§1. Introduction and summary. (For terminology, see §2) In [2] the following separation theorem is proved. Let K be spherically complete and let E be a locally convex space over K. If A ⊂ E is closed and absolutely convex and if \( x \in E \setminus A \) then there is an \( f \in E' \) such that

\[
(*) \quad |f(A)| < 1, \quad f(x) = 1.
\]

In order to obtain, 'real' separation one would prefer to have

\[
(**) \quad |f(A)| < 1, \quad f(x) = 1
\]

rather than (*). However, with the techniques used in [2], it is not clear how to arrive at such a result for densely valued fields K. The main obstruction is the fact that for an open absolutely convex, absorbing set A its associated seminorm \( q_A \),

\[
q_A(x) := \inf\{ |\lambda| : \lambda \in K : x \in \lambda A \} \quad (x \in E)
\]

does not determine A; one only has the rather 'vague' relation
\{x \in E : q_A(x) < 1\} \subset A \subset \{x \in E : q_A(x) \leq 1\}.

(Example. The open and closed convex sets \{(x_1, x_2) \in K^2 : |x_1| \leq 1, |x_2| \leq 1\}, \{(x_1, x_2) \in K^2 : |x_1| \leq 1, |x_2| < 1\}, \{(x_1, x_2) \in K^2 : |x_1| < 1, |x_2| \leq 1\} all have the associated (semi)norm \((x_1, x_2) \mapsto \max(|x_1|, |x_2|)\).

In this paper we shall extend the notion of a seminorm by admitting a larger range yielding the notion of a distinguishing seminorm (Definition 2). We shall prove that each absolutely convex absorbing set can be written as \(\{x \in E : p(x) < 1\}\) for some distinguishing seminorm \(p\) (Theorem 5). Next, we shall prove a Hahn Banach theorem for linear functions majorized by distinguishing seminorms (Theorem 6) and shall obtain, as a corollary, a strong separation theorem (with \((**)\) in place of \((*)\)). Here the notion of distinguishing seminorm is used only in the proof, not in the formulation (Theorem 8).

Note. This separation theorem can (for Banach spaces) also be obtained as a corollary of [1], Theorem 6.21. However, one techniques differ very much from the ones used in the (clever) proof of [1]. Also, the notion of a distinguishing seminorm may very well yield new applications.

§2. Terminology

Throughout \(K\) is a non-archimedean complete non-trivially valued field with valuation \(|\cdot|, |x| := \{\lambda \in K : \lambda \neq 0\}\). \(E\) is a vector space over \(K\).

A seminorm on \(E\) is a map \(q : E \to [0,\infty)\) satisfying (i) \(q(0) = 0\), (ii) \(q(\lambda x) = |\lambda|q(x)\) (\(x \in E, \lambda \in K\)), (iii) \(q(x+y) \leq q(x) \vee q(y)\) (\(x,y \in E\)) where \(\vee\) indicates 'maximum'.

A subset \(A\) of \(E\) is absolutely convex if it is a module over the ring \(\{\lambda \in K : |\lambda| \leq 1\}\), convex if it is an additive coset of an absolutely convex set, absorbing if \(\bigcup_{\lambda \in K} \lambda A = E\).
Each convex subset of $K$ has the form \( \{ \lambda \in K : |\lambda - a| < r \} \) or \( \{ \lambda \in K : |\lambda - a| \leq r \} \) for some \( a \in K \), \( r \in [0, \omega] \).

For a seminorm \( q \) the sets \( \{ x \in E : q(x) < 1 \} \) and \( \{ x \in E : q(x) \leq 1 \} \) are absolutely convex and absorbing. Conversely, for an absorbing, absolutely convex subset \( A \) of \( E \) the associated seminorm \( q_A \) defined by

\[
q_A(x) := \inf\{|\lambda| : \lambda \in K : x \in \lambda A\} \quad (x \in E)
\]

is a seminorm satisfying

\[
\{ x \in E : q_A(x) < 1 \} \subset A \subset \{ x \in E : q_A(x) \leq 1 \}.
\]

\( K \) is spherically complete if for any collection \( C \) of convex subsets for which \( A, B \in C \Rightarrow A \cap B \neq \emptyset \) we have \( nC = \emptyset \). A locally convex space is a \( K \)-vector space \( E \) with a topology induced by a collection of seminorms. Its dual space is denoted \( E' \).

§3. Distinguishing seminorms

We enlarge the set \([0, \omega]\) by giving each positive real number \( a \) an immediate predecessor \( a^- \). Formally

\[
V := [0, \omega) \cup (0, \omega)^-
\]

where \((0, \omega)^- := \{ a^- : a \in (0, \omega) \} \) is a second copy of \((0, \omega)\). Further we define \( 0^- := 0 \). The formulas

\[
\begin{align*}
 a^- < b & \iff a \leq b \quad (a, b \in (0, \omega)) \\
 a^- < b^- & \iff a < b^- \iff a < b \quad (a, b \in (0, \omega)) \\
 0 < a & \quad (a \in V, a \neq 0)
\end{align*}
\]

define a linear ordering \( < \) on \( V \) extending the usual ordering on \([0, \omega]\).
The projection map $\pi : V \to [0,\infty)$ is defined by
\[
\pi(a) := a \quad (a \in [0,\infty))
\]
\[
\pi(a^-) := a \quad (a \in [0,\infty)).
\]

Finally we extend the multiplication on $[0,\infty)$ to a multiplication $[0,\infty) \times V \to V$ by requiring $a \cdot b^- := (ab)^-$ for all $a,b \in [0,\infty)$.

Proposition 1 collects some direct consequences from the definitions.

**Proposition 1.**
(i) Let $b,c \in V$, $b \leq c$. Then $ab \leq ac$ for all $a \in [0,\infty)$.
(ii) $\pi(a \vee b) = \pi(a) \vee \pi(b)$ $(a,b \in V)$.
(iii) $\pi(ab) = a\pi(b)$ $(a \in [0,\infty), b \in V)$.

**Definition 2.** A distinguishing seminorm on $E$ is a map $p : E \to V$ satisfying

(i) $p(0) = 0$
(ii) $p(\lambda x) = |\lambda| p(x)$ $(\lambda \in K, x \in E)$
(iii) $p(x+y) \leq p(x) \vee p(y)$ $(x,y \in E)$.

If, in addition, $p(x) = 0$ implies $x = 0$ then $p$ is a distinguishing norm.

**Examples**
1. Any seminorm on $E$.
2. The map $(\xi_1,\xi_2) \mapsto |\xi_1| \vee |\xi_2|^{-1}$ ($(\xi_1,\xi_2) \in K^2)$ is a distinguishing norm on $K^2$.
3. Let $E$ be the space of all bounded $K$-valued functions on a set $X$.
Then
\[
\|f\|_\infty := \sup\{|f(x)| : x \in X} \quad (f \in E),
\]
where the supremum is taken in $V$, defines a distinguishing norm on $E$.

Observe that for $f \in E$ we have $\|f\|_\infty \leq 1$ if and only if $|f(x)| \leq 1$ for
all \( x \in E \) but

\[ \| f \|_\infty \sim < 1 \Leftrightarrow \| f \|_\infty \leq 1 \Leftrightarrow |f(x)| < 1 \text{ for all } x \in E \]

so that \( \| f \|_\infty = 1 \) if and only if \( 1 = \max\{|f(x)| : x \in X\} \).

**PROPOSITION 3.** Let \( p \) be a distinguishing seminorm on \( E \). Then \( \pi \circ p \) is a seminorm on \( E \).

**Proof.** The statement follows directly from Proposition 1 (ii), (iii).

**DEFINITION 4.** Let \( A \subset E \) be absorbing and absolutely convex, and let \( q_A \) be its associated seminorm. The distinguishing seminorm \( p_A \), associated to \( A \) is

\[ p_A(x) := \begin{cases} q_A(x) & \text{if } q_A(x) = \min\{|\lambda| : \lambda \in K, x \in \lambda A\} \\ q_A(x) & \text{otherwise.} \end{cases} \]

The following theorem shows that the associated distinguishing seminorm of an absorbing absolutely convex set \( A \) determines \( A \).

**THEOREM 5.** Let \( A \subset E \) be absorbing and absolutely convex. Let \( p_A \) and \( q_A \) be as in Definition 4. Then \( p_A \) is a distinguishing seminorm with \( \pi \circ p_A = q_A \). Further we have

\[ A = \{x \in E : p_A(x) < 1\} = \{x \in E : p_A(x) \leq 1\}. \]

**Proof.** Clearly \( \pi \circ p_A = q_A \). We first check the equality

\[ A = \{x \in E : p_A(x) < 1\}. \]

Let \( p_A(x) < 1 \). If \( p_A(x) < 1 \) then \( q_A(x) < 1 \), so \( x \in A \). If \( p_A(x) = 1 \) then \( 1 \in \{ |\lambda| : \lambda \in K, x \in \lambda A \} \) so that \( x \in \mu A \) for some \( \mu \in K, |\mu| = 1 \). By absolute convexity, \( x \in \mu^{-1} \mu A = A \). Conversely, let \( x \in A \). Then \( q_A(x) \leq 1 \) so that \( p_A(x) \leq q_A(x) \leq 1 \). If \( p_A(x) = 1 \) then

\[ 1 = \inf\{|\lambda| : x \in \lambda A\} \]

is not a minimum contradicting \( x \in A \). Hence,
\( p_A(x) < 1 \). Finally we prove the conditions (i), (ii), (iii) of Definition 2 for a distinguishing seminorm for \( p = p_A \). We have \( p_A(0) = 0^- = 0 \). To prove (ii) we may assume \( \lambda \neq 0 \) and \( p_A(x) \neq 0 \). If \( p_A(x) \in (0,\infty) \) then

\[
q_A(x) = \inf\{|x| : x \in \tau A}\]

is not a minimum. Then neither is

\[
q_A(\lambda x) = \inf\{||\lambda x| : x \in \tau A\} \quad \text{so that} \quad p_A(\lambda x) = q_A(\lambda x) = |\lambda|q_A(x) = |\lambda|p_A(x).
\]

If \( p_A(x) \in (0,\infty) \) then \( q_A(x) = \min\{|x| : x \in \tau A\} \). Then also

\[
q_A(\lambda x) = \min\{|\lambda x| : \lambda x \in \tau A\} \quad \text{so that} \quad p_A(\lambda x) = q_A(\lambda x) = \lambda q_A(x) = |\lambda|p_A(x).
\]

For the proof of the strong triangle inequality (iii) we may assume \( p_A(x+y) \neq 0 \). We distinguish two cases.

(i) \( p_A(x+y) \in (0,\infty) \). By Proposition 1 (ii) and \( \pi \circ p_A = q_A \), from

\[
\lambda_x(x+y) \leq q_A(x) \vee q_A(y)
\]

we obtain \( p_A(x+y) = q_A(x+y) = q_A(x) \vee q_A(y) \leq p_A(x) \vee p_A(y) \).

(ii) \( p_A(x+y) \notin (0,\infty) \). Then the valuation of \( K \) is dense. Assume

\[
p_A(y) \leq p_A(x).
\]

Suppose \( p_A(x+y) > p_A(x) \); we derive a contradiction. There is a \( \lambda \in K \) such that \( p_A(x+y) \geq |\lambda| > p_A(x) \). (In fact, if \( p_A(x) \neq p_A(x+y) \) then the interval \( (p_A(x), p_A(x+y)) \) contains infinitely many elements of \( |K| \), if \( p_A(x) = p_A(x+y) \) then \( p_A(x+y) \in |K| \) and we may choose \( \lambda \in K \) such that \( |\lambda| = p_A(x+y) \). Then \( p_A(\lambda^{-1}y) \leq p_A(\lambda^{-1}x) < 1 \) so that \( \lambda^{-1}y \in A \), \( \lambda^{-1}x \in A \) and, by absolute convexity, also \( \lambda^{-1}(x+y) \in A \) implying

\[
p_A(x+y) < |\lambda|, \text{ a contradiction.}
\]

§4. Hahn-Banach Theorem

For \( a \in K \), \( a \in V \) we write \( B(a,a) = \{ \lambda \in K : |\lambda-a| \leq a \} \).

**Theorem 6.** (Hahn-Banach Theorem) Let \( K \) be spherically complete, let \( p \)
be a distinguishing seminorm on a $K$-vector space $E$. Let $D$ be a $K$-linear subspace of $E$, let $f : D \to K$ be a $K$-linear map satisfying $|f(d)| \leq p(d)$ ($d \in D$). Then $f$ can be extended to a $K$-linear $\bar{f} : E \to K$ such that $|\bar{f}(x)| \leq p(x)$ ($x \in D$).

Proof. (It consists of checking that replacing of a seminorm by a distinguishing seminorm does not harm the well-known proof.) A simple application of Zorn's Lemma reduces the problem to the case $E = \{\lambda x + d : \lambda \in K, d \in D\}$ for some $x \in E \setminus D$. We are done if we can choose $\bar{f}(x) = \xi \in K$ such that

$$|\lambda \xi + f(d)| \leq p(\lambda x + d) \quad (\lambda \in K, d \in D).$$

As this condition is satisfied for $\lambda = 0$ and all $d \in D$ and since for $\lambda \neq 0$

$$p(\lambda x+d) = |\lambda| \ p(x+\lambda^{-1}d) \quad |\lambda \xi + f(d)| = |\lambda| \ |\xi + f(\lambda^{-1}d)|$$

it suffices, by Proposition 1 (i) to produce a $\xi \in K$ such that

$$|\xi - f(d)| \leq p(x-d) \quad (d \in D),$$

i.e. we must have that

$$\cap_{d \in D} B(f(d), p(x-d)) \neq \emptyset.$$

By spherical completeness it suffices to show that for any $d_1, d_2 \in D$ we have $B(f(d_1), p(x-d_1)) \cap B(f(d_2), p(x-d_2)) \neq \emptyset$. But this follows easily from $|f(d_1) - f(d_2)| \leq p(d_1 - d_2) \leq p(d_1 - x) \vee p(x - d_2)$.

A typical application: let $\mathbb{L}^\infty := \{ (\xi_1, \xi_2, \ldots) : \xi_n \in K \text{ for all } n, \sup_n |\xi_n| < \infty \}$.

Let $K$ be spherically complete. Then there is a linear function $g : \mathbb{L}^\infty \to K$
of norm 1 such that (i) \( g((\xi_1, \xi_2, \ldots)) = \lim_{n \to \infty} \xi_n \) if \( \lim_{n \to \infty} \xi_n \) exists and (ii) \( |g((\xi_1, \xi_2, \ldots))| < 1 \) if \( |\xi_n| < 1 \) for all \( n \in \mathbb{N} \).

(Proof: Choose in the above theorem \( D := c \) (the space of the convergent sequences), \( f((\xi_1, \xi_2, \ldots)) := \lim_{n \to \infty} (\xi_1, \xi_2, \ldots) \in c \), and \( p((\xi_1, \xi_2, \ldots)) := \sup |\xi_n| \), where the sup is taken in \( V \). Take \( g := \frac{1}{f} \).

§5. Separation of convex sets

Throughout §5, let \( E \) be a locally convex space over \( K \). We shall need the following observation.

**Proposition 7.** An open convex subset of \( E \) is closed.

**Proof.** Any convex set is a coset of an absolutely convex set. An open absolutely convex set is the complement of a union of cosets.

**Theorem 8.** Let \( K \) be spherically complete. Let \( A \subset E \) be closed, absolutely convex and let \( x \in E \setminus A \). Then there exists an \( f \in E' \) such that \( |f(A)| < 1 \) and \( f(x) = 1 \).

**Proof.** There is an absolutely convex open neighbourhood \( U \) of 0 such that \((x+U) \cap A = \emptyset\). Then \( U+A \) is absolutely convex, open, hence closed (Proposition 7). Further, \( x \notin U+A \). Thus, we may assume that \( A \) is open and closed. Then \( A \) is absorbing. Let \( p_A \) be the distinguishing seminorm of \( A \), let \( D := \{\lambda x : \lambda \in K\} \) and define \( g : D \to K \) by \( g(\lambda x) := \lambda \) (\( \lambda \in K \)). Then \( g(x) = 1 \). Since

\[
A = \{y \in E : p_A(y) < 1\}
\]

(Theorem 5) and \( x \notin A \) we have \( p_A(x) \geq 1 \) so that for \( \lambda \in K \)

\[
p_A(\lambda x) = |\lambda| p_A(x) \geq |\lambda| = |g(\lambda x)|, \text{ i.e. } |g| \leq p_A \text{ on } D.
\]

By Theorem 6 \( g \) extends to a linear \( f : E \to K \) such that \( |f(y)| \leq p_A(y) \) for all \( y \in E \).
We have \( f(x) = g(x) = 1 \) and, for \( y \in A \), \( |f(y)| \leq p_A(y) < 1 \). The continuity of \( f \) follows from the continuity of \( \pi \circ p_A \) and the inequality \( |f| \leq \pi \circ p_A \).

**COROLLARY 9.** Let \( K \) be spherically complete. Each closed convex set is weakly closed.

Let \( A, B \) be convex subsets of \( E \). If \( f : E \to K \) is a linear function then \( f(A) \) and \( f(B) \) are convex in \( K \). Hence, if \( f(A) \cap f(B) = \emptyset \) then \( \text{dist}(f(A), f(B)) > 0 \). With this in mind the following definition is quite natural.

**DEFINITION 10.** Two convex subsets \( A, B \) of \( E \) are separated by an \( f \in E' \) if \( f(A) \cap f(B) = \emptyset \).

If \( A \) and \( B \) are separated by \( f \in E' \) then, since \( \text{dist}(f(A), f(B)) > 0 \) there is an open convex neighbourhood \( U \) of \( 0 \) such that \( (A+U) \cap B = \emptyset \) (if \( E \) is a normed space this is equivalent to \( \text{dist}(A,B) > 0 \)). To prove the converse we need spherical completeness.

**THEOREM 11.** Let \( K \) be spherically complete. Let \( A, B \) be convex subsets of \( E \) and suppose there is an open convex neighbourhood \( U \) of \( A \) such that \( (A+U) \cap B = \emptyset \) (observe that this condition is satisfied if \( A \) is open).

Then \( A \) and \( B \) can be separated by some \( f \in E' \).

**Proof.** We may assume that \( A \) is open. Let \( C := A-B \). Then \( 0 \not\in C \), and

\[ C = U \ (A-b) \text{ is open, convex. Choose } c \in -C. \text{ Then } T := c + C \text{ is absolutely convex, open, hence closed and } c \not\in T. \text{ By Theorem 8 there is an } f \in E' \text{ such that } f(c) = 1, \ |f(T)| < 1. \text{ Thus, for each } a \in A, \ b \in B \text{ we have} \]

\[ 1 > |f(c + a - b)| = |1 + f(a) - f(b)|. \]

It follows that \( |f(a) - f(b)| = 1 \) for all \( a \in A, \ b \in B \). In particular, \( f(A) \cap f(B) = \emptyset. \)
References.
