in § 3 we assume that the valuation of K is dense. We shall extend the results of § 2 to locally convex spaces of countable type (i.e. for each continuous seminorm p the normed space $E_p := E/\mathrm{Ker}\ p$ with the norm induced by p is of countable type, see [4], Definition 4.3).
First we extend Lemma 2.6.

LEMMA 3.1.

Let E be a locally convex space of countable type over K. Let $A \subset E$ be absolutely convex and suppose that for each $f \in E'$ the restriction of $|f|$ to A has a maximum. Then A is a compactoid.

Proof.

Let U be a neighbourhood of 0 in E. There is a continuous seminorm p such that $\{x \in E : p(x) \leq 1\} \subset U$. Let $E_p^\wedge$ be the completion of E_p (see above). The canonical map $\pi_p : E \to E_p \subset E_p^\wedge$ is continuous. By Lemma 2.5, for each $f \in E_p^\wedge$ the restriction of $|f|$ to $\pi_p(A)$ has a maximum. Now $E_p^\wedge$ is of countable type and therefore has a base ([3], Theorem 3.16) so we may apply Lemma 2.6 and conclude that $\pi_p(A)$ is a compactoid in $E_p^\wedge$, hence in E_p. Since $\pi_p(U)$ is open in E_p there is a finite set $F \subset E$ such that $\pi_p(A) \subset \pi_p(U) + \mathrm{co}\ \pi_p F$. We have

$$A \subset U + \mathrm{co}\ F + \mathrm{Ker}\ p \subset U + \mathrm{co}\ F.$$

THEOREM 3.2.

Let X be a nonempty subset of a locally convex space E of countable type over K. The statements (α) - (ε) of Theorem 2.7 are equivalent.

Proof.

The proof of Theorem 2.7 applies with two modifications in the proof of $(\gamma) \Rightarrow (\varepsilon)$. The compactoidity of X follows from Lemma 3.1 (rather than Lemma 2.6) and the fact that E is a polar space is proved in [4], Theorem 4.4.

§ 4. DISCRETELY VALUED BASE FIELDS

One may wonder what happens to our results if we allow the valuation of K to be discrete. The next two Properties show the deviation from the previous theory.

PROPOSITION 4.1.

Let A *be an absolutely convex subset of a locally convex space* E *over a discretely valued* K. *The following are equivalent.*

(α) A *is a compactoid*.

(β) *Each continuous seminorm* $E \to [0,\infty]$ *has a maximum on* A.

Proof.

[5], Remark following Theorem 3.3.

PROPOSITION 4.2.

Let A,E,K *be as above. The following are equivalent.*

(α) A *is a compactoid for the weak topology*.

(β) *Each weakly continuous seminorm on* E *has a maximum in* A.

(γ) *For each* $f \in E'$, $\max_A |f|$ *exists*.

(δ) A *is bounded*.

Proof.

For (α) ⇔ (β) see [5], Remark following Theorem 3.3. (α) ⇔ (δ) is proved in [4]. (β) ⇒ (γ) is obvious. (γ) implies weak boundedness of A, hence boundedness ([4], Corollary 7.7), so that we have (γ) ⇒ (δ).

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