The purpose of this note is to prove the following theorem (for the definition of a $C^\infty$-function see below).

**THEOREM.** - Let $K$ be a complete non-archimedean valued field with characteristic zero. Let $X$ be a nonempty subset of $K$ without isolated points and let $f: X \rightarrow K$ be a $C^\infty$-function. Then there is a $C^\infty$-function $X \rightarrow K$ whose derivative is $f$.

First we quote some definitions and statements from [1] which are needed for the proof. Let $K$ and $X$ be as above.

**Definition ([1], p. 8 p. 78).** - Let $f: X \rightarrow K$. $f$ is differentiable if its derivative $a \mapsto f'(a) := \lim_{x \rightarrow a} (x - a)^{-1} (f(x) - f(a))$ $(a \in X)$ exists. For $n \in \mathbb{N}$, let $\nabla^n X := \{(y_1, y_2, \ldots, y_n) \in X^n ; y_i \neq y_j$ whenever $i \neq j\}$. The difference quotients $\xi_n f: \nabla^{n+1} X \rightarrow K$ $(n \in \{0, 1, 2, \ldots\})$ are given inductively by

$$\xi_0 f := f$$

and

$$\xi_n f(y_1, y_2, \ldots, y_{n+1}) := (y_1 - y_2)^{-1}(\xi_{n-1} f(y_1, y_3, \ldots, y_{n-1}) - \xi_{n-1} f(y_2, y_3, \ldots, y_{n+1}))$$

$$(y_1, y_2, \ldots, y_{n+1}) \in \nabla^{n+1} X, n \in \mathbb{N}.$$ 

$f$ is a $C^n$-function $(f \in C^n(X \rightarrow K))$ if $\xi_n f$ can (uniquely) be extended to a continuous function $\overline{\xi}_n f: X^{n+1} \rightarrow K$.

$f$ is a $C^\infty$-function if $f \in C^\infty(X \rightarrow K) := \cap_{n=0}^{\infty} C^n(X \rightarrow K)$.

**PROPOSITION ([1], p. 78, 86, 87, 116 and 123).** - Let $f: X \rightarrow K$. For each $n \in \mathbb{N}$ the function $\xi_n f$ is symmetric, $C^n(X \rightarrow K) \supset C^n(X \rightarrow K)$, if $f \in C^n(X \rightarrow K)$ then $f' \in C^{n+1}(X \rightarrow K)$ and $\xi_n f(a, a, \ldots, a) = f^{(n)}(a)/n!$ for each $a \in X$.

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if \( \lim_{x \to a} (x - y)^n (f(x) - f(y)) = 0 \) for each \( a \in X \) then \( f \in C^n(X \to K) \) and \( f' = 0 \). (locally) analytic functions are \( C^\infty \)-functions

**Definition ([1], p. 45 and 46).** - Let \( 0 < \rho < 1 \). For each \( n \in \mathbb{N} \), let \( R_n \) be a full set of representatives in \( X \) of the equivalence relation given by
\[ |x - y| < \rho^n \ (x, y \in X) \] such that \( R_1 \subset R_2 \subset ... \). Choose \( x_0 \in R_1 \). For each \( x \in X \), \( n \in \mathbb{N} \), let \( x_n \) be determined by the conditions \( x_n \in R_n \), \( |x - x_n| < \rho^n \).

**PROPOSITION ([1] Th. 11.2).** - Let \( n \in \mathbb{N} \), \( f \in C^n(X \to K) \). Set
\[ P_n f(x) := \sum_{m=0}^{n-1} \frac{f^{(j)}(x)}{(j + 1)!} (x_{m+1} - x_m)^{j+1} \ (x \in X) \].

Then \( P_n f \) is a \( C^n \)-antiderivative of \( f \).

**Proof of the theorem.** - We shall use the terminology of above.

Let \( j \in \{0, 1, 2, ...\} \). \( f^{(j)} \) is continuous hence locally bounded and there exists a partition of \( X \) into "closed" balls \( B_{j_1} \) (relative to \( X \)) of radius \( < 1 \) where \( i \) runs through some indexing set \( I_j \) such that \( f^{(j)} \) is bounded on each \( B_{j_1} \). For each \( i \in I_j \), we can choose \( m_{j_1} \in \mathbb{N} \) such that (recall that \( 0 < \rho < 1 \))
\[ \rho m_{j_1} \leq d(B_{j_1}) < 1, \ |f^{(j)}(x)| \rho m_{j_1} < (j + 1)^{i} \rho^{j} \ (x \in B_{j_1}) \].

Define \( F_j : X \to K \) as follows. If \( x \in X \), then \( x \in B_{j_1} \) for precisely one \( i \in I_j \). Set
\[ F_j(x) := \sum_{m=0}^{m_{j_1}} \frac{f^{(j)}(x_m)}{(j + 1)!} (x_{m+1} - x_m)^{j+1} \ .
\]
We shall prove that \( F := \sum_{j=0}^{\infty} F_j \) is a \( C^n \)-antiderivative of \( f \) by means of the following steps.

(i) Each \( F_j \) is well defined.

(ii) For each \( j \in \{0, 1, 2, ...\} \) and for all \( i \in I_j \),
\[ |F_j(x)| \leq \rho m_{j_1}^{j+1} \ (x \in B_{j_1}) \]
so that \( F \) is well defined.

(iii) \( \sum_{j=0}^{\infty} F_j \) is a \( C^n \)-antiderivative of \( f \) for each \( n \in \mathbb{N} \).

(iv) For each \( n \), \( \sum_{j=n+1}^{\infty} F_j \) is a \( C^n \)-function with zero derivative.

**Proof of (i).** - \( f^{(j)} \) is bounded on \( B_{j_1} \), and \( \lim_{m \to \infty} (x_{m+1} - x_m) = 0 \).

**Proof of (ii).** - Let \( x \in B_{j_1} \) and \( m \geq m_{j_1} \). Then by \((*)\),
\[ |x_{m+1} - x_m| \leq |x - x_m| \leq \rho^m \leq \rho m_{j_1} \leq d(B_{j_1}) \]
from which it follows that \( x_{m} \in B_{ji} \) and \( |x_{m+1} - x_{m}| \leq \rho^{|m|} \). Applying the second formula of (*) with \( x \) replaced by \( x_{m} \), we set

\[
\frac{f^{(j)}(x)}{(j+1)!} \left( x_{m+1} - x_{m} \right)^{j+1} \leq \rho^{j} \rho^{|m|} (j+1)! \rho^{m_{ji}(j+1)} = \rho^{m_{ji}(j+1)},
\]

and (ii) is proved.

Proof of (iii). - The function \( F_{j} \) and \( x \mapsto \sum_{m=0}^{\infty} f^{(j)}(x_{m}) (x_{m+1} - x_{m})^{j+1}/(j+1)! \) differ (on each \( B_{ji} \), hence globally) by a locally constant function. Summation from \( j=0 \) to \( j=n \) shows that \( \sum_{j=0}^{n} F_{j} = F_{n+1} \), \( f \) is locally constant. By the second proposition

\[
\sum_{j=0}^{n} F_{j} \in C^{n+1}(X \rightarrow K) \subset C^{n+1}(X \rightarrow K) \quad \text{and} \quad (\sum_{j=0}^{n} F_{j})' = f.
\]

Proof of (iv). - Set \( H := \sum_{j=n+1}^{\infty} F_{j} \). We shall prove that \( |H(x)-H(y)| \leq |x-y|^{n+1} \) for all \( x, y \in X \) which, by the first proposition implies (iv). To obtain the inequality if suffices to prove

\[
(\ast) \quad |F_{j}(x) - F_{j}(y)| \leq |x-y|^{n+1} \quad \text{for each } j \geq n+1.
\]

We consider several cases.

(a) \( x \in B_{ji} \), \( y \in B_{ji}' \), where \( j \neq i' \). Then by (\ast),

\[
|x-y| \geq d(B_{ji}) \geq \rho^{m_{ji}} \text{ so that } |x-y|^{n+1} \geq \rho^{m_{ji}(n+1)}.
\]

By (ii),

\[
|F_{j}(x)| \leq \rho^{m_{ji}(j+1)}.
\]

As \( jm_{ji} + j \geq (n+1)m_{ji} \), we have \( |F_{j}(x)| \leq |x-y|^{n+1} \). By symmetry, \( |F_{j}(y)| \leq |x-y|^{n+1} \), and (\ast) follows.

(b) There is \( i \) such that \( x, y \in B_{ji} \). We may assume \( x \neq y \), there exists an \( s \in \mathbb{N} \cup \{0\} \) such that (recall that \( d(B_{ji}) < 1 \))

\[
\rho^{s+1} \leq |x-y| < \rho^{s}.
\]

Then \( |x-y|^{n+1} \geq \rho^{s+1}(n+1) \). Consider two subcases.

(b.1) \( s < m_{ji} \). Then by (ii),

\[
|F_{j}(x)| \leq \rho^{m_{ji}(j+1)}
\]

and since \( jm_{ji} + j \geq (n+1)(s+1) + j \geq (s+1)(n+1) \), we have \( |F_{j}(x)| \leq |x-y|^{n+1} \). By symmetry \( |F_{j}(y)| \leq |x-y|^{n+1} \) and (\ast) follows.

(b.2) \( s \geq m_{ji} \). Then since \( x_{0} = y_{0}, \ldots \), \( x_{s} = y_{s} \), we have, for \( m=m_{ji}, \ldots, s-1, \)
\[
\frac{f^{(j)}(x_{m})}{(j+1)!} (x_{m+1} - x_{m})^{j+1} = \frac{f^{(j)}(y_{m})}{(j+1)!} (y_{m+1} - y_{m})^{j+1}
\]

so that

\[
F^{(j)}(x) - F^{(j)}(y) = \sum_{m \geq s} \frac{f^{(j)}(x_{m})}{(j+1)!} (x_{m+1} - x_{m})^{j+1} - \sum_{m \geq s} \frac{f^{(j)}(y_{m})}{(j+1)!} (y_{m+1} - y_{m})^{j+1}.
\]

If \( m \geq s \), we have by \((*)\) (observe that \( x_{m} \in B_{j+1} \))

\[
\left| \frac{f^{(j)}(x_{m})}{(j+1)!} (x_{m+1} - x_{m})^{j+1} \right| \leq \rho^{j-m_{j}+1+m(j+1)}
\]

\[
\left| \frac{f^{(j)}(y_{m})}{(j+1)!} (y_{m+1} - y_{m})^{j+1} \right| \leq \rho^{j-m_{j}+1+m(j+1)}
\]

and we find \( |F^{(j)}(x) - F^{(j)}(y)| \leq \rho^{j-m_{j}+s(j+1)} \). Using the fact that \( j \geq n+1 \) and our assumption \( s \geq m_{j} \), we obtain

\[
j - m_{j} + s(j+1) = (s+1) j + s - m_{j} \geq (s+1)(n+1).
\]

By consequence

\[
|F^{(j)}(x) - F^{(j)}(y)| \leq \rho^{(s+1)(n+1)} \leq |x - y|^{n+1}
\]

which finishes the proof.

Remark. - The above construction does not give us a linear antiderivation map \( C^{0}(X \rightarrow K) \rightarrow C^{0}(X \rightarrow K) \), and it is somewhat doubtful whether there exists a linear antiderivation map \( P : C^{0}(X \rightarrow K) \rightarrow C^{0}(X \rightarrow K) \) that is continuous with respect to a natural locally convex topology ([1], p. 119) on \( C^{0}(X \rightarrow K) \).

REFERENCES