WEAK C'-COMPACTNESS IN p-ADIC BANACH SPACES

by

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ABSTRACT. Let $X$ be a subset of a locally convex space $E$ of countable type over a nonarchimedean densely valued field $K$. If, for each continuous linear function $f : E \to K$, $\max_{X} |f|$ exists then $X$ is a compactoid in $E$. 
§ 1. PRELIMINARIES

Throughout, $K$ is a nonarchimedean nontrivially valued complete field with valuation $|\cdot|$. For fundamentals on Banach spaces and locally convex spaces over $K$ we refer to [3], [6], [4]. We recall a few definitions and facts and fix some notations. Let $E$ be a locally convex space over $K$. $E'$ is the linear space of all continuous linear functions $E \to K$. If $E = (E, \|\|)$ is a nonzero Banach space and $f \in E'$ then
\[ \|f\| := \sup \{|f(x)|/\|x\| : x \in E, x \neq 0\}. \]
$E$ is a polar space if there exists a base of continuous seminorms $p$ for which
\[ p = \sup \{|f| : f \in E', |f| \leq p\}. \]

PROPOSITION 1.1.

Let $E$ be a Banach space over $K$ with a base. Then $E$ is a polar space.

Proof.

By [3], Corollary 3.7 there exists a norm $\|\|$ inducing the topology of $E$ for which $E$ has an orthogonal base $\{e_i : i \in I\}$. It is not hard to see that we even may assume that $\{e_i : i \in I\}$ is orthonormal. For each $i \in I$, let $f_i$ be the $i$-th coordinate function. Let $x \in E$. Then
\[ x = \sum_{i \in I} f_i(x) e_i \quad \text{and} \quad \|x\| = \max_{i \in I} |f_i(x)|. \]

For a subset $X$ of a locally convex space $E$ over $K$ we denote by $\text{co} \, X$ its absolutely convex hull, by $[X]$ its $K$-linear span, by $\overline{X}$ its closure. Instead of $\text{co} \, X$ we write $\overline{\text{co}} \, X$. For an absolutely convex $A \subset E$ the formula
\[ p_A(x) = \inf \{ |\lambda| : x \in \lambda A \} \]
defines, on $[A]$, the seminorm $p_A$ associated to $A$. 
PROPOSITION 1.2.

Let $A \subset E$ be absolutely convex. Then

$$\{ x \in [A] : p_A(x) < 1 \} \subset A \subset \{ x \in [A] : p_A(x) \leq 1 \}.$$

Proof.

Straightforward.

An absolutely convex $A \subset E$ is edged if for each $x \in E$ the set

$$\{ |\lambda| : \lambda x \in A \}$$

is closed in $|K| := \{ |\lambda| : \lambda \in K \}$. $A$ is edged if and only if $A = \{ x \in [A] : p_A(x) \leq 1 \}$. A subset $X$ of $E$ is a compactoid if for each neighbourhood $U$ of 0 in $E$ there exists a finite set $F \subset E$ such that $X \subset U + \text{co } F$. For some elementary properties of compactoids, see [2].
§ 2. BANACH SPACES WITH A BASE

In § 2 we assume that the valuation of $K$ is dense.

**LEMMA 2.1.**

The closed absolutely convex hull of an orthonormal set in a Banach space over $K$ is edged.

**Proof.**

For an orthonormal set $\{e_i : i \in I\}$, set $A := \overline{\text{co}} \{e_i : i \in I\}$, $D := \overline{\{e_i : i \in I\}}$. Without any trouble one verifies that $D = [A]$, $p_A = || \cdot ||$ on $D$ and $A = \{x \in D : ||x|| \leq 1\}$.

**LEMMA 2.2.**

Let $E$ be a Banach space of countable type over $K$. Let $A$ be an absolutely convex neighbourhood of 0 in $E$ and suppose that, for each $f \in E'$, the restriction of $|f|$ to $A$ has a maximum.

(i) $A$ is bounded; $p_A$ is a norm $|| \cdot ||$ inducing the topology of $E$.

(ii) Let $\{e_i : i \in I\}$ be a maximal $|| \cdot ||$-orthonormal set in $A$.

$A_m := \{x \in A : \text{there is an } f \in E' \text{ with } |f(x)| = \max_{A} |f| \}$.

Then $A = \overline{\text{co}} \{e_i : i \in I\}$.

**Proof.**

(i) $A$ is weakly bounded hence bounded by [4], Corollary 7.7. The interior of $A$ is not empty and $A$ is an additive group so $A$ is open (and closed) and (i) follows.

(ii) $B := \overline{\text{co}} \{e_i : i \in I\}$ is contained in $A$. Suppose $B \neq A$; we shall prove
that \( \{ e_i : i \in I \} \) is not maximal yielding a contradiction. The set \( B \) is closed and edged (Lemma 2.1) so by [4], Proposition 4.8 and 3.4, there exists an \( f \in E' \) with \( |f| \leq 1 \) on \( B \) and \( |f(y)| > 1 \) for some \( y \in A \). Then \( \max_{A} |f| > 1 \) and after multiplying \( f \) by a suitable scalar we obtain \( g \in E' \) for which

\[
|g| < 1 \text{ on } B, \quad \max_{B} |g| = 1.
\]

From

\[
(*) \quad \{ x \in E : ||x|| < 1 \} \subseteq A \subseteq \{ x \in E : ||x|| \leq 1 \}
\]

(Proposition 1.2) it follows that \( ||g|| = 1 \). Choose an \( a \in A \) with \( |g(a)| = 1 \). We claim that \( \{a\} \cup \{ e_i : i \in I \} \) is an orthonormal set in \( A_m \). In fact, we have \( a \in A_m \). By \((*)\), \( ||a|| \leq 1 \) but also

\[
1 = |g(a)| \leq ||g|| ||a|| = ||a|| \quad \text{so that } ||a|| = 1.
\]

To prove orthogonality it suffices to show that for a \( K \)-linear combination \( z = \sum_{i \in I} \lambda_i e_i \) \((F \subseteq I, F \text{ finite})\) we have

\[
||a-z|| \geq ||a||.
\]

If \( \max |\lambda_i| > 1 \) this is an obvious consequence of the strong triangle inequality so assume \( \max |\lambda_i| \leq 1 \). Then \( z \in B \) so \( |g(z)| < 1 \) and

\[
||a|| = 1 = |g(a)| = |g(a-z)| \leq ||g|| ||a-z|| = ||a-z||
\]

which finishes the proof.

Remark.

The above proof works for a strongly polar ([4], Definition 3.5) Banach space \( E \), in particular for any Banach space over a spherically complete \( K \).

Two corollaries obtain. The first one we shall need later on. The
second one is rather surprising.

PROPOSITION 2.3.

Let $A$ be an absolutely convex subset of a finite dimensional space $E$ over $K$. The following are equivalent.

(a) For each $f \in E'$, $\max _A |f|$ exists.

(b) For each seminorm $p$ on $E$, $\max _A p$ exists.

(c) There exists a finite set $F \subset A$ with $A = \text{co } F$.

Proof.

$(c) \Rightarrow (b)$ is easy, $(b) \Rightarrow (a)$ is trivial. To prove $(a) \Rightarrow (c)$ we may assume $[A] = E$. Then $A$ is open (with respect to the unique Banach space topology of $E$). Lemma 2.2 (ii) yields a (finite) set $F$ with $A = \text{co } F = \text{co } F$ (the second equality because each convex set in $E$ is closed).

LEMMA 2.4.

Let $E, A$ be as in Lemma 2.2. Then $E$ is finite dimensional.

Proof.

Suppose $E$ is not finite dimensional. From Lemma 2.2 we infer that $E$ has an orthonormal base $\{e_i : i \in \mathbb{N}\}$ (with respect to the norm $|| \cdot ||$) and that $A = \text{co } \{e_i : i \in \mathbb{N}\}$ is the 'closed' unit ball of $(E, || \cdot ||)$. Now choose $\tau_1, \tau_2, \ldots$ in $K$ such that $0 < |\tau_1| < |\tau_2| < \ldots$, $\lim _{n \to \infty} |\tau_n| = 1$ and define $f \in E'$ by the formula

$$f(\sum _{i \in \mathbb{N}} \lambda_i e_i) = \sum _{i \in \mathbb{N}} \lambda_i \tau_i \quad (\lambda_i \in K, \lim _{i \to \infty} \lambda_i = 0).$$

We have $\sup _A |f| = 1$ but $|f(x)| < 1$ for each $x \in A$, a contradiction.

The proof of the next Lemma is left to the reader.
LEMMA 2.5.

Let $A$ be an absolutely convex subset of a locally convex space $E$ over $K$. Suppose $\max_{A} |f|$ exists for each $f \in E'$.

(i) For each $f \in E'$, $\max_{A} |f|$ exists.

(ii) If $T : E \to F$ is a continuous linear map into a locally convex space $F$ over $K$ then, for each $f \in F'$, $\max_{T(A)} |f|$ exists.

LEMMA 2.6.

Let $E$ be a Banach space over $K$ with a base. Let $A \subset E$ be absolutely convex and suppose that for each $f \in E'$ the restriction of $|f|$ to $A$ has a maximum. Then $A$ is a compactoid.

Proof.

We may assume ([3], Corollary 3.7) that $E$ has an orthogonal base.

Suppose $A$ is not a compactoid. By [3], Theorem 4.38, (5) $\Rightarrow$ (a) there exists an orthogonal sequence $e_1,e_2,\ldots$ in $A$ with $\inf \|e_n\| > 0$. Set $D := \{e_1,e_2,\ldots\}$, and choose a linear continuous projection $P : E \to D$ ([3], Corollary 3.18). By Lemma 2.5, for each $f \in D'$ the restriction of $|f|$ to $P(A)$ has a maximum. Also $P(A) = \overline{\text{co}} \{e_1,e_2,\ldots\}$ is open in $D$ and $D$ is infinite dimensional. But this is impossible (Lemma 2.4).

We now formulate the main Theorem of this section.

THEOREM 2.7.

Let $E$ be a Banach space with a base over (a densely valued) $K$. For a nonempty subset $X$ of $E$ the following are equivalent.

(a) For each $f \in E'$, $\max_{X} |f|$ exists.

(b) For each weakly continuous seminorm $p$, $\max_{X} p$ exists.

(c) For each weak neighbourhood $U$ of $0$ there exists a finite set $P \subset X$
such that $X \subseteq U + \text{co } F$.

(δ) For each continuous seminorm $p$ on $E$, $\max_X p$ exists.

(ε) For each neighbourhood $U$ of 0 there exists a finite set $F \subseteq X$ such that $X \subseteq U + \text{co } F$.

Proof.
Let (*) be any of the statements (α) - (ε). It is not hard to see that, for a nonempty set $Y \subseteq E$,

(*) holds for $X := Y \Leftrightarrow$ (*) holds for $X := \text{co } Y$

Therefore, to prove Theorem 2.7, we may assume that $X$ is absolutely convex. The equivalences (β) $\Rightarrow$ (γ) and (δ) $\Leftrightarrow$ (ε) are proved in [5], Theorem 3.3. Obviously, (δ) $\Rightarrow$ (β) $\Rightarrow$ (α). We shall prove (α) $\Rightarrow$ (β) and (γ) $\Rightarrow$ (ε). Suppose (α) and let $p$ be a weakly continuous seminorm on $E$. Then $\text{Ker } p$ has finite codimension. Let $\pi_p : E \to E/\text{Ker } p$ be the quotient map and let $\overline{p}$ be the norm on $E/\text{Ker } p$ induced by $p$. For each $f \in (E/\text{Ker } p)'$, $\max |f|$ exists (Lemma 2.5). Then also (Proposition 2.3)

$$\max_{\overline{p}} \frac{1}{\pi_p(X)}$$

exists. But $\{p(a) : a \in X\} = \{\overline{p}(x) : x \in \pi_p(X)\}$ and therefore

$$\max_X p$$

exists and we have (β). Now suppose (γ). Lemma 2.6 tells us that $X$ is a compactoid. $E$ is a polar space (Proposition 1.1) so by [4], Theorem 5.12, the weak topology and the norm topology coincide on $X$. Let $U$ be an absolutely convex neighbourhood of 0. There is a weak neighbourhood $V$ of 0 with $V \cap X \subseteq U \cap X$. By (γ) there is a finite set $F \subseteq X$ such that $X \subseteq V + \text{co } F$. Then also $X \subseteq (V \cap X) + \text{co } F \subseteq U + \text{co } F$ and (ε) is proved.
Remark.

In the terminology of [4], Theorem 2.7 entails that weak c'-compactness implies c'-compactness. Compare [1], Proposition 3a, where it is proved that weak c-compactness implies c-compactness.

PROBLEM.

Are (a) – (e) equivalent for a nonempty subset X of a strongly polar ([4], Definition 3.5) Banach space E (in particular, an arbitrary Banach space E over a spherically complete K)? The following example shows that just 'E is a polar space' is not enough.

EXAMPLE.

Let K be not spherically complete, let \( A = \{ x \in l^\infty : \| x \| \leq 1 \} \). Then for each \( f \in (l^\infty)' \), \( \max |f| \) exists although \( A \) is not a compactoid. [Since \( (l^\infty)' \cong c_0 \) ([3], Theorem 4.17) an \( f \in (l^\infty)' \) has the form

\[
x \mapsto \sum_{i \in \mathbb{N}} x_i a_i \quad \quad (x \in l^\infty)
\]

for some \( a \in c_0 \).]
§ 3. LOCALLY CONVEX SPACES OF COUNTABLE TYPE

Also in § 3 we assume that the valuation of K is dense. We shall extend the results of § 2 to locally convex spaces of countable type (i.e. for each continuous seminorm p the normed space \( E_p := E/\ker p \) with the norm induced by p is of countable type, see [4], Definition 4.3). First we extend Lemma 2.6.

**Lemma 3.1.**

Let \( E \) be a locally convex space of countable type over K. Let \( A \subseteq E \) be absolutely convex and suppose that for each \( f \in E' \) the restriction of \( |f| \) to \( A \) has a maximum. Then \( A \) is a compactoid.

**Proof.**

Let \( U \) be a neighbourhood of 0 in \( E \). There is a continuous seminorm \( p \) such that \( \{ x \in E : p(x) \leq 1 \} \subseteq U \). Let \( E_p^- \) be the completion of \( E_p \) (see above). The canonical map \( \pi_p : E \to E \subseteq E_p^- \) is continuous. By Lemma 2.5, for each \( f \in E'_p \) the restriction of \( |f| \) to \( \pi_p(A) \) has a maximum. Now \( E_p^- \) is of countable type and therefore has a base ([3], Theorem 3.16) so we may apply Lemma 2.6 and conclude that \( \pi_p(A) \) is a compactoid in \( E_p^- \), hence in \( E_p \). Since \( \pi_p(U) \) is open in \( E_p \), there is a finite set \( F \subseteq E \) such that \( \pi_p(A) \subseteq \pi_p(U) + \co \pi_p F \). We have

\[ A \subseteq U + \co F + \ker p \subseteq U + \co F. \]

**Theorem 3.2.**

Let \( X \) be a nonempty subset of a locally convex space \( E \) of countable type over K. The statements (a) - (e) of Theorem 2.7 are equivalent.
Proof.

The proof of Theorem 2.7 applies with two modifications in the proof of \((\gamma) \Rightarrow (\epsilon)\). The compactoidity of \(X\) follows from Lemma 3.1 (rather than Lemma 2.6) and the fact that \(E\) is a polar space is proved in [4], Theorem 4.4.
§ 4. DISCRETELY VALUED BASE FIELDS

One may wonder what happens to our results if we allow the valuation of $K$ to be discrete. The next two Properties show the deviation from the previous theory.

PROPOSITION 4.1.

Let $A$ be an absolutely convex subset of a locally convex space $E$ over a discretely valued $K$. The following are equivalent.

(a) $A$ is a compactoid.

(b) Each continuous seminorm $E \to [0,\infty]$ has a maximum on $A$.

Proof.

[5], Remark following Theorem 3.3.

PROPOSITION 4.2.

Let $A, E, K$ be as above. The following are equivalent.

(a) $A$ is a compactoid for the weak topology.

(b) Each weakly continuous seminorm on $E$ has a maximum in $A$.

(γ) For each $f \in E'$, $\max_{A} |f|$ exists.

(δ) $A$ is bounded.

Proof.

For (α) ↔ (β) see [5], Remark following Theorem 3.3. (α) ↔ (δ) is proved in [4]. (β) → (γ) is obvious. (γ) implies weak boundedness of $A$, hence boundedness ([4], Corollary 7.7), so that we have (γ) → (δ).
REFERENCES


