The purpose of this note is to prove the following theorem (for the definition of a \(C^\infty\)-function see below).

**Theorem.** Let \(K\) be a complete non-archimedean valued field with characteristic zero. Let \(X\) be a nonempty subset of \(K\) without isolated points and let \(f : X \rightarrow K\) be a \(C^\infty\)-function. Then there is a \(C^\infty\)-function \(X \rightarrow K\) whose derivative is \(f\).

First we quote some definitions and statements from [1] which are needed for the proof. Let \(K\) and \(X\) be as above.

**Definition ([1], p. 8).** - Let \(f : X \rightarrow K\). \(f\) is differentiable if its derivative \(a \mapsto f'(a) := \lim_{x \to a} (f(x) - f(a)) \) (\(a \in X\)) exists. For \(n \in \mathbb{N}\), let \(\nabla^n X := \{(y_1, y_2, \ldots, y_n) \in X^n ; y_i \neq y_j\text{ whenever }i \neq j\}\). The difference quotients \(\xi_n f : \nabla^{n+1} X \rightarrow K\) \((n \in \{0, 1, 2, \ldots\})\) are given inductively by

\[
\xi_0 f := f
\]

and

\[
\xi_n f(y_1, y_2, \ldots, y_{n+1}) := (y_1 - y_2)^{-1}(\xi_{n-1} f(y_1, y_3, \ldots, y_n) - \xi_{n-1} f(y_2, y_3, \ldots, y_{n+1}))
\]

\((y_1, y_2, \ldots, y_{n+1}) \in \nabla^{n+1} X, n \in \mathbb{N}\).

\(f\) is a \(C^n\)-function \((f \in C^n(X \rightarrow K))\) if \(\xi_n f\) can (uniquely) be extended to a continuous function \(\xi_n f : X^{n+1} \rightarrow K\).

\(f\) is a \(C^\infty\)-function if \(f \in C^\infty(X \rightarrow K) := \bigcap_{n=0}^{\infty} C^n(X \rightarrow K)\).

**Proposition ([1], p. 78, 86, 87, 116 and 123).** - Let \(f : X \rightarrow K\). For each \(n \in \mathbb{N}\) the function \(\xi_n f\) is symmetric, \(C^{n-1}(X \rightarrow K) \supseteq C^n(X \rightarrow K)\), if \(f \in C^n(X \rightarrow K)\) then \(f' \in C^{n-1}(X \rightarrow K)\) and \(\xi_n f(a, a, \ldots, a) = f^{(n)}(a)/n!\) for each \(a \in X\).

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if \( \lim_{y \to x} (x - y)^m (f(x) - f(y)) = 0 \) for each \( a \in A \) then \( f \in C^m(X \to \mathbb{K}) \) and \( f' = 0 \). (Locally) analytic functions are \( C^\infty \)-functions.

Definition ([1], p. 45 and 46). Let \( 0 < \rho < 1 \). For each \( n \in \mathbb{N} \), let \( R_n \) be a full set of representatives in \( X \) of the equivalence relation given by |\( x - y | < \rho^n \) (\( x, y \in X \)) such that \( R_1 \subseteq R_2 \subseteq \ldots \). Choose \( x_0 \in R_1 \). For each \( x \in X \), \( n \in \mathbb{N} \), let \( x_n \) be determined by the conditions \( x_n \in R_n, |x - x_n| < \rho^n \).

PROPOSITION ([1] Th. 11.2). Let \( n \in \mathbb{N} \), \( f \in C^{n-1}(X \to \mathbb{K}) \). Set

\[
P_n f(x) := \sum_{m=0}^{\infty} \frac{x^{m-1}}{(m+1)!} \frac{f^{(j)}(x)}{j!} (x_{m+1} - x_m)^{j+1} (x \in X).
\]

Then \( P_n f \) is a \( C^n \)-antiderivative of \( f \).

Proof of the theorem. We shall use the terminology of above.

Let \( j \in \{0, 1, 2, \ldots\} \). \( f^{(j)} \) is continuous hence locally bounded and there exists a partition of \( X \) into "closed" balls \( B_{j_i} \) (relative to \( X \)) of radius < 1 where \( i \) runs through some indexing set \( I_j \) such that \( f^{(j)} \) is bounded on each \( B_{j_i} \). For each \( i \in I_j \), we can choose \( m_{j_i} \in \mathbb{N} \) such that (recall that \( 0 < \rho < 1 \))

\[
\rho^{m_{j_i}} \leq \min(B_{j_{i+1}}) < 1, \quad \rho^{m_{j_i}} \leq |(j+1)^j| \rho^j (x \in B_{j_i})
\]

Define \( F_j : X \to \mathbb{K} \) as follows. If \( x \in X \), then \( x \in B_{j_i} \) for precisely one \( i \in I_j \). Set

\[
F_j(x) := \sum_{m=0}^{\infty} \frac{f^{(j)}(x)}{(j+1)!} (x_{m+1} - x_m)^{j+1}.
\]

We shall prove that \( F := \sum_{j=0}^{\infty} F_j \) is a \( C^n \)-antiderivative of \( f \) by means of the following steps.

(i) Each \( F_j \) is well defined.

(ii) For each \( j \in \{0, 1, 2, \ldots\} \) and for all \( i \in I_j \),

\[
|F_j(x)| \leq \rho^{m_{j_i}^j} (x \in B_{j_i})
\]

so that \( F \) is well defined.

(iii) \( \sum_{j=0}^{\infty} F_j \) is a \( C^n \)-antiderivative of \( f \) for each \( n \in \mathbb{N} \).

(iv) For each \( n \), \( \sum_{j=n+1}^{\infty} F_j \) is a \( C^n \)-function with zero derivative.

Proof of (i). \( f^{(j)} \) is bounded on \( B_{j_i} \), and \( \lim_{m \to \infty} (x_{m+1} - x_m) = 0 \).

Proof of (ii). Let \( x \in B_{j_i} \) and \( m \geq m_{j_i} \). Then by (\asterisk) ,

\[
|x_{m+1} - x_m| \leq |x - x_m| \leq \rho^m \leq \rho^{m_{j_i}} \leq \min(B_{j_i})
\]
from which it follows that $x_m \in B_{ji}$ and $|x_{m+1} - x_m| \leq \rho^{m_{ji}}$. Applying the second formula of (\*) with $x$ replaced by $x_m$, we set

$$f(j)(x_m) \left| (x_{m+1} - x_m)^{j+1} \right| \leq \rho^j \rho^{j_1} \rho^{m_{ji}(j+1)} = \rho^{j_{ji}+j},$$

and (ii) is proved.

**Proof of (iii).** - The function $F_j$ and $x \mapsto \sum_{m=0}^{\infty} f(j)(x_m)(x_{m+1} - x_m)^{j+1}/(j+1)$ differ (on each $B_{ji}$, hence globally) by a locally constant function. Summation from $j = 0$ to $j = n$ shows that $\sum_{j=0}^{n} F_j - F_{n+1}$ is locally constant. By the second proposition

$$\sum_{j=0}^{n} F_j \in C^{n+1}(X \to K) < C^{n}(X \to K),$$

and (iii) is proved.

**Proof of (iv).** - Set $H := \sum_{j=n+1}^{\infty} F_j$. We shall prove that $|H(x) - H(y)| < |x - y|^{n+1}$ for all $x$, $y \in X$ which, by the first proposition implies (iv). To obtain the inequality if suffices to prove

$(**) \quad |F_j(x) - F_j(y)| \leq |x - y|^{n+1}$ \quad (x, y \in X) \quad \text{for each} \quad j \geq n + 1.

We consider several cases.

(a) $x \in B_{ji}$, $y \in B_{i'j'}$, where $i \neq i'$. Then by (\*),

$$|x - y| > d(B_{ji}) \geq \rho^{m_{ji}} \quad \text{so that} \quad |x - y|^{n+1} \geq \rho^{m_{ji}(n+1)}.$$

By (ii),

$$|F_j(x)| \leq \rho^{j_{ji}+j}.$$

As $j_{ji} + j \geq (n + 1)m_{ji}$, we have $|F_j(x)| \leq |x - y|^{n+1}$. By symmetry, $|F_j(y)| \leq |x - y|^{n+1}$, and (**) follows.

(b) There is $i$ such that $x$, $y \in B_{ji}$. We may assume $x \neq y$, there exists an $s \in N \cup \{0\}$ such that (recall that $d(B_{ji}) < 1$)

$$\rho^{s+1} \leq |x - y| < \rho^s.$$

Then $|x - y|^{n+1} \geq \rho^{(s+1)(n+1)}$. Consider two subcases.

(b.1) $s < m_{ji}$. Then by (ii),

$$|F_j(x)| \leq \rho^{j_{ji}+j}$$

and since $j_{ji} + j \geq (n + 1)(s + 1) + j \geq (s + 1)(n + 1)$, we have $|F_j(x)| \leq |x - y|^{n+1}$. By symmetry, $|F_j(y)| \leq |x - y|^{n+1}$ and (**) follows.

(b.2) $s \geq m_{ji}$. Then since $x_0 = y_0$, ..., $x_s = y_s$, we have, for $m = m_{ji}$, ..., $s-1,$
so that

\[ F_j(x) - F_j(y) = \sum_{m \geq s} \frac{f^{(j)}(x_m)}{(j+1)!} (x_{m+1} - x_m)^{j+1} - \sum_{m \geq s} \frac{f^{(j)}(y_m)}{(j+1)!} (y_{m+1} - y_m)^{j+1}. \]

If \( m \geq s \), we have by (*) (observe that \( x_m \in B_j \))

\[ \frac{f^{(j)}(x_m)}{(j+1)!} (x_{m+1} - x_m)^{j+1} \leq \rho \]

\[ \frac{f^{(j)}(y_m)}{(j+1)!} (y_{m+1} - y_m)^{j+1} \leq \rho \]

and we find \( |F_j(x) - F_j(y)| \leq \rho \). Using the fact that \( j \geq n + 1 \) and our assumption \( s \geq m_j \), we obtain

\[ j - m_j + s(j + 1) = (s + 1) j + s - m_j \geq (s + 1)(n + 1). \]

By consequence

\[ |F_j(x) - F_j(y)| \leq \rho (s+1)(n+1) \leq |x - y|^{n+1} \]

which finishes the proof.

Remark. - The above construction does not give us a linear antiderivation map \( C^0(X \to K) \to C^0(X \to K) \), and it is somewhat doubtful whether there exists a linear antiderivation map \( P : C^0(X \to K) \to C^0(X \to K) \) that is continuous with respect to a natural locally convex topology ([1], p. 119) on \( C^0(X \to K) \).

REFERENCE