ULTRAMETRIC COMPACTOIDS OF FINITE TYPE

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Report 8634
August 1986

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Throughout, let K be a nonarchimedean nontrivially valued complete field with valuation $|\ \ |$.

§ 0. PRELIMINARIES, NOTATIONS

For fundamentals on Banach spaces, locally convex spaces over K we refer to [1], [6], [3].

Let E be a K-vector space. A subset A of E is absolutely convex if it is a submodule of E considered as a module over $\{\lambda \in K : |\lambda| \leq 1\}$. A nonempty set is convex if it is an additive coset of an absolutely convex set. For $X \subset E$ let co X be its absolutely convex hull (= the module generated by X), let [X] be its K-linear span. X is a finite dimensional set if $\dim [X] < \infty$.

Let E be a locally convex space over K. The closure of $X \subset E$ is denoted $\overline{X}$. Instead of $\overline{\text{co } X}$ we write $\overline{\text{co}}\ X$. The dual space E' of E is the K-linear space of all continuous linear maps $E \to K$. The weak topology on E is the weakest locally convex topology on E for which all elements of E' are continuous. For a normed space $E = (E, \| \ \|)$ and a nonempty bounded subset X of E we write

$$\operatorname{diam} X := \sup \{\|x-y\| : x \in X , y \in X\}.$$

§ 1. COMPACTOIDS OF FINITE TYPE

We recall the definition of compactoidity ([1], p. 134).

DEFINITION 1.1. An absolutely convex subset A of a locally convex space E over K is (a) compactoid if for each neighbourhood U of 0 in E there exist $n \in \mathbb{N}$ and $x_1,\dots,x_n \in E$ such that $A \subset U+\text{co}\{x_1,\dots,x_n\}$.

The purpose of this paper is to study the impact of the following innocent-looking modification of Definition 1.1.

DEFINITION 1.2. An absolutely convex subset A of a locally convex space E over K is a compactoid of finite type if for each neighbourhood U of 0 in E there exists a bounded finite dimensional (absolutely convex) set $F \subset A$ such that $A \subset U+F$.

Two remarks.

(i) A compactoid of finite type is, indeed, a compactoid in the sense of Definition 1.1. (One verifies easily that a bounded finite dimensional set lies in the absolutely convex hull of some finite set.)

(ii) Suppose the valuation of K is discrete. It is not hard to prove ([3], Lemma 8.1) that in this case we can without harm replace the expression '$x_1,\dots,x_n \in E$' in Definition 1.1 by '$x_1,\dots,x_n \in A$'. Thus, each compactoid is automatically of finite type. Therefore

FROM NOW ON IN THIS PAPER WE ASSUME THAT THE VALUATION OF K IS DENSE.

For the construction of a compactoid that is not of finite type we shall use the following simple lemma.

LEMMA 1.3. *Let $x_1, x_2, \ldots$ be linearly independent elements of a* K-*vector space. Let* F *be a finite dimensional absolutely convex subset of* $co\{x_1, x_2, \ldots\}$. *Then* $F \subset co\{x_1, \ldots, x_n\}$ *for some* n.

Proof. $[F]$ is a finite dimensional subspace of $[x_1, x_2, \ldots]$ so $F \subset [x_1, \ldots, x_n]$ for some n. From linear independence it follows easily that $co\{x_1, x_2, \ldots\} \cap [x_1, \ldots, x_n] = co\{x_1, \ldots, x_n\}$.

EXAMPLE 1.4. *There exists a compactoid* A *in* c_0 *that is not of finite type*.

Proof. Choose a two-sided sequence $(\lambda_n)_{n \in \mathbb{Z}}$ in K such that $|\lambda_n| < |\lambda_{n+1}|$ for each $n \in \mathbb{Z}$, $\lim_{n\to\infty} |\lambda_n| = 1$, $\lim_{n\to -\infty} |\lambda_n| = 0$. Let $e_1, e_2, \ldots$ be the standard unit vectors in c_0, define

$$x_n := \lambda_n e_1 + \lambda_{-n} e_{n+1} \qquad (n \in \mathbb{N})$$

and set

$$A := co\{x_1, x_2, \ldots\}.$$

A is a compactoid since $A \subset \{x \in c_0 : \|x\| \le |\lambda_{-n}|\} + co\{e_1, \ldots, e_n\}$ for each $n \in \mathbb{N}$. Now let F be a finite dimensional absolutely convex subset of A; we shall prove that A is not contained in U+F where $U := \{x \in c_0 : \|x\| \le |\lambda_1|\}$. Lemma 1.3 ($x_1, x_2, \ldots$ are linearly independent) yields $F \subset co\{x_1, \ldots, x_n\}$ for some n. For each $i \in \{1, \ldots, n\}$

$$\|x_i\| = \|\lambda_i e_1 + \lambda_{-i} e_{i+1}\| = |\lambda_i| \le |\lambda_n|$$

We see that the norm function is bounded by $|\lambda_n|$ on $co\{x_1, \ldots, x_n\}$, so certainly on F, hence also on U+F. Consequently, $x_{n+1} \in A\backslash U+F$.

The set A we just have constructed is not closed. (It is not hard to see that its closure is of finite type.) This is not completely accidental:

THEOREM 1.5. *Let* K *be spherically (= maximally) complete. Let* A *be a complete absolutely convex compactoid in a Hausdorff locally convex space over* K. *Then* A *is of finite type.*

Proof. A is bounded and c-compact ([4], Proposition 2.2). Now apply [4], Proposition 2.3.

PROBLEM. Let A be an absolutely convex complete compactoid in a Banach space E over a nonspherically complete K. Does it follow that A is of finite type ? (By [1] 4.S (viii) it suffices to consider $E = c_0$.)

Surprisingly we have

EXAMPLE 1.6. *Let* K *be not spherically complete. The unit ball* $\{x \in c_0 : \|x\| \leq 1\}$ *is, for the weak topology on* c_0, *a complete compactoid but not of finite type.*

Proof. From the reflexivity of c_0 ([1], Theorem 4.17) it follows that the weak topology is quasicomplete ([3], Theorem 9.6). As the unit ball is weakly bounded and weakly closed it is weakly complete. The other statements follow from the following example (where K is allowed to be spherically complete):

EXAMPLE 1.7. *The unit ball of* c_0 *is a weak compactoid but not of finite type.*

Proof. Weak compactoidity of $A := \{x \in c_0 : \|x\| \leq 1\}$ follows almost immediately from the fact that each weak neighbourhood of 0 contains a K-linear space with finite codimension. Now choose $\tau_1, \tau_2, \ldots \in K$ such that $0 < |\tau_1| < |\tau_2| < \ldots$, $\lim_{n\to\infty} |\tau_n| = 1$. There is a (unique) $f \in c_0'$ for which $f(e_n) = \tau_n$ for each $n \in \mathbb{N}$. Then $|f(x)| < 1$ for each $x \in A$. The set

$$U := \{x \in c_0 : |f(x)| \leq \tfrac{1}{2}\}$$

is a weak neighbourhood of 0. Let $F \subset A$ be finite dimensional; we prove that A is not contained in U+F. By Gruson's Theorem ([1], Theorem 5.9) the (finite dimensional) space [F] has an orthonormal base $z_1, \ldots, z_n$. From $F \subset A \cap [F] \subset co\{z_1, \ldots, z_n\}$ we obtain

$$\sup_F |f| \leq \max_{1 \leq i \leq n} |f(z_i)| < 1.$$

It follows that

$$\sup_{U+F} |f| < 1.$$

But

$$\sup_A |f| \geq \sup_n |f(e_n)| = \lim_{n\to\infty} |\tau_n| = 1$$

and therefore $A \not\subset U+F$.

§ 2. SOME GENERAL PROPERTIES

In § 2, E and E_i are (Hausdorff) locally convex spaces over K.

PROPOSITION 2.1. *If $A \subset E$ is a compactoid of finite type then so is $\overline{A}$.*

Proof. Let U be an absolutely convex neighbourhood of 0 in E. There is a finite dimensional bounded $F \subset A$ with $A \subset U+F$. Now $U+F$, and also its complement, is a union of cosets of the open additive group U. So $U+F$ is closed and $\overline{A} \subset U+F$.

PROPOSITION 2.2. *Let T be a continuous linear map of E_1 into E_2. If $A \subset E_1$ is a compactoid of finite type then so is $TA \subset E_2$.*

Proof. Let U be an absolutely convex neighbourhood of 0 in E_2. There is a finite dimensional bounded $F \subset A$ with $A \subset T^{-1}(U)+F$. Then $TF \subset TA$ is finite dimensional, bounded and $TA \subset U+TF$.

PROPOSITION 2.3. *Let $A_i \subset E_i$ $(i \in I)$ be compactoids of finite type. Then $A := \prod_i A_i$ is a compactoid of finite type in $E := \prod_i E_i$.*

Proof. Let U be a neighbourhood of 0 in E; we construct a finite dimensional $F \subset A$ with $A \subset U+F$. We may assume that $U = \prod_i U_i$ where, for each i, U_i is an absolutely convex neighbourhood of 0 in E_i and where $U_i = E_i$ except for i in some finite set $J \subset I$. For each $j \in J$, choose a finite dimensional $F_j \subset A_j$ such that $A_j \subset U_j+F_j$. If $i \in I\backslash J$, choose $F_i := (0) \subset A_i$. The set

$$F := \prod_i F_i$$

is finite dimensional and $F \subset A$. To prove $A \subset U+F$, let $a = (a_i)_{i \in I} \in A$. If $j \in J$, choose $u_j \in U_j$, $f_j \in F_j$ such that

$$a_j = u_j + f_j.$$

If $i \in I \setminus J$, take $u_i := a_i$ and $f_i := 0$. We obtain a decomposition

$$a = u + f$$

where $u = (u_i)_{i \in I} \in U$ and $f = (f_i)_{i \in I} \in F$.

Absolutely convex subsets of a compactoid of finite type may fail to be of finite type (Example 1.4: $\overline{A}$ is of finite type, A is not). In fact, each compactoid is a subset of some compactoid of finite type. ([5], Theorem 2.1. *Observe that* $\overline{co}\, X$ *(X compact) is of finite type.)* However, we do have results for special subsets (Proposition 2.4 and 2.5).

PROPOSITION 2.4. Let $A \subset E$ be a compactoid of finite type. Then so is $A^i := \bigcup_{|\lambda| < 1} \lambda A$.

Proof. An obvious verification yields

$$(S + T)^i = S^i + T^i$$

for absolutely convex $S, T \subset E$. Now let U be an absolutely convex neighbourhood of 0 in E. There is a bounded absolutely convex finite dimensional $F \subset A$ with $A \subset U+F$. Then $A^i \subset (U+F)^i = U^i+F^i \subset U+F^i$ and $F^i \subset A^i$.

PROPOSITION 2.5. Let $A \subset E$ be a compactoid of finite type, let $U \subset E$ be an absolutely convex neighbourhood of 0. Then $A \cap U$ is of finite type.

Proof. Let V be an absolutely convex neighbourhood of 0 in E. To prove that $A \cap U \subset V+F$ for some finite dimensional $F \subset A \cap U$ we may assume $V \subset U$. There is a finite dimensional bounded absolutely convex $G \subset A$ for which $A \subset V+G$. Set $F : G \cap U$. If $x \in A \cap U$ then $x = v+g$ for some $v \in V \subset U$ and $g \in G$. Then $g = x-v \in U$. Hence, $A \cap U \subset V+F$.

Proposition 2.4 leads to a 'dual' question. For an absolutely convex A set $A^e := \bigcap_{|\lambda| > 1} \lambda A$. If A is a compactoid of finite type, does it follow that A^e is of finite type ? This question is more difficult than the one for A^i. I can answer it only for spherically complete K. (Proposition 2.7.)

LEMMA 2.6. Let $S,T \subset E$ be absolutely convex. Suppose S is closed and T is c-compact. Then $(S+T)^e = S^e+T^e$.

Proof. We may assume $S = S^e$, $T = T^e$; we prove that $(S+T)^e \subset S+T$. Let $z \in (S+T)^e$. For each $\lambda \in K$, $|\lambda| > 1$ we have $z \in \lambda(S+T)$ i.e. $z-\lambda S$ meets λT. So, for each $\lambda \in K$, $|\lambda| > 1$ the convex set

$$V_\lambda := (z-\lambda S) \cap \lambda T$$

is a nonempty closed subset of λT, hence c-compact. If $1 < |\lambda| < |\mu|$ then $V_\lambda \subset V_\mu$. By c-compactness there is a $t \in \bigcap_{|\lambda| > 1} V_\lambda$. Then $t \in \bigcap_{|\lambda| > 1} \lambda T = T^e = T$ and $t \in z-\lambda S$ for all $\lambda \in K$, $|\lambda| > 1$, i.e. $z-t \in S^e = S$. It follows that $z \in S+T$.

PROPOSITION 2.7. Let K be spherically complete. If $A \subset E$ is a compactoid of finite type then so is $A^e := \bigcap_{|\lambda| > 1} \lambda A$.

Proof. Let U be an absolutely convex neighbourhood of 0 in E, let $\lambda \in K$, $0 < |\lambda| < 1$. There is a finite dimensional absolutely convex set $F \subset A$ such that $A \subset \lambda U+F$. Now λU is closed and F is c-compact (each convex subset of K^n is closed hence c-compact) so that we may apply the previous Lemma. We find

$$A^e \subset (\lambda U + F)^e = (\lambda U)^e + F^e \subset U + F^e$$

which proves Proposition 2.7.

PROBLEM. Is the statement about A in Proposition 2.7 true if K is not spherically complete ? (See [2], Example 5.4 for the difficulties one encounters with the identity $(S+T)^e = S^e+T^e$.)

PROPOSITION 2.8. _Let_ $A \subset E$ _be a compactoid of finite type. For each continuous seminorm_ p _on_ E _there exists a finite dimensional set_ $F \subset A$ _for which_ $\sup_A p = \sup_F p$.

Proof. We may suppose $s := \sup_A p > 0$. Set $U := \{x \in E : p(x) \le \frac{1}{2}s\}$. There is a finite dimensional absolutely convex set $F \subset A$ for which $A \subset U+F$. Then $A = A \cap U + F$. Now $p \le \frac{1}{2}s$ on $U \cap A$. It follows easily (strong triangle inequality) that $\sup_A p = \sup_F p$.

LEMMA 2.9. _Let_ K _be spherically complete, let_ $F \neq (0)$ _be a finite dimensional absolutely convex subset of some_ K-_vector space. Then there are onedimensional absolutely convex sets_ $F_1,\ldots,F_n$ _for some_ $n \in \mathbb{N}$ _such that_ $F = \sum_{i=1}^{n} F_i$.

Proof. [2], Corollary 2.13 (i).

Remark. Let K be not spherically complete. The unit ball of K_ν^2 (see [1], p. 68) is twodimensional but indecomposable.

COROLLARY 2.10. _Let_ K _be spherically complete, let_ $A \subset E$ _be a compactoid of finite type. For each continuous seminorm_ p _on_ E _there exists an_ $x \in E$ _such that_

$$\sup_A p = \sup_{Kx \cap A} p.$$

Proof. We may assume $p \neq 0$. By Proposition 2.8, $\sup_A p = \sup_F p$ for some finite dimensional absolutely convex $F \subset A$. By the Lemma, $F = \sum_{i=1}^{n} F_i$ for some onedimensional $F_1,\dots,F_n$. The strong triangle inequality yields $\sup_F p = \max_i \sup_{F_i} p = \sup_{F_j} p$ for some j. There is an $x \in E$ and an absolutely convex set $C \subset K$ such that $F_j = Cx$. Then $Kx \cap A \supset Cx$ and we have

$$\sup_F p = \sup_{Cx} p \leq \sup_{Kx \cap A} p \leq \sup_A p = \sup_F p.$$

§ 3. COMPACTOIDS OF FINITE TYPE IN NORMED SPACES

THEOREM 3.1. *Let* E *be a normed space over* K, *let* $A \subset E$ *be absolutely convex. The following are equivalent.*

(α) A *is a compactoid of finite type.*

(β) *There exist bounded finite dimensional absolutely convex sets* $F_0, F_1, \ldots$ *such that* $\lim_{n\to\infty} \operatorname{diam} F_n = 0$ *and*

$$\Sigma F_n \subset A \subset \overline{\Sigma F_n}$$

Proof.

(α) $\Rightarrow$ (β). For $n \in \mathbb{N}$, set $U_n := \{x \in E : \|x\| \le 2^{-n}\}$. There is a bounded finite dimensional absolutely convex $F_0 \subset A$ such that $A \subset U_1 + F_0$. Then $A = A \cap U_1 + F_0$. By Proposition 2.5, $A \cap U_1$ is a compactoid of finite type. So there exists a finite dimensional absolutely convex $F_1 \subset A \cap U_1$ such that $A \cap U_1 \subset U_2 + F_1$ i.e. $A \cap U_1 = A \cap U_2 + F_1$. We have $\operatorname{diam} F_1 \le 2^{-1}$ and $A = A \cap U_2 + F_1 + F_0$. Inductively we find bounded absolutely convex finite dimensional sets $F_0, F_1, \ldots$ with $\operatorname{diam} F_n \le 2^{-n}$ and

$$A = A \cap U_n + \sum_{i=1}^{n-1} F_i$$

for each $n \in \{1,2,\ldots\}$. As $\operatorname{diam} A \cap U_n \le 2^{-n}$, ($\beta$) follows.

(β) $\Rightarrow$ (α). Let $\varepsilon > 0$. There is an $N \in \mathbb{N}$ such that $\operatorname{diam} F_n \le \varepsilon$ for $n \ge N$. We see that

$$\sum_i F_i \subset \{x \in E : \|x\| \le \varepsilon\} + \sum_{i=0}^{n-1} F_i.$$

The set at the right hand side is closed so that

$$A \subset \{x \in E : \|x\| \le \varepsilon\} + \sum_{1=0}^{n-1} F_i$$

and we have (α).

<u>Two remarks</u>.

(i) For spherically complete K we have a more refined version of Theorem 3.1, namely we may replace (β) by:

(β)' <u>There exist</u> $e_1, e_2, \ldots$ <u>in</u> E <u>and absolutely convex sets</u> $C_1, C_2, \ldots \subset K$ <u>such that</u> $C_n e_n$ <u>is bounded for each</u> n, $\lim_{n\to\infty} (\text{diam } C_n) \, \|e_n\| = 0$ <u>and</u>

$$\Sigma\, C_n e_n \subset A \subset \overline{\Sigma\, C_n e_n}.$$

The proof is obvious (Lemma 2.9).

(ii) It is not hard to generalize Theorem 3.1 to the case of a metrizable absolutely convex set A in a locally convex space E. We leave the details to the reader.

§ 4. WEAK COMPACTOIDS OF FINITE TYPE

The main goal in this section is the following theorem, which is a generalization of Example 1.7. Recall that the valuation of K is dense.

THEOREM 4.1. *Let E be a Banach space with an orthogonal base over a spherically complete K. If $A \subset E$ is a compactoid of finite type for the weak topology then A is a compactoid (of finite type) for the norm topology.*

This Theorem is in contrast to

PROPOSITION 4.2. *Let E be a Banach space over a spherically complete K. An absolutely convex $A \subset E$ is a compactoid for the weak topology if and only if A is bounded for the norm topology.*

Proof. A weak compactoid is weakly bounded, hence bounded by [6], Théorème 4.21. Conversely, if A is norm bounded, let U be a weak neighbourhood of 0 in E. There is a closed K-subspace D of E of finite codimension with $D \subset U$. Let $\pi : E \to E/D$ be the quotient map. $\pi(A)$ is bounded in the finite dimensional space E/D, so a compactoid. As $\pi(u)$ is open in E/D there exist $x_1,\ldots,x_n \in E/D$ such that $\pi(A) \subset \pi(U)+\text{co}\{x_1,\ldots,x_n\}$. Let $y_1,\ldots,y_n \in E$ be such that $\pi(y_i) = x_i$ for each $i \in \{1,\ldots,n\}$. It is easy to see that $A \subset U+\text{co}\{y_1,\ldots,y_n\}$.

Theorem 4.1 and Proposition 4.2 indicate that, for a compactoid, to require it to be of finite type may be a severe restriction !

PROBLEM. Does the conclusion of Theorem 4.1 hold for arbitrary Banach spaces over a spherically complete K or for decent Banach spaces (e.g. c_0) over a nonspherically complete K ?

The proof of Theorem 4.1 requires several preparations. From now on in § 4 we assume that K is spherically complete.

For an element f of the dual E' of a Banach space E over K, let $\|f\|$ denote the operator norm of f defined by the usual formula
$\|f\| = \inf \{M : |f(x)| \le M\|x\| \text{ for all } x \in E\}$.

LEMMA 4.3. *Let* E *be a Banach space over* K. *Then*

$$\|f\| = \sup \{|f(x)| : \|x\| \le 1\} \qquad (f \in E').$$

Proof. We may assume dim E ≥ 1. Set $\|f\|_0 := \sup \{|f(x)| : \|x\| \le 1\}$.

If $x \in E$, $0 < \|x\| \le 1$ then $|f(x)| \le \dfrac{|f(x)|}{\|x\|} \le \|f\|$ so that $\|f\|_0 \le \|f\|$.

Conversely, let $x \in E$, $x \ne 0$. As the valuation of K is dense we can find $\lambda_1, \lambda_2, \ldots \in K$ such that $|\lambda_1| < |\lambda_2| < \ldots$, $\lim_{n\to\infty} |\lambda_n| = \|x\|^{-1}$.
Then $\|\lambda_n x\| \le 1$ whence $|f(\lambda_n x)| \le \|f\|_0$ for each n. We have

$$\frac{|f(x)|}{\|x\|} = \frac{|f(\lambda_n x)|}{\|\lambda_n x\|} \le \frac{\|f\|_0}{\|\lambda_n x\|}$$

which, after taking limits becomes $\dfrac{|f(x)|}{\|x\|} \le \|f\|_0$ and we obtain $\|f\| \le \|f\|_0$.

LEMMA 4.4. *Let* $\{D_i : i \in I\}$ *be an orthogonal system of closed subspaces* ([1], p. 166) *in a Banach space* E *over* K. *For each* $i \in I$, let $S_i \subset D_i$ *be closed, absolutely convex,* $[S_i] = D_i$, $S_i^e = S_i$ (*see the remark following Proposition* 2.5). *Then* $(\overline{\Sigma S_i})^e = \overline{\Sigma S_i}$.

Proof. Let $x \in (\overline{\Sigma S_i})^e$. Then $x \in \overline{\Sigma D_i}$ so x has a unique decomposition

$x = \sum_i d_i$ where $d_i \in D_i = [S_i]$ for each $i \in I$. For any $\lambda \in K$, $0 < |\lambda| < 1$, $\lambda x \in \overline{\Sigma S_i}$ so $\lambda x = \sum_i s_i$, where $s_i \in S_i$ for each i. But also $\lambda x = \sum_{i \in I} \lambda d_i$. By the uniqueness of the decomposition of λx we have $\lambda d_i = s_i$ i.e. $d_i \in \lambda^{-1} S_i$ for each i. This goes for any $\lambda \in K$, $0 < |\lambda| < 1$ and we find $d_i \in S_i^e = S_i$ for each i. It follows that $x \in \overline{\Sigma S_i}$.

LEMMA 4.5. *Let* S *be a closed absolutely convex subset of a Banach space* E *over* K *for which* $S = S^e$. *If* $x \in E \backslash S$ *there exists an* $f \in E'$ *with* $|\lambda| \le 1$ *on* S, $|f(x)| > 1$.

Proof. [6], Théorème 4.7a.

LEMMA 4.6. *Let* E *be a Banach space over* K *such that each maximal orthogonal system of vectors in* E *is an orthogonal base. Then* E *is finite dimensional.*

Proof. [1], Theorem 5.16 $(\alpha) \Rightarrow (\theta)$. (Recall that the valuation of K is dense.)

LEMMA 4.7. *Let* A *be an absolutely convex subset of a* K-*vector space. For the seminorm* p_A *associated to* A *defined on* [A] *by the formula*

$$p_A(x) = \inf \{|\lambda| : \lambda \in K \ , \ x \in \lambda A\} \qquad (x \in [A])$$

we have

$$\{x \in [A] : p_A(x) < 1\} \subset A \subset \{x \in [A] : p_A(x) \le 1\}.$$

Further, $A = A^e$ *if and only if*

$$A = \{x \in [A] : p_A(x) \le 1\}.$$

Proof. Left to the reader.

PROPOSITION 4.8. (Compare Example 1.7.) Let E be a Banach space over K and suppose there exists an open absolutely convex set $A \subset E$ that is a compactoid of finite type for the weak topology. Then E is finite dimensional.

Proof. By Proposition 2.7 we may assume that $A = A^e$. A is bounded by Proposition 4.2 and therefore the seminorm associated to A is norm $\| \ \|$ inducing the topology of E. Lemma 4.7 yields

$$A = \{x \in E : \|x\| \leq 1\}.$$

Now let $\{e_i : i \in I\}$ be a maximal orthogonal system in E, it suffices to prove that it is an orthogonal base for E (Lemma 4.6). For each $i \in I$ set $C_i := \{\lambda \in K : \lambda e_i \in A\}$. Then $C_i = \{\lambda \in K : |\lambda| \leq \|e_i\|^{-1}\}$ so that $(C_i e_i)^e = C_i e_i$. Set

$$B := \overline{\sum_{i \in I} C_i e_i}.$$

Obviously, $B \subset A$. It remains to prove that $B = A$. Suppose $B \neq A$. Lemma 4.4 tells us that $B = B^e$ so, by Lemma 4.5, there exists an $f \in E'$ such that $|f| \leq 1$ on B and $\sup_A |f| > 1$. According to Corollary 2.10 and Lemma 4.3 there exists an $x \in E$ such that

$$\|f\| = \sup_A |f| = \sup_{Kx \cap A} |f| > 1.$$

Without harm (suitable scalar multiplication) we may assume $x \in A$ and $|f(x)| > 1$. Now $x \neq e_i$ for each $i \in I$ and, by maximality, the system $\{x\} \cup \{e_i : i \in I\}$ is not orthogonal. So there exists a (finite) K-linear combination $z = \Sigma \lambda_i e_i$ for which $\|x-z\| < \|x\|$. Then $\max_i \|\lambda_i e_i\| = \|z\| = \|x\| \leq 1$, so that $\lambda_i \in C_i$ for each i whence $z \in B$ and $|f(z)| < 1$. We get

$$(*) \qquad |f(x)| = |f(x-z)| \leq \|f\| \ \|x-z\| < \|f\| \ \|x\|.$$

Now let $C := \{\lambda \in K : \lambda x \in A\}$. We have $\lambda \in C \Rightarrow \|\lambda x\| \le 1$. Hence,

$$(\text{diam } C)\ \|x\| \le 1.$$

On the other hand we obtain, using (*),

$$\|f\| = \sup_{\lambda \in C} |f(\lambda x)| = (\text{diam } C)\ |f(x)| < (\text{diam } C)\ \|f\|\ \|x\|$$

yielding

$$(\text{diam } C)\ \|x\| > 1,$$

a contradiction.

Proof of Theorem 4.1. Suppose A is not a compactoid for the norm topology. Then, by [1], Theorem 4.38, $(\eta) \Rightarrow (\alpha)$ there exists an orthogonal sequence $e_1, e_2, \ldots$ in A such that $\inf_n \|e_n\| > 0$. Set $D := \overline{[e_1, e_2, \ldots]}$. By [1], Corollary 3.18 there exists a continuous linear projection P of E onto D. Since $e_n \in PA$ for each n and $\inf_n \|e_n\| > 0$ we have that $\overline{PA}$ is open in D. But (P is also weakly continuous) the weak closure of $\overline{PA}$ is a compactoid of finite type in D for the weak topology of D (Propositions 2.2 and 2.1). This is impossible by Proposition 4.8. Thus, A is a compactoid for the norm topology. To see that A is also of finite type it suffices to observe that the weak and norm topology coincide on A ([3], Theorem 5.12).

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