THE CLOSED CONVEX HULL OF A COMPACT SET
IN A NON-ARCHIMEDEAN LOCALLY CONVEX SPACE

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ABSTRACT. For a complete absolutely convex set A in a locally convex space over a non-archimedean valued field K it is proved that

(i) A is the closed absolutely convex hull of a compact set if and only if A is isomorphic to some power of

\[ B(0,1) := \{ \lambda \in K : |\lambda| \leq 1 \}, \]

(ii) if the valuation of K is discrete and A is a compactoid (equivalently; A is c-compact and bounded) then A is the closed absolutely convex hull of a compact set,

(iii) the conclusion of (ii) is also true for any K if A is a metrizable pure compactoid,

(iv) if A is a compactoid it is isomorphic to a closed submodule of some power of B(0,1).

These results extend those (for a locally compact base field) of Carpentier ([1], Propositions 72,73). Corollary 1.8 is a non-archimedean approach to the Krein-Milman Theorem.
PRELIMINARIES. Throughout $K$ is a non-archimedean nontrivially valued complete field with valuation $|\cdot|$. For fundamentals on locally convex spaces $E$ over $K$ (which we assume to be Hausdorff) we refer to [8], [7], [3], [4], [1]. A set $A \subset E$ is absolutely convex if it is a $B(0,1)$-module. If $F$ is a locally convex space over $K$ and $A \subset E$, $B \subset F$ are absolutely convex then $\phi : A \to B$ is affine if it is a homeomorphism of $B(0,1)$-modules. We shall write $A = B$ if there exists an affine homeomorphism of $A$ onto $B$. For a set $X \subset E$, let $co X$ be its absolutely convex hull and $\overline{co} X$ be its closure.

An absolutely convex set $A \subset E$ is edged if for each $x \in E$ the set $
abla \{ |\lambda| : \lambda x \in A \}$ is closed in $|K| := \{ |\lambda| : \lambda \in K \}$ (or, equivalently, if $A = \{ x \in [A] : p_A(x) \leq 1 \}$, where $p_A$ is the Minkowski function, defined on the $K$-linear span $[A]$ of $A$ by the formula $p_A(x) = \inf \{ |\lambda| : x \in \lambda A \}$). It is easy to prove that if the valuation of $K$ is discrete each absolutely convex set is edged whereas, if the valuation of $K$ is dense, an absolutely convex $A \subset E$ is edged if and only if $\lambda x \in A$ for all $\lambda \in K$, $|\lambda| < 1$ implies $x \in A$.

For a subset $A$ of $E$, let $A^0 := \{ f \in E' : |f(x)| \leq 1 \text{ for all } x \in A \}$ (where $E'$ is the dual space of $E$) and let $A^\infty := \{ x \in E : |f(x)| \leq 1 \text{ for all } f \in A \}$. $A$ is a polar set if $A = A^\infty$.

A set $A \subset E$ is (a) compactoid if for each neighbourhood $U$ of $0$ there exist $n \in N$ and $x_1, \ldots, x_n \in E$ such that $A \subset U + co(x_1, \ldots, x_n)$; it is a pure compactoid if in the above we may choose $x_1, \ldots, x_n \in A$. If the valuation of $K$ is discrete each absolutely convex compactoid is pure (for example [4], Lemma 8.1), if the valuation of $K$ is dense $\{ \lambda \in K : |\lambda| < 1 \}$ is a compactoid in $E := K$ but not pure.
LEMMA 1.1 Let $E$ be a locally convex space over $K$. Let $A \subseteq E$ be a complete, absolutely convex, edged, absorbing compactoid. Assume that a seminorm on $E$ is continuous if its restriction to $A$ is continuous.

(i) $E$ is of countable type ([4], Definition 4.3).
(ii) $A$ is a polar set.
(iii) $E'$ is a Banach space over $K$ with respect to the norm $\| \cdot \|_A$ defined by $\|f\|_A = \sup \{ |f(x)| : x \in A \}$.
(iv) If $A = \text{co} \, X$ for some compact set $X \subseteq A$ then $(E', \| \cdot \|_A)$ has an orthonormal base.
(v) The canonical map $E \to (E', \| \cdot \|_A)'$ is a bijection.

Proof.

(i) [6], Proposition 4.3.
(ii) [4], Theorem 4.7.
(iii) As $A$ is absorbing $\| \cdot \|_A$ is a norm on $E'$. If $f_1, f_2, \ldots$ is a $\| \cdot \|_A$-Cauchy sequence in $E'$ then there is a linear $f : E \to K$ such that $f = \lim_{n \to \infty} f_n$ uniformly on $A$. Then $|f|$, restricted to $A$, is continuous. By assumption, $|f|$ is continuous. Hence, $f \in E'$ and $\lim_{n \to \infty} \|f - f_n\|_A = 0$.
(iv) Let $C(X \to K)$ be the Banach space of all continuous functions $X \to K$ with the supremum norm. For each $f \in E'$ we have $\|f\|_A = \sup_{A} |f| = \sup_{\text{co} \, X} |f| = \sup_{X} |f|$ so that the map $T : (E', \| \cdot \|_A) \to C(X \to K)$ given by $Tf := f|_{X}$ is a linear isometry. By [3], Theorem 5.22, $C(X \to K)$ has an
orthonormal base. Then so has its closed subspace \( \text{Im} T \) by Gruson's Theorem ([3], Theorem 5.9) and has \((E', || \cdot ||_A)\).

(iv) Contained in the proof of [6], Theorem 3.2 (the metrizability condition is not needed for part (ii) of that proof).

**Lemma 1.2** Let \( E, A, || \cdot ||_A \) be as in Lemma 1.1. Suppose \((E', || \cdot ||_A)\) has an orthonormal base \( \{f_i : i \in I\} \). Then \( A \simeq B(0,1)^I \).

**Proof.** The formula

\[
\phi(x) = (f_i(x))_{i \in I} \quad (x \in E)
\]

defines a continuous linear map \( \phi : E \to K^I \) (on \( K^I \) the product topology) sending \( A \) into \( B(0,1)^I \). We prove (i), (ii) below.

(i) \( \phi|_A \) is a homeomorphism into \( B(0,1)^I \). **Proof.** Let \( (x_j)_j \in J \) be a net in \( A \) for which \( \lim_j \phi(x_j) = 0 \) i.e. \( \lim_j f_i(x_j) = 0 \) for all \( i \in I \). Then \( \lim_j g(x_j) = 0 \) for all \( g \) in a \( || \cdot ||_A \) dense subset \( H \) of \( E' \). Let \( f \in E' \), \( \varepsilon > 0 \). There is a \( g \in H \) with \( ||f-g||_A < \varepsilon \).

For large \( j \)

\[
|f(x_j)| \leq \max \{ |f(x_j) - g(x_j)|, |g(x_j)| \} < \varepsilon
\]

so that \( \lim_j x_j = 0 \) weakly. But then \( \lim_j x_j = 0 \) for the initial topology of \( E \) ([4], Theorem 5.12).

(ii) \( \phi \) maps \( A \) onto \( B(0,1)^I \). **Proof.** Let \( z := (z_i)_i \in I \in B(0,1)^I \). Define \( h : (E', || \cdot ||_A)' \) by

\[
h(f_i) = z_i \quad (i \in I)
\]

By Lemma 1.1 (v) there exists an \( x \in E \) with \( f(x) = h(f) \) for all \( f \in E' \) i.e. with \( \phi(x) = z \). To prove that in fact \( x \in A^{\infty} = A \).
(Lemma 1.1 (ii)), let \( f \in E', f \in A^0 \). Then \( \| f \|_A \leq 1 \). There exist \( \lambda_i \in K \) for which \( f = \sum_{i \in I} \lambda_i f_i \) in the sense of \( \| \|_A \). By orthonormality
\[
\| f \|_A = \max_{i \in I} |\lambda_i| \leq 1.
\]
We see that \( |f(x)| \leq \max_{i \in I} |\lambda_i f_i(x)| = \max_{i \in I} |\lambda_i z_i| \leq 1 \). It follows that
\[
x \in A^\circ.
\]

**Proposition 1.3** Let \( X \) be a compact subset of a locally convex space \( E \) over \( K \). Then \( \overline{co} X \) is edged.

**Proof.** We may assume that the valuation of \( K \) is dense. Let \( z \in E \),
\( z \notin \overline{co} X \). There is ([6], Proposition 4.2) a continuous seminorm \( p \)
with \( p(z) = 1 \) and \( p < 1 \) on \( \overline{co} X \). By compactness, \( s := \sup \{ p(x) : x \in \overline{co} X \} \)
\[= \sup_{x \in X} p = \max_{x \in X} p < 1.\]
Hence, there is a \( \lambda \in K, |\lambda| < 1 \), such that
\[
p(\lambda z) > s \quad \text{i.e.} \quad \lambda z \notin \overline{co} X.
\]

**Theorem 1.4** Let \( A \) be a complete absolutely convex compactoid in a
locally convex space \( E \) over \( K \). The following are equivalent.

(a) There is a compact set \( X \subset A \) with \( A = \overline{co} X \).

(b) \( A = B(0,1)_I \) for some set \( I \).

**Proof.** (a) \( \Rightarrow \) (b). We may assume that \( E = [A] \). If we replace the ini-
tial topology \( \tau \) of \( E \) by the stronger locally convex topology \( \tau' \) gene-
erated by all seminorms \( p \) on \( E \) for which \( p|A \) is \( \tau \)-continuous then
\[
\tau = \tau' \quad \text{on} \quad A \quad \text{and} \quad A \quad \text{is} \quad \tau' \quad \text{-complete and a} \quad \tau' \quad \text{-compactoid ([6], Proposition 4.5). Therefore, to prove (b), we may assume} \quad \tau = \tau'. \quad \text{Now apply Proposition 1.3, Lemma 1.1 (iv), Lemma 1.2.}
\]

(a) \( \Rightarrow \) (b). Let \( e_i \in B(0,1)_I \) (\( i \in I \)) be given by
\[ \{ e_i \}_j = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases} \]

It is easily seen that \( Y := \{ 0 \} \cup \{ e_i : i \in I \} \) is compact and that 
\( B(0,1)^I = \overline{\text{co } Y} \).

**Theorem 1.5** Let the valuation of \( K \) be discrete. Let \( A \) be a complete absolutely convex compactoid in a locally convex space \( E \) over \( K \). (Or, equivalently, let \( A \) be bounded, absolutely convex and \( c \)-compact \(([6], \text{Corollary 2.5}) \).) Then there exists a compact set \( \overline{X} \subseteq \overline{A} \) with \( A = \overline{\text{co } X} \).

**Proof.** For the same reasons as in the previous proof we may assume that \( E = [A] \) and that a seminorm \( p \) on \( E \) is continuous if \( p|A \) is continuous. By Lemma 1.1 (iii), \( (E', \| \|_A) \) is a Banach space. As the valuation is discrete we have
\[ \| f \|_A = \sup_{x \in A} |f(x)| \in K \quad (f \in E') \]
Then by [3], Theorem 5.16, \( (E', \| \|_A) \) has an orthonormal base. Now apply Lemma 1.2 and Theorem 1.4.

For general \( K \) not every edged complete absolutely convex compactoid is the closed convex hull of a compact set. In fact, if the valuation of \( K \) is dense and \( r \in (0,\infty) \setminus K \) then \( A := \{ \lambda \in K : |\lambda| \leq r \} \) is edged but there is no compact set \( X \subseteq K \) for which \( A = \overline{\text{co } X} \). Indeed, we have the following.

**Proposition 1.6** Let \( X \) be a compact subset of a locally convex space \( E \) over \( K \). Then \( \overline{\text{co } X} \) is a pure compactoid.

**Proof.** Let \( U \) be an absolutely convex neighbourhood of \( 0 \) in \( E \). By com-
pactness there exist $x_1, \ldots, x_n \in X$ such that $X \subseteq U(x_i+U)$. Then $
abla x \subseteq U+\co\{x_1, \ldots, x_n\}$. The set $U+\co\{x_1, \ldots, x_n\}$ is an open additive subgroup of $E$, hence closed. It follows that $\co X \subseteq U+\co\{x_1, \ldots, x_n\}$.

Note. One can prove that each closed pure absolutely convex compactoid is edged.

For metrizable pure compactoids we have the following version of Theorem 1.5 for general $K$.

**THEOREM 1.7** Let $A \subseteq E$ be a complete absolutely convex pure compactoid that is metrizable. Then there is a sequence $e_1, e_2, \ldots$ in $A$ with $\lim_{n \to \infty} e_n = 0$ and $A = \co\{e_1, e_2, \ldots\}$.

**Proof.** The proof of [4], Proposition 8.2 applies with some minor modifications (as $A$ is pure the finite sets $F_1, F_2, \ldots$ constructed in that proof can be chosen in $A$ rather than in $\lambda A$).

**OPEN PROBLEM** Let $A$ be complete absolutely convex pure compactoid in a locally convex space $E$ over $K$. Does it follow that $A = \co X$ for some compact $X$?

The previous theory yields the following.

**COROLLARY 1.8** Let $A$ be a complete subset of a locally convex space $E$ over $K$ such that $A = \co X$ for some compact set $X$ (e.g. choose for $A$ any complete absolutely convex compactoid if the valuation of $K$ is discrete or any complete absolutely convex pure metrizable compactoid). Then there exists a linearly independent set.
\( Y = \{ e_i : i \in I \} \) in \( A \) such that

(i) \( Y \) is discrete,

(ii) for each neighbourhood \( U \) of 0 the set \( \{ i \in I : e_i \notin U \} \) is finite,

(iii) \( Y_0 := Y \cup \{0\} \) is compact,

(iv) \( A = \text{co} \ Y = \text{co} \ Y_0 \),

(v) for each \( (\lambda_i)_{i \in I} \in B(0,1)^I \), \( \sum_{i \in I} \lambda_i e_i \) converges and represents an element of \( A \),

(vi) each \( x \in A \) has a unique representation as a convergent sum

\[ x = \sum_{i \in I} \lambda_i e_i \]  

where \( \lambda_i \in B(0,1) \) for each \( i \in I \),

(vii) \( Y \) is a minimal element of \( \{ Z \subset E : A = \text{co} Z \} \)

(viii) \( Y_0 \) is a minimal element of \( \{ Z \subset E, Z \text{ is compact}, A = \text{co} Z \} \),

(ix) \( Y \) is a \( p_A \)-orthonormal set.

Proof. By Theorem 1.4 we may assume \( A = B(0,1)^I \). Choose \( \{ e_i : i \in I \} \) as in the second part of the proof of Theorem 1.4. We leave the details of checking (i)-(ix) to the reader.
THEOREM 2.1 Let $A$ be a compactoid in a locally convex space $E$ over $K$. Then there exists a locally convex space $F$ over $K$ containing $E$ as a subspace and a compact set $X \subset F$ such that $A \subset \text{co } X$.

Proof. For each continuous seminorm $p$ on $E$, let $E_p := E/\text{Ker}p$ with the norm induced by $p$. The natural maps $\pi_p : E \to E_p$ yield a linear homeomorphic embedding

$$\pi : E \to F := \prod_{p \in \Gamma} E_p$$

where $\Gamma$ is the collection of continuous seminorms of $E$. For each $p \in \Gamma$ the set $\pi_p(A)$ is a (metrizable) compactoid in $E_p$. By [4], Proposition 8.2 there is a compact set $X_p \subset E_p$ such that $\pi_p(A) \subset \text{co } X_p$. Without loss we may assume that $0 \in X_p$. We have

$$\pi(A) \subset \prod_{p \in \Gamma} \pi(A) \subset \prod_{p \in \Gamma} \text{co } X_p$$

We claim that $\prod_{p \in \Gamma} \text{co } X_p \subset \text{co } \prod_{p \in \Gamma} X_p$. (Then the theorem is proved with $X := \prod_{p \in \Gamma} X_p$.)

For $p \in \Gamma$ and $x \in X_p$ the element $f$ defined by

$$(*) \quad f(q) = \begin{cases} x & \text{if } q \in \Gamma, q = p \\ 0 & \text{if } q \in \Gamma, q \neq p \end{cases}$$

is in $\prod_{p \in \Gamma} X_p$. If $p \in \Gamma$ and $x \in \text{co } X_p$ then $f$, formally defined by $(*),$ is in $\text{co } \prod_{p \in \Gamma} X_p$. If $p_1, \ldots, p_n \in \Gamma$ and $x \in \text{co } X_{p_i}$ for $i \in \{1, \ldots, n\}$ the element $g$ defined by

$$(**) \quad g(q) = \begin{cases} x_{p_i} & \text{if } q \in \Gamma, i \in \{1, \ldots, n\}, q = p_i \\ 0 & \text{if } q \in \Gamma, \text{ otherwise} \end{cases}$$
is a finite sum of elements of \( \overline{\text{co}} \prod_{p \in \Gamma} X_p \), hence in \( \overline{\text{co}} \prod_{p \in \Gamma} X_p \). The elements of the type defined in (***) are dense in \( \prod_{p \in \Gamma} \overline{\text{co}} X_p \).

**COROLLARY 2.2** Let \( A \) be an absolutely convex compactoid in a locally convex space over \( K \). Then \( A \) is isomorphic to a submodule of some power of \( B(0,1) \).

**Proof.** By the previous theorem, \( A \subset \overline{\text{co}} X \) for some compact \( X \subset F \). We may suppose that \( F \) is complete. Now apply Theorem 1.4.

**Note to Theorem 2.1.** It is too optimistic to hope that in Theorem 2.1 we may require that \( F = E \) even when we allow \( X \) to be precompact. In fact, let \( K \) be not locally compact, let \( E = c_0 \), with the weak topology. \( A := \{ x \in E : \|x\| \leq 1 \} \) is a weak compactoid but according to [5], Proposition 3.3 there is no weakly precompact \( X \subset E \) for which \( A \subset \overline{\text{co}} X \). Observe that, if \( K \) is not spherically complete, \( A \) is even weakly complete ([4], Theorem 9.6), and that \( A \) is pure if the valuation of \( K \) is discrete.
REFERENCES


