The purpose of this note is to prove the following theorem (for the definition of a $C^\infty$-function see below).

**Theorem.** Let $K$ be a complete non-archimedean valued field with characteristic zero. Let $X$ be a nonempty subset of $K$ without isolated points and let $f : X \to K$ be a $C^\infty$-function. Then there is a $C^\infty$-function $X \to K$ whose derivative is $f$.

First we quote some definitions and statements from [1] which are needed for the proof. Let $K$ and $X$ be as above.

**Definition ([1], p. 8 or 75).** Let $f : X \to K$. $f$ is differentiable if its derivative $a \mapsto f'(a) := \lim_{x \to a}(f(x) - f(a))^{-1} = \lim_{x \to a}(f(x) - f(a) - f'(a)(x - a))^{-1}$ (a $\in X)$ exists. For $n \in \mathbb{N}$, let $\nabla^n X := \{(y_1, y_2, \ldots, y_n) \in X^n; y_i \neq y_j$ whenever $i \neq j\}$. The difference quotients $\delta_n f : \nabla^{n+1} X \to K$ ($n \in \{0, 1, 2, \ldots\}$) are given inductively by

$$\delta_0 f := f$$

and

$$\delta_n f(y_1, y_2, \ldots, y_{n+1}) := (y_1 - y_2)\delta_{n-1} f(y_1, y_3, \ldots, y_{n-1}) - \delta_{n-1} f(y_2, y_3, \ldots, y_{n+1})$$

where $((y_1, y_2, \ldots, y_{n+1}) \in \nabla^{n+1} X, n \in \mathbb{N})$.

$f$ is a $C^n$-function ($f \in C^n(X \to K)$) if $\delta_n f$ can (uniquely) be extended to a continuous function $\delta_n f : x^{n+1} \to K$.

$f$ is a $C^\infty$-function if $f \in C^\infty(X \to K) := \cap_{n=0}^{\infty} C^n(X \to K)$.

**Proposition ([1], p. 78, 86, 87, 116 and 123).** Let $f : X \to K$. For each $n \in \mathbb{N}$ the function $\delta_n f$ is symmetric, $C^{n+1}(X \to K) \supset C^n(X \to K)$, if $f \in C^n(X \to K)$ then $f' \in C^{n+1}(X \to K)$ and $\delta_n f(a, a, \ldots, a) = f^{(n)}(a)/n!$ for each $a \in X$.

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if \( \lim_{x \to y} a(x - y)^m (f(x) - f(y)) = 0 \) for each \( a \in X \) then \( f \in C^m(X \to K) \) and \( f' = 0 \). (Locally) analytic functions are \( C^\infty \)-functions.

Definition ([1], p. 45 and 46). - Let \( 0 < \rho < 1 \). For each \( n \in \mathbb{N} \), let \( R_n \) be a full set of representatives in \( X \) of the equivalence relation given by \(|x - y| < \rho^n \) \((x, y \in X)\) such that \( R_1 \subset R_2 \subset \ldots \). Choose \( x_0 \in R_1 \). For each \( \omega \in X \), \( n \in \mathbb{N} \), let \( x_n \) be determined by the conditions \( x_n \in R_n \), \(|x - x_n| < \rho^n \).

PROPOSITION ([1] Th. 11.2). - Let \( n \in \mathbb{N} \), \( f \in C^{n-1}(X \to K) \). Set

\[
P_n f(x) := \sum_{m=0}^{n} \sum_{j=0}^{n-1} \frac{f(j)(x)}{(j + 1)!} (x_{m+1} - x_m)^{j+1} (x \in X).
\]

Then \( P_n f \) is a \( C^n \)-antiderivative of \( f \).

Proof of the theorem. - We shall use the terminology of above.

Let \( j \in \{0, 1, 2, \ldots\} \). \( f(j) \) is continuous hence locally bounded and there exists a partition of \( X \) into "closed" balls \( B_{j_i} \) (relative to \( X \)) of radius \( < 1 \) where \( i \) runs through some indexing set \( I_j \) such that \( f(j) \) is bounded on each \( B_{j_i} \). For each \( i \in I_j \), we can choose \( m_{j_l} \in \mathbb{N} \) such that (recall that \( 0 < \rho < 1 \))

\[
(\ast) \quad \rho^{m_{j_l}} \leq \text{d}(B_{j_l}) < 1, \quad |f(j)(x)| \rho^{m_{j_l}} < |(j + 1)| \rho^j \quad (x \in B_{j_l}).
\]

Define \( F_j : X \to K \) as follows. If \( x \in X \), then \( x \in B_{j_l} \) for precisely one \( i \in I_j \). Set

\[
F_j(x) := \sum_{m \geq m_{j_l}} f(j)(x_m) (x_{m+1} - x_m)^{j+1}.
\]

We shall prove that \( F := \sum_{j=0}^{\infty} F_j \) is a \( C^\infty \)-antiderivative of \( f \) by means of the following steps.

(i) Each \( F_j \) is well defined.

(ii) For each \( j \in \{0, 1, 2, \ldots\} \) and for all \( i \in I_j \),

\[
|F_j(x)| \leq \rho^{m_{j_l}+j} \quad (x \in B_{j_l})
\]

so that \( F \) is well defined.

(iii) \( \sum_{j=0}^{n} F_j \) is a \( C^n \)-antiderivative of \( f \) for each \( n \in \mathbb{N} \).

(iv) For each \( n \), \( \sum_{j=n+1}^{\infty} F_j \) is a \( C^n \)-function with zero derivative.

Proof of (i). - \( f(j) \) is bounded on \( B_{j_l} \), and \( \lim_{m \to \infty} (x_{m+1} - x_m) = 0 \).

Proof of (ii). - Let \( x \in B_{j_l} \) and \( m \geq m_{j_l} \). Then by (\ast),

\[
|x_{m+1} - x_m| \leq |x - x_m| \leq \rho^m \leq \rho^{m_{j_l}} \leq \text{d}(B_{j_l}).
\]
from which it follows that \( x_m \in B_{j_1} \) and \( |x_{m+1} - x_m| \leq \rho^{m_{ji}} \). Applying the second formula of (*) with \( x \) replaced by \( x_m \), we get

\[
f(\frac{m_{ji}(j+1)}{m_{ji}+j}) (x_{m+1} - x_m)^{j+1} \leq \rho^{j} \rho^{m_{ji}} \rho^{m_{ji}(j+1)} = \rho^{m_{ji}+j},
\]
and (ii) is proved.

**Proof of (iii).** - The function \( F_j \) and \( x \mapsto \sum_{m=0}^{\infty} f(j)(x_m)(x_{m+1} - x_m)^{j+1}/(j+1) \) differ (on each \( B_{j_1} \), hence globally) by a locally constant function. Summation from \( j = 0 \) to \( j = n \) shows that \( \sum_{j=0}^{n} F_j - F_{n+1} \) is locally constant. By the second proposition

\[
\sum_{j=0}^{n} F_j \in C^1(X \to K) \subset C(X \to K) \text{ and } (\sum_{j=0}^{n} F_j)' = f.
\]

**Proof of (iv).** - Set \( H := \sum_{j=n+1}^{\infty} F_j \). We shall prove that \( |H(x) - H(y)| \leq |x - y|^{n+1} \) for all \( x, y \in X \) which, by the first proposition implies (iv). To obtain the inequality if suffices to prove

\[
(*) \quad |F_j(x) - F_j(y)| \leq |x - y|^{n+1} \quad (x, y \in X) \text{ for each } j \geq n + 1.
\]

We consider several cases.

(a) \( x \in B_{j_1} \), \( y \in B_{j_1'} \), where \( i \neq i' \). Then by (*),

\[
|x - y| \geq d(B_{j_1}) \geq \rho^{m_{ji}} \quad \text{so that} \quad |x - y|^{n+1} \geq \rho^{m_{ji(n+1)}}.
\]

By (ii),

\[
|F_j(x)| \leq \rho^{m_{ji}+j}.
\]

As \( m_{ji} + j \geq (n+1)m_{ji} \), we have \( |F_j(x)| \leq |x - y|^{n+1} \). By symmetry, \( |F_j(y)| \leq |x - y|^{n+1} \), and (**) follows.

(b) There is \( i \) such that \( x, y \notin B_{j_1} \). We may assume \( x \neq y \), there exists an \( s \in \mathbb{N} \cup \{0\} \) such that (recall that \( d(B_{j_1}) < 1 \))

\[
\rho^{s+1} \leq |x - y| < \rho^s.
\]

Then \( |x - y|^{n+1} \geq \rho^{(s+1)(n+1)} \). Consider two subcases.

(b.1) \( s < m_{ji} \). Then by (ii),

\[
|F_j(x)| \leq \rho^{m_{ji}+j}
\]

and since \( m_{ji} + j \geq (n+1)(s+1) + j \geq (s+1)(n+1) \), we have \( |F_j(x)| \leq |x - y|^{n+1} \). By symmetry, \( |F_j(y)| \leq |x - y|^{n+1} \) and (**) follows.

(b.2) \( s \geq m_{ji} \). Then since \( x_0 = y_0, \ldots, x_s = y_s \), we have, for \( m = m_{ji}, \ldots, s-1, \)
so that

\[ F_j(x) - F_j(y) = \sum_{m \geq s} \frac{f(j)(x_m)}{(j+1)!} (x_{m+1} - x_m)^{j+1} + \sum_{m \geq s} \frac{f(j)(y_m)}{(j+1)!} (y_{m+1} - y_m)^{j+1}. \]

If \( m \geq s \), we have by (\*) (observe that \( x_m \in B_{j+1} \))

\[ |\frac{f(j)(x_m)}{(j+1)!} (x_{m+1} - x_m)^{j+1}| \leq \rho^{j-m_j+1+m(j+1)} \]

and we find \( |F_j(x) - F_j(y)| \leq \rho^{j-m_j+s(j+1)} \). Using the fact that \( j \geq n + 1 \) and our assumption \( s \geq m_j \), we obtain

\[ j - m_j + s(j + 1) = (s + 1) j + s - m_j \geq (s + 1)(n + 1). \]

By consequence

\[ |F_j(x) - F_j(y)| \leq \rho^{(s+1)(n+1)} \leq |x - y|^{n+1} \]

which finishes the proof.

Remark. - The above construction does not give us a linear antiderivation map \( C^\infty(X \to K) \to C^\infty(X \to K) \), and it is somewhat doubtful whether there exists a linear antiderivation map \( P : C^\infty(X \to K) \to C^\infty(X \to K) \) that is continuous with respect to a natural locally convex topology ([1], p. 119) on \( C^\infty(X \to K) \).

REFERENCE