TOPOLOGICAL STABILITY OF p-ADIC COMPACTOIDS UNDER CONTINUOUS INJECTIONS

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Abstract. Let A be an absolutely convex bounded subset of a locally convex space (E,τ) over a complete non-archimedean valued field K. The equivalence of (α) and (β) below is proved (Theorems A&A').

(α) A *is a compactoid*.

(β) *If* τ' *is a locally convex topology on* E, *weaker than* τ, *and if there exists a* τ-*neighbourhood base of* 0 *in* E *consisting of absolutely convex* τ'-*closed sets then* $\tau = \tau'$ *on* A.

In the same spirit connections are derived between complete compactoidity (c-compactness) of A and a stronger version of (β) (Theorems B&B', Theorem 3.2), yielding also alternative proofs (Corollary 2.5 and Proposition 4.6) of two theorems of Gruson [2].

Preliminaries. Throughout K is a non-archimedean nontrivially valued complete field with valuation $| \ |$. For fundamentals on Banach spaces and locally convex spaces over K we refer to [4], [8]. We shall use the notations and terminology of [6]. For a subset X of a locally convex space E over K we denote its absolutely convex hull by co X, its K-linear hull by [X]. The closure of a set $Y \subset E$ is denoted $\overline{Y}$. Instead of $\overline{\text{co } X}$ we shall write $\overline{\text{co}}$ X.

Introduction. The well-known concepts of compactoidity and c-compactness (see Definition 1.1) are 'convexified' versions of (pre)compactness. Although (and because !) for non-locally compact K convex (pre)compact sets are trivial, one may put the general question as to whether 'convexified' versions of classical properties of (pre)compact sets hold for compactoids or c-compact sets. See, for example [7] for several fundamental compact-like properties of c-compact sets.
In this paper we consider the convexified form of 'a continuous injection f on a compact space X is a homeomorphism of X ont f(X)'. For an outline of the results see the abstract above. Facts of a more general nature, needed in this paper, have been put together in an Appendix (§4).

§ 1. STABILITY OF COMPACTOIDS

We recall the fundamental notions ([4],[7]).

1.1 DEFINITION. A subset X of a locally convex space E over K is (a) compactoid if for each neighbourhood U of 0 in E there exists a finite set $F \subset E$ such that $X \subset U + \text{co}\,F$. An (absolutely) convex subset A of E is c-compact if every collection of nonempty relatively closed convex subsets of A, having the finite intersection property, has a nonempty intersection.

(For a connection between these concepts see Proposition 4.6.) Our aim in this section is to prove the Theorems A&B.

1.2 LEMMA. Let B be a closed absolutely convex subset of a locally convex space E over K, let $a \in [B]$. Further, let $\lambda \in K$, where $|\lambda| > 1$ if the valuation of K is dense, $\lambda = 1$ if the valuation of K is discrete.

(i) If $(x_i)_{i \in I}$ is a net in $B+Ka$ converging to 0 then $x_i \in \lambda B$ for large i.

(ii) The closure of $B+\text{co}\{a\}$ is contained in $\lambda B+\text{co}\{a\}$.

Proof. (i) $C := \{\mu \in K : \mu a \in B\}$ is a nonzero absolutely convex subset of K. We may assume $C \neq K$ so that $r := \text{diam}\, C \in (0,\infty)$. We have

$$\{\mu \in K : |\mu| < r\} \subset C \subset \{\mu \in K : |\mu| \leq r\}$$

where, if the valuation of K is discrete, the second inclusion is an equality and $r \in |K|$. For each $i \in I$ we have a decomposition

$$(*) \qquad x_i = b_i + \lambda_i a \qquad (b_i \in B\ ,\ \lambda_i \in K)$$

To prove (i) we derive a contradiction from the following assumption. There exists an $\alpha\in K$, $|\alpha| > r$ and a directed cofinal $J \subset I$ such that $|\lambda_j| \geq |\alpha|$ for all $j\in J$.

From (*) we obtain

$$\alpha a = \lambda_j^{-1}\alpha x_j - \lambda_j^{-1}\alpha b_j. \qquad (j\in J)$$

As $|\lambda_j^{-1}\alpha| \leq 1$ we have $\lim_J \lambda_j^{-1}\alpha x_j = 0$ and $\lambda_j^{-1}\alpha b_j \in B$ for each $j\in J$ so that $\alpha a \in \overline{B} = B$. On the other hand, $|\alpha| > r$ so, $\alpha\notin C$, a contradiction.

(ii) Let $x\in \overline{B+co\{a\}}$. There is a net

$$i \mapsto x_i := b_i + \lambda_i a \qquad (b_i\in B,\ \lambda_i\in K,\ |\lambda_i| \leq 1)$$

converging to x. The net $(i,j)\mapsto x_i - x_j$ is in $B+co\{a\}$ and converges to 0. By (i) there is an $i_o \in I$ such that $x_i - x_j \in \lambda B$ for $i,j \geq i_o$. In particular

$$x_i \in x_{i_o} + \lambda B. \qquad (i \geq i_o)$$

The set $x_{i_o} + \lambda B$ is closed so that

$$x = \lim_I x_i \in x_{i_o} + \lambda B.$$

We see that $x\in b_{i_o} + \lambda_{i_o} a + \lambda B \subset \lambda B + \lambda_{i_o} a \subset \lambda B + co\{a\}$.

Remark. $B + co\{a\}$ is not always closed, see [5], 6.25.

1.3 LEMMA. *Let B,λ be as in Lemma 1.2. Let $a_1,\ldots,a_n \in [B]$. If $(x_i)_{i\in I}$ is a net in $B + co\{a_1,\ldots,a_n\}$ converging to 0 then $x_i \in \lambda B$ for large i.*

Proof. Let $\mu\in K$, $1 < |\mu|^{2n-1} \leq |\lambda|$ if the valuation of K is dense, $\mu = 1$ otherwise. We have

$$x_i \in \overline{B + co\{a_1,\ldots,a_{n-1}\}} + co\{a_n\} \qquad (i \in I)$$

so that by Lemma 1.2 (i)

$$x_i \in \mu(\overline{B + co\{a_1,\ldots,a_{n-1}\}})$$

for large i. By Lemma 1.2 (ii)

$$\overline{B + co\{a_1,\ldots,a_{n-1}\}} \subset \mu(\overline{B + co\{a_1,\ldots,a_{n-2}\}}) + co\{a_{n-1}\}$$

so that

$$x_i \in \mu^2(\overline{B + co\{a_1,\ldots,a_{n-2}\}}) + co\{\mu a_{n-1}\}$$

for large i. Again by Lemma 1.2 (i)

$$x_i \in \mu^3(\overline{B + co\{a_1,\ldots,a_{n-2}\}}), \text{ etc.}$$

Inductively we arrive at

$$x_i \in \mu^{2n-1} B \subset \lambda B$$

for large i.

1.4 THEOREM A. _Let_ X _be a compactoid in a locally convex space_ (E,τ) _over_ K. _Let_ τ' _be a locally convex topology on_ E, _weaker than_ τ. _Suppose there is a_ τ_-neighbourhood base of_ 0 _in_ E _consisting of absolutely convex_ τ'_-closed sets. Then_ $\tau = \tau'$ _on_ X.

Proof. We shall prove that $\tau = \tau'$ on $A := co\, X$. To this end, it suffices, as A is an additive group, to show that for a net $(x_i)_{i \in I}$ in A,

$$\tau'\text{-}\lim_I x_i = 0 \quad \text{implies} \quad \tau\text{-}\lim_I x_i = 0.$$

Let V be a τ-neighbourhood of 0 in E. There exists, by assumption, an absolutely convex τ'-closed τ-neighbourhood U of 0 with $U \subset V$. Let $\lambda \in K$, $|\lambda| > 1$. By compactoidity there exist $a_1, \ldots, a_n \in E$ such that

$$X \subset \lambda^{-1}U + co\{a_1, \ldots, a_n\}.$$

Then, by absolute convexity of $\lambda^{-1}U + co\{a_1, \ldots, a_n\}$, we have also

$$A \subset \lambda^{-1}U + co\{a_1, \ldots, a_n\}.$$

As $\lambda^{-1}U$ is τ'-closed and $a_1, \ldots, a_n \in [\lambda^{-1}U]$ we may apply Lemma 1.3 and conclude that $x_i \in \lambda\lambda^{-1}U \subset V$ for large i. It follows that $\tau\text{-}\lim_{I} x_i = 0$.

1.5 COROLLARY. *Let (E,τ) be a locally convex space over a spherically complete field K. Then, on τ-compactoids, the weak topology and the initial topology τ coincide.*

Proof. By [9], for absolutely convex sets, the proporties "weakly closed" and "τ-closed" are identical. Now apply Theorem A.

Remark. The conclusion of Corollary 1.5 holds also for polar locally convex spaces over a nonspherically complete field K. ([6], Theorem 5.12.)

1.6 THEOREM B. *Let A be an absolutely convex bounded c-compact subset of a Hausdorff locally convex space (E,τ) over K. Let τ' be a Hausdorff locally convex topology on E, weaker than τ. Then τ = τ' on A.*

Proof. Let $(x_i)_{i\in I}$ be a net in A converging to 0 for τ', let U be a τ-neighbourhood of 0 in E. We prove that $x_i \in U$ for large i. Let $\lambda \in K$, $|\lambda| > 1$. There exists an absolutely convex τ-neighbourhood V of 0 such that $\lambda^2 V \subset U$. By proposition 4.6 A is a τ-compactoid, so according to Katsaras' theorem ([3] or [6], Lemma 8.1) there exist $a_1,\ldots,a_n \in A$ such that

$$\lambda^{-1}A \subset V + co\{a_1,\ldots,a_n\}.$$

Then also

$$(*) \qquad \lambda^{-1}A \subset V \cap A + co\{a_1,\ldots,a_n\}.$$

By [7] $V \cap A$, being an absolutely convex τ-closed subset of A, is c-compact for τ, hence for τ' so that $V \cap A$ is τ'-closed. From (*) and Lemma 1.3 we have, for large i,

$$\lambda^{-1}x_i \in \lambda(V \cap A) \subset \lambda V.$$

We see that $x_i \in \lambda^2 V \subset U$ for large i.

Remark. Theorem B is only of interest if K is spherically complete. See, however, §3.

§ 2. CHARACTERIZATIONS OF COMPACTOIDS BY STABILITY

In this section we shall prove 'converses' to Theorems A&B (Theorems A'&B').

Remark. If, in Theorem A',(see2.3) we add the assumption that K is spherically complete or that E is a polar space over a nonspherically complete K, we may obtain a short proof of Theorem A' by choosing for τ' the weak topology.

For convenience we introduce the following terminology. Let τ_1, τ_2 be locally convex topologies on a K-vector space E. We say that τ_1 is closedly related to τ_2 if $\tau_1 \le \tau_2$ and if there exists a τ_2-neighbourhood base of 0 in E consisting of absolutely convex τ_1-closed sets.

2.1 LEMMA. *Let τ_1, τ_2 be locally convex topologies on a K-vector space E. The following are equivalent.*

(α) *τ_1 is closedly related to τ_2.*

(β) *The set P consisting of all τ_2-continuous q for which*

$$q = \sup \{p : p \text{ is a } \tau_1\text{-continuous seminorm}, \ p \le q\}$$

is a base of continuous seminorms for the topology τ_2.

Proof. (α) ⇒ (β). Let $\mathcal{U}$ be a τ_2-neighbourhood base of 0 consisting of absolutely convex τ_2-open and τ_1-closed sets. For each $U \in \mathcal{U}$ define

$$U^e := \begin{cases} U \text{ if the valuation of K is discrete} \\ \bigcap_{|\lambda|>1} \lambda U \text{ if the valuation of K is dense.} \end{cases}$$

Of course, $\mathcal{U}' := \{U^e : U \in \mathcal{U}\}$ is again a τ_2-neighbourhood base of 0 consisting of absolutely convex τ_2-open and τ_1-closed sets. But in addition we have for each $U \in \mathcal{U}'$

$$U = \{z \in E : p_U(z) \le 1\}$$

where p_U is the Minkowski function of U. Set

$$P' := \{p_U : U \in \mathcal{U}'\}.$$

Clearly P' is a base of continuous seminorms for τ_2. We shall prove that $P' \subset P$. (Then (β) follows.) Let $U \in \mathcal{U}'$. From Proposition 4.2 it follows that for each $x \in E \setminus U$ there exists a τ_1-continuous seminorm q_x such that $q_x(x) > 1$, $q_x \le 1$ on U, $q_x(z) \in \overline{|K|}$ ($z \in E$). (If the valuation of K is discrete we may obtain q_x by multiplying p of Proposition 4.2 by a suitable constant, if the valuation of K is dense there is, by definition of U^e, a $\lambda \in K$, $|\lambda| < 1$ such that $\lambda x \notin U$.) Now set

$$q := \sup \{q_x : x \in E \setminus U\}.$$

One verifies directly that $q = p_U$ which finishes the first part of the proof.

(β) $\Rightarrow$ (α). For each $p \in P$ set

$$U_p := \{x \in E : p(x) \le 1\}$$

and

$$\mathcal{U} := \{U_p : p \in P\}$$

One proves in a standard way that $\mathcal{U}$ is a τ_2-neighbourhood base of 0 consisting of τ_1-closed sets.

2.2 LEMMA. *Let τ be a locally convex topology on a K-vector space E, let D be a linear subspace of E, let τ_D be the restriction of τ to D. If τ'_D is a locally convex topology on D closedly related to τ_D then there is a locally convex topology τ' on E closedly related to τ such that $\tau'|D = \tau'_D$.*

Proof. Let Γ be the collection of all τ- continuous seminorms on E, let Γ_D (Γ'_D) be the collection of all τ_D- (τ'_D-) continuous seminorms on D. Define τ' to be the locally convex topology on E generated by

$$\Gamma' := \{p \in \Gamma : p|D \in \Gamma'_D\}.$$

Obviously $\tau' \leq \tau$, $\tau'|D = \tau'_D$ (Proposition 4.1), and Γ' is the set of all τ'- continuous seminorms. To complete the proof that τ' is closedly related to τ we shall construct a base P_E of continuous seminorms for τ such that each $p \in P_E$ is a supremum of τ'- continuous seminorms (Lemma 2.1). To this end, let

$$P_D := \{p \in \Gamma_D : p = \sup \{q \in \Gamma'_D : q \leq p\}\}$$

and

$$P_E := \{p \in \Gamma : p|D \in P_D\}.$$

By assumption and Lemma 2.1, P_D is a base of continuous seminorms for τ_D. To prove that P_E is a base for τ, let $p \in \Gamma$. Then $p|D \in \Gamma_D$. There is an $s \in P_D$ with $p \leq s$ on D. As $s \in \Gamma_D$ it extends to a $t \in \Gamma$. Set $q := \max(t,p)$. Then $q \in \Gamma$, $q|D = s \in P_D$ (so that $q \in P_E$) and $p \leq q$. Finally we prove that

$$p = \sup \{q \in \Gamma' : q \leq p\} \qquad (p \in P_E)$$

by constructing, for each $x \in E$, a $q \in \Gamma'$, $q \leq p$ such that $q(x)$ is close to $p(x)$. We distinguish two cases.

(i) $p(x-d) \geq p(x)$ for all $d \in D$. Then we take for q the quotient seminorm of p

$$q(y) = \inf\{p(y-d) : d \in D\} \qquad (y \in E)$$

We have $q \leq p$, $q(x) = p(x)$ and $q \in \Gamma'$ (since $q = 0$ on D).

(ii) There exists a $d \in D$ with $p(x-d) < p(x)$. Then $p(x) = p(d) \neq 0$ and there is a $c \in (0,1)$ with $p(x-d) = cp(x)$. Let $c' \in (0,1)$, $c' \geq c$. By definition $p|D \in P_D$ so there is a $q_1 \in \Gamma'_D$ with $q_1 \leq p$ on D and $q_1(d) > c'p(d)$. By Proposition 4.1, q_1 extends to a $q \in \Gamma$ with $q \leq p$ on E. Then $q \in \Gamma'$. We have

$$q(d) = q_1(d) > c'p(d) = c'p(x)$$

but also

$$q(x-d) \leq p(x-d) = cp(d) \leq c'p(d) < q_1(d) = q(d)$$

so that $q(x) = q(d) > c'p(x)$. As we may take c' close to 1 this completes the proof.

2.3 THEOREM A'. _Let_ A _be a bounded absolutely convex subset of a locally convex space_ (E,τ) _over_ K. _Suppose that for each locally convex topology_ $\tau' \leq \tau$ _on_ E, _for which there exists a_ τ- _neighbourhood base of_ 0 _consisting of absolutely convex_ τ'- _closed sets, we have_ $\tau = \tau'$ _on_ A. _Then_ A _is a compactoid in_ (E,τ).

Proof. By Proposition 4.4 it suffices to prove that, for any countable subset X of A, the set $B := \text{co}\, X$ is a compactoid in $D := [X]$, with the restriction topology τ_D of τ. Now D is of countable type so its weak topology τ'_D is closedly related to τ_D ([6], Proposition 5.2 (γ)). By Lemma 2.2 τ'_D is the restriction of a locally convex topology τ' on E that is closedly related to τ. By assumption $\tau = \tau'$ on A, hence $\tau_D = \tau'_D$ on B. Now B is bounded, hence a compactoid for the weak topology τ'_D. By Proposition 4.5 B is also a τ_D- compactoid.

Remark. The example E = A = K shows the relevance of the boundedness condition in Theorem A'. I do not know whether the absolute convexity of A can be dropped.

2.4 THEOREM B'. *Let A be a closed bounded absolutely convex subset of a Hausdorff locally convex space (E,τ) over K. Suppose that for each locally convex Hausdorff topology $\tau' \leq \tau$ on E we have $\tau = \tau'$ on A. Then A is a complete compactoid in (E,τ).*

Proof. Theorem A' yields compactoidity of A. Suppose A is not τ- complete; we construct a Hausdorff locally convex topology τ' on E with $\tau' \leq \tau$ on E and $\tau' \neq \tau$ on A. As A is closed there exists a τ- Cauchy net $(x_i)_{i\in I}$ in A that does not converge in (E,τ). Let τ' be the locally convex topology generated by the set P of all τ-continuous seminorm p for which $\lim_I p(x_i) = 0$. Obviously $\tau' \leq \tau$ and $\tau' \neq \tau$ on A.

It remains to be shown that τ' is Hausdorff. The net $(x_i)_{i\in I}$ converges to an x in the completion $(E,\tau)^\wedge$ of (E,τ), and $x \notin (E,\tau)$. Then for each $a\in E$, $a \neq 0$ the elements a, x are linearly independent and there exists a continuous seminorm q on $(E,\tau)^\wedge$ for which $q(x) = 0$, $q(a) \neq 0$ (consider the quotient map $(E,\tau)^\wedge \to (E,\tau)^\wedge/Kx$). Hence, $q|E \in P$. It follows that P separates the points of E.

2.5 COROLLARY. *Let A be a closed bounded absolutely convex subset of a Hausdorff locally convex space* (E,τ) *over a spherically complete field K. The following are equivalent.*

(α) *A is c-compact.*

(β) *A is a complete compactoid.*

(γ) *If* τ' *is a Hausdorff locally convex topology, weaker than* τ, *then* $\tau = \tau'$ *on A.*

Proof. Proposition 4.6, Theorems B&B'.

Remark. There exists noncomplete absolutely convex compactoids for which (γ) is true. (Let K be the spherical completion of $\mathbb{C}_p$ and set

$$A := \{(\xi_1,\xi_2,\ldots) \in c_0 : \sup_i |\xi_i| p^i < 1\}.$$

$$B := \{(\xi_1,\xi_2,\ldots) \in c_0 : \sup_i |\xi_i| p^i \leq 1\}.$$

B is c-compact, $pB \subset A \subset B$ so (γ) holds for A. However, A is not closed.)

§ 3. THEOREM B FOR NONSPHERICALLY COMPLETE K

Contrary to the Theorems A, A', B', Theorem B is a triviality for nonspherically complete K. The hope that absolutely convex complete compactoids are topologically stable under continuous linear injections (inspired by Theorem B') is too optimistic as the following example shows.

3.1 EXAMPLE. Let K be not spherically complete, let A be the 'closed' unit ball of c_0. Then A is a complete compactoid for the weak topology σ. However, for the topology σ' of coordinatewise convergence we have $\sigma' \leq \sigma$ but $\sigma \neq \sigma'$ on A.

Proof. As c_0 is reflexive ([4], Theorem 4.17) and A is weakly closed the first statement follows from [6], Theorem 9.6. The sequence $e_1, e_2, \ldots$ of the standard unit vectors does not converge weakly ([6], Proposition 4.11) but does converge coordinatewise (to 0).

On the positive side we have the following theorem (see also the remarks below).

3.2 THEOREM. Let A be an absolutely convex complete metrizable compactoid in a Hausdorff locally convex space (E,τ) over K. Let τ' be a Hausdorff locally convex topology on E, weaker than τ. Then $\tau = \tau'$ on A.

Proof. Without loss of generality we may assume E = [A] (then, by Proposition 4.3, (E,τ) is of countable type) and that A is edged

([6]) i.e. that, if the valuation of K is dense, $A = \bigcap_{|\lambda|>1} \lambda A$.

Let E'_A be the dual space of (E,τ) equipped with the topology of uniform convergence on A. This topology is induced by the single norm $f \mapsto \sup\{|f(x)| : x \in A\}$.

We first prove (i), (ii) below.

(i) E'_A <u>is strongly polar</u> ([6], Definition 3.5). <u>Proof</u>. By [6], Proposition 8.2 there is a sequence $e_1, e_2, \ldots$ in λA for some $\lambda \in K$, $|\lambda| > 1$ with $\lim_{n\to\infty} e_n = 0$ such that $A \subset \overline{\mathrm{co}}\{e_1, e_2, \ldots\} \subset \lambda A$. The formula

$$f \mapsto (f(e_1)\ ,\ f(e_2)\ ,\ldots) \qquad (f \in E')$$

defines a linear homeomorphism of E'_A into c_0. Now c_0 is of countable type hence ([6], Proposition 4.4) strongly polar, so are its subspaces ([6], Proposition 4.1) and, therefore, E'_A .

(ii) <u>The canonical map</u> $E \to (E'_A)'$ <u>is surjective</u>. <u>Proof</u>. By [6], Lemma 7.1 (ii) the map $E \to (E'_\sigma)'$ is bijective (where E'_σ is E', with the topology of pointwise convergence), so we shall prove that $(E'_A)' = (E'_\sigma)'$. To this end we shall check that the covering $\{\lambda A : \lambda \in K\}$ of E satisfies the conditions of [6], Proposition 7.4. Each λA is edged and a complete compactoid for τ and by [6], Theorem 5.13, also a complete compactoid for the weak topology $\sigma(E,E')$

Now we finish the proof as follows. Let σ (σ') be the weak topology of τ (τ'). By Corollary 1.5 and the remark following it we have $\sigma = \tau$ on A, $\sigma' = \tau'$ on A and $\sigma' \le \sigma$ on E. We prove that $\sigma = \sigma'$ on A. Let $F := (E,\sigma')'$. Then $F \subset E'$ and, as σ' is Hausdorff, F separates the points of E.

We claim that F is dense in E'_A. Indeed, if $\phi \in (E'_A)'$, $\phi = 0$ on F then, by (ii) there exists an $x \in E$ for which $\phi(f) = f(x)$ for all $f \in E'$ so that $f(x) = 0$ for all $f \in F$ i.e. $x = 0$. By (i) and [6], Corollary 4.9 the norm closure of F equals its weak closure which is E'. Now let $(x_i)_{i\in I}$ be a net in A converging to 0 for σ'. Then $\lim_I f(x_i) = 0$ for all $f \in F$. By the above for each $g \in E'$ there is a net in F converging to g uniformly on A. But then $\lim_I g(x_i) = 0$ so that $\lim_I x_i = 0$ for σ.

Remark 1. If K is spherically complete each locally convex space is strongly polar ([6], §4). So an obvious modification makes the above proof valid for a complete absolutely convex compactoid in a locally convex space over K yielding an alternative proof of Theorem B.

Remark 2. As a contrast to Theorem 3.2 we mention the following result of A. van Rooij ([5], Theorem 6.28). Let A be an absolutely convex subset of a Banach space E over a nonspherically complete field K. If for each K-Banach space F and each $T \in L(E,F)$ the set TA is closed then A is finite dimensional.

3.3 COROLLARY. Let A,B be absolutely convex compactoids in a Fréchet space (E,τ) over K.

(i) If A,B are closed and $A \cap B = (0)$ then A+B is closed.

(ii) If $\tau' \leq \tau$ is a Hausdorff locally convex topology on E then $\tau = \tau'$ on A.

Proof. (i) $A \times B$ is a complete compactoid in $E \times E$. As $[A] \cap [B] = 0$ addition is a continuous linear bijection:

$$[A] \times [B] \to [A+B]$$

sending $A \times B$ onto A+B. It follows from Theorem 3.2 that A+B is complete, hence closed in E. For (ii) apply Theorem 3.2 to the closure of A.

PROBLEM. For a nonspherically complete field K, characterize the absolutely convex complete compactoids A in a Hausdorff locally convex space (E,τ) over K for which $\tau = \tau'$ on A for any Hausdorff locally convex topology τ' on E that is weaker then τ.

§ 4. APPENDIX

4.1 PROPOSITION. Let D be a linear subspace of a K-vector space E. If p is a seminorm on E and q is a seminorm on D such that $q \leq p$ on D then q can be extended to a seminorm $\bar{q}$ on E for which $\bar{q} \leq p$ on E.

Proof. Set $\bar{q}(x) := \inf_{d \in D} \max(p(x-d), q(d)) \quad (x \in E)$.

4.2 PROPOSITION. Let A be a closed absolutely convex subset of a locally convex space E over K, let $x \in E \setminus A$. There exists a continuous seminorm p with $p(a) < 1$ for each $a \in A$ and $p(x) = 1$. If the valuation of K is discrete p can be chosen such that $p(z) \in |K|$ for each $z \in E$.

Proof., Let U be an open absolutely convex set that is maximal with respect to the properties $A \subset U$, $x \notin U$. Let q be the Minkowski function of U. Then q is continuous and

$$\{z \in E : q(z) < 1\} \subset U \subset \{z \in E : q(z) \leq 1\}.$$

If $q(x) > 1$ then $p := q(x)^{-1}q$ satisfies the requirements, so assume $q(x) = 1$. Set $p := q$. Obviously, $p(z) \in |K|$ for all $z \in E$ for a discretely values field K; we prove that $p(u) < 1$ for each $u \in U$. Suppose $p(u) = 1$ for some $u \in U$. The set $Ku+U$ is absolutely convex and it properly contains U so, by maximality, it contains x. There is a $\lambda \in K$ for which $x - \lambda u \in U$. As $x \notin U$ we must have $|\lambda| > 1$. As $p(x-\lambda u) \leq 1$ and $p(\lambda u) > 1$ we arrive at $p(x) > 1$, a contradiction.

4.3 PROPOSITION. If A is a compactoid in a locally convex space E over K then [A] is of countable type ([6], Definition 4.3).

Proof. Set $E = [A]$. For each continuous seminorm p, let π_p be the canonical map of E onto the normed space $E_p := E/\mathrm{Ker}\,p$. Then $\pi_p(A)$ is a compactoid in E_p, and absorbing. By [6], Proposition 8.2 there exist $e_1, e_2, \ldots \in E_p$ with $\pi_p(A) \subset \overline{\mathrm{co}}\,\{e_1, e_2, \ldots\}$ so that $E_p = \overline{[e_1, e_2, \ldots]}$ is of countable type. Then so is $\prod_p E_p$ ([6], Proposition 4.12 (iii)) and is E (being linearly homeomorphic to a subspace of $\prod_p E_p$).

4.4 PROPOSITION. *Let A be an absolutely convex subset of a locally convex space E over K. If each countable subset of A is a compactoid then A is a compactoid.*

Proof. For Banach spaces E this is proved in [4], Theorem 4.37. From [1], Lemma 1.3 it follows that the statement is true for normed spaces E. Now let E be locally convex. With π_p, E_p as in the proof of Proposition 4.3 we have that each $\pi_p(A)$ is a compactoid in E_p. But then $\prod_p \pi_p(A)$ is a compactoid ([1], Proposition 1.7) and so is A (being linearly homeomorphic to a subset of $\prod_p \pi_p(A)$).

4.5 PROPOSITION. *Let A be an absolutely convex subset of a K-vector space E. Let τ_1, τ_2 be locally convex topologies on E such that $\tau_1 = \tau_2$ on A. Then A is a τ_1-compactoid if and only if A is a τ_2-compactoid.*

Proof. Suppose A is a τ_1-compactoid. Let U be a τ_2-neighbourhood of 0. Let $\lambda \in K$, $0 < |\lambda| < 1$. There is an absolutely convex τ_2-neighbourhood U' of 0 with $U' \subset \lambda U$. There is an absolutely convex τ_2-neighbourhood V of 0 with $V \cap A \subset U' \cap A$. By Katsaras'theorem ([3], or [6], Lemma 8.1) there exist $a_1, \ldots, a_n \in A$ such that $\lambda A \subset V + \mathrm{co}\{a_1, \ldots, a_n\}$. Then also $\lambda A \subset V \cap A + \mathrm{co}\{a_1, \ldots, a_n\}$. We see that $A \subset \lambda^{-1}(V \cap A) + \mathrm{co}\{\lambda a_1, \ldots, \lambda a_n\} \subset U + \mathrm{co}\{\lambda a_1, \ldots, \lambda a_n\}$. It follows that A is a τ_2-compactoid.

4.6 PROPOSITION. *Let A be a bounded absolutely convex subset of a Hausdorff locally convex space E over a spherically complete field K. The following are equivalent.*

(α) A *is* c-*compact.*

(β) A *is a complete compactoid.*

Proof. [2], Proposition 4, p.93. A more elementary proof for Banach spaces E is given in [5], Theorem 6.15. From this the general statement can be obtained in a straightforward way by using the embedding $E \to \prod_p \hat{E_p}$ (where $\hat{E_p}$ is the completion of E_p) and the proporties of c-compact sets proved in [7].

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