SOME PROPERTIES OF C-COMPACT SETS

IN p-ADIC SPACES

by

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In this note we shall prove some properties of c-compact sets that may or may not be part of the 'folklore'. The concept of c-compactness, introduced by Springer [7], takes over the role played by convex-compact sets in Functional Analysis over $\mathbb{R}$ or $\mathbb{C}$ (or, any locally compact valued field).

Throughout, let $K$ be a nonarchimedean nontrivially valued field with valuation $|\cdot|$. We assume $K$ to be maximally (= spherically) complete. A subset $A$ of a $K$-linear space $E$ is absolutely convex if it is a submodule of $E$, considered as a module over the valuation ring $\{\lambda \in K : |\lambda| \leq 1\}$. A set $C \subseteq E$ is convex if it is either empty or an additive coset of an absolutely convex set. For a set $X \subseteq E$ we denote by $co X$ its absolutely convex hull, by $[X]$ its $K$-linear span.

From now on in this paper $E$ is a locally convex space over $K$ ([8]). We assume $E$ to be Hausdorff.
§ 1. DEFINITION AND FIRST PROPERTIES

DEFINITION 1.1. ([7]) Let \( C \subseteq E \) be a nonempty convex set. A convex filter on \( C \) is a filter of subsets of \( C \) that has a basis consisting of convex sets. \( C \) is c-compact if each convex filter on \( C \) has a cluster point in \( C \).

In other words, \( C \) is c-compact if and only if the following is true. Let \( C \) be a family of nonempty relatively closed convex subsets of \( C \) such that \( C_1, C_2 \in C \) implies \( C_1 \cap C_2 \in C \). Then \( \cap C \neq \emptyset \).

We quote the following properties, proved in [7].

PROPOSITION 1.2.

(i) \( K \) is c-compact.

(ii) A c-compact set is complete.

(iii) A nonempty closed convex subset of a c-compact set is c-compact.

(iv) Let \( (E_i)_{i \in I} \) be a family of Hausdorff locally convex spaces over \( K \). Suppose, for each \( i, C_i \in C \)-compact in \( E_i \). Then \( \bigcap_{i \in I} C_i \) is c-compact in \( \prod_{i \in I} E_i \).

(v) The image of a c-compact set under a continuous linear map is c-compact.

In [1] we find the following.

PROPOSITION 1.3.

(i) If \( K \) is locally compact then a bounded nonempty convex set \( C \subseteq E \) is c-compact if and only if it is convex and compact.

(ii) \( E \) is c-compact if and only if \( E \) is linearly homeomorphic to a
In § 3 (Theorem 3.3) we shall characterize arbitrary $c$-compact sets in the spirit of Proposition 1.3 (ii). But we conclude this first section with two statements that have nothing to do with the sequel. I just want to get rid of them.

**Proposition 1.4.**

A $c$-compact set is a Baire space.

**Proof.**

Let $U_1, U_2, \ldots$ be (relatively) open dense subsets of a $c$-compact set $C \subset E$. We prove that $\cap U_n \neq \emptyset$. There exists a nonempty open convex subset $B_1 \subset U_1$. As $U_2$ is dense we can find a nonempty open convex set $B_2 \subset B_1 \cap U_2$. Continuing this way we find nonempty open convex sets $B_1 \supset B_2 \supset \ldots$ with $B_n \subset \cap U_i$ for each $n$. The open sets $B_n$ are cosets of an additive group, hence closed. By $c$-compactness, $\cap B_n \neq \emptyset$. It follows that $\cap U_n \neq \emptyset$.

**Proposition 1.5.**

Let $X \subset E$ be closed, let $C \subset E$ be $c$-compact. Then $X+C$ is closed.

**Proof.**

Let $z \in X+C$ (the closure of $X+C$), let $U$ be the collection of all absolutely convex neighbourhoods of $0$. For each $U \in U$ the set $z+U$ intersects $X+C$ so

$$C_U := \{ c \in C : z-c \in X+U \}$$

is not empty. $X+U$ is a union of cosets of $U$, so is its complement.
Therefore, $X+U$ is closed and $C_u$ is closed in $C$. Further we have

$$C_u \cap C_v \supseteq C_u \cap v \quad (u,v \in U)$$

By $c$-compactness there exists a $c \in C$ such that

$$z-c \in \bigcap_{U \in \mathcal{U}} (X+U) = \overline{X} = X$$

i.e., $z \in X+c \subseteq X+C$.

**Remark.**

If the base field is not spherically complete there exist a complete absolutely convex compactoid $C \subseteq c_0$ and an element $a \in c_0$ such that $C+\text{co}\{a\}$ is not closed ([3], 6.25).
§ 2. LOCAL COMPACTOIDITY

DEFINITION 2.1. ([3], (6.7)) A subset $X$ of $E$ is a local compactoid if for each neighbourhood $U$ of 0 in $E$ there exists a finite dimensional $K$-linear subspace $D$ of $E$ such that $X \subset U + D$.

PROPOSITION 2.2.

Let $A$ be an absolutely convex subset of $E$. $A$ is c-compact if and only if $A$ is a complete local compactoid.

Proof.

For $E$ a Banach space this is proved in [3], 6.15. Now let $E$ be a locally convex.

(i) Assume $A$ is c-compact. By Proposition 1.2 (ii), $A$ is complete. To prove local compactoidity let $U$ be an absolutely convex neighbourhood of 0 in $E$. There is a continuous seminorm $p$ such that $\{x \in E : p(x) \leq 1\} \subset U$.

Let $\pi_p : E \to E_p$ be the quotient map where $E_p$ is the canonically normed space $E/Ker p$. Now $\pi_p(A)$ is c-compact (Proposition 1.2 (v)) so by the above it is a local compactoid in the completion $E_p^\wedge$ of $E_p$. By Corollary 6.15 of [3] we have $\pi_p(A) = R + T$ where $R$ is a compactoid and $T$ a finite dimensional subspace of $E_p^\wedge$. Then $T \subset E_p$. Now $\pi_p(U)$ is open in $E_p$ and by Katsaras' Theorem ([5], Lemma 8.1) there exist $x_1, \ldots, x_n \in [R]$ such that $R \subset \pi_p(U) + co(x_1, \ldots, x_n)$. Combining our knowledge on $R$ and $T$ we find a finite dimensional space $F \subset [\pi_p(A)]$ such that $\pi_p(A) \subset \pi_p(U) + F$. Choose a finite dimensional space $D \subset [A]$ such that $\pi_p(D) = F$. Then

$$A \subset U + D + Ker \pi_p \subset U + D.$$  

(ii) Let $A$ be a complete local compactoid. Let $\Gamma$ be the collection of all continuous seminorms on $E$. For each $p \in \Gamma$ we have that $\pi_p(A)$, and also $\pi_p(A)$, is a local compactoid in $E_p^\wedge$. 


As $E^\sim$ is a Banach space we know that $\prod_{p \in \Gamma} \pi_p(A)$ is $c$-compact. Then also $\prod_{p \in \Gamma} \pi_p(A)$ is a $c$-compact subset of $\prod_{p \in \Gamma} E^\sim$ (Proposition 1.1 (iv)).

The canonical map $E \to \prod_{p \in \Gamma} E^\sim$ sends $A$ homeomorphically and linearly into $A_0$. Its image is closed in $A_0$ because $A$ is complete. Then $A$ is $c$-compact (Proposition 1.2 (iii)).

The following Proposition may look innocent.

**Proposition 2.3.**

Let $A \subset E$ be absolutely convex and $c$-compact. For each neighbourhood $U$ of 0 there exists a finite dimensional absolutely convex set $F \subset A$ such that $A = U + F$.

(The crucial part is the phrase 'F $\subset A$'.) For the proof we use a lemma.

**Lemma 2.4.**

Let $A, U$ be absolutely convex subsets of $E$, where $t$ is closed, $A$ is $c$-compact. Let $x \in E$ be such that $A \subset U + Kx$. Then there exists an $y \in E$ and an absolutely convex $C \subset K$ such that $Cy \subset A$ and $A \subset U + Cy$.

**Proof.**

Let $C := \{c \in K : (U + cx) \cap A \neq \emptyset\}$. We have $A \subset U + Cx$, $C = \{c \in K : cx \in A + U\}$, so $C$ is absolutely convex. If $C = \{0\}$ then $A \subset U$ and we choose $y := 0$.

So assume $C \neq \{0\}$. For each $c \in C$, $c \neq 0$ define

$$H_c := c^{-1}(A \cap (cx + U)).$$

Each $H_c$ is a convex, closed, nonempty subset of $c^{-1}A$ hence $c$-compact.

Further, if $c, d \in C$, $0 < |c| \leq |d|$ then $H_d \subset H_c$. (Proof. Let $z \in H_d$. Then $dz \in A \cap (dx + U)$. By absolute convexity of $A$ and $U$,
\[
Cz = \frac{C}{d} \cdot dz \in A
\]
\[
Cz \in \frac{C}{d} (dx+U) \subset cx+ \frac{C}{d} U \subset cx+U.
\]

It follows that \(Cz \in A \cap (cx+U)\) i.e. \(z \in H_c\). By \(c\)-compactness there exists an \(y \in \subset H_c\). Let \(c \in C\), \(c \neq 0\). Then
\[
Cy \in C_{H_c} \subset A \cap (cx+U) \subset A.
\]

Also, \(Cy \in cx+U\) so that \(cx-cy \in U\). Let \(a \in A\). Then \(a = u+cx\) for some \(u \in U\), \(c \in C\). We see that \(a = u+cy+cx-cy \in cy+U\). It follows that \(A \subset U+Cy\).

**Proof of Proposition 2.3.**

We may assume that \(U\) is absolutely convex. By Proposition 2.2 \(A\) is a local compactoid so there exist \(x_1, \ldots, x_n \in E\) such that
\[
A \subset U+Kx_1+\ldots+Kx_n.
\]
By the Lemma, applied to \(U+Kx_1+\ldots+Kx_n\) in place of \(U\), there exist \(a y_1 \in E\) and an absolutely convex \(C_1 \in K\) such that
\[
C_1y_1 \subset A \quad \text{and}
\]
\[
A \subset U+C_1y_1+Kx_2+\ldots+Kx_n = (U+C_1y_1+Kx_2+\ldots, Kx_n) + Kx_2
\]
and we can continue. After \(n\) of these procedures we arrive at \(y_1, \ldots, y_n \in E\), absolutely convex \(C_1, \ldots, C_n \subset K\) such that \(C_i y_i \subset A\) for each \(i\) and \(A \subset U+C_1y_1+\ldots+C_ny_n\).

**Warning.**

The property of Proposition 2.3 is not shared by all absolutely convex local compactoids even when we require them to be closed! In fact we have:
EXAMPLE 2.5.

Let the valuation of $K$ be dense. Set

$$A = \{x \in c_0 : \|x\| \leq 1\}.$$ 

([4], p.47).

(i) $A$ is a closed (local) compactoid for the weak topology of $c_0$.

(ii) There exists a weak neighbourhood $U$ of 0 such that for any finite dimensional set $F \subseteq A$

$$A \nsubseteq U+F.$$

Proof.

(i) Let $U$ be a weak neighbourhood of 0. There exists a weakly continuous seminorm $p$ such that $\{x \in c_0 : p(x) \leq 1\} \subseteq U$. Then Ker$p$ has finite codimension. Choose a finite dimensional space $D \subseteq c_0$ with $\pi_p(D) = E_p$ (where as previously, $E_p := c_0/\text{Ker} \pi_p$ and $\pi_p : c_0 \rightarrow E_p$ is the quotient map). We have $A \subseteq \text{Ker}p+D \subseteq U+D$ (in fact, we have shown that each subset of $c_0$ is a local compactoid for the weak topology). To prove weak closedness of $A$, let $(x_i)_{i \in I}$ be a net in $A$ converging weakly to $x \in c_0$. By [4], Lemma 4.35 (i) there exists an $f \in c_0'$, $f \neq 0$ for which $|f(x)| = \|f\| \|x\|$. We have

$$\|f\| \|x\| = |f(x)| = \lim|f(x_i)| \leq \limsup \|f\| \|x_i\| \leq \|f\|$$

so that $\|x\| \leq 1$.

(ii) Choose $\tau_1, \tau_2, \ldots, \in K$, $0 < |\tau_1| < |\tau_2| < \ldots$, $\lim_n |\tau_n| = 1$. The formula

$$f(a_1, a_2, \ldots) = \sum_{i=1}^{\infty} a_i \tau_i$$

defines an element $f \in c_0'$. Observe that $\sup_{A} |f| = 1$ but $|f(x)| < 1$ for each $x \in A$. Set $U := \{x : |f(x)| \leq \frac{1}{2}\}$, let $F$ be any finite dimensional
set in $A$. We shall arrive at $A \not= U+F$ by showing that $\sup_{U+F}|f| < 1$. To this end it suffices to prove $\sup_F |f| < 1$. $[F]$ is a finite dimensional subspace of $c_0$ and therefore ([4], Theorem 5.9) has an orthonormal base $x_1, \ldots, x_n$. It is easily seen that

$$F' := \text{co} \{x_1, \ldots, x_n\} \supset F$$

and $\sup_{F'} |f| \leq \sup_{F} |f| = \max(|f(x_1)|, \ldots, |f(x_n)|) < 1$.

Remark.
The above construction works also for the case where the base field is not spherically complete. Then $A$ is even weakly complete! ([5], Theorem 9.6 and [4], Theorem 4.17).
§ 3. A REPRESENTATION THEOREM FOR C-COMPACT SETS

LEMMA 3.1.

Let $\lambda \in K, |\lambda| > 1$. Let $G \subset E$ be closed, absolutely convex, and let
$F \subset [G]$ be a finite dimensional set. If $(x_i)_i \in I$ is a net in $G+F$
converging to 0 then $x_i \in \lambda \cdot G$ for large $i$.

Proof.

[6], Lemma 1.3.

PROPOSITION 3.2.

(See also [2], Proposition 4, p. 93.) Let $A \subset E$ be absolutely convex,
c-compact. Let $\tau'$ be a Hausdorff locally convex topology on $E$, weaker
than the initial topology $\tau$. Then $\tau = \tau'$ on $A$.

Proof.

Let $(x_i)_i \in I$ be a net in $A$ converging to 0 for $\tau'$. Let $\lambda \in K, |\lambda| > 1$, let $U$ be an absolutely convex neighbourhood of 0 for $\tau$. Then $(\lambda^{-1}U) \cap A$
is c-compact in $(E,\tau)$ hence in $(E,\tau')$, so that $(\lambda^{-1}U) \cap A$ is $\tau'$-closed. There is (Proposition 2.3) a finite dimensional $F \subset A$ with $A \subset \lambda^{-1}U+F$. Then $A = (\lambda^{-1}U) \cap A + F$. Lemma 3.1 applies. It follows that $x_i \in$
$\lambda(\lambda^{-1}U) \cap A \subset U$ for large $i$, so $\lim x_i = 0$ in the sense of $\tau$.

THEOREM 3.3.

Let $A \subset E$ be absolutely convex. The following are equivalent.
(a) $A$ is c-compact.
(b) $A$ is isomorphic (as a topological module over $\{\lambda \in K : |\lambda| \leq 1\}$) to
a closed submodule of some power of $K$. 

Proof.

(β) ⇒ (α). This follows from Proposition 1.2, (i), (iv), (iii). Now suppose (α). The map

\[ x \mapsto (f(x))_f \in E' \]

is a continuous linear injection \( E \to K^{E'} \) (Hahn-Banach Theorem). According to Proposition 3.2 it is a homeomorphism, if restricted to \( A \), and (β) follows.
REFERENCES


