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WEAK C'-COMPACTNESS IN p-ADIC BANACH SPACES

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ABSTRACT. Let \( X \) be a subset of a locally convex space \( E \) of countable type over a nonarchimedean densely valued field \( K \). If, for each continuous linear function \( f : E \to K \), \( \max |f| \) exists then \( X \) is a compactoid in \( E \).
§ 1. PRELIMINARIES

Throughout, K is a nonarchimedean nontrivially valued complete field with valuation | |. For fundamentals on Banach spaces and locally convex spaces over K we refer to [3], [6], [4]. We recall a few definitions and facts and fix some notations. Let E be a locally convex space over K. E' is the linear space of all continuous linear functions \( E \rightarrow K \). If \( E = (E, \| \|) \) is a nonzero Banach space and \( f \in E' \) then \[ \| f \| : = \sup \{ | f(x) / \| x \| : x \in E, x \neq 0 \} \]. E is a polar space if there exists a base of continuous seminorms \( p \) for which \[ p = \sup \{ |f| : f \in E', |f| \leq p \} . \]

**PROPOSITION 1.1.**

Let \( E \) be a Banach space over K with a base. Then \( E \) is a polar space.

**Proof.**

By [3], Corollary 3.7 there exists a norm \( \| \| \) inducing the topology of \( E \) for which \( E \) has an orthogonal base \( \{ e_i : i \in I \} \). It is not hard to see that we even may assume that \( \{ e_i : i \in I \} \) is orthonormal. For each \( i \in I \), let \( f_i \) be the i-th coordinate function. Let \( x \in E \). Then \[ x = \sum_{i \in I} f_i(x) e_i \text{ and } \| x \| = \max_{i \in I} | f_i(x) | . \]

For a subset \( X \) of a locally convex space \( E \) over K we denote by \( \text{co} X \) its absolutely convex hull, by \([X]\) its K-linear span, by \( \overline{X} \) its closure. Instead of \( \text{co} X \) we write \( \overline{\text{co}} X \). For an absolutely convex \( A \subset E \) the formula \[ p_A(x) = \inf \{ |\lambda| : x \in \lambda A \} \]
defines, on \([A]\), the seminorm \( p_A \) associated to \( A \).
PROPOSITION 1.2.

Let $A \subset E$ be absolutely convex. Then

$$\{x \in [A] : p_A(x) < 1\} \subset A \subset \{x \in [A] : p_A(x) \leq 1\}.$$ 

**Proof.**

Straightforward.

An absolutely convex $A \subset E$ is **edged** if for each $x \in E$ the set

$$\{\lambda \in [\lambda] : \lambda x \in A\}$$

is closed in $[\lambda] := \{\lambda \in \lambda \subset X\}$. $A$ is edged if and only if $A = \{x \in [A] : p_A(x) \leq 1\}$. A subset $X$ of $E$ is a **compactoid** if for each neighbourhood $U$ of $0$ in $E$ there exists a finite set $F \subset E$ such that $X \subset U + \operatorname{co} F$. For some elementary properties of compactoids, see [2].
§ 2. BANACH SPACES WITH A BASE

In § 2 we assume that the valuation of $K$ is dense.

**Lemma 2.1.**

The closed absolutely convex hull of an orthonormal set in a Banach space over $K$ is edged.

**Proof.**

For an orthonormal set $\{e_i : i \in I\}$, set $A := \text{co} \{e_i : i \in I\}$, $D := \{e_i : i \in I\}$. Without any trouble one verifies that $D = [A]$, $\| \cdot \|$ on $D$ and $A = \{x \in D : \|x\| \leq 1\}$.

**Lemma 2.2.**

Let $E$ be a Banach space of countable type over $K$. Let $A$ be an absolutely convex neighbourhood of 0 in $E$ and suppose that, for each $f \in E'$, the restriction of $|f|$ to $A$ has a maximum.

(i) $A$ is bounded; $p_A$ is a norm $\| \cdot \|$ inducing the topology of $E$.

(ii) Let $\{e_i : i \in I\}$ be a maximal $\| \cdot \|$-orthonormal set in $E$.

Then $A = \text{co} \{e_i : i \in I\}$.

**Proof.**

(i) $A$ is weakly bounded hence bounded by [4], Corollary 7.7. The interior of $A$ is not empty and $A$ is an additive group so $A$ is open (and closed) and (i) follows.

(ii) $B := \text{co} \{e_i : i \in I\}$ is contained in $A$. Suppose $B \neq A$; we shall prove
that \( \{e_i : i \in I\} \) is not maximal yielding a contradiction. The set \( B \) is closed and edged (Lemma 2.1) so by [4], Proposition 4.8 and 3.4, there exists an \( f \in E' \) with \( |f| \leq 1 \) on \( B \) and \( |f(y)| > 1 \) for some \( y \in A \). Then \( \max_A |f| > 1 \) and after multiplying \( f \) by a suitable scalar we obtain \( g \in E' \) for which

\[
|g| < 1 \text{ on } B, \max_B |g| = 1.
\]

From

\[
(*) \quad \{x \in E : \|x\| < 1\} \subset A \subset \{x \in E : \|x\| \leq 1\}
\]

(Proposition 1.2) it follows that \( \|g\| = 1 \). Choose an \( a \in A \) with \( |g(a)| = 1 \). We claim that \( \{a\} \cup \{e_i : i \in I\} \) is an orthonormal set in \( A_m \). In fact, we have \( a \in A_m \). By (*) \( \|a\| \leq 1 \) but also

\[
1 = |g(a)| \leq \|g\| \|a\| = \|a\| \text{ so that } \|a\| = 1.
\]

To prove orthogonality it suffices to show that for a \( K \)-linear combination \( z = \sum_{i \in F} \lambda_i e_i \) \((F \subset I, F \text{ finite})\) we have

\[
\|a-z\| \geq \|a\|.
\]

If \( \max_i |\lambda_i| > 1 \) this is an obvious consequence of the strong triangle inequality so assume \( \max_i |\lambda_i| \leq 1 \). Then \( z \in B \) so \( |g(z)| < 1 \) and

\[
\|a\| = 1 = |g(a)| = |g(a-z)| \leq \|g\| \|a-z\| = \|a-z\|
\]

which finishes the proof.

Remark.

The above proof works for a strongly polar ([4], Definition 3.5) Banach space \( E \), in particular for any Banach space over a spherically complete \( K \).

Two corollaries obtain. The first one we shall need later on. The
second one is rather surprising.

PROPOSITION 2.3.

Let $A$ be an absolutely convex subset of a finite dimensional space $E$ over $K$. The following are equivalent.

(a) For each $f \in E'$, $\max_A |f|$ exists.

(b) For each seminorm $p$ on $E$, $\max_A p$ exists.

(γ) There exists a finite set $F \subset A$ with $A = \text{co } F$.

Proof.

(γ) ⇒ (β) is easy, (β) ⇒ (α) is trivial. To prove (α) ⇒ (γ) we may assume $[A] = E$. Then $A$ is open (with respect to the unique Banach space topology of $E$). Lemma 2.2 (ii) yields a (finite) set $F$ with $A = \text{co } F = \text{co } F$ (the second equality because each convex set in $E$ is closed).

LEMMA 2.4.

Let $E,A$ be as in Lemma 2.2. Then $E$ is finite dimensional.

Proof.

Suppose $E$ is not finite dimensional. From Lemma 2.2 we infer that $E$ has an orthonormal base $\{e_i : i \in \mathbb{N}\}$ (with respect to the norm $\|\|_1$) and that $A = \text{co } \{e_i : i \in \mathbb{N}\}$ is the 'closed' unit ball of $(E,\|\|_1)$. Now choose $\tau_1, \tau_2, \ldots$ in $K$ such that $0 < |\tau_1| < |\tau_2| < \ldots$, $\lim_{n \to \infty} |\tau_n| = 1$ and define $f \in E'$ by the formula

$$f\left( \sum_{i \in \mathbb{N}} \lambda_i e_i \right) = \sum_{i \in \mathbb{N}} \lambda_i \tau_i \quad (\lambda_i \in K, \lim_{i \to \infty} \lambda_i = 0).$$

We have $\sup_A |f| = 1$ but $|f(x)| < 1$ for each $x \in A$, a contradiction.

The proof of the next Lemma is left to the reader.
LEMMA 2.5.
Let $A$ be an absolutely convex subset of a locally convex space $E$ over $K$. Suppose $\max \limits_A |f|$ exists for each $f \in E'$.

(i) For each $f \in E'$, $\max \limits_A |f|$ exists.

(ii) If $T : E \to F$ is a continuous linear map into a locally convex space $F$ over $K$ then, for each $f \in F'$, $\max \limits_T |f|$ exists.

LEMMA 2.6.
Let $E$ be a Banach space over $K$ with a base. Let $A \subset E$ be absolutely convex and suppose that for each $f \in E'$ the restriction of $|f|$ to $A$ has a maximum. Then $A$ is a compactoid.

Proof.
We may assume ([3], Corollary 3.7) that $E$ has an orthogonal base.

Suppose $A$ is not a compactoid. By [3], Theorem 4.38, ($\zeta$) $\Rightarrow$ (a) there exists an orthogonal sequence $e_1', e_2', \ldots$ in $A$ with $\inf \limits_n \|e_n\| > 0$. Set $D := \{e_1', e_2', \ldots\}$, and choose a linear continuous projection $P : E \to D$ ([3], Corollary 3.18). By Lemma 2.5, for each $f \in D'$ the restriction of $|f|$ to $PA$ has a maximum. Also $PA > \text{co} \{e_1', e_2', \ldots\}$ is open in $D$ and $D$ is infinite dimensional. But this is impossible (Lemma 2.4).

We now formulate the main theorem of this section.

THEOREM 2.7.
Let $E$ be a Banach space with a base over (a densely valued) $K$. For a nonempty subset $X$ of $E$ the following are equivalent.

(a) For each $f \in E'$, $\max \limits_X |f|$ exists.

(b) For each weakly continuous seminorm $p$, $\max \limits_X p$ exists.

(c) For each weak neighbourhood $U$ of $0$ there exists a finite set $P \subset X$
such that \( X \subset U + co F \).

(\delta) For each continuous seminorm \( p \) on \( E \), \( \max_X p \) exists.

(\epsilon) For each neighbourhood \( U \) of \( 0 \) there exists a finite set \( F \subset X \) such that \( X \subset U + co F \).

**Proof.**

Let (*) be any of the statements (a) - (e). It is not hard to see that, for a nonempty set \( Y \subset E \),

\[
(*) \text{ holds for } X := Y \leftrightarrow (*) \text{ holds for } X := co Y
\]

Therefore, to prove Theorem 2.7, we may assume that \( X \) is absolutely convex. The equivalences \((\beta) \leftrightarrow (\gamma) \) and \((\delta) \leftrightarrow (\epsilon) \) are proved in [5], Theorem 3.3. Obviously, \((\delta) \Rightarrow (\beta) \Rightarrow (\alpha) \). We shall prove \((\alpha) \Rightarrow (\beta) \) and \((\gamma) \Rightarrow (\epsilon) \). Suppose (a) and let \( p \) be a weakly continuous seminorm on \( E \). Then \( \ker p \) has finite codimension. Let \( \pi_p : E \rightarrow E/\ker p \) be the quotient map and let \( \overline{p} \) be the norm on \( E/\ker p \) induced by \( p \). For each \( f \in (E/\ker p)' \), \( \max |f| \) exists (Lemma 2.5). Then also (Proposition 2.3)

\[
\max_{\pi_p(X)} \overline{p}
\]

exists. But \( \{p(a) : a \in X\} = \{\overline{p}(x) : x \in \pi_p(X)\} \) and therefore

\[
\max_X p
\]

exists and we have (\beta). Now suppose (\gamma). Lemma 2.6 tells us that \( X \) is a compactoid. \( E \) is a polar space (Proposition 1.1) so by [4], Theorem 5.12, the weak topology and the norm topology coincide on \( X \). Let \( U \) be an absolutely convex neighbourhood of \( 0 \). There is a weak neighbourhood \( V \) of \( 0 \) with \( V \cap X \subset U \cap X \). By (\gamma) there is a finite set \( F \subset X \) such that \( X \subset V + co F \). Then also \( X \subset (V \cap X) + co F \subset U + co F \) and (\epsilon) is proved.
Remark.

In the terminology of [4], Theorem 2.7 entails that weak c'-compactness implies c'-compactness. Compare [1], Proposition 3a, where it is proved that weak c-compactness implies c-compactness.

PROBLEM.

Are (a) - (e) equivalent for a nonempty subset X of a strongly polar ([4], Definition 3.5) Banach space E (in particular, an arbitrary Banach space E over a spherically complete K)? The following example shows that just 'E is a polar space' is not enough.

EXAMPLE.

Let K be not spherically complete, let \( A = \{ x \in l^\infty : \| x \| \leq 1 \} \). Then for each \( f \in (l^\infty)' \), \( \max_A |f| \) exists although A is not a compactoid. [Since \( (l^\infty)' \cong c_0 \) ([3], Theorem 4.17) an \( f \in (l^\infty)' \) has the form

\[
\sum_{i} x_i a_i \quad \text{for some } a \in c_0.
\]

\( \Rightarrow \) \( x \in l^\infty \)
§ 3. LOCALLY CONVEX SPACES OF COUNTABLE TYPE

Also in § 3 we assume that the valuation of \( K \) is dense. We shall extend the results of § 2 to locally convex spaces of countable type (i.e. for each continuous seminorm \( p \) the normed space \( E_p := E/\text{Ker} \ p \) with the norm induced by \( p \) is of countable type, see [4], Definition 4.3).

First we extend Lemma 2.6.

**LEMMA 3.1.**

Let \( E \) be a locally convex space of countable type over \( K \). Let \( A \subset E \) be absolutely convex and suppose that for each \( f \in E' \) the restriction of \( |f| \) to \( A \) has a maximum. Then \( A \) is a compactoid.

**Proof.**

Let \( U \) be a neighbourhood of 0 in \( E \). There is a continuous seminorm \( p \) such that \( \{ x \in E : p(x) \leq 1 \} \subset U \). Let \( E_p^- \) be the completion of \( E_p \) (see above). The canonical map \( \pi_p^0 : E \to E_p^- \subset E^- \) is continuous. By Lemma 2.5, for each \( f \in E_p^- \) the restriction of \( |f| \) to \( \pi_p^0(A) \) has a maximum. Now \( E_p^- \) is of countable type and therefore has a base ([3], Theorem 3.16) so we may apply Lemma 2.6 and conclude that \( \pi_p^0(A) \) is a compactoid in \( E_p^- \), hence in \( E_p \). Since \( \pi_p(U) \) is open in \( E_p \) there is a finite set \( F \subset E \) such that

\[
\pi_p^0(A) \subset \pi_p^0(U) + \text{co} \pi_p^0(F).
\]

We have

\[
A \subset U + \text{co} F + \text{Ker} \ p \subset U + \text{co} F.
\]

**THEOREM 3.2.**

Let \( X \) be a nonempty subset of a locally convex space \( E \) of countable type over \( K \). The statements (a) - (c) of Theorem 2.7 are equivalent.
Proof.

The proof of Theorem 2.7 applies with two modifications in the proof of $(\gamma) \Rightarrow (c)$. The compactoidity of $X$ follows from Lemma 3.1 (rather than Lemma 2.6) and the fact that $E$ is a polar space is proved in [4], Theorem 4.4.
§ 4. DISCRETELY VALUED BASE FIELDS

One may wonder what happens to our results if we allow the valuation of $K$ to be discrete. The next two Properties show the deviation from the previous theory.

PROPOSITION 4.1.

Let $A$ be an absolutely convex subset of a locally convex space $E$ over a discretely valued $K$. The following are equivalent.

(a) $A$ is a compactoid.

(b) Each continuous seminorm $E \to [0,\infty]$ has a maximum on $A$.

Proof.

[5], Remark following Theorem 3.3.

PROPOSITION 4.2.

Let $A,E,K$ be as above. The following are equivalent.

(a) $A$ is a compactoid for the weak topology.

(b) Each weakly continuous seminorm on $E$ has a maximum in $A$.

(γ) For each $f \in E'$, $\max_A |f|$ exists.

(δ) $A$ is bounded.

Proof.

For (α) ⇔ (β) see [5], Remark following Theorem 3.3. (α) ⇔ (δ) is proved in [4]. (β) ⇒ (γ) is obvious. (γ) implies weak boundedness of $A$, hence boundedness ([4], Corollary 7.7), so that we have (γ) ⇔ (δ).
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