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In this note we shall prove some properties of c-compact sets that may or may not be part of the 'folklore'. The concept of c-compactness, introduced by Springer [7], takes over the role played by convex-compact sets in Functional Analysis over \( \mathbb{R} \) or \( \mathbb{C} \) (or, any locally compact valued field).

Throughout, let \( K \) be a nonarchimedean nontrivially valued field with valuation \( | \cdot | \). We assume \( K \) to be maximally (= spherically) complete. A subset \( A \) of a \( K \)-linear space \( E \) is absolutely convex if it is a submodule of \( E \), considered as a module over the valuation ring \( \{ \lambda \in K : |\lambda| \leq 1 \} \).

A set \( C \subseteq E \) is convex if it is either empty or an additive coset of an absolutely convex set. For a set \( X \subseteq E \) we denote by \( \text{co} X \) its absolutely convex hull, by \( [X] \) its \( K \)-linear span.

From now on in this paper \( E \) is a locally convex space over \( K \) ([8]). We assume \( E \) to be Hausdorff.
§ 1. DEFINITION AND FIRST PROPERTIES

DEFINITION 1.1. ([7]) Let $C \subseteq E$ be a nonempty convex set. A convex filter on $C$ is a filter of subsets of $C$ that has a basis consisting of convex sets. $C$ is $c$-compact if each convex filter on $C$ has a cluster point in $C$.

In other words, $C$ is $c$-compact if and only if the following is true. Let $C$ be a family of nonempty relatively closed convex subsets of $C$ such that $C_1, C_2 \in C$ implies $C_1 \cap C_2 \in C$. Then $\cap C \neq \emptyset$.

We quote the following properties, proved in [7].

PROPOSITION 1.2.

(i) $K$ is $c$-compact.

(ii) A $c$-compact set is complete.

(iii) A nonempty closed convex subset of a $c$-compact set is $c$-compact.

(iv) Let $(E_i)_{i \in I}$ be a family of Hausdorff locally convex spaces over $K$. Suppose, for each $i$, $C_i$ is $c$-compact in $E_i$. Then $\bigcap_{i \in I} C_i$ is $c$-compact in $\bigcap_{i \in I} E_i$.

(v) The image of a $c$-compact set under a continuous linear map is $c$-compact.

In [1] we find the following.

PROPOSITION 1.3.

(i) If $K$ is locally compact then a bounded nonempty convex set $C \subseteq E$ is $c$-compact if and only if it is convex and compact.

(ii) $E$ is $c$-compact if and only if $E$ is linearly homeomorphic to a
In § 3 (Theorem 3.3) we shall characterize arbitrary c-compact sets in the spirit of Proposition 1.3 (ii). But we conclude this first section with two statements that have nothing to do with the sequel. I just want to get rid of them.

PROPOSITION 1.4.

A c-compact set is a Baire space.

Proof.

Let $U_1, U_2, \ldots$ be (relatively) open dense subsets of a c-compact set $C \subseteq E$. We prove that $\bigcap_n U_n \neq \emptyset$. There exists a nonempty open convex subset $B_1 \subseteq U_1$. As $U_2$ is dense we can find a nonempty open convex set $B_2 \subseteq B_1 \cap U_2$. Continuing this way we find nonempty open convex sets $B_n \subseteq \bigcap_{i=1}^n U_i$ for each $n$. The open sets $B_n$ are cosets of an additive group, hence closed. By c-compactness, $\bigcap_n B_n \neq \emptyset$. It follows that $\bigcap_n U_n \neq \emptyset$.

PROPOSITION 1.5.

Let $X \subseteq E$ be closed, let $C \subseteq E$ be c-compact. Then $X+C$ is closed.

Proof.

Let $z \in \overline{X+C}$ (the closure of $X+C$), let $\mathcal{U}$ be the collection of all absolutely convex neighbourhoods of 0. For each $U \in \mathcal{U}$ the set $z+U$ intersects $X+C$ so

$$C_U := \{ c \in C : z-c \in X+U \}$$

is not empty. $X+U$ is a union of cosets of $U$, so is its complement.
Therefore, \(X+U\) is closed and \(C_U\) is closed in \(C\). Further we have

\[C_U \cap C_V \supseteq C_{U \cap V} \quad (U, V \in U)\]

By \(c\)-compactness there exists a \(c \in C\) such that

\[z-c \in \bigcap \{X+U \mid U \in U\} = X = X\]

i.e., \(z \in X+c \subseteq X+C\).

**Remark.**

If the base field is not spherically complete there exist a complete absolutely convex compactoid \(C \subseteq C_0\) and an element \(a \in C_0\) such that \(C+a\) is not closed ([3], 6.25).
§ 2. LOCAL COMPACTOIDITY

DEFINITION 2.1. ([3], (6.7)) A subset $X$ of $E$ is a local compactoid if for each neighbourhood $U$ of 0 in $E$ there exists a finite dimensional $K$-linear subspace $D$ of $E$ such that $X \subseteq U + D$.

PROPOSITION 2.2.

Let $A$ be an absolutely convex subset of $E$. $A$ is $c$-compact if and only if $A$ is a complete local compactoid.

Proof.

For $E$ a Banach space this is proved in [3], 6.15. Now let $E$ be a locally convex.

(i) Assume $A$ is $c$-compact. By Proposition 1.2 (ii), $A$ is complete. To prove local compactoidity let $U$ be an absolutely convex neighbourhood of 0 in $E$. There is a continuous seminorm $p$ such that $\{x \in E : p(x) \leq 1\} \subseteq U$.

Let $\pi_p : E \to E_p$ be the quotient map where $E_p$ is the canonically normed space $E/\ker p$. Now $\pi_p(A)$ is $c$-compact (Proposition 1.2 (v)) so by the above it is a local compactoid in the completion $E^\pi$ of $E_p$. By Corollary 6.15 of [3] we have $\pi_p(A) = R + T$ where $R$ is a compactoid and $T$ a finite dimensional subspace of $E^\pi_p$. Then $T \subseteq E_p$. Now $\pi_p(U)$ is open in $E_p$ and by Katsaras' Theorem ([5], Lemma 8.1) there exist $x_1, \ldots, x_n \in [R]$ such that $R \subseteq \pi_p(U) + \text{co}(x_1, \ldots, x_n)$. Combining our knowledge on $R$ and $T$ we find a finite dimensional space $F \subseteq [\pi_p(A)]$ such that $\pi_p(A) \subseteq \pi_p(U) + F$. Choose a finite dimensional space $D \subseteq [A]$ such that $\pi_p(D) = F$. Then

$$A \subseteq U + D + \ker \pi_p \subseteq U + D.$$  

(ii) Let $A$ be a complete local compactoid. Let $\Gamma$ be the collection of all continuous seminorms on $E$. For each $p \in \Gamma$ we have that $\pi_p(A)$, and also $\pi_p(A)$, is a local compactoid in $E^\pi_p$. 
As $E^*$ is a Banach space we know that $\prod_{\beta} \pi_\beta(A)$ is $c$-compact. Then also $A_0 := \prod_{\beta} \pi_\beta(A)$ is a $c$-compact subset of $\prod_{\beta} E^*$ (Proposition 1.1 (iv)).

The canonical map $E \to \prod_{\beta} E^*$ sends $A$ homeomorphically and linearly into $A_0$. Its image is closed in $A_0$ because $A$ is complete. Then $A$ is $c$-compact (Proposition 1.2 (iii)).

The following Proposition may look innocent.

**PROPOSITION 2.3.**

Let $A \subset E$ be absolutely convex and $c$-compact. For each neighbourhood $U$ of $0$ there exists a finite dimensional absolutely convex set $F \subset A$ such that $A \subset U + F$.

(The crucial part is the phrase 'F \subset A'.) For the proof we use a lemma.

**LEMMA 2.4.**

Let $A, U$ be absolutely convex subsets of $E$, where $U$ is closed, $A$ is $c$-compact. Let $x \in E$ be such that $A \subset U + Kx$. Then there exists an $y \in E$ and an absolutely convex $C \subset K$ such that $Cy \subset A$ and $A \subset U + Cy$.

**Proof.**

Let $C := \{ c \in K : (U + cx) \cap A \neq \emptyset \}$. We have $A \subset U + CX$, $C = \{ c \in K : cx \in A + U \}$, so $C$ is absolutely convex. If $C = (0)$ then $A \subset U$ and we choose $y := 0$.

So assume $C \neq (0)$. For each $c \in C$, $c \neq 0$ define

$$H_c := c^{-1}(A \cap (cx + U)).$$

Each $H_c$ is a convex, closed, nonempty subset of $c^{-1}A$ hence $c$-compact. Further, if $c,d \in C$, $0 < |c| \leq |d|$ then $H_d \subset H_c$. (Proof. Let $z \in H_d$. Then $dz \in A \cap (dx + U)$. By absolute convexity of $A$ and $U$,
\[ cz = \frac{c}{d} \cdot dz \in A \]
\[ cz \in \frac{c}{d} (dx+U) \subseteq cx + \frac{c}{d} U \subseteq cx+U. \]

It follows that \( cz \in A \cap (cx+U) \) i.e. \( z \in H_c \). By \( c \)-compactness there exists an \( y \in \cap H_c \). Let \( c \in C, c \neq 0 \). Then
\[ cy \in ch_c \subseteq A \cap (cx+U) \subseteq A. \]

Also, \( cy \in cx+U \) so that \( cx-cy \in U \). Let \( a \in A \). Then \( a = u+cx \) for some \( u \in U, c \in C \). We see that \( a = u+cy+cx-cy \in cy+U \). It follows that
\[ A \subseteq u+cy+U. \]

**Proof of Proposition 2.3.**

We may assume that \( U \) is absolutely convex. By Proposition 2.2 \( A \) is a local compactoid so there exist \( x_1, \ldots, x_n \in E \) such that
\[ A \subseteq U+Kx_1+\ldots+Kx_n. \]
By the Lemma, applied to \( U+Kx_2+\ldots+Kx_n \) in place of \( U \), there exist a \( y_1 \in E \) and an absolutely convex \( C_1 \in K \) such that
\[ C_1 y_1 \subseteq A \]
and
\[ A \subseteq U + C_1 y_1 + Kx_2 + \ldots + Kx_n = (U + C_1 y_1 + Kx_3 + \ldots, Kx_n) + Kx_2 \]
and we can continue. After \( n \) of these procedures we arrive at
\[ y_1, \ldots, y_n \in E, \text{ absolutely convex } C_1, \ldots, C_n \subseteq K \text{ such that } C_i y_i \subseteq A \]
for each \( i \) and \( A \subseteq U+C_1 y_1+\ldots+C_n y_n \).

**Warning.**

The property of Proposition 2.3 is not shared by all absolutely convex local compactoids even when we require them to be closed! In fact we have:
EXAMPLE 2.5.

Let the valuation of \( K \) be dense. Set

\[
A = \{ x \in c_0 : \| x \| \leq 1 \}.
\]

([4], p.47).

(i) \( A \) is a closed (local) compactoid for the weak topology of \( c_0 \).

(ii) There exists a weak neighbourhood \( U \) of 0 such that for any finite dimensional set \( F \subset A \)

\[ A \not\subset U+F. \]

Proof.

(i) Let \( U \) be a weak neighbourhood of 0. There exists a weakly continuous seminorm \( p \) such that \( \{ x \in c_0 : p(x) \leq 1 \} \subset U \). Then \( \ker p \) has finite codimension. Choose a finite dimensional space \( D \subset c_0 \) with \( \pi_p(D) = E_p \) (where as previously, \( E_p := c_0/\ker p \) and \( \pi_p : c_0 \rightarrow E_p \) is the quotient map). We have \( A \subset \ker p + D \subset U + D \) (in fact, we have shown that each subset of \( c_0 \) is a local compactoid for the weak topology). To prove weak closedness of \( A \), let \( (x_i)_{i \in I} \) be a net in \( A \) converging weakly to \( x \in c_0 \). By [4], Lemma 4.35 (i) there exists an \( f \in c_0^* \) \( f \not= 0 \) for which \( |f(x_i)| = \| f \| \| x_i \| \). We have

\[
|f(x)| = \lim_{i \to \infty} |f(x_i)| \leq \limsup \| f \| \| x_i \| \leq \| f \|
\]

so that \( \| x \| \leq 1 \).

(ii) Choose \( \tau_1, \tau_2, \ldots, \in K, 0 < |\tau_1| < |\tau_2| < \ldots, \lim_{n \to \infty} |\tau_n| = 1 \). The formula

\[
f(a_1, a_2, \ldots) = \sum_{i=1}^{\infty} a_i \tau_i
\]

defines an element \( f \in c_0^* \). Observe that \( \sup f = 1 \) but \( |f(x)| < 1 \) for each \( x \in A \). Set \( U := \{ x : |f(x)| \leq \frac{1}{2} \} \), let \( F \) be any finite dimensional
set in $A$. We shall arrive at $A \not\subset U + F$ by showing that $\sup_{U + F}|f| < 1$. To this end it suffices to prove $\sup_F|f| < 1$. $[F]$ is a finite dimensional subspace of $c_0$ and therefore $([4]$, Theorem 5.9) has an orthonormal base $x_1, \ldots, x_n$. It is easily seen that

$$F' := \text{co} \{x_1, \ldots, x_n\} \supset F$$

and $\sup_{F'}|f| = \sup_F|f| = \max(|f(x_1)|, \ldots, |f(x_n)|) < 1$.

Remark.
The above construction works also for the case where the base field is not spherically complete. Then $A$ is even weakly complete! $([5]$, Theorem 9.6 and $[4]$, Theorem 4.17)
§ 3. A REPRESENTATION THEOREM FOR C-COMPACT SETS

LEMMA 3.1.
Let \( \lambda \in K, |\lambda| > 1 \). Let \( G \subseteq E \) be closed, absolutely convex, and let \( F \subseteq [G] \) be a finite dimensional set. If \( \{x_i\}_{i \in I} \) is a net in \( G+F \) converging to 0 then \( x_i \in \lambda G \) for large \( i \).

Proof.
[6], Lemma 1.3.

PROPOSITION 3.2.
(See also [2], Proposition 4, p. 93.) Let \( A \subseteq E \) be absolutely convex, c-compact. Let \( \tau' \) be a Hausdorff locally convex topology on \( E \), weaker than the initial topology \( \tau \). Then \( \tau = \tau' \) on \( A \).

Proof.
Let \( \{x_i\}_{i \in I} \) be a net in \( A \) converging to 0 for \( \tau' \). Let \( \lambda \in K, |\lambda| > 1 \), let \( U \) be an absolutely convex neighbourhood of 0 for \( \tau \). Then \( (\lambda^{-1}U) \cap A \) is c-compact in \((E,\tau)\) hence in \((E,\tau')\), so that \( (\lambda^{-1}U) \cap A \) is \( \tau' \)-closed.
There is (Proposition 2.3) a finite dimensional \( F \subseteq A \) with \( A \subseteq \lambda^{-1}U+F \).
Then \( A = (\lambda^{-1}U) \cap A + F \). Lemma 3.1 applies. It follows that \( x_i \in \lambda(\lambda^{-1}U) \cap A \subseteq U \) for large \( i \), so \( \lim x_i = 0 \) in the sense of \( \tau \).

THEOREM 3.3.
Let \( A \subseteq E \) be absolutely convex. The following are equivalent.
(a) \( A \) is c-compact.
(b) \( A \) is isomorphic (as a topological module over \( \{\lambda \in K : |\lambda| \leq 1\} \)) to a closed submodule of some power of \( K \).
Proof.

$(\beta) \Rightarrow (\alpha)$. This follows from Proposition 1.2, (i), (iv), (iii). Now suppose $(\alpha)$. The map

$$x \mapsto (f(x))_{f \in E'}$$

is a continuous linear injection $E \to K^E$ (Hahn-Banach Theorem).

According to Proposition 3.2 it is a homeomorphism, if restricted to $A$, and $(\beta)$ follows.
REFERENCES


