THE CLOSED CONVEX HULL OF A COMPACT SET IN A NON-ARCHIMEDEAN LOCALLY CONVEX SPACE

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Report 8646
October 1986

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ABSTRACT. For a complete absolutely convex set A in a locally convex space over a non-archimedean valued field K it is proved that

(i) A is the closed absolutely convex hull of a compact set if and only if A is isomorphic to some power of $B(0,1) := \{\lambda \in K : |\lambda| \leq 1\}$,

(ii) if the valuation of K is discrete and A is a compactoid (equivalently; A is c-compact and bounded) then A is the closed absolutely convex hull of a compact set,

(iii) the conclusion of (ii) is also true for any K if A is a metrizable pure compactoid,

(iv) if A is a compactoid it is isomorphic to a closed submodule of some power of $B(0,1)$.

These results extend those (for a locally compact base field) of Carpentier ([1], Propositions 72,73). Corollary 1.8 is a non-archimedean approach to the Krein-Milman Theorem.

PRELIMINARIES. Throughout K is a non-archimedean nontrivially valued complete field with valuation $|\ |$. For fundamentals on locally convex spaces E over K (which we assume to be Hausdorff) we refer to [8], [7], [3], [4], [1]. A set $A \subset E$ is <u>absolutely convex</u> if it is a $B(0,1)$-module. If F is a locally convex space over K and $A \subset E$, $B \subset F$ are absolutely convex then $\phi : A \to B$ is <u>affine</u> if it is a homeomorphism of $B(0,1)$-modules. We shall write $A \simeq B$ if there exists an affine homeomorphism of A onto B. For a set $X \subset E$, let co X be its absolutely convex hull and $\overline{\text{co}}\, X$ be its closure.

An absolutely convex set $A \subset E$ is <u>edged</u> if for each $x \in E$ the set $\{|\lambda| : \lambda x \in A\}$ is closed in $|K| := \{|\lambda| : \lambda \in K\}$ (or, equivalently, if $A = \{x \in [A] : p_A(x) \le 1\}$, where p_A is the Minkowski function, defined on the K-linear span $[A]$ of A by the formula $p_A(x) = \inf \{|\lambda| : x \in \lambda A\}$). It is easy to prove that if the valuation of K is discrete each absolutely convex set is edged whereas, if the valuation of K is dense, an absolutely convex $A \subset E$ is edged if and only if $\lambda x \in A$ for all $\lambda \in K$, $|\lambda| < 1$ implies $x \in A$.

For a subset A of E, let $A^{\circ} := \{f \in E' : |f(x)| \le 1 \text{ for all } x \in A\}$ (where E' is the dual space of E) and let $A^{\circ\circ} := \{x \in E : |f(x)| \le 1$ for all $f \in A^{\circ}\}$. A is a <u>polar set</u> if $A = A^{\circ\circ}$.

A set $A \subset E$ is (<u>a</u>) <u>compactoid</u> if for each neighbourhood U of 0 there exist $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in E$ such that $A \subset U + \text{co}\{x_1, \ldots, x_n\}$; it is a <u>pure compactoid</u> if in the above we may choose $x_1, \ldots, x_n \in A$. If the valuation of K is discrete each absolutely convex compactoid is pure (for example [4], Lemma 8.1), if the valuation of K is dense $\{\lambda \in K : |\lambda| < 1\}$ is a compactoid in $E := K$ but not pure.

§ 1 COMPLETE COMPACTOIDS

LEMMA 1.1 *Let E be a locally convex space over K. Let $A \subset E$ be a complete, absolutely convex, edged, absorbing compactoid. Assume that a seminorm on E is continuous if its restriction to A is continuous.*

(i) *E is of countable type* ([4], Definition 4.3).

(ii) *A is a polar set.*

(iii) *E' is a Banach space over K with respect to the norm $\| \ \|_A$ defined by* $\|f\|_A = \sup \{|f(x)| : x \in A\}$.

(iv) *If* $A = \overline{\mathrm{co}}\, X$ *for some compact set* $X \subset A$ *then* $(E', \| \ \|_A)$ *has an orthonormal base.*

(v) *The canonical map* $E \to (E', \| \ \|_A)'$ *is a bijection.*

Proof.

(i) [6], Proposition 4.3.

(ii) [4], Theorem 4.7.

(iii) As A is absorbing $\| \ \|_A$ is a norm on E'. If $f_1, f_2, \ldots$ is a $\| \ \|_A$ -Cauchy sequence in E' then there is a linear $f : E \to K$ such that $f = \lim_{n\to\infty} f_n$ uniformly on A. Then $|f|$, restricted to A, is continuous. By assumption, $|f|$ is continuous. Hence, $f \in E'$ and $\lim_{n\to\infty} \|f-f_n\|_A = 0$.

(iv) Let $C(X \to K)$ be the Banach space of all continuous functions: $X \to K$ with the supremum norm. For each $f \in E'$ we have

$$\|f\|_A = \sup_A |f| = \sup_{\mathrm{co}\, X} |f| = \sup_X |f|$$

so that the map $T : (E', \| \ \|_A) \to C(X \to K)$ given by $Tf := f|X$ is a linear isometry. By [3], Theorem 5.22, $C(X \to K)$ has an orthonormal base. Then so has its closed subspace Im T by Gruson's Theorem ([3], Theorem 5.9) and has $(E', \| \ \|_A)$.

(iv) Contained in the proof of [6], Theorem 3.2 (the metrizability condition is not needed for part (ii) of that proof).

LEMMA 1.2 Let E, A, $\| \ \|_A$ be as in Lemma 1.1. Suppose $(E', \| \ \|_A)$ has an orthonormal base $\{f_i : i \in I\}$. Then $A \approx B(0,1)^I$.

Proof. The formula

$$\Phi(x) = (f_i(x))_{i \in I} \qquad (x \in E)$$

defines a continuous linear map $\Phi : E \to K^I$ (on K^I the product topology) sending A into $B(0,1)^I$. We prove (i), (ii) below.

(i) $\Phi|A$ is a homeomorphism into $B(0,1)^I$. Proof. Let $(x_j)_{j \in J}$ be a net in A for which $\lim_j \Phi(x_j) = 0$ i.e. $\lim_j f_i(x_j) = 0$ for all $i \in I$. Then $\lim_j g(x_j) = 0$ for all g in a $\| \ \|_A$ dense subset H of E'. Let $f \in E'$, $\varepsilon > 0$. There is a $g \in H$ with $\|f-g\|_A < \varepsilon$. For large j

$$|f(x_j)| \le \max\,(|f(x_j)-g(x_j)|\ ,\ |g(x_j)|) < \varepsilon$$

so that $\lim_j x_j = 0$ weakly. But then $\lim_j x_j = 0$ for the initial topology of E ([4], Theorem 5.12).

(ii) Φ maps A onto $B(0,1)^I$. Proof. Let $z := (z_i)_{i \in I} \in B(0,1)^I$. Define $h \in (E', \| \ \|_A)'$ by

$$h(f_i) = z_i \qquad (i \in I)$$

By Lemma 1.1 (v) there exists an $x \in E$ with $f(x) = h(f)$ for all $f \in E'$ i.e. with $\Phi(x) = z$. To prove that in fact $x \in A^{\circ\circ} = A$

(Lemma 1.1 (ii)), let $f \in E'$, $f \in A^{\circ}$. Then $\|f\|_A \leq 1$. There exist $\lambda_i \in K$ for which $f = \sum_{i \in I} \lambda_i f_i$ in the sense of $\| \ \|_A$. By orthonormality

$$\|f\|_A = \max |\lambda_i| \leq 1.$$

We see that $|f(x)| \leq \max_i |\lambda_i f_i(x)| = \max_i |\lambda_i z_i| \leq 1$. It follows that $x \in A^{\circ\circ}$.

PROPOSITION 1.3 *Let* X *be a compact subset of a locally convex space* E *over* K. *Then* $\overline{co}$ X *is edged*.

Proof. We may assume that the valuation of K is dense. Let $z \in E$, $z \notin \overline{co}\, X$. There is ([6], Proposition 4.2) a continuous seminorm p with $p(z) = 1$ and $p < 1$ on $\overline{co}\, X$. By compactness, $s := \sup\{p(x) : x \in \overline{co}\, X\} = \sup_X p = \max_X p < 1$. Hence, there is a $\lambda \in K$, $|\lambda| < 1$, such that $p(\lambda z) > s$ i.e. $\lambda z \notin \overline{co}\, X$.

THEOREM 1.4 *Let* A *be a complete absolutely convex compactoid in a locally convex space* E *over* K. *The following are equivalent*.

(α) *There is a compact set* $X \subset A$ *with* $A = \overline{co}\, X$.

(β) $A \simeq B(0,1)^I$ *for some set* I.

Proof. (α) ➡ (β). We may assume that $E = [A]$. If we replace the initial topology τ of E by the stronger locally convex topology τ' generated by all seminorms p on E for which $p|A$ is τ-continuous then $\tau = \tau'$ on A and A is τ'-complete and a τ'-compactoid ([6], Proposition 4.5). Therefore, to prove (β), we may assume $\tau = \tau'$. Now apply Proposition 1.3, Lemma 1.1 (iv), Lemma 1.2.

(α) ➡ (β). Let $e_i \in B(0,1)^I$ ($i \in I$) be given by

$$(e_i)_j = \begin{cases} 1 \text{ if } j = i \\ 0 \text{ if } j \neq i \end{cases}$$

It is easily seen that $Y := \{0\} \cup \{e_i : i \in I\}$ is compact and that $B(0,1)^I = \overline{\text{co}}\, Y$.

THEOREM 1.5 Let the valuation of K be discrete. Let A be a complete absolutely convex compactoid in a locally convex space E over K. (Or, equivalently, let A be bounded, absolutely convex and c-compact ([6], Corollary 2.5).) Then there exists a compact set $X \subset A$ with $A = \overline{\text{co}}\, X$.

Proof. For the same reasons as in the previous proof we may assume that $E = [A]$ and that a seminorm p on E is continuous if $p|A$ is continuous. By Lemma 1,1 (iii), $(E', \|\ \|_A)$ is a Banach space. As the valuation is discrete we have

$$\|f\|_A = \sup_{x \in A} |f(x)| \in |K| \qquad (f \in E')$$

Then by [3], Theorem 5.16, $(E', \|\ \|_A)$ has an orthonormal base. Now apply Lemma 1.2 and Theorem 1.4.

For general K not every edged complete absolutely convex compactoid is the closed convex hull of a compact set. In fact, if the valuation of K is dense and $r \in (0,\infty)\setminus|K|$ then $A := \{\lambda \in K : |\lambda| \leq r\}$ is edged but there is no compact set $X \subset K$ for which $A = \overline{\text{co}}\, X$. Indeed, we have the following.

PROPOSITION 1.6 Let X be a compact subset of a locally convex space E over K. Then $\overline{\text{co}}\, X$ is a pure compactoid.

Proof. Let U be an absolutely convex neighbourhood of 0 in E. By compactness there exist $x_1,\ldots,x_n \in X$ such that $X \subset \bigcup_i (x_i+U)$. Then $\mathrm{co}\, X \subset U+\mathrm{co}\{x_1,\ldots,x_n\}$. The set $U+\mathrm{co}\{x_1,\ldots,x_n\}$ is an open additive subgroup of E, hence closed. It follows that $\overline{\mathrm{co}}\, X \subset U+\mathrm{co}\{x_1,\ldots,x_n\}$.

Note. One can prove that each closed pure absolutely convex compactoid is edged.

For metrizable pure compactoids we have the following version of Theorem 1.5 for general K.

THEOREM 1.7 *Let $A \subset E$ be a complete absolutely convex pure compactoid that is metrizable. Then there is a sequence $e_1,e_2,\ldots$ in A with $\lim_{n\to\infty} e_n = 0$ and $A = \overline{\mathrm{co}}\{e_1,e_2,\ldots\}$.*

Proof. The proof of [4], Proposition 8.2 applies with some minor modifications (as A is pure the finite sets $F_1,F_2,\ldots$ constructed in that proof can be chosen in A rather than in λA).

OPEN PROBLEM Let A be complete absolutely convex pure compactoid in a locally convex space E over K. Does it follow that $A = \overline{\mathrm{co}}\, X$ for some compact X ?

The previous theory yields the following.

COROLLARY 1.8 *Let A be a complete subset of a locally convex space E over K such that $A = \overline{\mathrm{co}}\, X$ for some compact set X (e.g. choose for A any complete absolutely convex compactoid if the valuation of K is discrete or any complete absolutely convex pure metrizable compactoid). Then there exists a linearly independent set*

$Y = \{e_i : i \in I\}$ *in* A *such that*

(i) Y *is discrete,*

(ii) *for each neighbourhood* U *of* 0 *the set* $\{i \in I : e_i \notin U\}$ *is finite,*

(iii) $Y_0 := Y \cup \{0\}$ is compact,

(iv) $A = \overline{co}\, Y = \overline{co}\, Y_0$,

(v) *for each* $(\lambda_i)_{i \in I} \in B(0,1)^I$, $\sum_{i \in I} \lambda_i e_i$ *converges and represents an element of* A,

(vi) *each* $x \in A$ *has a unique representation as a convergent sum* $x = \sum_{i \in I} \lambda_i e_i$ *where* $\lambda_i \in B(0,1)$ *for each* $i \in I$,

(vii) Y *is a minimal element of* $\{Z \subset E : A = \overline{co}\, Z\}$

(viii) Y_0 *is a minimal element of* $\{Z \subset E,\ Z$ *is compact*, $A = \overline{co}\, Z\}$,

(ix) Y *is a* p_A*-orthonormal set.*

Proof. By Theorem 1.4 we may assume $A = B(0,1)^I$. Choose $\{e_i : i \in I\}$ as in the second part of the proof of Theorem 1.4. We leave the details of checking (i)-(ix) to the reader.

§ 2 GENERAL COMPACTOIDS

THEOREM 2.1 _Let A be a compactoid in a locally convex space E over K. Then there exists a locally convex space F over K containing E as a subspace and a compact set_ $X \subset F$ _such that_ $A \subset \overline{co}\, X$.

Proof. For each continuous seminorm p on E, let $E_p := E/\mathrm{Ker}\,p$ with the norm induced by p. The natural maps $\pi_p : E \to E_p$ yield a linear homeomorphic embedding

$$\pi : E \to F := \prod_{p\in\Gamma} E_p$$

where Γ is the collection of continuous seminorms of E. For each $p \in \Gamma$ the set $\pi_p(A)$ is a (metrizable) compactoid in E_p. By [4], Proposition 8.2 there is a compact set $X_p \subset E_p$ such that $\pi_p(A) \subset \overline{co}\, X_p$. Without loss we may assume that $0 \in X_p$. We have

$$\pi(A) \subset \prod_{p\in\Gamma} \pi(A) \subset \prod_{p\in\Gamma} \overline{co}\, X_p$$

We claim that $\prod_{p\in\Gamma} \overline{co}\, X_p \subset \overline{co} \prod_{p\in\Gamma} X_p$. (Then the theorem is proved with $X := \prod_{p\in\Gamma} X_p$.) For $p \in \Gamma$ and $x \in X_p$ the element f defined by

$$(*) \qquad f(q) = \begin{cases} x & \text{if } q \in \Gamma,\ q = p \\ 0 & \text{if } q \in \Gamma,\ q \neq p \end{cases}$$

is in $\prod_{p\in\Gamma} X_p$. If $p \in \Gamma$ and $x \in \overline{co}\, X_p$ then f, formally defined by (*), is in $\overline{co} \prod_{p\in\Gamma} X_p$. If $p_1,\dots,p_n \in \Gamma$ and $x_{p_i} \in \overline{co}\, X_{p_i}$ for $i \in \{1,\dots,n\}$ the element g defined by

$$(**) \qquad g(q) = \begin{cases} x_{p_i} & \text{if } q \in \Gamma,\ i \in \{1,\dots,n\},\ q = p_i \\ 0 & \text{if } q \in \Gamma, \text{ otherwise} \end{cases}$$

is a finite sum of elements of $\overline{\text{co}}\ \prod_{p\in\Gamma} X_p$, hence in $\overline{\text{co}}\ \prod_{p\in\Gamma} X_p$. The elements of the type defined in (**) are dense in $\prod_{p\in\Gamma} \overline{\text{co}}\ X_p$.

COROLLARY 2.2 *Let A be an absolutely convex compactoid in a locally convex space over K. Then A is isomorphic to a submodule of some power of* B(0,1).

Proof. By the previous theorem, $A \subset \overline{\text{co}}\ X$ for some compact $X \subset F$. We may suppose that F is complete. Now apply Theorem 1.4.

Note to Theorem 2.1. It is too optimistic to hope that in Theorem 2,1 we may require that F = E even when we allow X to be precompact. In fact, let K be not locally compact, let $E = c_0$, with the weak topology. $A := \{x \in E : \|x\| \le 1\}$ is a weak compactoid but according to [5], Proposition 3.3 there is no weakly precompact $X \subset E$ for which $A \subset \overline{\text{co}}\ X$. Observe that, if K is not spherically complete, A is even weakly complete ([4], Theorem 9.6), and that A is pure if the valuation of K is discrete.

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