A COMPLEMENTARY VARIANT OF C-COMPACTNESS
IN p-ADIC FUNCTIONAL ANALYSIS

by

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ABSTRACT. An absolutely convex subset $A$ of a locally convex space $E$ over a nonarchimedean valued field is $c'$-compact in $E$ if each continuous seminorm on $E$, restricted to $A$, has a maximum. Various descriptions of $c'$-compactness are given revealing its close analogy to $c$-compactness ($\S$ 4).

PRELIMINARIES. Throughout let $K$ be a nonarchimedean nontrivially valued field with valuation $|\ |$. For fundamentals on Banach spaces and locally convex spaces over $K$ we refer to [2], [5], [4]. For a subset $X$ of a locally convex space $E$ over $K$ we denote its absolutely convex hull by $\text{co } X$, its linear span by $[X]$. The closure of $X$ is $\overline{X}$. Instead of $\text{co } X$ we write $\overline{X}$. A convex set is an (additive) coset of an absolutely convex set.
§ 1. ELEMENTARY PROPERTIES

In § 1 let $E$ be a locally convex space over $K$ and let $A \subset E$ be absolutely convex. The next proposition will justify the use of the term 'c'-compact' rather than 'c'-compact in $E$.

PROPOSITION 1.1.

$A$ is c'-compact in $[A]$ if and only if $A$ is c'-compact in $E$.

Proof.

Only the 'if' part needs a proof. Let $A$ be c'-compact in $E$, let $p$ be a continuous seminorm on $[A]$. There is a continuous seminorm $q$ on $E$ such that $p \leq q$ on $[A]$. The formula

$$\overline{p}(x) = \inf_{y \in [A]} \max(p(y), q(x-y))$$

defines a continuous seminorm $\overline{p}$ on $E$ whose restriction to $[A]$ is $p$.

It follows that $\max_A \overline{p} = \max_A p$ exists so that $A$ is c'-compact in $[A]$.

PROPOSITION 1.2.

$A$ is c'-compact if and only if $\overline{A}$ is c'-compact.

Proof.

By the isosceles triangle principle

$$\{p(a) \in (0,\omega) : a \in A\} = \{p(a) \in (0,\omega) : a \in \overline{A}\}$$

for each continuous seminorm $p$.

We see that c'-compact sets need not be closed or complete. The next proposition furnishes examples of c'-compact sets.
PROPOSITION 1.3.

If \( X \subset E \) is precompact then \( \text{co } X \) is \( c'\)-compact.

\textbf{Proof.}

Let \( p \) be a continuous seminorm on \( E \). If \( z \in \text{co } X \) then there exist \( x_1, \ldots, x_n \in X \) and \( \lambda_1, \ldots, \lambda_n \in K \) with \( |\lambda_i| \leq 1 \) for each \( i \in \{1, \ldots, n\} \) such that \( z = \sum_{i=1}^{n} \lambda_i x_i \). Then

\[
p(z) \leq \max_{i} p(\lambda_i x_i) \leq \max_{i} p(x_i).
\]

It follows that \( \sup_{\text{co } X} p = \sup_{X} p \). We complete the proof by showing that \( \tilde{p} \) has a maximum on \( X \). We may assume that \( s := \sup_{X} p > 0 \). Set

\[
U := \{x \in E : p(x) < \frac{s}{4s}\}.
\]

By precompactness there exist \( x_1, \ldots, x_n \in X \) such that

\[
X \subset \bigcup_{i=1}^{n} (x_i + U).
\]

For each \( i \in \{1, \ldots, n\} \) we have either \( p(x_i) < \frac{s}{4} \) (then \( p < \frac{s}{4} \) on \( x_i + U \)) or \( p(x_i) \geq \frac{s}{4} \) (then \( p \) is constant on \( x_i + U \)). Hence,

\[
\sup_{X} p = \max_{i} p(x_i) = \max_{X} p.
\]

PROPOSITION 1.4.

Let \( F \) be a locally convex space over \( K \), let \( T : E \to F \) be a continuous linear map. If \( A \subset E \) is \( c'\)-compact then so is \( TA \).

\textbf{Proof.}

If \( p \) is a continuous seminorm on \( F \) then \( p \circ T \) is a continuous seminorm on \( E \).
Remark.

If $K$ is spherically complete, \( \{ \lambda \in K : |\lambda| < 1 \} \) is $c$-compact but not $c'$-compact if the valuation is dense. If $K$ is not spherically complete, \( \{ \lambda \in K : |\lambda| \leq 1 \} \) is $c'$-compact (and complete) but not $c$-compact.
§ 2. C'-COMPACTNESS IN BANACH SPACES

Throughout § 2, $E$ is a Banach space over $K$ with norm $\| \cdot \|$, and except for Corollary 2.8 $A$ is a $c'$-compact subset of $E$. Our first goal is to show that $[A]$ is of countable type (Proposition 2.3).

**LEMMA 2.1.**

Let $p$ be a continuous seminorm on $E$, not vanishing on $A$. Then every $p$-orthogonal set in

$$A_p := \{ x \in A : p(x) = \max_A p \}$$

is finite.

**Proof.**

We may assume $\max p = 1$. Let $\{ e_i : i \in I \}$ be a maximal $p$-orthogonal set in $A$. Suppose $\exists N < I$, we derive a contradiction. $p$ is a norm on $D := [e_i : i \in I]$ and $\{ e_i : i \in I \}$ is a $p$-orthonormal (algebraic) base for $D$. Choose real numbers $\rho_1, \rho_2, \ldots$ such that $0 < \rho_1 < \rho_2 < \ldots$ and $\lim_{n \to \infty} \rho_n = 1$, and consider the seminorm $q$ on $D$ defined by the formula

$$q(\Sigma \lambda_i e_i) = \max_{i \in \mathbb{N}} |\lambda_i| \rho_i \quad (\lambda_i \in K, \{ i : \lambda_i \neq 0 \} \text{ finite}).$$

As $q \leq p$ on $D$, $q$ is continuous. Observe that $q(e_i) < 1$ for each $i \in I$.

Set $B := \text{co} \{ e_i : i \in I \}$. We have $\sup B q = 1$, but $q(b) < 1$ for each $b \in B$. The formula

$$\overline{q}(x) = \inf_{d \in D} \max(q(d), p(x-d))$$

defines a continuous seminorm $\overline{q}$ on $E$ extending $q$ for which $\overline{q} \leq p$. We
shall arrive at the desired contradiction by showing that \( q \) does not have a maximum on \( A \). As \( B \subset A \) we have

\[
1 = \sup_B q = \sup_B \overline{q} \leq \sup_A \overline{q} \leq \sup_A p = 1,
\]
whence

\[
\sup_A \overline{q} = 1.
\]

Now we shall prove that \( \overline{q}(a) < 1 \) for each \( a \in A \). If \( a \in A \setminus A_p \) then \( \overline{q}(a) \leq p(a) < 1 \) so assume \( a \in A_p \). Then \( a \) is not \( p \)-orthogonal to \( D \) and there exists a finite set \( F \subset I \) and a map \( i \mapsto \lambda_i \in K \) (\( i \in F \)) such that

\[
p(a - \sum_{i \in F} \lambda_i e_i) < p(a) = 1
\]

By the isosceles triangle principle, \( p(\sum_{i \in F} \lambda_i e_i) = p(a) = 1 \);

\( p \)-orthonormality yields \( \max_{i \in F} |\lambda_i| = 1 \). Therefore,

\[
q(\sum_{i \in F} \lambda_i e_i) \leq \max_{i \in F} |\lambda_i| q(e_i) < 1
\]

leading to

\[
\max_{i \in F} (q(\sum_{i \in F} \lambda_i e_i), p(a - \sum_{i \in F} \lambda_i e_i)) < 1.
\]

Using the definition of \( \overline{q} \) we arrive at \( \overline{q}(a) < 1 \).

**COROLLARY 2.2.**

Let \( p \) be a continuous seminorm on \( E \), not vanishing on \( A \). There is a finite dimensional subspace \( D \) of \( E \) such that for the quotient seminorm \( q \) defined by

\[
q(x) = \inf \{ p(x-d) : d \in D \}
\]

we have
Proof.

Let \( \{e_1, \ldots, e_n\} \) be a maximal \( p \)-orthogonal set in \( A_p \) (Lemma 2.1) and set \( D := \langle e_1, \ldots, e_n \rangle \). Suppose \( x \in A \) and \( q(x) = \max_A p \). Then \( x \in A_p \). For each \( d \in D \) we have

\[
p(x-d) \geq q(x) = \max_A p \geq p(x)
\]

so that \( x \) is \( p \)-orthogonal to \( \langle e_1, \ldots, e_n \rangle \), a contradiction. Hence,

\( q(x) < \max_A p \) for each \( x \in A \).

**Proposition 2.3.**

\( [A] \) is of countable type.

Proof.

For each subspace \( D \) of countable type the formula

\[
p_D(x) = \inf \{ \|x-d\| : d \in D \}
\]

defines a continuous seminorm \( p_D \) on \( E \). We set

\[
x_D := \max_A p_D
\]

\( R = \{x_D : D \text{ is a subspace of countable type}\} \),

Then \( R \subseteq [0, \infty) \). We have (i), (ii) below.

(i) **If** \( t_1, t_2, \ldots \) **are in** \( R \) **then there exists a** \( t \in R \) **with** \( t \leq \inf t_n \).

**Proof.** Let \( t_n = x_{D_n} \) (\( n \in \mathbb{N} \)) where each \( D_n \) is a subspace of countable type. Then \( D := \langle D_n : n \in \mathbb{N} \rangle \) is of countable type. Obviously, \( p_D \leq p_{D_n} \) for each \( n \in \mathbb{N} \), so \( t := x_D \leq \inf x_{D_n} = \inf t_n \).

(ii) **If** \( t \in R \), \( t > 0 \) **then there exists an** \( s \in R \) **with** \( s < t \).
Proof. Let \( t = r_D \) where \( D \) is a subspace of countable type. By Corollary 2.2 there is a finite dimensional space \( F \subset E \) such that

\[
\max_A q < \max_A p_D = r_D
\]

where

\[
q(x) = \inf \{ p_D(x-y) : y \in F \} \quad (x \in E)
\]

It is easily seen that

\[
q(x) = \inf \{ ||x-z|| : z \in F+D \} \quad (x \in E)
\]

i.e. \( q = p_{F+D} \). Now \( F+D \) is of countable type and (ii) is proved with \( s := r_{F+D} \).

From (i), (ii) above we obtain \( 0 \in R \). So there exists a subspace \( D \) of countable type with \( r_D = 0 \) i.e. \( p_D = 0 \) on \( A \) implying \( [A] \subset \overline{D} \). It follows that \( [A] \) is of countable type.

Our next step is to prove that \( A \) is a compactoid (Corollary 2.6).

**Lemma 2.4.**

Every \( || \mid \mid \) -orthogonal sequence in \( A \) tends to 0.

**Proof.**

We may assume \( E = [A] \). Then \( E \) is of countable type (Proposition 2.3).

Suppose we had an orthogonal sequence \( e_1, e_2, \ldots \) in \( A \) with

\( s := \inf ||e_n|| > 0 \). Set \( D := [e_1, e_2, \ldots] \). By [2], Theorem 3.16 (v)

there exists a continuous linear projection \( P : E \rightarrow D \). By Proposition 1.4 \( PA \) is \( c' \)-compact. We have \( e_n \in PA \) for each \( n \in N \) and \([PA] = D \).

We therefore may also assume that \( E = D \) i.e. that \([A] = [e_1, e_2, \ldots] \).
For each $n \in \mathbb{N}$ set

$$D_n := [e_1, \ldots, e_n]$$

$$p_n(x) = \inf \{ ||x-d|| : d \in D_n \} \quad (x \in E)$$

Each $p_n$ is a continuous seminorm, $p_1 \geq p_2 \geq \ldots$ and by assumption,

$$\lim_{n \to \infty} p_n(x) = 0 \text{ for each } x \in \overline{[e_1, e_2, \ldots]} = E.$$ By orthogonality,

$$p_n(e_j) = ||e_j|| \geq s \text{ for each } n \in \mathbb{N}.$$ Let

$$s_n := \max_{A} p_n \quad (n \in \mathbb{N})$$

Then $s_n \geq s$ for each $n$. Choose real numbers $\rho_1, \rho_2, \ldots$ such that

$$0 < \rho_1 < \rho_2 < \ldots, \lim_{n \to \infty} \rho_n = 1.$$ The formula

$$p(x) = \sup_{n \in \mathbb{N}} s_n^{-1} p_n(x) \rho_n \quad (x \in E)$$

defines a seminorm on $E$. For each $x \in E$ we have

$$s_n^{-1} p_n(x) \rho_n \leq s_n^{-1} \rho_1 = s_n^{-1} ||x|| \quad (n \in \mathbb{N})$$

so $p$ is continuous. For each $n \in \mathbb{N}$ we have

$$\max_{x \in A} s_n^{-1} p_n(x) \rho_n = \rho_n$$

yielding $\sup_{n} p = 1.$

By $c'$-compactness there is an $a \in A$ with $p(a) = 1.$ As, for each $n,$

$$s_n^{-1} p_n(a) \rho_n \leq \rho_n \not= 1 \text{ there must be infinitely many } n \in \mathbb{N} \text{ for which}$$

$$s_n^{-1} p_n(a) \rho_n \geq \frac{1}{2}. \text{ Then also } p_n(a) \rho_n \geq \frac{1}{2} \text{ is for infinitely many } n \text{ which}$$

is in conflict to $\lim_{n \to \infty} p_n(a) = 0.$


PROPOSITION 2.5.

Let $E$ be of countable type. For each $s \in (0,1)$ there exist a norm
∥∥' on E such that \( s \parallel x\parallel' \leq \parallel x\parallel \leq \parallel x\parallel' \) for all \( x \in E \) and for which \((E, \parallel \cdot \parallel')\) has an orthogonal base.

**Proof.**

The statement is an easy consequence of [2], Theorem 3.16 (ii).

**COROLLARY 2.6.**

\( A \) is a compactoid.

**Proof.**

We may assume that \( E \) is of countable type. By Proposition 2.5 we may even assume that \( E \) has an orthogonal base. Now apply Lemma 2.4 and [2], Theorem 4.38, (η) ⇒ (α).

**THEOREM 2.7.**

For each \( s \in (0,1) \) there exists an \( s \)-orthogonal sequence \( e_1, e_2, \ldots \) in \( A \) for which \( \lim_{n \to \infty} e_n = 0 \) such that

\[
\text{co} \{e_1, e_2, \ldots\} \subset A \subset \text{co} \{e_1, e_2, \ldots\}. 
\]

If, in addition, \([A]\) has an orthogonal base then the sequence \( e_1, e_2, \ldots \) can be chosen to be orthogonal.

**Proof.**

We may assume that \([A]\) = \( E \). By Proposition 2.5 it suffices to prove only the second statement. We shall construct an orthogonal sequence \( e_1, e_2, \ldots \) in \( A \) such that for each \( a \in A \) and \( n \in \mathbb{N} \) there exists a \( b \in \text{co} \{e_1, \ldots, e_n\} \) with \( \|a-b\| \leq \|e_{n+1}\| \). (This proves the Theorem since, by Lemma 2.4, \( \lim_{n \to \infty} \|e_n\| = 0 \).) There is an \( e_1 \in A \) with \( \|e_1\| = \max \{\|x\| : x \in A\} \). By [2], Lemma 4.35, \( E \) is the orthogonal
direct sum of $K_\mathbf{e}_1$ and some subspace $D_1$. For each $a \in A$, $a = \lambda_1 e_1 + d_1$
($\lambda_1 \in K_1$, $d_1 \in D$), we have

$$\|a\| = \max (\|\lambda_1 e_1\|, \|d_1\|) \geq \|\lambda_1 e_1\| = \|\lambda_1\| e_1$$

so that $|\lambda_1| \leq 1$ (if $e_1 = 0$, choose $\lambda_1 = 0$). It follows that

$d_1 \in D_1 \cap A$. Therefore, $A$ decomposes into an orthogonal sum of $K_\mathbf{e}_1$ and

$D_1 \cap A$, so $D_1 \cap A$ is $c'$-compact by Proposition 1.4. There exists an

$e_2 \in D_1 \cap A$ with $\|e_2\| = \max \{\|x\| : x \in D_1 \cap A\}$. Then

$$\|a_1 - \lambda_1 e_1\| \leq \|e_2\|$$. In its turn, $D_1$ decomposes into an orthogonal sum

of $K_\mathbf{e}_2$ and a space $D_2$ such that $D_2 \cap A$ is $c'$-compact. Let $e_3 \in D_2 \cap A$

with $\|e_3\| = \max \{\|x\| : x \in D_2 \cap A\}$. Then $\|a_1 - \lambda_1 e_1 - \lambda_2 e_2\| \leq \|e_3\|$ for

some $\lambda_2 \in K$, $|\lambda_2| \leq 1$, etc.

**COROLLARY 2.8.** Let $A$ be an absolutely convex subset of a $K$-Banach space.

The following statements (a)-(d) are equivalent.

(a) $A$ is $c'$-compact.

(b) There exists a compact set $X$ with $\text{co} \: X \subset A \subset \text{co} \: X$.

(γ) There exists a sequence $e_1, e_2, \ldots$ in $A$ with $\lim_{n \to \infty} e_n = 0$ such that

$\text{co} \{e_1, e_2, \ldots\} \subset A \subset \text{co} \{e_1, e_2, \ldots\}$.

(δ) For each $s \in (0,1)$ there exists an $s$-orthogonal sequence $e_1, e_2, \ldots$

in $A$ for which $\text{co} \{e_1, e_2, \ldots\} \subset A \subset \text{co} \{e_1, e_2, \ldots\}$.

If $K$ is spherically complete, (a)-(δ) are equivalent to:

(c) There exists an orthogonal sequence $e_1, e_2, \ldots$ in $A$ such that

$\text{co} \{e_1, e_2, \ldots\} \subset A \subset \text{co} \{e_1, e_2, \ldots\}$.

**Proof.**

(c) $\Rightarrow$ (δ) $\Rightarrow$ (γ) $\Rightarrow$ (β) are obvious. (β) $\Rightarrow$ (a) follows from Propositions

1,2 and 1.3. The first part of Theorem 2.7 yields (a) $\Rightarrow$ (δ). If $K$ is

spherically complete each Banach space of countable type has an
orthogonal base ([2], Lemma 5.5) and \((\alpha) \Rightarrow (\epsilon)\) is a consequence of the second part of Theorem 2.7.
§ 3. OTHER CHARACTERIZATIONS OF C'-COMPACTNESS

In this section there is no need to restrict ourselves to Banach spaces so in § 3, let \( E \) be a locally convex space over \( K \).

DEFINITION 3.1.

An absolutely convex subset \( A \subset E \) is a pure compactoid if for each neighbourhood \( U \) of 0 there exist a finite set \( F \subset A \) such that \( A \subset U + \text{co } F \).

(The difference with the definition of 'ordinary' compactoidity ([2], p. 134) lies in the fact that we require \( F \subset A \) rather then \( F \subset E \).)

DEFINITION 3.2.

A function \( \phi : E \to \mathbb{R} \) is convex if \( n \in \mathbb{N}, x_1, \ldots, x_n \in E, \lambda_1, \ldots, \lambda_n \in K, |\lambda_i| \leq 1 \) for each \( i \), \( \sum \lambda_i = 1 \) imply

\[
\phi(\sum \lambda_i x_i) \leq \max_i |\lambda_i| \phi(x_i).
\]

(Example: \( x \mapsto p(x-a) \) for \( a \in E \) and a seminorm \( p \).)

THEOREM 3.3.

For an absolutely convex \( A \subset E \) the following statements are equivalent.

(a) \( A \) is a pure compactoid.

(b) If \( U \) is a covering of \( A \) by (convex) open sets then there exist \( n \in \mathbb{N} \) and \( U_1, \ldots, U_n \in U \) such that \( A \subset \text{co} \left( \bigcup_{i=1}^{n} U_i \right) \).

(c) Let \( U_1 \subset U_2 \subset \ldots \) be open convex sets covering \( A \). Then \( A \subset U_n \) for some \( n \).

(d) Each continuous convex function \( \phi : E \to \mathbb{R} \), restricted to \( A \), has a maximum.
(ε) A is $c'$-compact.

Proof.

(a) ⇒ (β). There is a $U_0 \in U$ with $0 \in U_0$. There exist $x_1, \ldots, x_m \in A$ such that $A \subseteq U_0 + \text{co}(x_1, \ldots, x_m)$. Let $U_1, \ldots, U_m \in U$ be with $x_i \in U_i$ for each $i \in \{1, \ldots, m\}$. Then $A \subseteq U_0 + \text{co}(U_1) \subseteq \text{co}(U_0 + U_i)$. To prove $i=0$.

(β) ⇒ (γ) we may assume that $0 \in U_1$ so that all $U_i$ are absolutely convex. By (β) there exists a finite set $F \subseteq N$ such that $A \subseteq \text{co}(U_i)$. But then $A \subseteq \text{co} U_n = U_n$ where $n = \max F$.

(γ) Suppose $s$ is not a value of $\phi|A$. Choose $s_1 < s_2 < \ldots$, $\lim_{n \to \infty} s_n = s$ and set $U_n := \{x \in E : \phi(x) < s_n\}$ (n \in N)

Each $U_n$ is open. As $\phi$ is convex, $U_n$ is convex. Further we have $U_1 \subseteq U_2 \subseteq \ldots$ and, by assumption, the $U_n$ cover $A$. By (γ), $A \subseteq U_n$ for some $n$ implying $\phi < s_n$ on $A$, a contradiction.

(δ) ⇒ (ε) is trivial.

Finally we prove (ε) ⇒ (α). First, assume that $E$ is a Banach space.

From Corollary 2.8 we obtain a compact set $X \subseteq E$ with $\text{co} X \subseteq A \subseteq \text{co} X$.

Let $U$ be an absolutely convex neighbourhood of $0$ in $E$. By compactness there exist $x_1, \ldots, x_n \in X$ such that $X \subseteq \bigcup_{i=1}^n (x_i + U)$. Then $\text{co} X \subseteq U + \text{co}(x_1, \ldots, x_n)$. As the latter set is an open additive subgroup of $E$ it is also closed and we have $A \subseteq \text{co} X \subseteq U + \text{co}(x_1, \ldots, x_n)$.

Now let $E$ be a locally convex space, and let $U$ be an absolutely convex neighbourhood of $0$ in $E$. There is a continuous seminorm $p$ such that
\{x \in E : p(x) \leq 1\} \subset U. Let \(\pi_p : E \to E_p\) be the canonical quotient map, where \(E_p\) is the completion of the space \(E_p := E/\text{Ker}p\) with the norm induced by \(p\). By Proposition 1.4 the set \(\pi_p(A)\) is \(c'\)-compact and we just have proved that it is a pure compactoid in \(E_p\), hence in \(E\).

Since \(\pi_p(U)\) is open in \(E\) there exists a finite set \(F \subset \pi_p(A)\) such that \(\pi_p(A) \subset \pi_p(U) + \text{co} F\). Choose a finite set \(F' \subset A\) such that \(\pi_p(F') = F\). We then have \(\pi_p(A) \subset \pi_p(U + \text{co} F')\) and, as \(\text{Ker} \pi_p \subset U\),

\[A \subset U + \text{co} F' + \text{Ker} \pi_p \subset U + \text{co} F'.\]

**Remark.**

The following statements are easy to prove.

(i) If the valuation of \(K\) is discrete the properties (a) - (e) of Theorem 3.3 are equivalent to 'A is a compactoid'.

(ii) If \(K\) is locally compact the properties (a) - (e) of Theorem 3.3 are equivalent to 'A is precompact'.

§ 4. C'-'COMPACTNESS VERSUS C-COMPACTNESS

First we extend Theorem 3.3 to convex sets. For a subset \( X \) of a K-linear space, let \( c(X) \) be its convex hull.

**THEOREM 4.1.**

Let \( C \) be a convex subset of a locally convex space \( E \) over \( K \). The following are equivalent.

(a) For each neighbourhood of 0 in \( E \) there exists a finite set \( F \subset C \) with \( C \subset U + c(F) \).

(b) If \( U \) is a covering of \( C \) by (convex) open sets then there exist \( n \in \mathbb{N} \) and \( U_1, \ldots, U_n \in U \) such that \( C \subset c\left( \bigcup_{i=1}^{n} U_i \right) \).

(c) Let \( U_1 \subset U_2 \subset \ldots \) be open convex sets covering \( C \). Then \( C \subset U_n \) for some \( n \).

(d) Each continuous convex function \( \phi : E \to \mathbb{R} \), restricted to \( C \), has a maximum.

(e) For each continuous seminorm \( p \) and each \( a \in C \), \( \max \{ p(x-a) : x \in C \} \) exists.

**Proof.**

Straightforward.

A convex set \( C \subset E \) is \( c' \)-compact if it satisfies (a) - (e) of Theorem 4.1.

The following theorem explains the term 'complementary' in the title of this paper. (Compare Theorem 4.1.)

**THEOREM 4.2.**

Let \( C \) be a convex subset of a Banach space \( E \) over \( K \). The following are
equivalent.

(β)' If \( C \) is a collection of closed convex sets in \( E \) such that \( n \subseteq C \) does not meet \( C \) then there exist \( n \in \mathbb{N} \) and \( C_1, \ldots, C_n \subseteq C \) such that \( C \cap \bigcap_{i=1}^{n} C_i = \emptyset \).

(γ)' Let \( C_1 \supseteq C_2 \supseteq \ldots \) be closed convex sets in \( E \) such that \( C \cap \bigcap_{i=1}^{n} C_i = \emptyset \). Then \( C \cap C_n = \emptyset \) for some \( n \).

(δ)' Each continuous convex function \( \phi : E \to \mathbb{R} \), restricted to \( C \), has a minimum.

(ε)' \( C \) is c-compact.

Proof.

(ε)' \( \Rightarrow \) (β)' is just the definition of c-compactness ([4]), (β)' \( \Rightarrow \) (γ)' is obvious. The proof of (γ)' \( \Rightarrow \) (δ)' runs similar to the one of (γ) \( \Rightarrow \) (δ) of Theorem 3.3. To prove (δ)' \( \Rightarrow \) (ε)' it suffices, by [1] Theorem 6.15 (ζ) \( \Rightarrow \) (a), to show that \( C \) is spherically complete relative to any norm \( \| \| \) defining the topology of \( E \). Thus, let \( B_1 \supseteq B_2 \supseteq \ldots \) be balls in \( C \) where

\[
B_n = \{ x \in C : \| x - c_n \| \leq r_n \} \quad (n \in \mathbb{N})
\]

for some \( c_1, c_2, \ldots \) in \( C \) and \( r_1 \geq r_2 \geq \ldots \). Let \( (E^\sim, \| \|) \) be the spherical completion ([2], Theorem 4.43) of \( (E, \| \|) \) and consider for each \( n \in \mathbb{N} \)

\[
B_n^\sim := \{ x \in E^\sim : \| x - c_n \| \leq r_n \}
\]

These \( B_n^\sim \) form a nested sequence of balls in \( E^\sim \) so there exists a \( z \in \bigcap_{n} B_n^\sim \). The function \( \phi : x \mapsto \| x - z \| \) (\( x \in E \)) is convex and attains a minimum on \( C \), say in \( c \in C \). As \( \phi(c_n) \leq r_n \) (\( n \in \mathbb{N} \)) we have \( \phi(c) \leq \inf_{n} r_n \). For each \( n \in \mathbb{N} \)
We see that $c \in B_n$ for each $n$ and it follows that $A$ is spherically complete for $\| \cdot \|$.

Remarks.

(1) Theorem 4.2 is only of interest if $K$ is spherically complete ([4], (2.1)). I do not know whether the properties $(\beta)' - (e)'$ are equivalent for a convex set $C$ in a locally convex space $E$. (Of course one has some obvious implications.)

(2) The notion of $c$-compactness may be viewed as a 'convexification' of the intersection property for closed sets in a compact space, whereas $c'$-compactness can be seen as a 'convexification' of the 'open covering' definition of compactness. (Theorem 4.1 $(\beta)$, Theorem 4.2 $(\beta)'$.)

(3) In a future paper [3] we shall discuss the relation between weak and strong $c'$-compactness.

REFERENCES


