Distinguishing non-archimedean seminorms

by

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§1. Introduction and summary. (For terminology, see §2) In [2] the following separation theorem is proved. Let $K$ be spherically complete and let $E$ be a locally convex space over $K$. If $A \subset E$ is closed and absolutely convex and if $x \in E \setminus A$ then there is an $f \in E'$ such that

\[ |f(A)| < 1 \quad \text{and} \quad f(x) = 1. \]

In order to obtain, 'real' separation one would prefer to have

\[ |f(A)| < 1 \quad \text{and} \quad f(x) = 1 \]

rather than (*). However, with the techniques used in [2], it is not clear how to arrive at such a result for densely valued fields $K$. The main obstruction is the fact that for an open absolutely convex, absorbing set $A$ its associated seminorm $q_A$,

\[ q_A(x) := \inf\{|\lambda| : \lambda \in K : x \in \lambda A\} \quad (x \in E) \]

does not determine $A$; one only has the rather 'vague' relation
\{x \in E : q_A(x) < 1\} \subset \subset \{x \in E : q_A(x) \leq 1\}.

(Example. The open and closed convex sets \{(\xi_1, \xi_2) \in K^2 : |\xi_1| \leq 1, |\xi_2| \leq 1\},
\{(\xi_1, \xi_2) \in K^2 : |\xi_1| \leq 1, |\xi_2| < 1\}, \{(\xi_1, \xi_2) \in K^2 : |\xi_1| < 1, |\xi_2| < 1\} all
have the associated (semi)norm \((\xi_1, \xi_2) \mapsto \max(|\xi_1|, |\xi_2|)).

In this paper we shall extend the notion of a seminorm by admitting
a larger range yielding the notion of a distinguishing seminorm
(Definition 2). We shall prove that each absolutely convex absorbing set
can be written as \(\{x \in E : p(x) < 1\}\) for some distinguishing seminorm \(p\)
(Theorem 5). Next, we shall prove a Hahn Banach theorem for linear
functions majorized by distinguishing seminorms (Theorem 6) and shall
obtain, as a corollary, a strong separation theorem (with \((**)\) in place
of \((*)\)). Here the notion of distinguishing seminorm is used only in the
proof, not in the formulation (Theorem 8).

Note. This separation theorem can (for Banach spaces) also be obtained
as a corollary of [1], Theorem 6.21. However, one techniques differ very
much from the ones used in the (clever) proof of [1]. Also, the notion
of a distinguishing seminorm may very well yield new applications.

§2. Terminology

Throughout \(K\) is a non-archimedian complete non-trivially valued field
with valuation \(||, | \cdot |, |X| := \{|\lambda| : \lambda \in X\}\). \(E\) is a vector space over \(K\).
A seminorm on \(E\) is a map \(q : E \to [0,\infty)\) satisfying (i) \(q(0) = 0,\)
(ii) \(q(\lambda x) = |\lambda|q(x) \quad (x \in E, \lambda \in K),\)
(iii) \(q(x+y) \leq q(x) \vee q(y) \quad (x, y \in E)\)
where \(\vee\) indicates 'maximum'.
A subset \(A\) of \(E\) is absolutely convex if it is a module over the ring
\(\{\lambda \in K : |\lambda| \leq 1\},\)
convex if it is an additive coset of an absolutely
convex set, absorbing if \(\bigcup_{\lambda \in K} \lambda A = E.\)
Each convex subset of $K$ has the form $\{\lambda \in K : |\lambda - a| < r\}$ or $\{\lambda \in K : |\lambda - a| \leq r\}$ for some $a \in K$, $r \in [0, \omega]$.

For a seminorm $q$ the sets $\{x \in E : q(x) < 1\}$ and $\{x \in E : q(x) \leq 1\}$ are absolutely convex and absorbing. Conversely, for an absorbing, absolutely convex subset $A$ of $E$ the associated seminorm $q_A$ defined by

$$q_A(x) := \inf\{|\lambda| : \lambda \in K : x \in \lambda A\} \quad (x \in E)$$

is a seminorm satisfying

$$\{x \in E : q_A(x) < 1\} \subseteq A \subseteq \{x \in E : q_A(x) \leq 1\}.$$ 

$K$ is spherically complete if for any collection $C$ of convex subsets for which $A, B \in C \Rightarrow A \cap B \neq \emptyset$ we have $\cap C = \emptyset$. A locally convex space is a $K$-vector space $E$ with a topology induced by a collection of seminorms. Its dual space is denoted $E'$.

§3. Distinguishing seminorms

We enlarge the set $[0, \omega)$ by giving each positive real number $a$ an immediate predecessor $a^-$. Formally

$$V := [0, \omega) \cup (0, \omega)^-$$

where $(0, \omega)^- := \{a^- : a \in (0, \omega)\}$ is a second copy of $(0, \omega)$. Further we define $0^- := 0$. The formulas

$$a^- < b^- \Rightarrow a \leq b$$

$$a^- < b^- \Rightarrow a < b^- \Rightarrow a < b$$

$$0 < a$$

define a linear ordering $<$ on $V$ extending the usual ordering on $[0, \omega)$.
The projection map $\pi : V \to [0,\infty)$ is defined by

$$
\pi(a) := a \quad (a \in [0,\infty))
$$

Finally we extend the multiplication on $[0,\infty)$ to a multiplication $[0,\infty) \times V \to V$ by requiring $a \cdot b^- := (ab)^-$ for all $a, b \in [0,\infty)$.

Proposition 1 collects some direct consequences from the definitions.

**Proposition 1.**

(i) Let $b, c \in V$, $b \leq c$. Then $ab \leq ac$ for all $a \in [0,\infty)$.

(ii) $\pi(a \vee b) = \pi(a) \vee \pi(b)$, $(a, b \in V)$.

(iii) $\pi(ab) = a\pi(b)$, $(a \in [0,\infty)$, $b \in V)$.

**Definition 2.** A distinguishing seminorm on $E$ is a map $p : E \to V$ satisfying

(i) $p(0) = 0$

(ii) $p(\lambda x) = |\lambda| p(x)$, $(\lambda \in K, x \in E)$

(iii) $p(x+y) \leq p(x) \vee p(y)$, $(x, y \in E)$.

If, in addition, $p(x) = 0$ implies $x = 0$ then $p$ is a distinguishing norm.

**Examples**

1. Any seminorm on $E$.

2. The map $(\xi_1, \xi_2) \mapsto |\xi_1| \vee |\xi_2|^-$ ($(\xi_1, \xi_2) \in K^2$) is a distinguishing norm on $K^2$.

3. Let $E$ be the space of all bounded $K$-valued functions on a set $X$.

Then

$$
\| f \|_\infty := \sup \{|f(x)| : x \in X\} \quad (f \in E),
$$

where the supremum is taken in $V$, defines a distinguishing norm on $E$.

Observe that for $f \in E$ we have $\| f \|_\infty \leq 1$ if and only if $|f(x)| \leq 1$ for
all \( x \in E \) but
\[
\| f \|_\infty < 1 \iff \| f \|_{\infty} < 1 \iff |f(x)| < 1 \text{ for all } x \in E
\]
so that \( \| f \|_{\infty} = 1 \) if and only if \( 1 = \max\{|f(x)| : x \in X\} \).

**Proposition 3.** Let \( p \) be a distinguishing seminorm on \( E \). Then \( \pi \circ p \) is a seminorm on \( E \).

**Proof.** The statement follows directly from Proposition 1 (ii), (iii).

**Definition 4.** Let \( A \subset E \) be absorbing and absolutely convex, and let \( q_A \) be its associated seminorm. The distinguishing seminorm \( p_A \), associated to \( A \) is
\[
p_A(x) := \begin{cases} q_A(x) & \text{if } q_A(x) = \min\{|\lambda| : \lambda \in K, x \in \lambda A\} \\ q_A(x) & \text{otherwise.} \end{cases}
\]

The following theorem shows that the associated distinguishing seminorm of an absorbing absolutely convex set \( A \) determines \( A \).

**Theorem 5.** Let \( A \subset E \) be absorbing and absolutely convex. Let \( p_A \) and \( q_A \) be as in Definition 4. Then \( p_A \) is a distinguishing seminorm with \( \pi \circ p_A = q_A \). Further we have
\[
A = \{ x \in E : p_A(x) < 1 \} = \{ x \in E : p_A(x) \leq 1 \}.
\]

**Proof.** Clearly \( \pi \circ p_A = q_A \). We first check the equality
\[
A = \{ x \in E : p_A(x) < 1 \}. \text{ Let } p_A(x) < 1. \text{ If } p_A(x) < 1^- \text{ then } q_A(x) < 1, \text{ so } x \in A. \text{ If } p_A(x) = 1^- \text{ then } 1 \in \{ |\lambda| : \lambda \in K, x \in \lambda A\} \text{ so that } x \in \mu A \text{ for some } \mu \in K, |\mu| = 1. \text{ By absolute convexity, } x \in \mu^{-1} \mu A = A. \text{ Conversely, let } x \in A. \text{ Then } q_A(x) \leq 1 \text{ so that } p_A(x) \leq q_A(x) \leq 1. \text{ If } p_A(x) = 1 \text{ then } 1 = \inf\{|\lambda| : x \in \lambda A\} \text{ is not a minimum contradicting } x \in A. \text{ Hence, }
\( p_A(x) < 1 \). Finally we prove the conditions (i), (ii), (iii) of Definition 2 for a distinguishing seminorm for \( p = p_A \). We have \( p_A(0) = 0^- = 0 \). To prove (ii) we may assume \( \lambda \neq 0 \) and \( p_A(x) \neq 0 \). If \( p_A(x) \in (0,\infty) \) then

\[
q_A(x) = \inf\{|x| : x \in \tau A\} \text{ is not a minimum. Then neither is }
q_A(\lambda x) = \inf\{|\lambda x| : \lambda x \in \tau A\}
\text{ so that } p_A(\lambda x) = q_A(\lambda x) = |\lambda|q_A(x) = |\lambda|p_A(x).
\]

If \( p_A(x) \in (0,\infty)^- \) then \( q_A(x) = \min\{|x| : x \in \tau A\} \). Then also

\[
q_A(\lambda x) = \min\{|\lambda x| : \lambda x \in \tau A\} \text{ so that }
p_A(\lambda x) = q_A(\lambda x)^- = (|\lambda|q_A(x))^- = |\lambda|q_A(x)^- = |\lambda|p_A(x).
\]

For the proof of the strong triangle inequality (iii) we may assume \( p_A(x+y) \neq 0 \). We distinguish two cases.

(i) \( p_A(x+y) \in (0,\infty)^- \). By Proposition 1 (ii) and \( \pi \circ p_A = q_A \), from

\[
q_A(x+y) \leq q_A(x) \vee q_A(y)
\]

we obtain \( p_A(x+y) = q_A(x+y)^- \leq q_A(x)^- \vee q_A(y)^- \leq p_A(x) \vee p_A(y) \).

(ii) \( p_A(x+y) \in (0,\infty) \). Then the valuation of \( K \) is dense. Assume

\[
p_A(y) \leq p_A(x). \text{ Suppose } p_A(x+y) > p_A(x); \text{ we derive a contradiction. There is a } \lambda \in K \text{ such that } p_A(x+y) \geq |\lambda| > p_A(x). \text{ (In fact, if } p_A(x) \neq p_A(x+y)^- \text{ then the interval } (p_A(x), p_A(x+y)) \text{ contains infinitely many elements of } |K|, \text{ if } p_A(x) = p_A(x+y)^- \text{ then } p_A(x+y) \in |K| \text{ and we may choose } \lambda \in K \text{ such that } |\lambda| = p_A(x+y).) \text{ Then } p_A(\lambda^{-1} y) = p_A(\lambda^{-1} x) < 1 \text{ so that } \lambda^{-1} y \in A,\lambda^{-1} x \in A \text{ and, by absolute convexity, also } \lambda^{-1}(x+y) \in A \text{ implying } \]

\[
p_A(x+y) < |\lambda|, \text{ a contradiction.}
\]

§4. Hahn-Banach Theorem

For \( a \in K \), \( a \in V \) we write \( B(a, a) = \{\lambda \in K : |\lambda-a| \leq a\} \).

**Theorem 6.** (Hahn-Banach Theorem) Let \( K \) be spherically complete, let \( p \)
be a distinguishing seminorm on a $K$-vector space $E$. Let $D$ be a $K$-linear subspace of $E$, let $f : D \to K$ be a $K$-linear map satisfying $|f(d)| \leq p(d)$ ($d \in D$). Then $f$ can be extended to a $K$-linear $\tilde{f} : E \to K$ such that $|\tilde{f}(x)| \leq p(x)$ ($x \in D$).

**Proof.** (It consists of checking that replacing of a seminorm by a distinguishing seminorm does not harm the well-known proof.) A simple application of Zorn's Lemma reduces the problem to the case $E = \{\lambda x + d : \lambda \in K, d \in D\}$ for some $x \in E \setminus D$. We are done if we can choose $\tilde{f}(x) = \xi \in K$ such that

$$|\lambda \xi + f(d)| \leq p(\lambda x + d) \quad (\lambda \in K, d \in D).$$

As this condition is satisfied for $\lambda = 0$ and all $d \in D$ and since for $\lambda \neq 0$

$$p(\lambda x + d) = |\lambda| \, p(x + \lambda^{-1} d)$$

$$|\lambda \xi + f(d)| = |\lambda| \, |\xi + f(\lambda^{-1} d)|$$

it suffices, by Proposition 1 (i) to produce a $\xi \in K$ such that

$$|\xi - f(d)| \leq p(x - d) \quad (d \in D),$$

i.e. we must have that

$$\cap_{d \in D} B(f(d), p(x - d)) \neq \emptyset.$$

By spherical completeness it suffices to show that for any $d_1, d_2 \in D$ we have $B(f(d_1), p(x - d_1)) \cap B(f(d_2), p(x - d_2)) \neq \emptyset$. But this follows easily from $|f(d_1) - f(d_2)| \leq p(d_1 - d_2) \leq p(d_1 - x) \vee p(x - d_2)$.

A typical application: let $\ell^\infty := \{((\xi_1, \xi_2, \ldots)) : \xi_n \in K$ for all $n$, $\sup|\xi_n| < \infty\}$. Let $K$ be spherically complete. Then there is a linear function $g : \ell^\infty \to K$
of norm 1 such that (i) \( g((\xi_1, \xi_2, \ldots)) = \lim_{n \to \infty} \xi_n \) if \( \lim_{n \to \infty} \xi_n \) exists and 
(ii) \( |g((\xi_1, \xi_2, \ldots))| < 1 \) if \( |\xi_n| < 1 \) for all \( n \in \mathbb{N} \).

(Proof: Choose in the above theorem \( D := \mathbb{C} \) (the space of the convergent sequences), \( f((\xi_1, \xi_2, \ldots)) := \lim_{n \to \infty} ((\xi_1, \xi_2, \ldots) \in \mathbb{C}) \), and \( p((\xi_1, \xi_2, \ldots)) := \sup|\xi_n| \), where the sup is taken in \( \mathbb{V} \). Take \( g := \frac{1}{f} \).)

§5. Separation of convex sets

Throughout §5, let \( E \) be a locally convex space over \( K \). We shall need the following observation.

**Proposition 7.** An open convex subset of \( E \) is closed.

**Proof.** Any convex set is a coset of an absolutely convex set.

An open absolutely convex set is the complement of a union of cosets.

**Theorem 8.** Let \( K \) be spherically complete. Let \( A \subset E \) be closed, absolutely convex and let \( x \in E \setminus A \). Then there exists an \( f \in E' \) such that \( |f(A)| < 1 \) and \( f(x) = 1 \).

**Proof.** There is an absolutely convex open neighbourhood \( U \) of \( 0 \) such that \((x+U) \cap A = \emptyset \). Then \( U+A \) is absolutely convex, open, hence closed (Proposition 7). Further, \( x \notin U+A \). Thus, we may assume that \( A \) is open and closed. Then \( A \) is absorbing. Let \( p_A \) be the distinguishing seminorm of \( A \), let \( D := \{\lambda x : \lambda \in K\} \) and define \( g : D \to K \) by \( g(\lambda x) := \lambda \) (\( \lambda \in K \)). Then \( g(x) = 1 \). Since

\[
A = \{y \in E : p_A(y) < 1\}
\]

(Theorem 5) and \( x \notin A \) we have \( p_A(x) \geq 1 \) so that for \( \lambda \in K \)

\[
p_A(\lambda x) = |\lambda| p_A(x) \geq |\lambda| = |g(\lambda x)|, \text{ i.e. } |g| \leq p_A \text{ on } D.
\]

By Theorem 6 \( g \) extends to a linear \( f : E \to K \) such that \( |f(y)| \leq p_A(y) \) for all \( y \in E \).
We have \( f(x) = g(x) = 1 \) and, for \( y \in A \), \( |f(y)| \leq p_A(y) < 1 \). The continuity of \( f \) follows from the continuity of \( \pi \circ p_A \) and the inequality \( |f| \leq \pi \circ p_A \).

**COROLLARY 9.** Let \( K \) be spherically complete. Each closed convex set is weakly closed.

Let \( A, B \) be convex subsets of \( E \). If \( f : E \to K \) is a linear function then \( f(A) \) and \( f(B) \) are convex in \( K \). Hence, if \( f(A) \cap f(B) = \emptyset \) then \( \text{dist}(f(A), f(B)) > 0 \). With this in mind the following definition is quite natural.

**DEFINITION 10.** Two convex subsets \( A, B \) of \( E \) are separated by an \( f \in E' \) if \( f(A) \cap f(B) = \emptyset \).

If \( A \) and \( B \) are separated by \( f \in E' \) then, since \( \text{dist}(f(A), f(B)) > 0 \) there is an open convex neighbourhood \( U \) of \( 0 \) such that \( (A+U) \cap B = \emptyset \) (if \( E \) is a normed space this is equivalent to \( \text{dist}(A,B) > 0 \)). To prove the converse we need spherical completeness.

**THEOREM 11.** Let \( K \) be spherically complete. Let \( A, B \) be convex subsets of \( E \) and suppose there is an open convex neighbourhood \( U \) of \( A \) such that \( (A+U) \cap B = \emptyset \) (observe that this condition is satisfied if \( A \) is open).

Then \( A \) and \( B \) can be separated by some \( f \in E' \).

**Proof.** We may assume that \( A \) is open. Let \( C := A-B \). Then \( 0 \in C \), and

\[ C = U \ (A-b) \text{ is open, convex. Choose } c \in -C. \text{ Then } T := c + C \text{ is absolutely convex, open, hence closed and } c \notin T. \text{ By Theorem 8 there is an } f \in E' \text{ such that } f(c) = 1, \ |f(T)| < 1. \text{ Thus, for each } a \in A, \ b \in B \text{ we have } \]

\[ 1 > |f(c + a - b)| = |1 + f(a) - f(b)|. \]

It follows that \( |f(a) - f(b)| = 1 \) for all \( a \in A, \ b \in B \). In particular, \( f(A) \cap f(B) = \emptyset \).
References.
