UNIQUENESS OF THE BANACH ALGEBRA TOPOLOGY FOR NON-ARCHIMEDEAN ALGEBRAS

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1. Introduction.

An algebra A over $\mathbb{R}$ or $\mathbb{C}$ is said to have a unique Banach algebra topology if any two Banach algebra norms on A are equivalent. Johnson's theorem [2] is very satisfactory; it states that a semi-simple algebra over $\mathbb{R}$ or $\mathbb{C}$ has this property.

In this note we are concerned with the non-archimedean analogue. Thus, let A be an algebra over a complete non-archimedean valued field K. We say that A has UBAT (a unique Banach algebra topology) if each two (non-archimedean) Banach algebra norms on A are equivalent. Our problem is to find reasonable conditions on A implying the UBAT property.

It is known [3] that in the non-archimedean case semi-simple algebras (even commutative fields) may fail to have UBAT. In fact, we have

1.1 EXAMPLE. *Let p be a prime. Let $\mathbb{C}_p$ be the completion (with respect to the natural valuation $|\ \ |$) of the algebraic closure of the field $\mathbb{Q}_p$ of the p-adic numbers. Then $(\mathbb{C}_p, |\ \ |)$ is a valued field and a $\mathbb{Q}_p$-Banach algebra. There exists a valuation $|\ \ |'$ on $\mathbb{C}_p$, not equivalent to $|\ \ |$, for which $(\mathbb{C}_p, |\ \ |')$ is also a $\mathbb{Q}_p$-Banach algebra.*

PROOF. It is well known that $\mathbb{C}_p$ is algebraically closed. Let I be a maximal set of algebraically independent elements over $\mathbb{Q}_p$. Then $\mathbb{Q}_p \subset \mathbb{Q}_p(I) \subset \mathbb{C}_p$, $\mathbb{C}_p$ is the algebraic closure of $\mathbb{Q}_p(I)$, $I \neq \emptyset$. Fix $X \in I$, and define $\sigma : \mathbb{Q}_p(I) \to \mathbb{Q}_p(I)$ by $\sigma(X) = pX$ and $\sigma(Y) = Y$ for $Y \in I$, $Y \neq X$. Then σ is an endomorphism $\mathbb{Q}_p(I) \to \mathbb{Q}_p(I)$ that can be extended to an endomorphism $\tilde{\sigma} : \mathbb{C}_p \to \mathbb{C}_p$. It is easy to see that $\tilde{\sigma}$ is also a $\mathbb{Q}_p$-algebra homomorphism. Define $|\ \ |'$ via

$$|x|' := |\tilde{\sigma}(x)| \qquad (x \in \mathbb{C}_p) .$$

Then $|\ \ |'$ is not equivalent to $|\ \ |$ since $|X^n|' = |p|^n|X^n|$, so there is no $c > 0$ for which $|\ \ |' \geq c|\ \ |$. The rest is obvious. ∎

With 1.1 in mind it is rather surprising that we can prove that a K-Banach algebra whose norm is multiplicative and that is not a field has UBAT (see 4.7). Further results are:

Tate algebras without nilpotents $\neq 0$ have UBAT. (4.4)

$L(E)$ has UBAT if E is a well-behaved Banach space. (5.4).

For background information on non-archimedean fields, Banach spaces and algebras we refer to [5].

In the sequel K is a non-archimedean non-trivially valued complete field.

Instead of "A is a K-Banach algebra with respect to the norms $\|\ \ \|_1$, and $\|\ \ \|_2$" we will sometimes use the expression "$(A, \|\ \ \|_1, \|\ \ \|_2)$ is bicomplete".

2. Algebras of functions.

Theorem 2.1 is more or less contained in [3].

Let X be a nonempty set. For $f \in K^X$ set

$\| f \|_\infty := \sup\{|f(x)| : x \in X\}$ (possibly ∞). A <u>function algebra</u> is a K-algebra that is, for some X, (algebraically isomorphic to) a sub-algebra of K^X. Without much effort we can prove

2.1 THEOREM. *Let F be a function algebra. Then*

(i) F has UBAT

(ii) *If* $\| \ \|$ *is a Banach algebra norm on F then* $\| \ \| \geq \| \ \|_\infty$.

PROOF. Let $\| \ \|$ be a Banach algebra norm on F. Let $a \in X$. The map $f \mapsto f(a)$ $(f \in F)$ is a homomorphism: $F \to K$, so by [5] it has norm ≤ 1: $|f(a)| \leq \| f \|$. It follows that $\| \ \|_\infty \leq \| \ \|$. Now let $\| \ \|_1$ and $\| \ \|_2$ be two Banach algebra norms on F. We prove that the identity: $(F, \| \ \|_1) \to (F, \| \ \|_2)$ is continuous. Let $f, f_1, f_2, \ldots \in F$ such that $\| f_n \|_1 \to 0$, $\| f_n - f \|_2 \to 0$. By the foregoing, $\| f_n \|_\infty \to 0$, $\| f_n - f \|_\infty \to 0$, so $f = 0$. Continuity follows after applying the closed graph theorem. ∎

3. *The separating seminorm.*

(This is a non-archimedean version of [4], (2.5.1))

3.1 DEFINITION. *Let* $\| \ \|_1$ *and* $\| \ \|_2$ *be norms on a K-vector space E. The function* $\Delta : E \to \mathbb{R}$ *defined by*

$$\Delta(s) := \inf\{\max(\| x \|_1, \| y \|_2) : x+y = s\} \qquad (s \in E)$$

is called the separating seminorm of $\| \ \|_1$ *and* $\| \ \|_2$.

One easily checks that Δ is the largest among the (non-archimedean) seminorms that are $\leq \| \ \|_1$ and $\leq \| \ \|_2$. As in [4] we have

3.2 LEMMA *In case* $\| \ \|_1$ *and* $\| \ \|_2$ *are complete norms on E then*:

Δ *is a norm* $\Leftrightarrow \| \ \|_1 \sim \| \ \|_2$.

3.3 LEMMA. *Let A be a normed K-algebra with respect to $\| \ \|_1$ and $\| \ \|_2$, and let Δ be its separating seminorm. Then Ker Δ is a two-sided ideal that is closed with respect to both norms. In fact, we have for* $s,t \in A$:

$$\Delta(st) \le \Delta(s) \max(\|t\|_1, \|t\|_2)$$
$$\Delta(st) \le \Delta(t) \max(\|s\|_1, \|s\|_2).$$

The proofs of 3.2 and 3.3 are elementary and similar to the ones in [4], (2.5) and are omitted.∎

3.4 DEFINITION. *For a linear subspace D of an algebra A that is normed by $\| \ \|_1$, $\| \ \|_2$ we set for* $d \in D$:

$$\Delta_D(d) := \inf\{\max(\|x\|_1, \|y\|_2) : x,y \in D, x+y = d\}$$

(Δ_D *is the separating seminorm of the restriction of* $\| \ \|_1$ *and* $\| \ \|_2$ *to* D).

We have the following elementary facts concerning the behaviour of Δ with respect to subalgebras and quotients:

3.5 LEMMA. *With the notations as above we have*

(i) $\Delta_D \ge \Delta|D$, *so* Ker $\Delta_D \subset$ Ker $\Delta \cap D$.

(ii) *Let D be a left ideal, then for* $x \in A$, $t \in D$:

$$\Delta_D(xt) \le \Delta(x) \max(\|t\|_1, \|t\|_2)$$
$$\Delta_D(xt) \le \max(\|x\|_1, \|x\|_2)\ \Delta_D t), \text{ so}$$

Ker Δ_D *is a left ideal in* A, *satisfying*

$$(\text{Ker } \Delta) \cdot D \subset \text{Ker } \Delta_D \subset \text{Ker } \Delta \cap D.$$

PROOF. (i) For $t \in D$ we have $\Delta(t) = \inf\{\max(\|x\|_1, \|y\|_2) : x+y = t, x,y \in A\} \le \inf\{\max(\|x\|_1, \|y\|_2) : x+y = t, x,y \in D\} = \Delta_D(t)$.

(ii) $\Delta_D(xt) = \inf_{z\in D} \max(\|z\|_1, \|xt-z\|_2) \leq \inf_{y\in A} \max(\|yt\|_1, \|xt-yt\|_2)$

$\leq \inf_{y\in A} \max(\|y\|_1\|t\|_1, \|x-y\|_2\|t\|_2) \leq \Delta(x) \max(\|t\|_1, \|t\|_2)$.

Also, $\Delta_D(xt) = \inf_{z\in D} \max(\|z\|_1, \|xt-z\|_2) \leq$

$\leq \inf_{d\in D} \max(\|xd\|_1, \|xt-xd\|_2) \leq \max(\|x\|_1, \|x\|_2) \cdot \Delta_D(t)$. ∎

3.6 LEMMA. Let $(A, \| \ \|_1, \| \ \|_2)$ be bicomplete. Suppose D is a closed linear subspace with respect to both $\| \ \|_1$ and $\| \ \|_2$, and suppose that the quotient norms on A/D are equivalent. Then $\mathrm{Ker}\ \Delta \subset D$.

PROOF. Let $\Delta(x) = 0$ for some $x \in A$. Then there are $x_1, x_2, \ldots$ in A such that $\|x-x_n\|_1 \to 0$, $\|x_n\|_2 \to 0$. Let $\pi : A \to A/D$ be the quotient map. Then $\lim \pi(x_n) = \pi(x)$ for the first quotient norm and $\lim \pi(x_n) = 0$ for the second one. Hence, $\pi(x) = 0$ i.e., $x \in D$. ∎

A subset of a K-algebra A is called universally closed if it is closed with respect to each Banach algebra topology on A. (In case A has no Banach algebra topology then, by definition, each subset of A is universally closed). Examples of universally closed sets are

(i) $\emptyset$, A, singletons, finite dimensional linear subspaces.

(ii) For each set $X \subset A$ its commutant $X' := \{y \in A : yx = xy \text{ for all } x \in X\}$, in particular, the center of A.

(iii) For each $X \subset A$ the left and right annihilator of X:

$${}^{\perp}X := \{y \in A : yx = 0 \text{ for all } x \in X\}$$

$$X^{\perp} := \{y \in A : xy = 0 \text{ for all } x \in X\}.$$

(iv) For each idempotent e of A the left ideal Ae, the right ideal eA, the subalgebra eAe.

(v) Maximal modular left, right, two-sided ideals.

(vi) If A is unitary, the set of the non-invertible elements of A.

We proceed by stating some corollaries of the lemmas 3.5, and 3.6.

3.7 LEMMA. Let $(A, \| \ \|_1, \| \ \|_2)$ be bicomplete and let e be an idempotent in A. Then $\Delta|eA$ is equivalent to Δ_{eA}

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$\Delta|eAe$ is equivalent to Δ_{eAe}.

PROOF. For $s \in eA$ we have $\Delta_{eA}(s) = \Delta_{eA}(es) \leq$ (bij 3.5) $\leq \max(\|e\|_1, \|e\|_2)\ \Delta(s) \leq \max(\|e\|_1, \|e\|_2)\ \Delta_{eA}(s)$. The other proofs are similar. ∎

3.8 LEMMA. Let $(A, \| \ \|_1, \| \ \|_2)$ be bicomplete, and let I be a universally closed left ideal, that, as a K-algebra, has UBAT. Then Ker $\Delta \subset {}^{\perp}I$.

PROOF. $(I, \| \ \|_1, \| \ \|_2)$ is bicomplete, I has UBAT, so $\| \ \|_1 \sim \| \ \|_2$ on I. Thus Δ_I is a norm: Ker $\Delta_I = (0)$. By 3.5, (Ker Δ) $\cdot$ I $= (0)$ so Ker $\Delta \subset {}^{\perp}I$. ∎

3.9 THEOREM. Let I be a universally closed two-sided ideal in a K-algebra A. Suppose that $I \cap {}^{\perp}I \cap I^{\perp} = (0)$ (this is true, for example, if for any two-sided ideal J in A, $J^2 = (0)$ implies $J = (0)$). Then, if I, A/I have UBAT then so has A.

PROOF. By 3.6 and 3.5, if $(A, \| \ \|_1, \| \ \|_2)$ is bicomplete then Ker $\Delta \subset I \cap {}^{\perp}I \cap I^{\perp} = (0)$: Δ is a norm, so $\| \ \|_1 \sim \| \ \|_2$. ∎

3.10 THEOREM. ([3], (1.1)) Let A be a K-algebra. Suppose the intersection of the maximal modular left (right, two-sided) ideals *with finite codimension is zero. Then A has UBAT.*

PROOF. Maximal modular left (right, two-sided) ideals are universally closed. Now apply 3.6. ∎

4. Topological zero divisors.

In this section we will show that in many cases, for a bicomplete algebra, the ideal Ker Δ consists only of topological zero divisors. (Compare [4] (2.5.6)). We first consider unitary algebras.

Let A be a K-Banach algebra with identity 1. Set

$$A^i := \{x \in A : x^{-1} \text{ exists}\}$$

Then A^i is open.

Let us call $T(A) := \overline{A^i}$.

An element $x \in A$ is called a *strong two-sided topological zero divisor* iff there exist $s_1, s_2, \ldots \in A$ such that $\inf_n \|s_n\| > 0$ and $\lim_{n\to\infty} s_n x = \lim_{n\to\infty} x s_n = 0$.

4.1 LEMMA. *Let A be a K-Banach algebra with unit. Then*

$$x \in T(A) \setminus A^i \Rightarrow x \text{ is a strong two-sided topological zero divisor.}$$

Proof: Let $x \in T(A)\setminus A^i$. Then there are $x_n \in A^i$ such that $\lim_{n\to\infty} x_n = x$. Then we claim that $\|x_n^{-1}\|$ is unbounded. Suppose namely that $\sup_n \|x_n^{-1}\| = M < \infty$ then for $n,m \in \mathbb{N}$.

$\|x_n^{-1}-x_m^{-1}\| = \|x_n^{-1}(x_n-x_m)x_m^{-1}\| \le M^2\|x_n-x_m\|$, so $y := \lim x_n^{-1}$ exists.

But then $xy = yx = 1$: x would be invertible, a contradiction.

By taking a suitable subsequence, assume $\lim_{n\to\infty}\|x_n^{-1}\| = \infty$.

There are $\lambda_n \in K$, $c_1, c_2 \in \mathbb{R}^+$ such that

$$c_1 \leq \frac{\| x_n^{-1} \|}{|\lambda_n|} \leq c_2 \qquad (n \in \mathbb{N}).$$

Then $\lim\limits_{n\to\infty} |\lambda_n| = \infty$ and

$$\frac{xx_n^{-1}}{\lambda_n} = \frac{(x-x_n)x_n^{-1}}{\lambda_n} + \frac{1}{\lambda_n} \to 0 \qquad (\text{if } n \to \infty)$$

hence $xs_n \to 0$, where $s_n := \lambda_n^{-1} x_n^{-1}$.

Analogously,

$$s_n x = \frac{x_n^{-1}x}{\lambda_n} = \frac{x_n^{-1}(x-x_n)}{\lambda_n} + \frac{1}{\lambda_n} \to 0 \qquad (\text{if } n \to \infty) \ .$$

Thus indeed, x is a strong two-sided topological zero divisor in the above sense. ∎

For a bicomplete algebra with unit $(A, \| \ \|_1, \| \ \|_2)$ let us define $T_1(A)$ (resp. $T_2(A)$) to be the closure of A^i with respect to $\| \ \|_1$ (resp. $\| \ \|_2$). We have

4.2 LEMMA. *Let* $(A, \| \ \|_1, \| \ \|_2)$ *be a bicomplete algebra with unit.* *Then* $\operatorname{Ker} \Delta \subset T_1(A) \cap T_2(A)$.

PROOF. Choose $\lambda_1, \lambda_2, \ldots \in K$ such that $|\lambda_n| \geq n$ $(n \in \mathbb{N})$. Let $\Delta(x) = 0$ for some $x \in A$. Let $n \in \mathbb{N}$. Then $\Delta(\lambda_n x) = 0$, so there is a sequence $x_1, x_2, \ldots$ in A such that $\lim\limits_{k\to\infty} \| x_k \|_1 = 0$, $\lim\limits_{k\to\infty} \| \lambda_n x - x_k \|_2 = 0$. So $1-x_k$ is invertible for large k. It follows that $1-\lambda_n x \in T_2(A)$, hence so is $x-\lambda_n^{-1}$. Now $x = \lim\limits_{n\to\infty}(x-\lambda_n^{-1})$ (with respect to $\| \ \|_2$), so $x \in T_2(A)$. Similarly, $x \in T_1(A)$. ∎

Thus we have the following alternative.

4.3 THEOREM. *Let* $(A, \| \ \|_1, \| \ \|_2)$ *be a bicomplete algebra with unit.* *and with separating seminorm* Δ. *Then we have either* (i) *or* (ii):

(i) $\operatorname{Ker} \Delta = A$, $A = T_1(A) = T_2(A)$. If an element of A is not invertible then it is a strong two-sided topological zero divisor with respect to both norms.

(ii) $\operatorname{Ker} \Delta$ is a proper ideal. $\operatorname{Ker} \Delta$ consists only of strong two-sided topological zero divisors with respect to both norms .

NOTE. In contrast to the classical theory, case (i) can occur. In fact the separating seminorm of $|\ \ |$ and $|\ \ |'$ in Example (1.1) must be zero.

An example of case (i) in which A is not a field can easily be made. Let $A := \mathbb{C}_p \times \mathbb{C}_p$ with pointwise operations. Let

$$\left.\begin{aligned} \|(a_1,a_2)\| &:= \max(|a_1|,|a_2|) \\ \|(a_1,a_2)\|' &:= \max(|a_1|',|a_2|') \end{aligned}\right\} \quad ((a_1,a_2) \in A)$$

Then $(A, \|\ \ \|, \|\ \ \|')$ is a bicomplete $\mathbb{C}_p$-algebra, is not a field. The separating seminorm is zero. (Bij 3.5 (i), with $D := (0) \times \mathbb{C}_p$, we have $\Delta(0,1) = 0$. Similarly, $\Delta(1,0) = 0$ so $\Delta = 0$).

A Tate algebra is a quotient of $K\{X_1,\ldots,X_n\}$, where the latter is the algebra of formal power series in $X_1,\ldots,X_n$ of which the coefficients tend to zero. (see [1] and [6]). We have the following application of 4.3.

4.4 THEOREM. Let $(A, \|\ \ \|_1, \|\ \ \|_2)$ be a bicomplete Tate algebra with separating seminorm Δ. Then $\operatorname{Ker} \Delta$ consists of only nilpotent elements. In particular, a Tate algebra without nilpotents $\neq 0$ has UBAT.

PROOF. Since A is noetherian ([3] 1.5) each ideal in A is universally closed. Let P be a prime ideal of A. Then A/P is a noetherian Banach algebra with respect to both quotient norms, (again denoted by $\|\ \|_1$ and $\|\ \|_2$). Now A/P has maximal ideals of finite codimension ([6] (4.5)), so the separating seminorm of the norms on A/P is nonzero by 3.6. Theorem 4.2 (ii) tells us that its kernel consists only of topological zero divisors with respect to both norms.

On the other hand for any $x \in A/P$, $x \neq 0$ the map $t \mapsto tx$ $(t \in A/P)$ is a bijection of A/P onto the principal ideal I generated by x (A/P has no zero divisors). I is universally closed in A/P, the norms $tx \mapsto \|t\|_1$ and $tx \mapsto \|tx\|_1$ $(t \in A/P)$ on I are complete, the latter is majorized by the first. By the open mapping theorem they are equivalent: there is a $c > 0$ such that $\|tx\|_1 \geq c\|t\|_1$ $(t \in A/P)$. It follows that x is not a topological zero with respect to $\|\ \|_1$.

Combining the results of the two previous paragraphs we conclude that $\|\ \|_1$ and $\|\ \|_2$ *induce equivalent quotient norms on* A/P. By 3.6, Ker Δ is contained in the intersection of all prime ideals of A, hence consists only of nilpotents. ∎

Next we turn to K-algebras A without unit. Application of 4.3 to A_1 where A_1 is the usual unitary extension of A does not seem to lead to interesting results. We follow a different path.

An element x of a normed K-algebra A is called a <u>two-sided topological zero divisor</u> if there are sequences $s_1, s_2, \ldots$, $t_1, t_2, \ldots$ such that $\inf\|s_n\| > 0$, $\inf\|t_n\| > 0$, $\lim s_n x = \lim x t_n = 0$.

We have the following analog of 4.3

4.5 THEOREM. <u>Let</u> $(A, \|\ \|_1, \|\ \|_2)$ <u>be a bicomplete</u> K-<u>algebra without a unit, and with separating seminorm</u> Δ. <u>Then we have either</u> (i) <u>or</u> (ii):

(i) Ker Δ = A. A <u>has a one-sided unit</u>.

(ii) Ker Δ <u>consists only of two-sided topological zero divisors with respect to both norms.</u>

PROOF. We prove that if we have not (ii) then we have (i). Hence suppose we have $s \in A$ for which $\Delta(s) = 0$ and such that s is not a two-sided topological zero divisor with respect to both norms. Without loss, assume that the map $x \mapsto xs$ $(x \in A)$ is a homeomorphism of A onto As with respect to $\| \ \|_1$. Now let A_1 be the usual unitary extension of A . Define for i = 1,2

$$\|(\lambda,x)\|_i := \max(|\lambda|,\|x\|_i) \qquad (\lambda \in K,\ x \in A)$$

Then $(A_1,\| \ \|_1,\| \ \|_2)$ is bicomplete and, by 3.5, $\Delta_{A_1}(s) = 0$. Since A is a maximal ideal in A_1 of codimension 1, Ker $\Delta_{A_1} \neq A_1$ (3.6). Hence by 4.3 (ii) there are $(\lambda_n,x_n) \in A_1$ such that $\lim_{n\to\infty} s(\lambda_n,x_n) = \lim_{n\to\infty}(\lambda_n,x_n)s = 0$ in the sense of $\| \ \|_1$ and such that $c := \inf\|(\lambda_n,x_n)\|_1 > 0$. If for some subsequence $\mu_1,\mu_2,\ldots$ of $\lambda_1,\lambda_2,\ldots$ we had $\lim \mu_n = 0$ then $\|sy_n\|_1 \to 0$, $\|y_ns\|_1 \to 0$, $\|y_n\| \geq c$ for some subsequence $y_1,y_2,\ldots$ of $x_1,x_2,\ldots$, contradicting our assumption on s. Hence we may assume $\inf_n|\lambda_n| > 0$. From

$$\lim_{n\to\infty}(\lambda_n s + x_n s) = 0 \qquad \text{(in the sense of } \| \ \|_1)$$

we arrive at

$$\lim_{n\to\infty}(s + \frac{x_n}{\lambda_n} s) = 0 \qquad \text{(in the sense of } \| \ \|_1)$$

It follows that $s \in \overline{As}$ (here the closure if meant with respect to $\| \ \|_1$). But As is closed, hence there is $e \in A$ for which s = es. For each $x \in A$ we have $(xe-x)s = 0$ and since s is no left zero divisor, $xe-x = 0$. We conclude that e is a one-sided unit for A. We proceed to prove that $\Delta(e) = 0$ which will finish the proof. The algebra eAe is universally closed in A, e is a unit in eAe and s = es = ese $\in$ eAe. We have $\Delta(s) = 0$, so by 3.7, $\Delta_{eAe}(s) = 0$. Since s is not a left topological zero divisor in A it is certainly not in eAe. Applying 4.3 to eAe we see that we are in case (i): $\Delta_{eAe} = 0$. It follows that $\Delta(e) = 0$. ∎

In order to be able to conclude for certain algebras to be in case (i), we briefly look at K-algebras A without unit but having a one-sided unit e, say xe = x for all x $\in$ A. Consider $e^{\perp} := \{y \in A : ey = 0\}$. It is perfectly easy to see from $x = (x-ex) + ex$ ($x \in A$) that $A = e^{\perp} \oplus eA$. Since eA = eAe is an algebra with a two-sided unit, we have $e^{\perp} \neq (0)$. $e^{\perp}$ is a two-sided ideal for which $Ae^{\perp} = (0)$. In particular all products in $e^{\perp}$ are zero. Therefore:

4.6 COROLLARY. *Let $(A, \| \ \|_1, \| \ \|_2)$ be a bicomplete K-algebra without unit. Suppose one of the following conditions holds.*

(i) A *is commutative.*

(ii) A *has no one-sided unit.*

(iii) *For a two-sided ideal* J *in* A, $J^2 = (0)$ *implies* $J = (0)$.

(iv) ${}^{\perp}A = (0)$, $A^{\perp} = (0)$.

Then Ker Δ *contains only two-sided topological zero divisors with respect to both norms.*

An application:

4.7 THEOREM. *Let $(A, \| \ \|)$ be a K-Banach algebra whose norm is multiplicative. If* A *is not a (skew) field then* A *has* UBAT.

PROOF. Let $\| \ \|'$ be some Banach algebra norm on A and let Δ be the separating seminorm of $\| \ \|$ and $\| \ \|'$. Since $\| \ \|$ is multiplicative, A has no topological zero divisors with respect to $\| \ \|$, except 0. If A has no unit, apply 4.6 (use (iii) or (iv)) to arrive at Ker $\Delta = (0)$.

If A has a unit we may use 4.3: case (i) would imply that A is a (skew) field which is forbidden and case (ii) leads again to Ker $\Delta = (0)$. ∎

5. The uniqueness of the norm topology of $L(E)$.

In this section E is a K-Banach space, $L(E)$ is the K-algebra of all continuous linear operators $E \to E$, and A is a K-Banach algebra.

Let E be a (left) A-module with structure map $(a,\xi) \mapsto a\xi$ $(a \in A,\ \xi \in E)$. We say that E is 2-fold transitive if for each $\xi_1, \xi_2, \eta_1, \eta_2 \in E$, where ξ_1, ξ_2 are linearly independent, there is $a \in A$ such that $a\xi_1 = \eta_1$, $a\xi_2 = \eta_2$.
By the density lemma of Jacobson we then have n-fold transitivity for each $n \in \mathbb{N}$ i.e., if $\xi_1, \ldots, \xi_n \in E$ are linearly independent and $\eta_1, \ldots, \eta_n \in E$ then there exists $a \in A$ such that $a\xi_i = \eta_i$ $(i = 1, \ldots, n)$.

The following is essentially what remains of the proof of Johnsons theorem [2] in the non-archimedean case.

5.1 LEMMA. Let E be a 2-fold transitive A-module such that the maps $\xi \mapsto a\xi$ $(\xi \in E)$ are continuous for each $a \in A$. (Or, equivalently, in the corresponding representation $a \mapsto T_a$ all the T_a are in $L(E)$). Then there exists $M > 0$ such that

$$\|a\xi\| \le M\|a\|\ \|\xi\| \qquad (a \in A,\ \xi \in E)$$

PROOF. By the uniform boundedness principle it suffices to show that the structure map $(a,\xi) \mapsto a\xi$ $(a \in A,\ \xi \in E)$ is separately continuous i.e., we have to show that for each $\xi \in E$ the map $a \mapsto a\xi$ $(a \in A)$ is continuous. By 2-fold transitivity (in fact, irreducibility) these maps are continuous either for all $\xi \in E$, $\xi \neq 0$ or for no such ξ.

First assume $\dim_K E = \infty$. We assume that $a \mapsto a\xi$ is continuous only in case $\xi = 0$ and shall derive a contradiction. Choose independent $\xi_1, \xi_2, \ldots \in E$ such that $1 \le \|\xi_i\| \le 2$ for all i, and set $J_i := \{a \in A: a\xi_i = 0\}$ $(i = 1,2,\ldots)$. Each J_i is a maximal modular left ideal of A (if $x\xi_i = \xi_i$ then x is an identity modulo J_i), hence closed in A. For each $m \ge 2$ we have

$$(*) \qquad A = (J_1 \cap J_2 \cap \ldots \cap J_{m-1}) + J_m$$

(By the m-fold transitivity there is $x \in A$ such that $x\xi_1 = x\xi_2 = \ldots = x\xi_{m-1} = 0$, $x\xi_m \neq 0$, hence $x \in (J_1 \cap J_2 \cap \ldots \cap J_{m-1})$, $x \notin J_m$. Now J_m is maximal and (*) follows). The addition map $(J_1 \cap \ldots \cap J_{m-1}) \times J_m \to A$ is continuous and surjective hence open by Banach's open mapping theorem. So there is $\gamma > 0$ such that we can write each $a \in A$ as b+c where $b \in J_1 \cap \ldots \cap J_{m-1}$, $c \in J_m$ $\|b\| \le \gamma\|a\|$, $\|c\| \le \gamma\|a\|$. With the help of this one can choose inductively $x_1, x_2, \ldots \in A$ such that for each $n \in \mathbb{N}$, $n \ge 2$

$$\|x_n\| \le 2^{-n}; \; x_n \in J_1 \cap \ldots \cap J_{n-1}; \; \|x_n\xi_n\| \ge n + \left\|\sum_{i=2}^{n-1} x_i\xi_n\right\|$$

using also the discontinuity at 0 of $x \mapsto x\xi_n$.

Set $z := \sum_{i=2}^{\infty} x_i \in A$. Since for $n \in \mathbb{N}$, $n \ge 2$ we have $\sum_{i>n} x_i \in J$ we get
$$\|z\xi_n\| = \|(x_2 + \ldots + x_n)\xi_n\| \ge \|x_n\xi_n\| - \left\|\sum_{i=2}^{n-1} x_i\xi_n\right\| \ge n.$$
Thus, $\lim_{n\to\infty} \|z\xi_n\| = \infty$. But the sequence $\xi_1, \xi_2, \ldots$ is bounded, so this conflicts with the continuity of $\xi \mapsto z\xi$ $(\xi \in E)$.

If, finally, $\dim_K E < \infty$ the map $a \mapsto a\xi$ $(a \in A)$ can be decomposed:

$$A \to A/I \overset{\sim}{\to} E$$

where A/I is equipped with the quotient norm and where $I := \{x \in A: x\xi = 0\}$. It follows that $a \mapsto a\xi$ $(a \in A)$ is continuous.

5.2 THEOREM. <u>Let</u> $(B, \|\ \|_1, \|\ \|_2)$ <u>be a bicomplete K-algebra, and suppose</u> E <u>is a 2-fold transitive B-module such that the map</u>

$\xi \mapsto b\xi$ $(\xi \in E)$ *is continuous for each* $b \in B$. *Set*

$$I_E := \{x \in B : x\xi = 0 \text{ for all } \xi \in E\}.$$

Then $\operatorname{Ker} \Delta \subset I_E$ *where* Δ *is the separating seminorm of* $\| \ \|_1$ *and* $\| \ \|_2$.

PROOF. Let $b \notin I_E$. Then there is $\xi \in E$ such that $b\xi \neq 0$. Lemma 5.1 yields the existence of $M > 0$ such that

$$\left.\begin{aligned} \|x\xi\| &\le M\|x\|_1 \ \|\xi\| \\ \|x\xi\| &\le M\|x\|_2 \ \|\xi\| \end{aligned}\right\} \quad (x \in B,\ \xi \in E).$$

The seminorm $p : x \mapsto M^{-1}\|\xi\|^{-1}\|x\xi\|$ $(x \in B)$ satisfies $p \le \| \ \|_1$, $p \le \| \ \|_2$, $p(b) \neq 0$. So $0 < p(b) \le \Delta(b)$. It follows that $\operatorname{Ker} \Delta \subset I_E$. ∎

5.3 COROLLARY. *Let* E *have the property that for each independent* $\xi_1, \xi_2 \in E$ *and* η_1, η_2 *there exists* $T \in L(E)$ *such that* $T\xi_1 = \eta_1$, $T\xi_2 = \eta_2$. *Then* $L(E)$ *has* UBAT.

PROOF. E is a 2-fold transitive $L(E)$-module under $(T,\xi) \mapsto T\xi$ $(T \in L(E), \xi \in E)$, satisfying the continuity condition of 5.2. $I_E = \{T \in L(E) : T\xi = 0 \text{ for all } \xi \in E\} = (0)$. Hence for each two Banach algebra norms the separating seminorm is a norm, so the norms are equivalent. ∎

Fianlly we indicate a class of Banach spaces E for which $L(E)$ has UBAT. For the notions used below see [5].

5.4 THEOREM. *Let* E *be a* K-*Banach space. Each of the following conditions implies that* $L(E)$ *has a unique Banach algebra topology.*

(i) K *is spherically complete.*

(ii) E *has a base. (In particular,* E *is of countable type.)*

(iii) E *is the dual of some* K-*Banach space.*

(iv) E *is spherically complete.*

PROOF. We shall first prove: if the elements of the dual E' separate the points of E then $L(E)$ has UBAT, which takes care of (i), (ii) and (iii). Let $\xi_1,\xi_2 \in E$ be independent and $\eta_1,\eta_2 \in E$. There are $f,g \in E'$ such that $f_i(\xi_j) = \delta_{ij}$ $(i,j \in \{1,2\})$. The map

$$\xi \mapsto f_1(\xi)\eta_1 + f_2(\xi)\eta_2$$

is in $L(E)$ and sends ξ_i into η_i $(i = 1,2)$. Now apply 5.3.

Finally we prove (v). Let $\xi_1,\xi_2 \in E$ be independent and $\eta_1,\eta_2 \in E$. Let A be the map $\lambda_1\xi_1+\lambda_2\xi_2 \mapsto \lambda_1\eta_1+\lambda_2\eta_2$ $(\lambda_1,\lambda_2 \in K)$, $A : D \to E$ where D is the subspace of E spanned by ξ_1 and ξ_2. By the sperical completeness of E, A can be extended to an element of $L(E)$. Now apply 5.3. ∎

PROBLEM: *Do there exist K-Banach spaces E for which* $L(E)$ *admits inequivalent Banach algebra norms?*

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