UNIQUENESS OF THE BANACH ALGEBRA TOPOLOGY

FOR NON-ARCHIMEDEAN ALGEBRAS

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1. Introduction.

An algebra $A$ over $\mathbb{R}$ or $\mathbb{C}$ is said to have a unique Banach algebra topology if any two Banach algebra norms on $A$ are equivalent. Johnson's theorem \[2\] is very satisfactory; it states that a semi-simple algebra over $\mathbb{R}$ or $\mathbb{C}$ has this property.

In this note we are concerned with the non-archimedean analogue. Thus, let $A$ be an algebra over a complete non-archimedean valued field $K$. We say that $A$ has UBAT (a unique Banach algebra topology) if each two (non-archimedean) Banach algebra norms on $A$ are equivalent. Our problem is to find reasonable conditions on $A$ implying the UBAT property.

It is known \[3\] that in the non-archimedean case semi-simple algebras (even commutative fields) may fail to have UBAT. In fact, we have

1.1 EXAMPLE. Let $p$ be a prime. Let $\mathbb{C}_p$ be the completion (with respect to the natural valuation $|\cdot|$) of the algebraic closure of the field $\mathbb{Q}_p$ of the $p$-adic numbers. Then $(\mathbb{C}_p, |\cdot|)$ is a valued field and a $\mathbb{Q}_p$-Banach algebra. There exists a valuation $|\cdot'|$ on $\mathbb{C}_p$, not equivalent to $|\cdot|$, for which $(\mathbb{C}_p, |\cdot'|)$ is also a $\mathbb{Q}_p$-Banach algebra.
PROOF. It is well known that $\mathbb{C}$ is algebraically closed. Let $I$ be a maximal set of algebraically independent elements over $\mathbb{Q}$. Then $\mathbb{Q}_p \subset \mathbb{Q}_p(I) \subset \mathbb{C}_p$, $\mathbb{C}_p$ is the algebraic closure of $\mathbb{Q}_p(I)$, $I \neq \emptyset$. Fix $x \in I$, and define $\sigma : \mathbb{Q}_p(I) \to \mathbb{Q}_p(I)$ by $\sigma(x) = px$ and $\sigma(y) = y$ for $y \in I$, $y \neq x$. Then $\sigma$ is an endomorphism $\mathbb{Q}_p(I) \to \mathbb{Q}_p(I)$ that can be extended to an endomorphism $\tilde{\sigma} : \mathbb{C}_p \to \mathbb{C}_p$. It is easy to see that $\tilde{\sigma}$ is also a $\mathbb{Q}_p$-algebra homomorphism. Define $| \cdot |'$ via

$$|x|' := |\tilde{\sigma}(x)| \quad (x \in \mathbb{C}_p).$$

Then $| \cdot |'$ is not equivalent to $| \cdot |$ since $|x^n|' = |p^n|x^n|$, so there is no $c > 0$ for which $| \cdot |' \geq c| \cdot |$. The rest is obvious. 

With 1.1 in mind it is rather surprising that we can prove that a $K$-Banach algebra whose norm is multiplicative and that is not a field has UBAT (see 4.7). Further results are:

- Tate algebras without nilpotents $\neq 0$ have UBAT. (4.4)
- $L(E)$ has UBAT if $E$ is a well-behaved Banach space. (5.4).

For background information on non-archimedean fields, Banach spaces and algebras we refer to [5].

In the sequel $K$ is a non-archimedean non-trivially valued complete field.

Instead of "$A$ is a $K$-Banach algebra with respect to the norms $\| \cdot \|_1$ and $\| \cdot \|_2$ we will sometimes use the expression

"($A$, $\| \cdot \|_1$, $\| \cdot \|_2$) is bicomplete".

2. **Algebras of functions.**

Theorem 2.1 is more or less contained in [3].

Let $X$ be a nonempty set. For $f \in K^X$ set
23-03

\[ \| f \|_\infty := \sup\{|f(x)| : x \in X\} \] (possibly \( \infty \)). A function algebra is a K-algebra that is, for some X, (algebraically isomorphic to) a sub-algebra of \( K^X \). Without much effort we can prove

2.1 THEOREM. Let \( F \) be a function algebra. Then

(i) \( F \) has UBAT

(ii) If \( \| \| \) is a Banach algebra norm on \( F \) then

\[ \| \| \geq \| \|_\infty . \]

PROOF. Let \( \| \| \) be a Banach algebra norm on \( F \). Let \( a \in X \). The map \( f \mapsto f(a) \) \( (f \in F) \) is a homomorphism: \( F \to K \), so by [5] it has norm \( \leq 1: \| f(a) \| \leq \| f \| \). It follows that \( \| \| \leq \| \|_\infty \). Now let \( \| \|_1 \) and \( \| \|_2 \) be two Banach algebra norms on \( F \). We prove that the identity: \( (F, \| \|_1) \to (F, \| \|_2) \) is continuous. Let \( f, f_1, f_2, \ldots \in F \) such that \( \| f_n \|_1 \to 0, \| f_n - f \|_2 \to 0 \). By the foregoing, \( \| f_n \|_\infty \to 0, \| f_n - f \|_\infty \to 0 \), so \( f = 0 \). Continuity follows after applying the closed graph theorem.

3. The separating seminorm.

(This is a non-archimedean version of [4], (2.5.1))

3.1 DEFINITION. Let \( \| \|_1 \) and \( \| \|_2 \) be norms on a K-vector space \( E \).

The function \( \Delta : E \to \mathbb{R} \) defined by

\[ \Delta(s) := \inf\{\max(\| x \|_1, \| y \|_2) : x + y = s\} \quad (s \in E) \]

is called the separating seminorm of \( \| \|_1 \) and \( \| \|_2 \).

One easily checks that \( \Delta \) is the largest among the (non-archimedean) seminorms that are \( \leq \| \|_1 \) and \( \leq \| \|_2 \). As in [4] we have

3.2 LEMMA In case \( \| \|_1 \) and \( \| \|_2 \) are complete norms on \( E \) then:

\( \Delta \) is a norm \( \leftrightarrow \| \|_1 \sim \| \|_2 \).
3.3 LEMMA. Let $A$ be a normed $K$-algebra with respect to $\| \cdot \|_1$ and $\| \cdot \|_2$, and let $\Delta$ be its separating seminorm. Then $\text{Ker } \Delta$ is a two-sided ideal that is closed with respect to both norms. In fact, we have for $s,t \in A$:

$$\Delta(st) \leq \Delta(s) \max(\| t \|_1, \| t \|_2)$$

$$\Delta(st) \leq \Delta(t) \max(\| s \|_1, \| s \|_2).$$

The proofs of 3.2 and 3.3 are elementary and similar to the ones in [4], (2.5) and are omitted.

3.4 DEFINITION. For a linear subspace $D$ of an algebra $A$ that is normed by $\| \cdot \|_1$, $\| \cdot \|_2$ we set for $d \in D$:

$$\Delta_D(d) := \inf \{ \max(\| x \|_1, \| y \|_2) : x, y \in D, x+y = d \}$$

($\Delta_D$ is the separating seminorm of the restriction of $\| \cdot \|_1$ and $\| \cdot \|_2$ to $D$).

We have the following elementary facts concerning the behaviour of $\Delta$ with respect to subalgebras and quotients:

3.5 LEMMA. With the notations as above we have

(i) $\Delta_D \geq \Delta|_D$, so $\text{Ker } \Delta_D \supset \text{Ker } \Delta \cap D$.

(ii) Let $D$ be a left ideal, then for $x \in A$, $t \in D$:

$$\Delta_D(xt) \leq \Delta(x) \max(\| t \|_1, \| t \|_2)$$

$$\Delta_D(xt) \leq \max(\| x \|_1, \| x \|_2 \Delta_D t), \text{ so } \text{Ker } \Delta_D \text{ is a left ideal in } A, \text{ satisfying }$$

$$\text{Ker } \Delta \cdot D \subset \text{Ker } \Delta_D \subset \text{Ker } \Delta \cap D.$$

PROOF. (i) For $t \in D$ we have $\Delta(t) = \inf \{ \max(\| x \|_1, \| y \|_2) : x+y = t, x,y \in A \} \leq \inf \{ \max(\| x \|_1, \| y \|_2) : x+y = t, x,y \in D \} = \Delta_D(t)$. 


(ii) $\Delta_D(xt) = \inf_{z \in D} \max(\|z\|_1, \|xt-z\|_2) \leq \inf_{y \in A} \max(\|yt\|_1, \|xt-yt\|_2)$
\[ \leq \inf_{y \in A} \max(\|y\|_1, \|t\|_1, \|x-y\|_2, \|t\|_2) \leq \Delta(x) \max(\|t\|_1, \|t\|_2). \]

Also, $\Delta_D(xt) = \inf_{z \in D} \max(\|z\|_1, \|xt-z\|_2) \leq \inf_{y \in A} \max(\|y\|_1, \|t\|_1, \|x-y\|_2, \|t\|_2) \leq \max(\|x\|_1, \|x\|_2) \cdot \Delta_D(t).$

3.6 LEMMA. Let $(A, \|\cdot\|_1, \|\cdot\|_2)$ be bicomplete. Suppose $D$ is a closed linear subspace with respect to both $\|\cdot\|_1$ and $\|\cdot\|_2$, and suppose that the quotient norms on $A/D$ are equivalent. Then $\ker \Delta \subset D$.

PROOF. Let $\Delta(x) = 0$ for some $x \in A$. Then there are $x_1, x_2, \ldots$ in $A$ such that $\|x-x_n\|_1 \to 0$, $\|x_n\|_2 \to 0$. Let $\pi : A \to A/D$ be the quotient map. Then $\lim \pi(x_n) = \pi(x)$ for the first quotient norm and $\lim \pi(x_n) = 0$ for the second one. Hence, $\pi(x) = 0$ i.e., $x \in D$. 

A subset of a $K$-algebra $A$ is called universally closed if it is closed with respect to each Banach algebra topology on $A$. (In case $A$ has no Banach algebra topology then, by definition, each subset of $A$ is universally closed). Examples of universally closed sets are

(i) $\emptyset$, $A$, singletons, finite dimensional linear subspaces.

(ii) For each set $X \subset A$ its commutant $X' := \{y \in A : yx = xy \text{ for all } x \in X\}$, in particular, the center of $A$.

(iii) For each $X \subset A$ the left and right annihilator of $X$:

$X^\perp := \{y \in A : yx = 0 \text{ for all } x \in X\}$

$X^{\perp} := \{y \in A : xy = 0 \text{ for all } x \in X\}$.

(iv) For each idempotent $e$ of $A$ the left ideal $Ae$, the right ideal $eA$, the subalgebra $eAe$. 
(v) Maximal modular left, right, two-sided ideals.

(vi) If A is unitary, the set of the non-invertible elements of A.

We proceed by stating some corollaries of the lemmas 3.5, and 3.6.

3.7 LEMMA. Let \((A, || \cdot ||_1, || \cdot ||_2)\) be bicomplete and let \(e\) be an idempotent in A. Then \(A|eA\) is equivalent to \(A_{eA}\)
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\(A|eA\) is equivalent to \(A_{eA}\).

PROOF. For \(s \in eA\) we have \(\Delta_{eA}(s) = \Delta_{eA}(es) \leq (bij 3.5) \leq \max(||e||_1, ||e||_2) \Delta(s) \leq \max(||e||_1, ||e||_2) \Delta_{eA}(s)\). The other proofs are similar.

3.8 LEMMA. Let \((A, || \cdot ||_1, || \cdot ||_2)\) be bicomplete, and let I be a universally closed left ideal, that, as a K-algebra, has UBAT.

Then \(\ker A \subseteq \frac{1}{I}\).

PROOF. \((I, || \cdot ||_1, || \cdot ||_2)\) is bicomplete, I has UBAT, so \(|| \cdot ||_1 \psi || \cdot ||_2\) on I. Thus \(\Delta_{I}\) is a norm: \(\ker A_{\frac{1}{I}}\) (by 3.5, \((\ker A) \cdot I = (0))\) so \(\ker A \subseteq \frac{1}{I}\).

3.9 THEOREM. Let I be a universally closed two-sided ideal in a K-algebra A. Suppose that \(I \cap \frac{1}{I} \cap \frac{1}{I} = (0)\) (this is true, for example, if for any two-sided ideal \(J\) in A, \(J^2 = (0)\) implies \(J = (0))\). Then, if I, A/I have UBAT then so has A.

PROOF. By 3.6 and 3.5, if \((A, || \cdot ||_1, || \cdot ||_2)\) is bicomplete then \(\ker A \subseteq I \cap \frac{1}{I} \cap \frac{1}{I} = (0)\); A is a norm, so \(|| \cdot ||_1 \psi || \cdot ||_2\).

3.10 THEOREM. ([3], (1.1)) Let A be a K-algebra. Suppose the intersection of the maximal modular left (right, two-sided) ideals
with finite codimension is zero. Then $A$ has UBAT.

**PROOF.** Maximal modular left (right, two-sided) ideals are universally closed. Now apply 3.6. ■

4. **Topological zero divisors.**

In this section we will show that in many cases, for a bicomplete algebra, the ideal $\ker A$ consists only of topological zero divisors. (Compare [4] (2.5.6)). We first consider unitary algebras.

Let $A$ be a $K$-Banach algebra with identity 1. Set

$$A^i := \{x \in A : x^{-1} \text{ exists}\}$$

Then $A^i$ is open.

Let us call $T(A) := A^i$.

An element $x \in A$ is called a **strong two-sided topological zero divisor** iff there exist $s_1, s_2, \ldots \in A$ such that $\inf_n \|s_n\| > 0$ and

$$\lim_{n \to \infty} s_n x = \lim_{n \to \infty} x s_n = 0.$$ 

**4.1 LEMMA.** Let $A$ be a $K$-Banach algebra with unit. Then

$$x \in T(A) \setminus A^i \Rightarrow x \text{ is a strong two-sided topological zero divisor.}$$

**Proof:** Let $x \in T(A) \setminus A^i$. Then there are $x_n \in A^i$ such that $\lim_{n \to \infty} x_n = x$.

Then we claim that $\|x_n^{-1}\|$ is unbounded. Suppose namely that

$$\sup_n \|x_n^{-1}\| = M < \infty \text{ then for } n, m \in \mathbb{N}.$$ 

$$\|x_n^{-1} - x_m^{-1}\| = \|x_n^{-1}(x_n - x_m)x_m^{-1}\| \leq M^2 \|x_n - x_m\|, \text{ so } y := \lim_{n \to \infty} x_n^{-1} \text{ exists.}$$

But then $xy = yx = 1 : x$ would be invertible, a contradiction.

By taking a suitable subsequence, assume $\lim_{n \to \infty} \|x_n^{-1}\| = \infty$.

There are $\lambda_n \in \mathbb{K}, c_1, c_2 \in \mathbb{R}^+$ such that
Then \( \lim_{n \to \infty} \lambda_n = 0 \) and
\[
\frac{x_n^{-1}}{\lambda_n} = \frac{(x-x_n)x_n^{-1}}{\lambda_n} + \frac{1}{\lambda_n} + 0 \quad (\text{if } n \to \infty)
\]
hence \( x_n \to 0 \), where \( s_n := \lambda_n^{-1} x_n^{-1} \).

Analogously,
\[
s_n x = \frac{x_n^{-1} x}{\lambda_n} = \frac{x_n^{-1} (x-x_n)}{\lambda_n} + \frac{1}{\lambda_n} + 0 \quad (\text{if } n \to \infty).
\]

Thus indeed, \( x \) is a strong two-sided topological zero divisor in the above sense.

For a bicomplete algebra with unit \((A, \| \|_1, \| \|_2)\) let us define \( T_1(A) \) (resp. \( T_2(A) \)) to be the closure of \( A \) with respect to \( \| \|_1 \) (resp. \( \| \|_2 \)). We have

4.2 LEMMA. Let \((A, \| \|_1, \| \|_2)\) be a bicomplete algebra with unit.
Then \( \ker \Delta \subset T_1(A) \cap T_2(A) \).

PROOF. Choose \( \lambda_1, \lambda_2, \ldots \in K \) such that \( |\lambda_n| \geq n \) \((n \in \mathbb{N})\). Let \( \Delta(x) = 0 \) for some \( x \in A \). Let \( n \in \mathbb{N} \). Then \( \Delta(\lambda_n x) = 0 \), so there is a sequence \( x_1, x_2, \ldots \) in \( A \) such that \( \lim_{k \to \infty} \| x_k \|_1 = 0 \), \( \lim_{k \to \infty} \| \lambda_n x-x_k \|_2 = 0 \). So \( 1-x_k \)
is invertible for large \( k \). It follows that \( 1-\lambda_n x \in T_2(A) \), hence so is \( x^{-1} \). Now \( x = \lim_{n \to \infty} (x^{-1} \lambda_n) \) (with respect to \( \| \|_2 \)), so \( x \in T_2(A) \).

Similarly, \( x \in T_1(A) \).

Thus we have the following alternative.

4.3 THEOREM. Let \((A, \| \|_1, \| \|_2)\) be a bicomplete algebra with unit.
and with separating seminorm \( \Delta \). Then we have either (i) or (ii):
(i) \( \text{Ker} \, \Delta = A, \, A = T_1(A) = T_2(A) \). If an element of \( A \) is not invertible then it is a strong two-sided topological zero divisor with respect to both norms.

(ii) \( \text{Ker} \, \Delta \) is a proper ideal. \( \text{Ker} \, \Delta \) consists only of strong two-sided topological zero divisors with respect to both norms.

**NOTE.** In contrast to the classical theory, case (i) can occur. In fact the separating seminorm of \(| \cdot |\) and \(| \cdot |'\) in Example (1.1) must be zero.

An example of case (i) in which \( A \) is not a field can easily be made. Let \( A := \mathbb{C} \times \mathbb{C} \) with pointwise operations. Let
\[
\| (a_1, a_2) \| := \max(|a_1|, |a_2|) \quad ((a_1, a_2) \in A)
\]
\[
\| (a_1, a_2) \|' := \max(|a_1'|, |a_2'|) \quad ((a_1, a_2) \in A)
\]

Then \( (A, \| \cdot \|, \| \cdot \|') \) is a bicomplete \( \mathbb{Q}_p \)-algebra, is not a field.

The separating seminorm is zero. (Bij 3.5 (i), with \( D := (0) \times \mathbb{C}_p \), we have \( \Delta(0,1) = 0 \). Similarly, \( \Delta(1,0) = 0 \) so \( \Delta = 0 \).

A Tate algebra is a quotient of \( K[X_{1},...,X_{n}] \), where the latter is the algebra of formal power series in \( X_{1},...,X_{n} \) of which the coefficients tend to zero. (see [1] and [6]). We have the following application of 4.3.

**4.4 THEOREM.** Let \((A, \| \cdot \|, \| \cdot \|')\) be a bicomplete Tate algebra with separating seminorm \( \Delta \). Then \( \text{Ker} \, \Delta \) consists of only nilpotent elements. In particular, a Tate algebra without nilpotents \( \neq 0 \) has UBAT.

**PROOF.** Since \( A \) is noetherian ([3] 1.5) each ideal in \( A \) is universally closed. Let \( \mathcal{P} \) be a prime ideal of \( A \). Then \( A/\mathcal{P} \) is a noetherian Banach
algebra with respect to both quotient norms, (again denoted by $\| \|_1$ and $\| \|_2$). Now $A/P$ has maximal ideals of finite codimension ([6] (4.5)), so the separating seminorm of the norms on $A/P$ is nonzero by 3.6. Theorem 4.2 (ii) tells us that its kernel consists only of topological zero divisors with respect to both norms.

On the other hand for any $x \in A/P$, $x \not= 0$ the map $t\mapsto tx$ ($t \in A/P$) is a bijection of $A/P$ onto the principal ideal $I$ generated by $x$ ($A/P$ has no zero divisors). $I$ is universally closed in $A/P$, the norms $tx \mapsto \| t \|_1$ and $tx \mapsto \| tx \|_1$ ($t \in A/P$) on $I$ are complete, the latter is majorized by the first. By the open mapping theorem they are equivalent: there is a $c > 0$ such that $\| tx \|_1 \geq c \| t \|_1$ ($t \in A/P$). It follows that $x$ is not a topological zero with respect to $\| \|_1$.

Combining the results of the two previous paragraphs we conclude that $\| \|_1$ and $\| \|_2$ induce equivalent quotient norms on $A/P$. By 3.6, $\ker \Delta$ is contained in the intersection of all prime ideals of $A$, hence consists only of nilpotents.

Next we turn to $K$-algebras $A$ without unit. Application of 4.3 to $A_1$ where $A_1$ is the usual unitary extension of $A$ does not seem to lead to interesting results. We follow a different path.

An element $x$ of a normed $K$-algebra $A$ is called a two-sided topological zero divisor if there are sequences $s_1, s_2, \ldots$, $t_1, t_2, \ldots$ such that $\inf_n \| s_n \| > 0$, $\inf_n \| t_n \| > 0$, $\lim_n s_n x = \lim_n x t_n = 0$.

We have the following analog of 4.3

4.5 THEOREM. Let $(A, \| \|_1', \| \|_2')$ be a bicomplete $K$-algebra without a unit, and with separating seminorm $\Delta$. Then we have either (i) or (ii):

(i) $\ker \Delta = A$. $A$ has a one-sided unit.
(ii) Ker A consists only of two-sided topological zero divisors with respect to both norms.

PROOF. We prove that if we have not (ii) then we have (i). Hence suppose we have $s \in A$ for which $\Delta(s) = 0$ and such that $s$ is not a two-sided topological zero divisor with respect to both norms. Without loss, assume that the map $x \mapsto xs$ ($x \in A$) is a homeomorphism of A onto $A_s$ with respect to $\| \|_1$. Now let $A_1$ be the usual unitary extension of A.

Define for $i = 1,2$

$$\| (\lambda, x) \|_i := \max(\| \lambda \|, \| x \|_1)$$

($\lambda \in K, x \in A$)

Then $(A_1, \| \|_1, \| \|_2)$ is bicomplete and, by 3.5, $\Delta_{A_1}(s) = 0$. Since $A$ is a maximal ideal in $A_1$ of codimension 1, $\text{Ker} \Delta_{A_1} \neq A_1$ (3.6). Hence by 4.3 (ii) there are $(\lambda_n, x_n) \in A_1$ such that

$$\lim_{n \to \infty} s(\lambda_n, x_n) = \lim_{n \to \infty} (\lambda_n, x_n)s = 0$$

in the sense of $\| \|_1$ and such that

$$c := \inf \| (\lambda_n, x_n) \|_1 > 0.$$ 

If for some subsequence $\mu_1, \mu_2, ...$ of $\lambda_1, \lambda_2, ...$

we had $\lim \mu_n = 0$ then $\| sy_n \|_1 \to 0$, $\| y_n s \|_1 \to 0$, $\| y_n \|_1 \geq c$ for some subsequence $y_1, y_2, ...$ of $x_1, x_2, ...$, contradicting our assumption on $s$.

Hence we may assume $\inf |\lambda_n| > 0$. From

$$\lim_{n \to \infty} (\lambda_n s + x_n s) = 0$$

(in the sense of $\| \|_1$)

we arrive at

$$\lim_{n \to \infty} (s + \frac{x_n}{\lambda_n} s) = 0$$

(in the sense of $\| \|_1$)

It follows that $s \in \overline{A_s}$ (here the closure if meant with respect to $\| \|_1$).

But $A_s$ is closed, hence there is $e \in A$ for which $s = es$. For each $x \in A$ we have $(xe-x)s = 0$ and since $s$ is no left zero divisor, $xe-x = 0$.

We conclude that $e$ is a one-sided unit for A. We proceed to prove that $\Delta(e) = 0$ which will finish the proof. The algebra $eAe$ is universally
closed in A, e is a unit in eAe and s = es = ese ∈ eAe. We have
$\Delta(s) = 0$, so by 3.7, $\Delta_{eAe}(s) = 0$. Since $s$ is not a left topological zero divisor in A it is certainly not in eAe. Applying 4.3 to eAe we see that we are in case (i): $\Delta_{eAe} = 0$. It follows that $\Delta(e) = 0$.

In order to be able to conclude for certain algebras to be in case (i), we briefly look at K-algebras A without unit but having a one-sided unit e, say xe = x for all $x \in A$. Consider $e^\perp = \{ y \in A : ey = 0 \}$. It is perfectly easy to see from $x = (x-ex) + ex (x \in A)$ that $A = e^\perp \otimes eA$. Since eA = eAe is an algebra with a two-sided unit, we have $e^\perp \neq (0)$. $e^\perp$ is a two-sided ideal for which $Ae^\perp = (0)$. In particular all products in $e^\perp$ are zero. Therefore:

4.6 COROLLARY. Let $(A, || | |, || | |, || | |)$ be a bicomplete K-algebra without unit. Suppose one of the following conditions holds.

(i) A is commutative.
(ii) A has no one-sided unit.
(iii) For a two-sided ideal J in A, $J^2 = (0)$ implies $J = (0)$.
(iv) $A^\perp = (0), A^\perp = (0)$.

Then Ker $\Delta$ contains only two-sided topological zero divisors with respect to both norms.

An application:

4.7 THEOREM. Let $(A, || | |)$ be a K-Banach algebra whose norm is multiplicative. If A is not a (skew) field then A has UBAT.

PROOF. Let || | | be some Banach algebra norm on A and let $\Delta$ be the separating seminorm of || | | and || | |. Since || | | is multiplicative, A has no topological zero divisors with respect to || | |, except 0. If A has no unit, apply 4.6 (use (iii) or (iv)) to arrive at Ker $\Delta = (0)$. 
If $A$ has a unit we may use 4.3: case (i) would imply that $A$ is a (skew) field which is forbidden and case (ii) leads again to $\ker A = \{0\}$.

5. **The uniqueness of the norm topology of $L(E)$**.

In this section $E$ is a $K$-Banach space, $L(E)$ is the $K$-algebra of all continuous linear operators $E \to E$, and $A$ is a $K$-Banach algebra.

Let $E$ be a (left) $A$-module with structure map $(a, \xi) \mapsto a \xi$ ($a \in A$, $\xi \in E$). We say that $E$ is **2-fold transitive** if for each $\xi_1, \xi_2, \eta_1, \eta_2 \in E$, where $\xi_1, \xi_2$ are linearly independent, there is a $c \in A$ such that $a \xi_1 = \eta_1$, $a \xi_2 = \eta_2$.

By the density lemma of Jacobson we then have $n$-fold transitivity for each $n \in \mathbb{N}$, i.e., if $\xi_1, \ldots, \xi_n \in E$ are linearly independent and $\eta_1, \ldots, \eta_n \in E$ then there exists a $c \in A$ such that $a \xi_i = \eta_i$ ($i = 1, \ldots, n$).

The following is essentially what remains of the proof of Johnson's theorem [2] in the non-archimedean case.

5.1 **LEMMA.** Let $E$ be a 2-fold transitive $A$-module such that the maps $\xi \mapsto a \xi$ ($\xi \in E$) are continuous for each $a \in A$. (Or, equivalently, in the corresponding representation $a \mapsto T_a$ all the $T_a$ are in $L(E)$). Then there exists $M > 0$ such that

$$||a \xi|| \leq M||a|| ||\xi|| \quad (a \in A, \xi \in E)$$

**PROOF.** By the uniform boundedness principle it suffices to show that the structure map $(a, \xi) \mapsto a \xi$ ($a \in A$, $\xi \in E$) is separately continuous i.e., we have to show that for each $\xi \in E$ the map $a \mapsto a \xi$ ($a \in A$) is continuous. By 2-fold transitivity (in fact, irreducibility) these maps are continuous either for all $\xi \in E$, $\xi \neq 0$ or for no such $\xi$.

First assume $\dim_k E = \infty$. We assume that $a \mapsto a \xi$ is continuous only in
case $\xi = 0$ and shall derive a contradiction. Choose independent $\xi_1, \xi_2, \ldots \in E$ such that $1 \leq ||\xi_i|| \leq 2$ for all $i$, and set $J_i := \{a \in A : a\xi_i = 0\}$ $(i = 1, 2, \ldots)$. Each $J_i$ is a maximal modular left ideal of $A$ (if $x\xi_i = \xi_i$ then $x$ is an identity modulo $J_i$), hence closed in $A$. For each $m \geq 2$ we have

$$(*) \quad A = (J_1 \cap J_2 \cap \ldots \cap J_{m-1}) + J_m$$

(By the $m$-fold transitivity there is $x \in A$ such that $x\xi_1 = x\xi_2 = \ldots = x\xi_{m-1} = 0$, $x\xi_m \neq 0$, hence $x \in (J_1 \cap J_2 \cap \ldots \cap J_{m-1})$, $x \notin J_m$. Now $J_m$ is maximal and

$(*)$ follows). The addition map $(J_1 \cap J_2 \cap \ldots \cap J_{m-1}) \times J_m \rightarrow A$ is continuous and surjective hence open by Banach's open mapping theorem. So there is $\gamma > 0$ such that we can write each $a \in A$ as $b+c$ where $b \in J_1 \cap \ldots \cap J_{m-1}$, $c \in J_m$ $\|b\| \leq \gamma \|a\|$, $\|c\| \leq \gamma \|a\|$. With the help of this one can choose inductively $x_1, x_2, \ldots \in A$ such that for each $n \in \mathbb{N}$, $n \geq 2$

$$\|x_n\| \leq 2^{-n}; \quad x_n \in J_1 \cap \ldots \cap J_{n-1}; \quad \|x_n\xi_n\| \geq n + \sum_{i=2}^{n-1} \|x_i\xi_n\|$$

using also the discontinuity at $0$ of $x \mapsto x\xi_n$.

Set $z := \sum x_i \xi_i \in A$. Since for $n \in \mathbb{N}$, $n \geq 2$ we have $\sum x_i \xi_i \in J_n$ we get

$$\|z\xi_n\| = \| (x_2 + \ldots + x_n) \xi_n\| \geq \|x_n\xi_n\| - \| \sum_{i=2}^{n-1} x_i \xi_n\| \geq n.$$ 

Thus, $\lim_{n \to \infty} \|z\xi_n\| = \infty$. But the sequence $\xi_1, \xi_2, \ldots$ is bounded, so this conflicts with the continuity of $x \mapsto z\xi$ ($\xi \in E$).

If, finally, $\dim_K E < \infty$ the map $a \mapsto a\xi$ ($a \in A$) can be decomposed:

$$A + A/I \rightarrow E$$

where $A/I$ is equipped with the quotient norm and where $I := \{x \in A : x\xi = 0\}$. It follows that $a \mapsto a\xi$ ($a \in A$) is continuous.

5.2 THEOREM. Let $(B, \| \|_1, \| \|_2)$ be a bicomplete $K$-algebra, and suppose $E$ is a 2-fold transitive $B$-module such that the map
\( \xi \mapsto b\xi \ (\xi \in E) \) is continuous for each \( b \in B \). Set

\[
I_E := \{ x \in B : x\xi = 0 \text{ for all } \xi \in E \}.
\]

Then \( \text{Ker } \Delta \subset I_E \) where \( \Delta \) is the separating seminorm of \( || \cdot ||_1 \) and \( || \cdot ||_2 \).

**Proof.** Let \( b \notin I_E \). Then there is \( \xi \in E \) such that \( b\xi \neq 0 \).

Lemma 5.1 yields the existence of \( M > 0 \) such that

\[
\begin{align*}
||x\xi|| &\leq M||x||_1 ||\xi|| \\
||x\xi|| &\leq M||x||_2 ||\xi||
\end{align*}
\]

(\( x \in B, \xi \in E \)).

The seminorm \( p : x \mapsto M^{-1}||\xi||^{-1}||x\xi|| \) (\( x \in B \)) satisfies \( p \leq || \cdot ||_1 \),

\( p \leq || \cdot ||_2 \), \( p(b) \neq 0 \). So \( 0 < p(b) \leq \Delta(b) \). It follows that \( \text{Ker } \Delta \subset I_E \).

5.3 **Corollary.** Let \( E \) have the property that for each independent \( \xi_1, \xi_2 \in E \) and \( \eta_1, \eta_2 \) there exists \( T \in L(E) \) such that \( T\xi_1 = \eta_1 \),

\( T\xi_2 = \eta_2 \). Then \( L(E) \) has UBAT.

**Proof.** \( E \) is a 2-fold transitive \( L(E) \)-module under \( (T, \xi) \mapsto T\xi \) \( (T \in L(E), \xi \in E) \), satisfying the continuity condition of 5.2. \( I_E = \{ T \in L(E) : T\xi = 0 \text{ for all } \xi \in E \} = \{0\} \). Hence for each two Banach algebra norms the separating seminorm is a norm, so the norms are equivalent.

Finally we indicate a class of Banach spaces \( E \) for which \( L(E) \) has UBAT.

For the notions used below see \([5]\).

5.4 **Theorem.** Let \( E \) be a \( K \)-Banach space. Each of the following conditions implies that \( L(E) \) has a unique Banach algebra topology.

(i) \( K \) is spherically complete.

(ii) \( E \) has a base. (In particular, \( E \) is of countable type.)

(iii) \( E \) is the dual of some \( K \)-Banach space.

(iv) \( E \) is spherically complete.
PROOF. We shall first prove: if the elements of the dual $E'$ separate the points of $E$ then $l(E)$ has UBAT, which takes care of (i), (ii) and (iii). Let $\xi_1, \xi_2 \in E$ be independent and $\eta_1, \eta_2 \in E$. There are $f, g \in E'$ such that $f_1(\xi_j) = \delta_{1j}$ ($i, j \in \{1, 2\}$). The map

$$\xi \mapsto f_1(\xi) \eta_1 + f_2(\xi) \eta_2$$

is in $l(E)$ and sends $\xi_i$ into $\eta_i$ ($i = 1, 2$). Now apply 5.3.

Finally we prove (v). Let $\xi_1, \xi_2 \in E$ be independent and $\eta_1, \eta_2 \in E$. Let $A$ be the map $\lambda_1 \xi_1 + \lambda_2 \xi_2 \mapsto \lambda_1 \eta_1 + \lambda_2 \eta_2$ ($\lambda_1, \lambda_2 \in K$), $A : D \rightarrow E$ where $D$ is the subspace of $E$ spanned by $\xi_1$ and $\xi_2$. By the spherical completeness of $E$, $A$ can be extended to an element of $l(E)$. Now apply 5.3.

PROBLEM: Do there exist $K$-Banach spaces $E$ for which $l(E)$ admits inequivalent Banach algebra norms?

REFERENCES


