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$C^\infty$ -ANTIDERIVATIVES OF p-ADIC  $C^\infty$ -FUNCTIONS

by Wilhelm H. SCHIKHOF (\*)

[Universiteit Nijmegen]

The purpose of this note is to prove the following theorem (for the definition of a  $C^\infty$ -function see below).

**THEOREM.** - Let  $K$  be a complete non-archimedean valued field with characteristic zero. Let  $X$  be a nonempty subset of  $K$  without isolated points and let  $f : X \rightarrow K$  be a  $C^\infty$ -function. Then there is a  $C^\infty$ -function  $X \rightarrow K$  whose derivative is  $f$ .

First we quote some definitions and statements from [1] which are needed for the proof. Let  $K$  and  $X$  be as above.

Definition ([1], p. 8 and 75). - Let  $f : X \rightarrow K$ .  $f$  is differentiable if its derivative  $a \mapsto f'(a) := \lim_{x \rightarrow a} (x - a)^{-1} (f(x) - f(a))$  ( $a \in X$ ) exists. For  $n \in \mathbb{N}$ , let  $\nabla^n X := \{(y_1, y_2, \dots, y_n) \in X^n; y_i \neq y_j \text{ whenever } i \neq j\}$ . The difference quotients  $\delta_n f : \nabla^{n+1} X \rightarrow K$  ( $n \in \{0, 1, 2, \dots\}$ ) are given inductively by

$$\delta_0 f := f$$

and

$$\delta_n f(y_1, y_2, \dots, y_{n+1})$$

$$:= (y_1 - y_2)^{-1} (\delta_{n-1} f(y_1, y_3, \dots, y_{n-1}) - \delta_{n-1} f(y_2, y_3, \dots, y_{n-1}))$$

$((y_1, y_2, \dots, y_{n+1}) \in \nabla^{n+1} X, n \in \mathbb{N})$ .

$f$  is a  $C^n$ -function ( $f \in C^n(X \rightarrow K)$ ) if  $\delta_n f$  can (uniquely) be extended to a continuous function  $\bar{\delta}_n f : X^{n+1} \rightarrow K$ .

$f$  is a  $C^\infty$ -function if  $f \in C^\infty(X \rightarrow K) := \bigcap_{n=0}^\infty C^n(X \rightarrow K)$ .

**PROPOSITION** ([1], p. 78, 86, 87, 110 and 123). - Let  $f : X \rightarrow K$ . For each  $n \in \mathbb{N}$  the function  $\delta_n f$  is symmetric,  $C^{n-1}(X \rightarrow K) \supset C^n(X \rightarrow K)$ , if  $f \in C^n(X \rightarrow K)$  then  $f' \in C^{n-1}(X \rightarrow K)$  and  $\bar{\delta}_n f(a, a, \dots, a) = f^{(n)}(a)/n!$  for each  $a \in X$ ,

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(\*) Wilhelm H. SCHIKHOF, Mathematisch Instituut, Katholieke Universiteit, Tournooiveld, NIJMEGEN (Pays-Bas).

if  $\lim_{x, y \rightarrow a} (x - y)^{-n} (f(x) - f(y)) = 0$  for each  $a \in X$  then  $f \in C^n(X \rightarrow K)$  and  $f' = 0$ . (Locally) analytic functions are  $C^\infty$ -functions

Definition ([1], p. 45 and 46). - Let  $0 < \rho < 1$ . For each  $n \in \mathbb{N}$ , let  $R_n$  be a full set of representatives in  $X$  of the equivalence relation given by  $|x - y| < \rho^n$  ( $x, y \in X$ ) such that  $R_1 \subset R_2 \subset \dots$ . Choose  $x_0 \in R_1$ . For each  $x \in X$ ,  $n \in \mathbb{N}$ , let  $x_n$  be determined by the conditions  $x_n \in R_n$ ,  $|x - x_n| < \rho^n$ .

PROPOSITION ([1] Th. 11.2). - Let  $n \in \mathbb{N}$ ,  $f \in C^{n-1}(X \rightarrow K)$ . Set

$$P_n f(x) := \sum_{m=0}^{\infty} \sum_{j=0}^{n-1} \frac{f^{(j)}(x_m)}{(j+1)!} (x_{m+1} - x_m)^{j+1} \quad (x \in X).$$

Then  $P_n f$  is a  $C^n$ -antiderivative of  $f$ .

Proof of the theorem. - We shall use the terminology of above.

Let  $j \in \{0, 1, 2, \dots\}$ .  $f^{(j)}$  is continuous hence locally bounded and there exists a partition of  $X$  into "closed" balls  $B_{ji}$  (relative to  $X$ ) of radius  $< 1$  where  $i$  runs through some indexing set  $I_j$  such that  $f^{(j)}$  is bounded on each  $B_{ji}$ . For each  $i \in I_j$ , we can choose  $m_{ji} \in \mathbb{N}$  such that (recall that  $0 < \rho < 1$ )

$$(*) \quad \rho^{m_{ji}} \leq d(B_{ji}) < 1, \quad |f^{(j)}(x)| \rho^{m_{ji}} < |(j+1)!| \rho^j \quad (x \in B_{ji}).$$

Define  $F_j : X \rightarrow K$  as follows. If  $x \in X$ , then  $x \in B_{ji}$  for precisely one  $i \in I_j$ . Set

$$F_j(x) := \sum_{m \geq m_{ji}} \frac{f^{(j)}(x_m)}{(j+1)!} (x_{m+1} - x_m)^{j+1}.$$

We shall prove that  $F := \sum_{j=0}^{\infty} F_j$  is a  $C^\infty$ -antiderivative of  $f$  by means of the following steps.

(i) Each  $F_j$  is well defined.

(ii) For each  $j \in \{0, 1, 2, \dots\}$  and for all  $i \in I_j$ ,

$$|F_j(x)| \leq \rho^{j m_{ji} + j} \quad (x \in B_{ji})$$

so that  $F$  is well defined.

(iii)  $\sum_{j=0}^n F_j$  is a  $C^n$ -antiderivative of  $f$  for each  $n \in \mathbb{N}$ .

(iv) For each  $n$ ,  $\sum_{j=n+1}^{\infty} F_j$  is a  $C^n$ -function with zero derivative.

Proof of (i). -  $f^{(j)}$  is bounded on  $B_{ji}$ , and  $\lim_{m \rightarrow \infty} (x_{m+1} - x_m) = 0$ .

Proof of (ii). - Let  $x \in B_{ji}$  and  $m \geq m_{ji}$ . Then by (\*),

$$|x_{m+1} - x_m| \leq |x - x_m| \leq \rho^m \leq \rho^{j m_{ji}} \leq d(B_{ji})$$

from which it follows that  $x_m \in B_{ji}$  and  $|x_{m+1} - x_m| \leq \rho^{m_{ji}}$ . Applying the second formula of (\*) with  $x$  replaced by  $x_m$ , we get

$$\left| \frac{f^{(j)}(x_m)}{(j+1)!} (x_{m+1} - x_m)^{j+1} \right| \leq \rho^j \rho^{-m_{ji}} \rho^{m_{ji}(j+1)} = \rho^{jm_{ji}+j},$$

and (ii) is proved.

Proof of (iii). - The function  $F_j$  and  $x \mapsto \sum_{m=0}^{\infty} f^{(j)}(x_m) (x_{m+1} - x_m)^{j+1} / (j+1)!$  differ (on each  $B_{ji}$ , hence globally) by a locally constant function. Summation from  $j=0$  to  $j=n$  shows that  $\sum_{j=0}^n F_j - P_{n+1} f$  is locally constant. By the second proposition

$$\sum_{j=0}^n F_j \in C^{n+1}(X \rightarrow K) \subset C^n(X \rightarrow K) \quad \text{and} \quad (\sum_{j=0}^n F_j)' = f.$$

Proof of (iv). - Set  $H := \sum_{j=n+1}^{\infty} F_j$ . We shall prove that  $|H(x) - H(y)| \leq |x - y|^{n+1}$  for all  $x, y \in X$  which, by the first proposition implies (iv). To obtain the inequality it suffices to prove

$$(**) \quad |F_j(x) - F_j(y)| \leq |x - y|^{n+1} \quad (x, y \in X) \quad \text{for each } j \geq n+1.$$

We consider several cases.

(a)  $x \in B_{ji}$ ,  $y \in B_{j i'}$ , where  $i \neq i'$ . Then by (\*),

$$|x - y| \geq d(B_{ji}) \geq \rho^{m_{ji}} \quad \text{so that} \quad |x - y|^{n+1} \geq \rho^{m_{ji}(n+1)}.$$

By (ii),

$$|F_j(x)| \leq \rho^{jm_{ji}+j}.$$

As  $jm_{ji} + j \geq (n+1)m_{ji}$ , we have  $|F_j(x)| \leq |x - y|^{n+1}$ . By symmetry,  $|F_j(y)| \leq |x - y|^{n+1}$ , and (\*\*) follows.

(b) There is  $i$  such that  $x, y \in B_{ji}$ . We may assume  $x \neq y$ , there exists an  $s \in \mathbb{N} \cup \{0\}$  such that (recall that  $d(B_{ji}) < 1$ )

$$\rho^{s+1} \leq |x - y| < \rho^s.$$

Then  $|x - y|^{n+1} \geq \rho^{(s+1)(n+1)}$ . Consider two subcases.

(b.1)  $s < m_{ji}$ . Then by (ii),

$$|F_j(x)| \leq \rho^{jm_{ji}+j}$$

and since  $jm_{ji} + j \geq (n+1)(s+1) + j \geq (s+1)(n+1)$ , we have  $|F_j(x)| \leq |x - y|^{n+1}$ . By symmetry  $|F_j(y)| \leq |x - y|^{n+1}$  and (\*\*) follows.

(b.2)  $s \geq m_{ji}$ . Then since  $x_0 = y_0, \dots, x_s = y_s$ , we have, for  $m = m_{ji}, \dots, s-1$ ,

$$\frac{f^{(j)}(x_m)}{(j+1)!} (x_{m+1} - x_m)^{j+1} = \frac{f^{(j)}(y_m)}{(j+1)!} (y_{m+1} - y_m)^{j+1}$$

so that

$$F_j(x) - F_j(y) = \sum_{m \geq s} \frac{f^{(j)}(x_m)}{(j+1)!} (x_{m+1} - x_m)^{j+1} - \sum_{m \geq s} \frac{f^{(j)}(y_m)}{(j+1)!} (y_{m+1} - y_m)^{j+1}.$$

If  $m \geq s$ , we have by (\*) (observe that  $x_m \in B_{ji}$ )

$$\left| \frac{f^{(j)}(x_m)}{(j+1)!} (x_{m+1} - x_m)^{j+1} \right| \leq \rho^{j-m_{ji}+m(j+1)}$$

$$\left| \frac{f^{(j)}(y_m)}{(j+1)!} (y_{m+1} - y_m)^{j+1} \right| \leq \rho^{j-m_{ji}+m(j+1)}$$

and we find  $|F_j(x) - F_j(y)| \leq \rho^{j-m_{ji}+s(j+1)}$ . Using the fact that  $j \geq n+1$  and our assumption  $s \geq m_{ji}$ , we obtain

$$j - m_{ji} + s(j+1) = (s+1)j + s - m_{ji} \geq (s+1)(n+1).$$

By consequence

$$|F_j(x) - F_j(y)| \leq \rho^{(s+1)(n+1)} \leq |x - y|^{n+1}$$

which finishes the proof.

Remark. - The above construction does not give us a linear antiderivation map  $C^\infty(X \rightarrow K) \rightarrow C^\infty(X \rightarrow K)$ , and it is somewhat doubtful whether there exists a linear antiderivation map  $P: C^\infty(X \rightarrow K) \rightarrow C^\infty(X \rightarrow K)$  that is continuous with respect to a natural locally convex topology ([1], p. 119) on  $C^\infty(X \rightarrow K)$ .

#### REFERENCE

- [1] SCHIKHOF (W. H.). - Non-archimedean calculus. - Mathematisch instituut, Nijmegen, 1978 (Lecture Notes. Report 7812).