PDF hosted at the Radboud Repository of the Radboud University Nijmegen

The following full text is a publisher's version.

For additional information about this publication click this link.
http://hdl.handle.net/2066/57044

Please be advised that this information was generated on 2018-11-12 and may be subject to change.
DUALITY OF PROJECTIVE LIMIT SPACES AND INDUCTIVE LIMIT SPACES OVER A NONSPHERICALLY COMPLETE NONARCHIMEDEAN FIELD

WIM H. SCHIKHOF AND YASUO MORITA

(Received October 14, 1985)

Abstract. A duality theorem of projective and inductive limit spaces over a nonspherically complete valued field is obtained under a certain condition, and topologies of spaces of locally analytic functions are studied.

Introduction. Morita obtained in [5] a duality theorem of projective limit spaces and inductive limit spaces over a spherically complete nonarchimedean valued field, and Schikhof studied in [8] locally convex spaces over a nonspherically complete nonarchimedean valued field. In this paper, we use the results of [8] and study the duality of such spaces over a nonspherically complete nonarchimedean valued field.

The duality theorem of [5] was obtained as a generalization of the results of Komatsu [3] by Morita using the theory of van Tiel [10] about locally convex spaces over a spherically complete nonarchimedean valued field. There the following two facts are used essentially: (i) The Mackey topology is the topology of uniform convergence on weakly e-compact sets; (ii) Any absolutely convex weakly e-compact set is strongly closed. Though we can generalize the notion of e-compactness to our case, it is difficult to obtain good analogues of these two facts over a nonspherically complete valued field. Hence we restrict our attention to a more restricted class than in [5], and prove a duality theorem by making use of van der Put's duality theorem of sequence spaces $c_0 = \{(a_n, a_2, a_3, \cdots) \in \mathbb{K}^\omega; |a_n| \rightarrow 0 (m \rightarrow \cdots)\}$ and $l^\omega = \{(b_1, b_2, b_3, \cdots) \in \mathbb{K}^\omega; \sup |b_m| < \infty\}$ over a nonspherically complete valued field $\mathbb{K}$.

We prove a general duality theorem over such a field in Section 1, and apply the theorem to some examples in Section 2. In particular, we generalize the results of Morita [6] to any complete nonarchimedean valued field, and give a positive answer to a question of P. Robba.

We use the notation and terminology of Schikhof [8] throughout this paper.
1. Duality theorem.

1.1. Let $K$ be a complete field with a nontrivial nonarchimedean valuation $| |$. We assume in Section 1 that $K$ is not spherically complete. For each positive integer $m$, let $(X_m, | |_m)$ and $(Y_m, | |_m)$ be Banach spaces over $K$. We assume that $X_m$ is of countable type. Hence $X_m$ is reflexive (cf. e.g. van Rooij [7, Corollary 4.18]). Let $$(x, y) : X_m \times Y_m \to K$$ be a nondegenerate bicontinuous $K$-bilinear form such that $X_m$ and $Y_m$ become mutually dual locally convex spaces with respect to $(x, y)_m$. Let $\{u_{m,n} : X_n \to X_m (m < n)\}$ be a projective system, and $\{v_{n,m} : Y_m \to Y_n (m < n)\}$ be an inductive system such that (i) the $u_{m,n}$'s are $K$-linear continuous maps, (ii) the $v_{n,m}$'s are $K$-linear continuous injective maps, and (iii) the equality $(u_{m,n}(x_n), y_m)_m = (x_n, v_{n,m}(y_m))_n$ holds for any $x_n \in X_n$ and $y_m \in Y_m$. Let $(X, u_m)$ be the locally convex projective limit of $\{X_m, u_{m,n}\}$ and let $(Y, v_m)$ be the locally convex inductive limit of $\{Y_m, v_{n,m}\}$. We assume further that (iv) the projection map $u_m : X \to X_m$ has a dense image for each $m$.

By definition, any element $x$ of the projective limit $X$ can be written as $x = (x_m)$ with $x_m \in X_m$ satisfying $u_{m,n}(x_n) = x_m$ for any $m$ and $n$ with $m < n$, and any element $y$ of the inductive limit space $Y$ can be written as $y = v_m(y_m)$ with some $y_m \in Y_m$. By our assumption (iii), the equality $(u_{m,n}(x_n), y_m)_m = (x_n, v_{n,m}(y_m))_n$ holds for any $m < n$. Hence $(u_{m}(x), y_m)$ does not depend on a special choice of $y$ with $y = v_m(y_m) \in v_m(Y_m)$, and we can define a pairing $$(x, y) : X \times Y \to K$$ by $(x, y) = (u_{m}(x), y_m)_m$ with such a $y_m \in Y_m$. It is easy to see that this pairing $(x, y)$ is $K$-bilinear. Since the projection map $u_m : X \to X_m$ is continuous, our pairing $(x, y)$ is bicontinuous on $X \times Y_m$ for each $m$. Hence, by the universal mapping property of the inductive limit topology, $(x, y)$ is bicontinuous on $X \times Y$.

Let $x = (x_m)$ be a nonzero element of $X$. Then $x_m \neq 0$ for some $m$. Since $(x, y)_m$ is nondegenerate, $(x_m, y_m)_m \neq 0$ for some $y_m \in Y_m$. Hence $(x, y) = (x_m, y_m)_m \neq 0$ for some $y = v_m(y_m) \in v_m(Y_m) \subset Y$. Let $y = v_m(y_m)$ be a nonzero element of $Y$. Then $\{x_m \in X_m ; (x_m, y_m)_m \neq 0\}$ is a non-empty open subset of $X_m$. Since the image of the projection map $u_m : X \to X_m$ is dense, there is an element $x = (x_m) \in X$ such that $(x, y) = (x_m, y_m)_m \neq 0$. Therefore our pairing $(x, y)$ is nondegenerate. Hence we have proved the following:
DUALITY OVER A COMPLETE NONARCHIMEDEAN FIELD

Proposition 1. Let $X = \text{proj lim } X_n$ and $Y = \text{ind lim } Y_n$ be as before. Then we have a nondegenerate bicontinuous $K$-bilinear form

$$(\cdot, \cdot) : X \times Y \to K.$$

1.2.

Lemma 1. Let $E$ and $F$ be locally convex $K$-vector spaces, and let

$$(\cdot, \cdot) : E \times F \to K$$

be a nondegenerate bicontinuous $K$-bilinear form. Let $\sigma(E, F)$ be the weakest locally convex topology on $E$ such that $E \ni e \mapsto (e, f) \in K$ is continuous for each $f \in F$. Then for any continuous $K$-linear form $L : E \to K$ with respect to $\sigma(E, F)$, there exists an element $f \in F$ such that

$$L(e) = (e, f)$$

holds for any $e \in E$. In particular, $(E, \sigma(E, F))' = F$.

Proof. Let $L : E \to K$ be as in the lemma. Since $L$ is continuous, there are a finite number of elements $f_1, \cdots, f_n$ in $F$ such that for all $e \in E$

$$|L(e)| \leq \sup_{1 \leq i \leq n} |(e, f_i)|.$$

Let $E^* = \{ e \in E ; (e, f_i) = 0 \text{ for any } i = 1, \cdots, n \}$. Then $E^*$ is contained in the kernel of $L$, and $L$ factors through $E/E^*$. Since $(\cdot, \cdot)$ induces a nondegenerate $K$-bilinear form on $(E/E^*) \times (Kf_1 + \cdots + Kf_n)$, the algebraic dual of $E/E^*$ can be identified with $Kf_1 + \cdots + Kf_n$. Hence there are $a_1, \cdots, a_n \in K$ such that

$$L(e) = \left( e, \sum_{i=1}^n a_i f_i \right)$$

holds for any $e \in E$. Then $f = \sum a_i f_i$ satisfies the condition of the lemma.

We apply this lemma to our case. Let $E = X$ and $F = Y$. Then our bilinear form $(\cdot, \cdot)$ satisfies the condition of the lemma. Let $\sigma(X, Y)$ (resp. $\sigma(Y, X)$) be the weakest locally convex topology on $X$ (resp. on $Y$) such that $X \ni x \mapsto (x, y) \in K$ is continuous for each $y \in Y$ (resp. $Y \ni y \mapsto (x, y) \in K$ is continuous for each $x \in X$). Then it follows from Lemma 1 that $(X, \sigma(X, Y))' = Y$ and $(Y, \sigma(Y, X))' = X$.

Let $\tau(X)$ be the projective limit topology on $X$, and let $\tau(Y)$ be the inductive limit topology of $Y$. Since our pairing is bicontinuous, $\sigma(X, Y) \leq \tau(X)$ and $\sigma(Y, X) \leq \tau(Y)$. Hence we have

$$Y = (X, \sigma(X, Y))' \subset (X, \tau(X))'$$

and

$$X = (Y, \sigma(Y, X))' \subset (Y, \tau(Y))'.$$
Let $f: X \rightarrow K$ be a $K$-linear continuous map with respect to $\tau(X)$. Then $f^{-1}(\{x \in X; |f(x)| < 1\})$ is open in $X$. It follows from the definition of the projective limit topology that there exist a positive integer $m$ and a positive number $\varepsilon$ such that $f^{-1}(\{x \in X; |f(x)| < 1\}) \supset \{x \in X; |u_m(x)|_m < \varepsilon\}$. This shows that $u_m(X) \supset u_m(x) \mapsto f(x) \in K$ is continuous. Since $u_m(X)$ is dense in $X_m$, this map can be extended to a continuous $K$-linear map $f_m: X_m \rightarrow K$. Since $X_m$ and $Y_m$ are mutually dual with respect to $(\cdot, \cdot)_m$, there is an element $y_m \in Y_m$ such that $f_m(x_m) = (x_m, y_m)_m$ holds for any $x_m \in X_m$. Then $f(x) = f_m(x_m) = (x_m, y_m)_m = (x, y)$ holds for any $x = (x_m) \in X$ with $y = v_m(y_m) \in v_m(Y_m) \subset Y$. Therefore $(X, \tau(X))' = Y$.

Let $g: Y \rightarrow K$ be a continuous $K$-linear map with respect to $\tau(Y)$. Since the natural injection $v_m: Y_m \rightarrow Y$ is continuous, $g$ induces a continuous map $g_m: Y_m \rightarrow K$ for each $m$. Since $X_m$ is the dual of $Y_m$, there is a unique element $x_m \in X_m$ such that $g_m(y_m) = (x_m, y_m)_m$ holds for any $y_m \in Y_m$. If $n > m$, then $(x_n, y_m)_m = g_m(y_m) = g(v_m(y_m)) = g(v_{n,m}(y_m)) = (x_n, v_{n,m}(y_m))_n = (v_{n,m}(x_n), y_m)_n$ holds for any $y_m \in Y_m$. Since the pairing $(\cdot, \cdot)_m$ is nondegenerate, $u_{n,m}(x_n) = x_n$ holds for $n > m$. Hence $x = (x_m)$ is an element of $\text{proj lim } X_m = X$ such that $g(y) = (x, y)$ holds for any $y \in \text{ind lim } Y_m = Y$. Hence $(Y, \tau(Y))' = X$. Therefore we have proved the following:

**Proposition 2.** We have $(X, \tau(X))' = Y$ and $(Y, \tau(Y))' = X$ as sets.

1.3. Since each $X_m$ is a Banach space of countable type, it follows from Schikhof [8, 4.12] that $X = \text{proj lim } X_m$ is a Fréchet space of countable type. Since $K$ is not spherically complete, it follows from [8, Corollary 9.8] that $X$ is reflexive. Hence we have the following:

**Proposition 3.** $X$ is a Fréchet space of countable type. In particular, $X$ is reflexive.

Let $y$ be a nonzero element of $Y$. Since the pairing $(\cdot, \cdot): X \times Y \rightarrow K$ is nondegenerate, there is an element $x$ of $X$ such that $(x, y) \neq 0$. Then $|(x, y)| \neq 0$. Since $p_x(y) = |(x, y)|$ is a continuous seminorm for $\sigma(Y, X)$, it follows that $(Y, \sigma(Y, X))$ is Hausdorff. Since $\tau(Y)$ is stronger than $\sigma(Y, X)$, $(Y, \tau(Y))$ is also Hausdorff. Hence we have proved the following:

**Proposition 4.** $(Y, \tau(Y))$ is a Hausdorff space.

**Remark.** If the maps $u_{m,n}$'s are compact maps, then we can show that $X = \text{proj lim } X_m$ is a nuclear Montel space. In general, since each $Y_m$ is barreled, $Y = \text{ind lim } Y_m$ is also barreled.

Now we can prove the following key lemma:
Proposition 5. The strong topology $b(Y, X)$ on $(X, \tau(X))' = Y$ and the inductive limit topology $\tau(Y)$ of $Y$ coincide.

Proof. Since any bounded set of $(X, \tau(X))$ is contained in a bounded set of the form $B = \{x = (x_m) \in X; |x_m| \leq M_m\}$ with a sequence $(M_m)$ of positive numbers, the subsets of $Y$ of the form $U_b = \{y \in Y; |(x, y)| \leq 1$ for all $x \in B\}$ make a fundamental system of neighbourhoods of $0 \in Y$ with respect to $b(Y, X)$. Since the pairing $(x, y)_m : X_m \times Y_m \rightarrow K$ makes $X_m$ and $Y_m$ into mutually dual Banach spaces, for any positive number $M_m$, there is a positive number $N_m$ such that, if $y_m \in Y_m$ satisfies $|y_m| \leq N_m$, then the condition $|(x_m, y_m)_m| \leq 1$ is satisfied for any $x_m \in X_m$ with $|x_m| \leq M_m$. Then $y = v_m(y_m) \in v_m(Y_m)$ is contained in $U_b$ if $|y_m| \leq N_m$. Hence $U_b$ contains

$$\bigcup_{m \geq 1} v_m(\{y_m \in Y_m; |y_m| \leq N_m\}).$$

Since the subsets of $Y$ of this form make a fundamental system of neighbourhoods of $0 \in Y$ with respect to $\tau(Y)$, we have $b(Y, X) \leq \tau(Y)$. Since we can prove the opposite inequality $\tau(Y) \leq b(Y, X)$ in the same way, the strong topology $b(Y, X)$ and the inductive limit topology $\tau(Y)$ of $Y$ coincide.

Since $(X, \tau(X))$ is reflexive, the following corollary follows from Proposition 5.

Corollary. $(Y, \tau(Y))$ is reflexive, and the strong dual space of $(Y, \tau(Y))$ is isomorphic to $(X, \tau(X))$.

Since $X$ is a Fréchet space, $X$ is bornologic (cf. proofs of van Tiel [10, Théorèmes 3.17 and 4.30]). It follows from Schikhof [8, Proposition 6.8] that $(Y, \tau(Y)) \simeq (X, \tau(X))', b(Y, X))$ is complete. Therefore we have proved the following theorem:

Theorem 1. Let $X = \text{proj lim} X_m$, $Y = \text{ind lim} Y_m$ and $(x, y) : X \times Y \rightarrow K$ be as in 1.1. Then $X$ is a Fréchet space of countable type, $Y$ is Hausdorff and complete and the pairing $(x, y)$ makes $X$ and $Y$ into mutually dual locally convex spaces over $K$.

2. Examples.

2.1. Let $k$ be an algebraically closed field with a nontrivial non-archimedean complete valuation $|\ |$. Let $P'(k) = k \cup \{\infty\}$ be the one-dimensional projective space over $k$, let $K$ be a complete subfield of $k$, and let $C$ be a compact subset of $K$. Put $V = P'(k)$. Let $\{r_n\}_{n=1}^\infty$ be a
strictly decreasing sequence in \(|K^*|\) such that \(r_n \to 0\) \((n \to \infty)\). Then, for each \(n\), \(C\) is covered by a finite number of open balls of the form

\[
C_{n,i} = \{z \in k; |z - c_{n,i}| < r_n\} \quad (c_{n,i} \in C).
\]

We assume that (i) \(C\) is covered by \(C_{n,i}\) \((i = 1, \ldots, l_n)\) and (ii) the \(C_{n,i}\)'s are mutually disjoint. Put

\[
C_n = \prod_{i=1}^{l_n} C_{n,i}.
\]

Then \(C = \cap C_n\).

Let \(f\) be a \(k\)-valued function on \(V - C = \{z \in V; z \notin C\}\). Then \(f\) is called a \(K\)-analytic function on \(V - C\) if and only if the restriction of \(f\) to each \(V - C_n\) is given by a convergent series of the form

\[
f_n(z) = a_{\omega_0} + \sum_{i=1}^{l_n} \sum_{m=1}^{\infty} a^{(i)}_m(z - c_{n,i})^m
\]

with \(a_{\omega_0}, a^{(i)}_m \in K\) and \(|a^{(i)}_m| r_n^m \to 0\) \((m \to -\infty)\) (cf. Morita [4], Gerritzen-van der Put [1], etc.). Let \(\mathcal{O}_K(V - C_n)\) be the space of all functions \(f_n; V - C_n \to K\) of this form. Then the equality

\[
\max_\omega (|a_{\omega_0}|, \max_m |a^{(i)}_m| r_n^m) = \max_{z \in V - C_n} |f_n(z)|
\]

holds. If we define a norm \(|f_n|_n\) by this formula, then the \(K\)-vector space \(\mathcal{O}_K(V - C_n)\) becomes a complete Banach space with this norm. Further, we can identify the quotient space \(\mathcal{O}_K(V - C_n)/K\) \((K = \{f_n(z) = a_{\omega_0}; a_{\omega_0} \in K\})\) with the subspace \(\{\sum_{i=1}^{l_n} \sum_{m=1}^{\infty} a^{(i)}_m(z - c_{n,i})^m; a^{(i)}_m \in K, |a^{(i)}_m| r_n^m \to 0\} (m \to -\infty)\) of \(\mathcal{O}_K(V - C_n)\). Since the set of all finite sums of this form is dense in \(\mathcal{O}_K(V - C_n)/K\), \(\mathcal{O}_K(V - C_n)/K\) is a Banach space of countable type.

Let \(\mathcal{O}_K(V - C)\) be the set of all \(K\)-analytic functions on \(V - C\), and put \(\mathcal{B}_K(C) = \mathcal{O}_K(V - C)/K\). Then \(\mathcal{B}_K(C)\) can be identified with the locally convex projective limit of the \(\mathcal{O}_K(V - C_n)/K\) with respect to the restriction maps. Obviously the restriction maps \(u_{n,i}; \mathcal{O}_K(V - C_n)/K \to \mathcal{O}_K(V - C_n)/K\) \((n < l)\) are \(K\)-linear and continuous. Since any finite sum of the form \(\sum_i \sum_m a^{(i)}_m(z - c_{n,i})^m\) \((a^{(i)}_m \in K)\) is in \(\mathcal{O}_K(V - C)/K = \mathcal{B}_K(C)\), the image of the projection map \(\mathcal{B}_K(C) \to \mathcal{O}_K(V - C_n)/K\) is dense for each \(n\).

Put

\[
\mathcal{O}_{b,K}(C_n) = \left\{ g(z) = \sum_{i=1}^{l_n} \sum_{m=0}^{\infty} b^{(i)}_m(z - c_{n,i})^m; b^{(i)}_m \in K, \sup_{0 \leq m < \infty} |b^{(i)}_m| r_n^m < +\infty \right\}.
\]

Then \(\mathcal{O}_{b,K}(C_n)\) becomes a Banach space with

\[
|g(z)|_n = \sup_{0 \leq m < \infty} \sup_{i \leq l_n} |b^{(i)}_m| r_n^m.
\]
If \( n < l \) and \( |c_{n,i} - c_{l,i}| < r_n \), then \( \sum_{i=1}^{l_n} \sum_{m=1}^{\infty} b_{m}^{(i)}(z - c_{n,i})^m \) can be written in the form \( \sum_{i=1}^{l_n} \sum_{m=1}^{\infty} b_{m}^{(i)}(z - c_{l,i})^m \) with \( b_{m}^{(i)} \in K \), and we have \( \sup_{0 \leq m < +\infty} |b_{m}^{(i)}| r_m^{-m} = \sup_{0 \leq m < +\infty} |b_{m}^{(i)}| r_m^{-m} \). Hence we have an injective \( K \)-linear continuous map \( u_{i,n}: \mathcal{O}_{b,K}(C_n) \to \mathcal{O}_{b,K}(C_l) \) \((n < l)\). Let \( \mathcal{A}_K(C) \) be the locally convex inductive limit space of the Banach spaces \( \mathcal{O}_K(C_n) \).

For any
\[
A(z) = \sum_{i=1}^{l_n} \sum_{m=1}^{\infty} a_m^{(i)}(z - c_{n,i})^m \in \mathcal{O}_K(V - C_n) / K
\]
and
\[
B(z) = \sum_{i=1}^{l_n} \sum_{m=1}^{\infty} b_m^{(i)}(z - c_{l,i})^m \in \mathcal{O}_{b,K}(C_l),
\]
we define
\[
(f(z), g(z))_n = \sum_{i=1}^{l_n} \sum_{m=1}^{\infty} a_m^{(i)} b_m^{(i)}.
\]
Since \( |a_m^{(i)}| r_m^{m} \to 0 \) \((m \to \infty)\), and since the \( |b_m^{(i)}| r_m^{m} \)'s are bounded, this pairing \(( , )_n\) is a well defined bicontinuous \( K \)-bilinear nondegenerate pairing. If \( n < l \) and \( f(z) \in \mathcal{O}_K(V - C_l) / K \), then \( u_{n,l} f(z) \in \mathcal{O}_K(V - C_n) / K \) and \( f(z) g(z) \) is a \( K \)-analytic function on \( C_n - C_l \). Since \( (u_{n,l} f(z), g(z))_n \) can be regarded as the sum of residues of \( f(z) g(z) \) in \( C_l \), it is equal to \((f(z), v_{l,n} g(z))_l\).

For any \( f(z) \in \mathcal{B}(C) \) and \( g(z) \in \mathcal{A}_K(C) \), we choose a positive integer \( n \) and a unique element \( g_n(z) \) of \( \mathcal{O}_{b,K}(C_n) \) such that \( g(z) = v_n g_n(z) \), and we define
\[
(f(z), g(z)) = (u_n f(z), g_n(z)).
\]
Then it follows from the arguments in 1.1 that this pairing \(( , )\) is well defined, bicontinuous, \( K \)-bilinear and nondegenerate.

Now we have the following theorem:

**Theorem 2.** Let \( C, \mathcal{B}_K(C), \mathcal{A}_K(C) \) and
\[
( , ) : \mathcal{B}_K(C) \times \mathcal{A}_K(C) \to K
\]
be as before. Then \( \mathcal{B}_K(C) \) is a Fréchet space of countable type, \( \mathcal{A}_K(C) \) is a complete Hausdorff space, \(( , )\) is a bicontinuous \( K \)-bilinear nondegenerate pairing, and \( \mathcal{B}_K(C) \) and \( \mathcal{A}_K(C) \) become mutually dual locally convex spaces with respect to \(( , )\).

**Proof.** Let \( r_n = |d| \) \((d \in K)\). Then the mapping
\[
\mathcal{O}_K(V - C_n) / K \ni \sum_{i=1}^{l_n} \sum_{m=1}^{\infty} a_m^{(i)}(z - c_{n,i})^m
\]
\[
\mapsto (a_1^{(1)} d^{-1}, a_2^{(1)} d^{-1}, \ldots, a_1^{(i)} d^{-2}, a_2^{(i)} d^{-2}, \ldots, a_1^{(m)} d^{-m}, a_2^{(m)} d^{-m}, \ldots) \in c_0
\]
and
$\mathcal{O}_{k}(C_n) \ni \sum_{t=1}^{l_n} \sum_{m=0}^{\infty} b_m^{(t)}(x - a_n)^m$

\[ \mapsto (b_0^{(1)}, b_0^{(2)}, \ldots, b_1^{(1)}d, b_1^{(2)}d, \ldots, b_m^{(1)}d^m, b_m^{(2)}d^m, \ldots) \in l^\infty \]

are $K$-linear isometries of Banach spaces, and compatible with the pairing $(\cdot, \cdot)$ and the standard pairing $\langle \cdot, \cdot \rangle$ of $c_0$ and $l^\infty$ up to a constant factor $d^{-1}$, where

$\alpha_0 = \{(a_1, a_2, a_3, \cdots) \in K, |a_m| \to 0 (m \to \infty)\}$

$\beta_0 = \{(b_1, b_2, a_3, \cdots) \in K, \sup_{1 \leq m < \infty} |b_m| < \infty\}$

$l^\infty = \{(b_1, b_2, \cdots) \in K, \sup_{1 \leq m < \infty} |b_m| < \infty\}$

If $K$ is not spherically complete, then, by a theorem of van der Put (cf. e.g. van Rooij [7, p. 111]), $c_0$ and $l^\infty$ are mutually dual Banach spaces with respect to $\langle \cdot, \cdot \rangle$. Hence $\mathcal{O}_K(V - C_n)/K$ and $\mathcal{O}_{n,K}(C_n)$ are also mutually dual Banach spaces with respect to $\langle \cdot, \cdot \rangle$. Therefore we apply Theorem 1 to our case and obtain Theorem 2 in this case.

If $K$ is spherically complete, then this duality of $c_0$ and $l^\infty$ does not hold. But we can use Lemma 3.5 and Theorems in Morita [5, 3-1] in this case. Since we can prove our theorem as in [5, 3-3~3-4], we omit the details.

**Remark.** We can also show in the same way that the space $\mathcal{A}(U)$, Ind$(P, G, \chi)$ and $D_1$ of Morita [6, III] are complete Hausdorff spaces over any complete nonarchimedean field $k$. Further, we can construct the holomorphic discrete series $\pi_s$ of Morita [6, I] and prove the duality of $\pi_s$ and $T_s$ (cf. Morita [6, II, Theorem 3]) over any complete nonarchimedean field.

2.2. Let $C = \{0\}$ and let $d$ be an element of $K^*$ whose absolute value is smaller than 1. Then

$\mathcal{B}_K(C) = \{(a_{-1}, a_{-2}, a_{-3}, \cdots) \in K^*; \text{for any positive integer } n, |a_md^{m}| \to 0\}$

$\mathcal{A}_K(C) = \{(b_0, b_1, b_2, \cdots) \in K^*; \text{for some positive integer } n, \sup_{m} |b_md^{m}| < \infty\}$

$$\langle (a_{-1}, a_{-2}, a_{-3}, \cdots), (b_0, b_1, b_2, \cdots) \rangle = a_{-1}b_0 + a_{-2}b_1 + a_{-3}b_2 + \cdots.$$ 

The duality of Theorem 2 in this case for a nonspherically complete field was first proved by Schikhof by means of the results of De Grande-De
2.3. Let $K$ be a field with a complete nontrivial nonarchimedean valuation $| |$. Let $(r_n)_{n=1}^\infty$ be a strictly increasing sequence in $|K^*|$ such that $r_n \to 1$ ($n \to \infty$). Let $W$ be the $K$-vector space consisting of all Laurent series $\sum_{m=-\infty}^{+\infty} a_m z^m$ ($a_m \in K$) such that $|a_m| r^m \to 0$ ($m \to +\infty$) for any $r$ with $0 < r < 1$, and $|a_m| r^m \to 0$ ($m \to -\infty$) for some $r$ with $0 < r < 1$. Then $W$ is the direct sum $W_1 \oplus W_2$ of two subspaces:

$W_1 = \left\{ \sum_{m=0}^{+\infty} a_m z^m; a_m \in K, |a_m| r^m \to 0$ ($m \to +\infty$) for any $r$ with $0 < r < 1 \right\}$

$W_2 = \left\{ \sum_{m=-\infty}^{-1} b_m z^m; b_m \in K, |b_m| r^m \to 0$ ($m \to -\infty$) for some $r$ with $0 < r < 1 \right\}$.

Put

$W_{1,n} = \left\{ \sum_{m=0}^{+\infty} a_m z^m; a_m \in K, |a_m| r^m \to 0$ ($m \to +\infty$) \right\}$

and

$W_{2,n} = \left\{ \sum_{m=-\infty}^{-1} b_m z^m; b_m \in K, \sup_{m} |b_m| r^m < \infty \right\}$.

Then they become Banach spaces with the following norms:

$$\left| \sum_{m=0}^{+\infty} a_m z^m \right|_{1,n} = \sup_{m} |a_m| r^m \quad \text{and}$$

$$\left| \sum_{m=-\infty}^{-1} b_m z^m \right|_{2,n} = \sup_{m} |b_m| r^m .$$

Put

$$\left( \sum_{m=0}^{+\infty} a_m z^m, \sum_{m=-\infty}^{-1} b_m z^m \right) = \sum_{m+n=-1} a_m b_n .$$

Let $d$ be an element of $K$ with $|d| = r_n$. Then

$$\sum_{m=0}^{+\infty} a_m z^m \mapsto (a_0, a_1 d^1, a_2 d^2, \ldots) \quad \text{and}$$

$$\sum_{m=-\infty}^{-1} b_m z^m \mapsto (b_{-1} d^{-1}, b_{-2} d^{-2}, b_{-3} d^{-3}, \ldots)$$

induce isometries $W_{1,n} \tilde{\to} a_0$ and $W_{2,n} \tilde{\to} l^\infty$ preserving the pairings up to a constant factor. Hence $W_{1,n}$ is of countable type, and $W_{1,n}$ and $W_{2,n}$ become mutually dual Banach spaces with respect to $(,)_n$. Further $W_1$ and $W_2$ can be identified with proj lim $W_{1,n}$ and ind lim $W_{2,n}$ with respect to the natural maps $\nu_{n,i} : \sum a_m z^m \mapsto \sum a_m z^m$ ($n < l$) and $\nu_{1,n} : \sum b_m z^m \mapsto \sum b_m z^m$ ($n < l$). Let $\tau_1$ and $\tau_2$ be the projective limit topology of $W_1$ and the
inductive limit topology of $W$. By Morita [5, Lemma 3.5], the $v_{i,n}$'s are $c$-compact maps and the projective system \{ $W_{1,n}$, $u_{n,i}$ \} can be replaced by a cofinal system \{ $W_{1,m}$, $u'_{n,i}$ \} so that the resulting maps $u'_{n,i}$ are also $c$-compact if $K$ is spherically complete (cf. the arguments in [5, 3-3-3-3-4]). Since $W_i$ contains all finite sums of the form $\sum_{m=0}^{+\infty} a_m z^m$, the image of the projection map $u_n: W_i \rightarrow W_{1,n}$ is dense for each $n$. Hence it follows from Theorem 1 of this paper and theorems in Morita [5, 3-1] that (i) $W_i$ is a Fréchet space of countable type, (ii) $W_i$ is a complete Hausdorff space, and (iii) the pairing

$$(\sum_{m=0}^{+\infty} a_m z^m, \sum_{m=-1}^{+\infty} b_m z^m) = \sum_{m+n=-1} a_m b_m$$

makes $W_i$ and $W_2$ into mutually dual spaces. Hence the direct sum $W = W_i \oplus W_2$ is a complete Hausdorff space, and the inner product

$$(\sum_{m=0}^{+\infty} a_m z^m, \sum_{m=-\infty}^{+\infty} b_m z^m) = \sum_{m+n=-1} a_m b_n$$

makes $W$ into a self dual space. This selfduality of $W$ was conjectured by P. Robba.

REFERENCES

DUALITY OVER A COMPLETE NONARCHIMEDEAN FIELD


DEPARTMENT OF MATHEMATICS AND MATHEMATICAL INSTITUTE
Katholieke Universiteit Tōhoku University
6525 ED Nijmegen Sendai 980
THE NETHERLANDS JAPAN