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UNIQUENESS OF THE BANACH ALGEBRA TOPOLOGY

FOR NON-ARCHIMEDEAN ALGEBRAS

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1. Introduction.

An algebra $A$ over $\mathbb{R}$ or $\mathbb{C}$ is said to have a unique Banach algebra topology if any two Banach algebra norms on $A$ are equivalent. Johnson's theorem [2] is very satisfactory; it states that a semi-simple algebra over $\mathbb{R}$ or $\mathbb{C}$ has this property.

In this note we are concerned with the non-archimedean analogue. Thus, let $A$ be an algebra over a complete non-archimedean valued field $K$. We say that $A$ has UBAT (a unique Banach algebra topology) if each two (non-archimedean) Banach algebra norms on $A$ are equivalent. Our problem is to find reasonable conditions on $A$ implying the UBAT property.

It is known [3] that in the non-archimedean case semi-simple algebras (even commutative fields) may fail to have UBAT. In fact, we have

1.1 EXAMPLE. Let $p$ be a prime. Let $\mathbb{C}_p$ be the completion (with respect to the natural valuation $| |$) of the algebraic closure of the field $\mathbb{Q}_p$ of the $p$-adic numbers. Then $(\mathbb{C}_p, | |)$ is a valued field and a $\mathbb{Q}_p$-Banach algebra. There exists a valuation $| |'$ on $\mathbb{C}_p$, not equivalent to $| |$, for which $(\mathbb{C}_p, | |')$ is also a $\mathbb{Q}_p$-Banach algebra.
PROOF. It is well known that $\mathbb{C}$ is algebraically closed. Let $I$ be a maximal set of algebraically independent elements over $\mathbb{Q}$. Then $\mathbb{Q}_p \subset \mathbb{Q}_p(I) \subset \mathbb{C}_p$, $\mathbb{C}_p$ is the algebraic closure of $\mathbb{Q}_p(I)$, $I \neq \emptyset$. Fix $x \in I$, and define $\sigma : \mathbb{Q}_p(I) \to \mathbb{Q}_p(I)$ by $\sigma(x) = px$ and $\sigma(y) = y$ for $y \in I$, $y \neq x$. Then $\sigma$ is an endomorphism $\mathbb{Q}_p(I) \to \mathbb{Q}_p(I)$ that can be extended to an endomorphism $\tilde{\sigma} : \mathbb{C}_p \to \mathbb{C}_p$. It is easy to see that $\tilde{\sigma}$ is also a $\mathbb{Q}_p$-algebra homomorphism. Define $| \cdot |'$ via

$$|x|^' := |\tilde{\sigma}(x)| \quad (x \in \mathbb{C}_p).$$

Then $| \cdot |'$ is not equivalent to $| \cdot |$ since $|x^n|^' = |p|^n|x^n|$, so there is no $c > 0$ for which $| \cdot |' \geq c | \cdot |$. The rest is obvious.

With 1.1 in mind it is rather surprising that we can prove that a $\mathbb{K}$-Banach algebra whose norm is multiplicative and that is not a field has UBAT (see 4.7). Further results are:

- Tate algebras without nilpotents $\neq 0$ have UBAT. (4.4)
- $L(E)$ has UBAT if $E$ is a well-behaved Banach space. (5.4).

For background information on non-archimedean fields, Banach spaces and algebras we refer to [5].

In the sequel $\mathbb{K}$ is a non-archimedean non-trivially valued complete field.

Instead of "$A$ is a $\mathbb{K}$-Banach algebra with respect to the norms $|| \cdot ||_1'$ and $|| \cdot ||_2$" we will sometimes use the expression "$(A, || \cdot ||_1', || \cdot ||_2)$ is bicomplete".

2. Algebras of functions.

Theorem 2.1 is more or less contained in [3].

Let $X$ be a nonempty set. For $f \in K^X$ set
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\[ \| f \|_\infty := \sup\{|f(x)| : x \in X\} \] (possibly \( \infty \)). A function algebra is a K-algebra that is, for some \( X \), (algebraically isomorphic to) a sub-algebra of \( \mathcal{K}^X \). Without much effort we can prove

2.1 THEOREM. Let \( F \) be a function algebra. Then

(i) \( F \) has UBAT

(ii) If \( \| \cdot \| \) is a Banach algebra norm on \( F \) then

\[ \| \cdot \| \geq \| \cdot \|_\infty. \]

PROOF. Let \( \| \cdot \| \) be a Banach algebra norm on \( F \). Let \( a \in X \). The map

\[ f \mapsto f(a) \quad (f \in F) \]

is a homomorphism: \( F \to \mathcal{K} \), so by [5] it has norm \( \leq 1 \):

\[ |f(a)| \leq \| f \|. \] It follows that \( \| \| \leq \| \| \). Now let \( \| \|_1 \) and \( \| \|_2 \) be two Banach algebra norms on \( F \). We prove that the identity:

\[ (F, \| \|_1) \to (F, \| \|_2) \]

is continuous. Let \( f, f_1, f_2, \ldots \in F \) such that

\[ \| f_n \|_1 \to 0, \quad \| f_n - f \|_2 \to 0. \] By the foregoing, \( \| f_n \|_\infty \to 0, \quad \| f_n - f \|_\infty \to 0 \),

so \( f = 0 \). Continuity follows after applying the closed graph theorem.

3. The separating seminorm.

(This is a non-archimedean version of [4], (2.5.1))

3.1 DEFINITION. Let \( \| \|_1 \) and \( \| \|_2 \) be norms on a K-vector space \( E \).

The function \( \Delta : E \to \mathbb{R} \) defined by

\[ \Delta(s) := \inf\{\max(\| x \|_1, \| y \|_2) : x + y = s \} \quad (s \in E) \]

is called the separating seminorm of \( \| \|_1 \) and \( \| \|_2 \).

One easily checks that \( \Delta \) is the largest among the (non-archimedean) seminorms that are \( \leq \| \|_1 \) and \( \leq \| \|_2 \). As in [4] we have

3.2 LEMMA In case \( \| \|_1 \) and \( \| \|_2 \) are complete norms on \( E \) then:

\( \Delta \) is a norm \( \leftrightarrow \| \|_1 \sim \| \|_2 \).
3.3 LEMMA. Let $A$ be a normed $K$-algebra with respect to $\| \cdot \|_1$ and $\| \cdot \|_2$, and let $\Delta$ be its separating seminorm. Then $\ker \Delta$ is a two-sided ideal that is closed with respect to both norms. In fact, we have for $s, t \in A$:

\[
\Delta(st) \leq \Delta(s) \max(\|t\|_1, \|t\|_2)
\]

\[
\Delta(st) \leq \Delta(t) \max(\|s\|_1, \|s\|_2).
\]

The proofs of 3.2 and 3.3 are elementary and similar to the ones in [4], (2.5) and are omitted.

3.4 DEFINITION. For a linear subspace $D$ of an algebra $A$ that is normed by $\| \cdot \|_1, \| \cdot \|_2$ we set for $d \in D$:

\[
\Delta_D(d) := \inf\{\max(\|x\|_1, \|y\|_2) : x, y \in D, x + y = d\}
\]

($\Delta_D$ is the separating seminorm of the restriction of $\| \cdot \|_1$ and $\| \cdot \|_2$ to $D$).

We have the following elementary facts concerning the behaviour of $\Delta$ with respect to subalgebras and quotients:

3.5 LEMMA. With the notations as above we have

(i) $\Delta_D \geq \Delta|D$,

(ii) Let $D$ be a left ideal, then for $x \in A, t \in D$:

\[
\Delta_D(xt) \leq \Delta(x) \max(\|t\|_1, \|t\|_2)
\]

\[
\Delta_D(xt) \leq \max(\|x\|_1, \|x\|_2\cdot \Delta_D t),
\]

so $\ker \Delta_D$ is a left ideal in $A$, satisfying

($\ker \Delta$) $\cdot D \subset \ker \Delta_D \subset \ker \Delta \cap D$.

PROOF. (i) For $t \in D$ we have $\Delta(t) = \inf\{\max(\|x\|_1, \|y\|_2) : x + y = t, x, y \in A\} \leq \inf\{\max(\|x\|_1, \|y\|_2) : x + y = t, x, y \in D\} = \Delta_D(t)$. 

(ii) $\Delta_D(xt) = \inf_{z \in D} \max(|| z ||_1, || xt-z ||_2) \leq \inf_{y \in A} \max(|| yt ||_1, || xt-yt ||_2) \\
\leq \inf_{y \in A} \max(|| yt ||_1, || x-yt ||_2, || t ||_2) \leq \Delta(x) \max(|| t ||_1, || t ||_2).$

Also, $\Delta_D(xt) = \inf_{z \in D} \max(|| z ||_1, || xt-z ||_2) \leq \\
\leq \inf_{d \in D} \max(|| x-d ||_1, || xt-xd ||_2) \leq \max(|| x ||_1, || x ||_2) \cdot \Delta_D(t).$

3.6 LEMMA. Let $(A, || ||_1, || ||_2)$ be bicomplete. Suppose $D$ is a closed linear subspace with respect to both $|| ||_1$ and $|| ||_2$, and suppose that the quotient norms on $A/D$ are equivalent. Then $\text{Ker} \Delta \subseteq D$. 

PROOF. Let $\Delta(x) = 0$ for some $x \in A$. Then there are $x_1, x_2, \ldots$ in $A$ such that $||x-x_n||_1 \rightarrow 0$, $||x_n||_2 \rightarrow 0$. Let $\pi : A \rightarrow A/D$ be the quotient map. Then $\lim \pi(x_n) = \pi(x)$ for the first quotient norm and $\lim \pi(x_n) = 0$ for the second one. Hence, $\pi(x) = 0$ i.e., $x \in D.$

A subset of a $K$-algebra $A$ is called universally closed if it is closed with respect to each Banach algebra topology on $A$. (In case $A$ has no Banach algebra topology then, by definition, each subset of $A$ is universally closed). Examples of universally closed sets are

(i) $\emptyset$, $A$, singletons, finite dimensional linear subspaces.
(ii) For each set $X \subseteq A$ its commutant $X' := \{y \in A : yx = xy \text{ for all } x \in X\}$, in particular, the center of $A$.
(iii) For each $X \subseteq A$ the left and right annihilator of $X$:
$X^L := \{y \in A : yx = 0 \text{ for all } x \in X\}$
$X^R := \{y \in A : yx = 0 \text{ for all } x \in X\}$.
(iv) For each idempotent $e$ of $A$ the left ideal $eA$, the right ideal $eA$, the subalgebra $eA$. 
(v) Maximal modular left, right, two-sided ideals.

(vi) If $A$ is unitary, the set of the non-invertible elements of $A$.

We proceed by stating some corollaries of the lemmas 3.5, and 3.6.

3.7 LEMMA. Let $(A, \| \cdot \|_1, \| \cdot \|_2)$ be bicomplete and let $e$ be an idempotent in $A$. Then

$\Delta|_{eA}$ is equivalent to $\Delta_{eA}$

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PROOF. For $s \in eA$ we have $\Delta_{eA}(s) = \Delta_{eA}(es) \leq (\text{bij } 3.5) \leq \max(\| e \|_1, \| e \|_2) \Delta(s) \leq \max(\| e \|_1, \| e \|_2) \Delta_{eA}(s)$. The other proofs are similar.

3.8 LEMMA. Let $(A, \| \cdot \|_1, \| \cdot \|_2)$ be bicomplete, and let $I$ be a universally closed left ideal, that, as a $K$-algebra, has UBAT. Then $\text{Ker } \Delta \subset \frac{1}{I}$.

PROOF. $(I, \| \cdot \|_1, \| \cdot \|_2)$ is bicomplete, $I$ has UBAT, so $\| \cdot \|_1 \preceq \| \cdot \|_2$ on $I$. Thus $\Delta_{\frac{1}{I}}$ is a norm: $\text{Ker } \Delta_{\frac{1}{I}} = (0)$. By 3.5, $(\text{Ker } \Delta) \cdot I = (0)$, so $\text{Ker } \Delta \subset \frac{1}{I}$.

3.9 THEOREM. Let $I$ be a universally closed two-sided ideal in a $K$-algebra $A$. Suppose that $I \cap \frac{1}{I} \cap I = (0)$ (this is true, for example, if for any two-sided ideal $J$ in $A$, $J^2 = (0)$ implies $J = (0)$). Then, if $I$, $A/I$ have UBAT then so has $A$.

PROOF. By 3.6 and 3.5, if $(A, \| \cdot \|_1, \| \cdot \|_2)$ is bicomplete then $\text{Ker } \Delta \subset I \cap \frac{1}{I} \cap I = (0)$ : $\Delta$ is a norm, so $\| \cdot \|_1 \preceq \| \cdot \|_2$.

3.10 THEOREM. ([3], (1.1)) Let $A$ be a $K$-algebra. Suppose the intersection of the maximal modular left (right, two-sided) ideals
with finite codimension is zero. Then $A$ has UBAT.

**Proof.** Maximal modular left (right, two-sided) ideals are universally closed. Now apply 3.6. $\blacksquare$

4. **Topological zero divisors.**

In this section we will show that in many cases, for a bicomplete algebra, the ideal $\text{Ker } A$ consists only of topological zero divisors. (Compare [4] (2.5.6)). We first consider unitary algebras.

Let $A$ be a $K$-Banach algebra with identity $1$. Set

$$A^i := \{x \in A : x^{-1} \text{ exists} \}$$

Then $A^i$ is open.

Let us call $T(A) := A^i$.

An element $x \in A$ is called a **strong two-sided topological zero divisor** iff there exist $s_1, s_2, \ldots \in A$ such that $\inf \|s_n\| > 0$ and

$$\lim s_n x = \lim xs_n = 0.$$  

4.1 **Lemma.** Let $A$ be a $K$-Banach algebra with unit. Then

$$x \in T(A) \setminus A^i \Rightarrow x \text{ is a strong two-sided topological zero divisor.}$$

**Proof:** Let $x \in T(A) \setminus A^i$. Then there are $x_n \in A^i$ such that $\lim_{n \to \infty} x_n = x$.

Then we claim that $\|x_n^{-1}\|$ is unbounded. Suppose namely that

$$\sup_{n} \|x_n^{-1}\| = M < \infty \text{ then for } n, m \in \mathbb{N}.$$  

$$\|x_n^{-1} - x_m^{-1}\| = \|x_n^{-1}(x_n^{-1} - x_m^{-1})x_m^{-1}\| \leq M^2 \|x_n^{-1} - x_m^{-1}\|,$$  

so $y := \lim x_n^{-1}$ exists.

But then $xy = yx = 1 : x$ would be invertible, a contradiction.

By taking a suitable subsequence, assume $\lim_{n \to \infty} \|x_n^{-1}\| = \infty$.

There are $\lambda_n \in K$, $c_1, c_2 \in \mathbb{R}^+$ such that
Then $\lim_{n \to \infty} \lambda_n = \infty$ and
\[
\frac{x x_n^{-1}}{\lambda_n} = \frac{(x-x_n) x_n^{-1}}{\lambda_n} + \frac{1}{\lambda_n} + 0 \quad \text{if } n \to \infty
\]
hence $x_{s_n} \to 0$, where $s_n := \lambda_n^{-1} x_n^{-1}$.

Analogously,
\[
\frac{x x_n^{-1}}{\lambda_n} = \frac{x_n^{-1}(x-x_n)}{\lambda_n} + \frac{1}{\lambda_n} + 0 \quad \text{if } n \to \infty.
\]

Thus indeed, $x$ is a strong two-sided topological zero divisor in the above sense. □

For a bicomplete algebra with unit $(A, || | |)$ let us define $T^1(A)$ (resp. $T^2(A)$) to be the closure of $A^1$ with respect to $|| | |_1$ (resp. $|| | |_2$). We have

4.2 LEMMA. Let $(A, || | |_1, || | |_2)$ be a bicomplete algebra with unit. Then $\ker \Delta \subset T^1(A) \cap T^2(A)$.

PROOF. Choose $\lambda_1, \lambda_2, \ldots \in K$ such that $|\lambda_n| \geq n$ ($n \in \mathbb{N}$). Let $\Delta(x) = 0$ for some $x \in A$. Let $n \in \mathbb{N}$. Then $\Delta(\lambda_n x) = 0$, so there is a sequence $x_1, x_2, \ldots$ in $A$ such that $\lim_{k \to \infty} || x_k ||_1 = 0$, $\lim_{k \to \infty} || x_n - x_k ||_2 = 0$. So $1-x_k$ is invertible for large $k$. It follows that $1-\lambda_n x \in T^2(A)$, hence so is $x^{-1}_n$. Now $x = \lim_{n \to \infty} (x^{-1}_n)$ (with respect to $|| | |_2$), so $x \in T^2(A)$.

Similarly, $x \in T^1(A)$. □

Thus we have the following alternative.

4.3 THEOREM. Let $(A, || | |_1, || | |_2)$ be a bicomplete algebra with unit and with separating seminorm $\Delta$. Then we have either (i) or (ii):
(i) Ker $\Delta = A$, $A = T_1(A) = T_2(A)$. If an element of $A$ is not invertible then it is a strong two-sided topological zero divisor with respect to both norms.

(ii) Ker $\Delta$ is a proper ideal. Ker $\Delta$ consists only of strong two-sided topological zero divisors with respect to both norms.

NOTE. In contrast to the classical theory, case (i) can occur. In fact the separating seminorm of $| |$ and $| |'$ in Example (1.1) must be zero.

An example of case (i) in which $A$ is not a field can easily be made. Let $A := \mathbb{C} \times \mathbb{C}$ with pointwise operations. Let

$$\| (a_1, a_2) \| := \max(\|a_1\|, \|a_2\|) \quad ((a_1, a_2) \in A)$$

Then $(A, \| \|, \| \|')$ is a bicomplete $\mathbb{Q}$-algebra, is not a field.

A Tate algebra is a quotient of $K\{X_1, \ldots, X_n\}$, where the latter is the algebra of formal power series in $X_1, \ldots, X_n$ of which the coefficients tend to zero. (see [1] and [6]). We have the following application of 4.3.

4.4 THEOREM. Let $(A, \| \|_1, \| \|_2)$ be a bicomplete Tate algebra with separating seminorm $\Delta$. Then Ker $\Delta$ consists of only nilpotent elements. In particular, a Tate algebra without nilpotents $\neq 0$ has UBAT.

PROOF. Since $A$ is noetherian ([3] 1.5) each ideal in $A$ is universally closed. Let $P$ be a prime ideal of $A$. Then $A/P$ is a noetherian Banach
algebra with respect to both quotient norms, (again denoted by $|| \cdot ||_1$ and $|| \cdot ||_2$). Now $A/P$ has maximal ideals of finite codimension ([6] (4.5)), so the separating seminorm of the norms on $A/P$ is nonzero by 3.6. Theorem 4.2 (ii) tells us that its kernel consists only of topological zero divisors with respect to both norms.

On the other hand for any $x \in A/P$, $x \neq 0$ the map $t \mapsto tx$ ($t \in A/P$) is a bijection of $A/P$ onto the principal ideal $I$ generated by $x$ ($A/P$ has no zero divisors). $I$ is universally closed in $A/P$, the norms $tx \mapsto || t ||_1$ and $tx \mapsto || tx ||_1$ ($t \in A/P$) on $I$ are complete, the latter is majorized by the first. By the open mapping theorem they are equivalent: there is a $c > 0$ such that $|| tx ||_1 \geq c || t ||_1$ ($t \in A/P$). It follows that $x$ is not a topological zero with respect to $|| \cdot ||_1$.

Combining the results of the two previous paragraphs we conclude that $|| \cdot ||_1$ and $|| \cdot ||_2$ induce equivalent quotient norms on $A/P$. By 3.6, Ker $A$ is contained in the intersection of all prime ideals of $A$, hence consists only of nilpotents.

Next we turn to $K$-algebras $A$ without unit. Application of 4.3 to $A_1$ where $A_1$ is the usual unitary extension of $A$ does not seem to lead to interesting results. We follow a different path.

An element $x$ of a normed $K$-algebra $A$ is called a two-sided topological zero divisor if there are sequences $s_1, s_2, \ldots, t_1, t_2, \ldots$ such that $\inf_n || s_n || > 0$, $\inf_n || t_n || > 0$, $\lim_n s_n x = \lim_n x t_n = 0$.

We have the following analog of 4.3

4.5 THEOREM. Let $(A, || \cdot ||_1', || \cdot ||_2')$ be a bicomplete $K$-algebra without a unit, and with separating seminorm $\Delta$. Then we have either

(i) or (ii):

(i) Ker $\Delta = A$. $A$ has a one-sided unit.
(ii) Ker $\Delta$ consists only of two-sided topological zero
divisors with respect to both norms.

PROOF. We prove that if we have not (ii) then we have (i). Hence
suppose we have $s \in A$ for which $\Delta(s) = 0$ and such that $s$ is not a two-
sided topological zero divisor with respect to both norms. Without loss,
assume that the map $x \mapsto xs$ ($x \in A$) is a homeomorphism of $A$ onto $As$ with
respect to $\|\|_1$. Now let $A_1$ be the usual unitary extension of $A$.
Define for $i = 1, 2$
$$\| (\lambda, x) \|_i := \max(\| \lambda \|, \| x \|_i) \quad (\lambda \in K, x \in A)$$
Then $(A_1, \| \|_1, \| \|_2)$ is bicomplete and, by 3.5, $\Delta_{A_1}(s) = 0$. Since $A$
is a maximal ideal in $A_1$ of codimension 1, Ker $\Delta_{A_1} \neq A_1$ (3.6). Hence
by 4.3 (ii) there are $(\lambda_n, x_n) \in A_1$ such that $\lim_n s(\lambda_n, x_n) =$
$\lim_n (\lambda_n, x_n)s = 0$ in the sense of $\|\|_1$ and such that
$c := \inf \| (\lambda_n, x_n) \|_1 > 0$. If for some subsequence $\mu_1, \mu_2, \ldots$
of $\lambda_1, \lambda_2, \ldots$
we had $\lim_n \mu_n = 0$ then $\| y_n \|_1 \to 0$, $\| y_n \|_1 \to 0$, $\| y_n \|_i \geq c$ for some
subsequence $y_1, y_2, \ldots$ of $x_1, x_2, \ldots$, contradicting our assumption on $s$.
Hence we may assume $\inf_n |\lambda_n| > 0$. From
$$\lim_n (\lambda_n, x_n)s = 0 \quad \text{(in the sense of $\|\|_1$)}$$
we arrive at
$$\lim_n (\frac{x_n}{\lambda_n})s = 0 \quad \text{(in the sense of $\|\|_1$)}$$
It follows that $s \in \overline{As}$ (here the closure if meant with respect to $\|\|_1$).
But $As$ is closed, hence there is $e \in A$ for which $s = es$. For each
$x \in A$ we have $(xe-x)s = 0$ and since $s$ is no left zero divisor, $xe-x = 0$.
We conclude that $e$ is a one-sided unit for $A$. We proceed to prove that
$\Delta(e) = 0$ which will finish the proof. The algebra $eAe$ is universally
closed in A, e is a unit in eAe and s = es = ese ∈ eAe. We have
\[ \Delta(s) = 0, \text{ so by 3.7, } \Delta_{eAe}(s) = 0. \]
Since s is not a left topological zero divisor in A it is certainly not in eAe. Applying 4.3 to eAe we see that we are in case (i): \( \Delta_{eAe} = 0 \). It follows that \( \Delta(e) = 0 \).

In order to be able to conclude for certain algebras to be in case (i), we briefly look at K-algebras A without unit but having a one-sided unit e, say xe = x for all \( x \in A \). Consider \( e^\perp := \{ y \in A : ey = 0 \} \).

It is perfectly easy to see from \( x = (x-ex) + ex (x \in A) \) that \( A = e^\perp \otimes eAe \). Since eA = eAe is an algebra with a two-sided unit, we have \( e^\perp \neq (0) \). \( e^\perp \) is a two-sided ideal for which \( Ae^\perp = (0) \). In particular all products in \( e^\perp \) are zero. Therefore:

4.6 COROLLARY. Let \( (A, \| \|_1, \| \|_2) \) be a bicomplete K-algebra without unit. Suppose one of the following conditions holds.

(i) A is commutative.

(ii) A has no one-sided unit.

(iii) For a two-sided ideal \( J \) in \( A \), \( J^2 = (0) \) implies \( J = (0) \).

(iv) \( \perp A = (0), A^\perp = (0) \).

Then \( \ker \Delta \) contains only two-sided topological zero divisors with respect to both norms.

An application:

4.7 THEOREM. Let \( (A, \| \|) \) be a K-Banach algebra whose norm is multiplicative. If A is not a (skew) field then A has UBAT.

PROOF. Let \( \| \|' \) be some Banach algebra norm on A and let \( \Lambda \) be the separating seminorm of \( \| \| \) and \( \| \|' \). Since \( \| \| \) is multiplicative, A has no topological zero divisors with respect to \( \| \| \), except 0. If A has no unit, apply 4.6 (use (iii) or (iv)) to arrive at \( \ker \Delta = (0) \).
If $A$ has a unit we may use 4.3: case (i) would imply that $A$ is a (skew) field which is forbidden and case (ii) leads again to $\text{Ker } A = \{0\}$.

5. The uniqueness of the norm topology of $L(E)$.

In this section $E$ is a $K$-Banach space, $L(E)$ is the $K$-algebra of all continuous linear operators $E \to E$, and $A$ is a $K$-Banach algebra.

Let $E$ be a (left) $A$-module with structure map $(a, \xi) \mapsto a\xi$ ($a \in A, \xi \in E$). We say that $E$ is 2-fold transitive if for each $\xi_1, \xi_2, \eta_1, \eta_2 \in E$, where $\xi_1, \xi_2$ are linearly independent, there is a $c A$ such that $a\xi_1 = \eta_1, a\xi_2 = \eta_2$.

By the density lemma of Jacobson we then have $n$-fold transitivity for each $n \in \mathbb{N}$ i.e., if $\xi_1, \ldots, \xi_n \in E$ are linearly independent and $\eta_1, \ldots, \eta_n \in E$ then there exists a $c A$ such that $a\xi_i = \eta_i$ ($i = 1, \ldots, n$).

The following is essentially what remains of the proof of Johnsons theorem [2] in the non-archimedean case.

5.1 LEMMA. Let $E$ be a 2-fold transitive $A$-module such that the maps $\xi \mapsto a\xi$ ($\xi \in E$) are continuous for each $a \in A$. (Or, equivalently, in the corresponding representation $a \mapsto T_a$ all the $T_a$ are in $L(E)$). Then there exists $M > 0$ such that

$$||a\xi|| \leq M||a||||\xi|| \quad (a \in A, \xi \in E)$$

PROOF. By the uniform boundedness principle it suffices to show that the structure map $(a, \xi) \mapsto a\xi$ ($a \in A, \xi \in E$) is separately continuous i.e., we have to show that for each $\xi \in E$ the map $a \mapsto a\xi$ ($a \in A$) is continuous. By 2-fold transitivity (in fact, irreducibility) these maps are continuous either for all $\xi \in E, \xi \neq 0$ or for no such $\xi$.

First assume $\dim_E E = \infty$. We assume that $a \mapsto a\xi$ is continuous only in
case $\xi = 0$ and shall derive a contradiction. Choose independent 
$\xi_1, \xi_2, \ldots \in E$ such that $1 \leq \|\xi_i\| \leq 2$ for all $i$, and set 
$J_i := \{a \in A : a\xi_i = 0\}$ (i = 1, 2, ...). Each $J_i$ is a maximal modular left 
ideal of $A$ (if $x\xi_i = \xi_i$ then $x$ is an identity modulo $J_i$), hence closed 
in $A$. For each $m \geq 2$ we have 

$$(*) \quad A = (J_1 \cap J_2 \cap \ldots \cap J_{m-1}) + J_m$$

(By the $m$-fold transitivity there is $x \in A$ such that $x\xi_1 = x\xi_2 = \ldots = x\xi_{m-1} = 0$, $x\xi_m \neq 0$, hence $x \in (J_1 \cap J_2 \cap \ldots \cap J_{m-1})$, $x \notin J_m$. Now $J_m$ is maximal and 
$(*)$ follows). The addition map $(J_1 \cap \ldots \cap J_{m-1}) \times J_m \to A$ is continuous and 
surjective hence open by Banach's open mapping theorem. So there is $\gamma > 0$ 
such that we can write each $a \in A$ as $b + c$ where $b \in J_1 \cap \ldots \cap J_{m-1}$, $c \in J_m$ 
$satisfying $\|b\| < \gamma \|a\|$, $\|c\| < \gamma \|a\|$. With the help of this one can choose inductively 
$x_1, x_2, \ldots \in A$ such that for each $n \in \mathbb{N}$, $n \geq 2$

$$\|x_n\| \leq 2^{-n}; \quad x_n \in J_1 \cap \ldots \cap J_{n-1}; \quad \|x_n \xi_n\| \geq n + \sum_{i=2}^{n-1} x_i \xi_i \xi_n$$

using also the discontinuity at 0 of $x \mapsto x\xi_n$.

Set $z := \Sigma x_i \in A$. Since for $n \in \mathbb{N}$, $n \geq 2$ we have $\Sigma x_i \in J$ we get 

$$\|z\xi_n\| = \|x_n \xi_n\| \geq \|x_n \xi_n\| - \|\Sigma x_i \xi_i \xi_n\| \geq n.$$ 

Thus, $\lim_{n \to \infty} \|z\xi_n\| = \infty$. But the sequence $\xi_1, \xi_2, \ldots$ is bounded, so this 
conticts with the continuity of $\xi \mapsto z\xi$ ($\xi \in E$).

If, finally, $\dim_K E < \infty$ the map $a \mapsto a\xi$ ($a \in A$) can be decomposed:

$$A + A/I \sim E$$

where $A/I$ is equipped with the quotient norm and where $I := \{x \in A : x\xi = 0\}$. It follows that $a \mapsto a\xi$ ($a \in A$) is continuous.

5.2 THEOREM. Let $(B, \|\cdot\|_1, \|\cdot\|_2)$ be a bicomplete $K$-algebra, and 

suppose $E$ is a 2-fold transitive $B$-module such that the map
\[ \xi \mapsto b\xi \ (\xi \in E) \text{ is continuous for each } b \in B. \text{ Set} \]

\[ I_E := \{ x \in B : x\xi = 0 \text{ for all } \xi \in E \}. \]

Then \( \ker A \subseteq I_E \) where \( A \) is the separating seminorm of \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \).

\textbf{PROOF.} Let \( b \notin I_E \). Then there is \( \xi \in E \) such that \( b\xi \neq 0 \).

Lemma 5.1 yields the existence of \( M > 0 \) such that

\[ \| x\xi \| \leq M\| x \|_1 \| \xi \| \quad \| x\xi \| \leq M\| x \|_2 \| \xi \| \]

\((x \in B, \xi \in E)\).

The seminorm \( p : x \mapsto M^{-1}\| \xi \|^{-1}\| x\xi \| \ (x \in B) \) satisfies \( p \leq \| \cdot \|_1 \), \( p \leq \| \cdot \|_2 \), \( p(b) \neq 0 \). So \( 0 < p(b) \leq A(b) \). It follows that \( \ker A \subseteq I_E \).

5.3 \textbf{COROLLARY.} Let \( E \) have the property that for each independent \( \xi_1, \xi_2 \in E \) and \( \eta_1, \eta_2 \) there exists \( T \in L(E) \) such that \( T\xi_1 = \eta_1 \), \( T\xi_2 = \eta_2 \). Then \( L(E) \) has UBAT.

\textbf{PROOF.} \( E \) is a 2-fold transitive \( L(E) \)-module under \((T, \xi) \mapsto T\xi \ (T \in L(E), \xi \in E) \), satisfying the continuity condition of 5.2. \( I_E = \{ T \in L(E) : T\xi = 0 \text{ for all } \xi \in E \} = \{ 0 \} \). Hence for each two Banach algebra norms the separating seminorm is a norm, so the norms are equivalent.

Finally we indicate a class of Banach spaces \( E \) for which \( L(E) \) has UBAT.

For the notions used below see [5].

5.4 \textbf{THEOREM.} Let \( E \) be a K-Banach space. Each of the following conditions implies that \( L(E) \) has a unique Banach algebra topology.

(i) \( K \) is spherically complete.

(ii) \( E \) has a base. (In particular, \( E \) is of countable type.)

(iii) \( E \) is the dual of some K-Banach space.

(iv) \( E \) is spherically complete.
PROOF. We shall first prove: if the elements of the dual $E'$ separate the points of $E$ then $l(E)$ has UBAT, which takes care of (i), (ii) and (iii). Let $\xi_1, \xi_2 \in E$ be independent and $\eta_1, \eta_2 \in E$. There are $f, g \in E'$ such that $f_1(\xi_j) = \delta_{ij} \ (i, j \in \{1, 2\})$. The map

$$\xi \mapsto f_1(\xi) \eta_1 + f_2(\xi) \eta_2$$

is in $l(E)$ and sends $\xi_i$ into $\eta_i \ (i = 1, 2)$. Now apply 5.3.

Finally we prove (v). Let $\xi_1, \xi_2 \in E$ be independent and $\eta_1, \eta_2 \in E$.

Let $A$ be the map $\lambda_1 \xi_1 + \lambda_2 \xi_2 \mapsto \lambda_1 \eta_1 + \lambda_2 \eta_2 \ (\lambda_1, \lambda_2 \in K)$, $A : D \rightarrow E$ where $D$ is the subspace of $E$ spanned by $\xi_1$ and $\xi_2$. By the spherical completeness of $E$, $A$ can be extended to an element of $l(E)$. Now apply 5.3. \[\]

PROBLEM: Do there exist $K$-Banach spaces $E$ for which $l(E)$ admits inequivalent Banach algebra norms?

REFERENCES


