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UNIQUENESS OF THE BANACH ALGEBRA TOPOLOGY FOR NON-ARCHIMEDEAN ALGEBRAS

W.H. Schikhof

1. Introduction.

An algebra $A$ over $\mathbb{R}$ or $\mathbb{C}$ is said to have a unique Banach algebra topology if any two Banach algebra norms on $A$ are equivalent. Johnson's theorem [2] is very satisfactory; it states that a semi-simple algebra over $\mathbb{R}$ or $\mathbb{C}$ has this property.

In this note we are concerned with the non-archimedean analogue. Thus, let $A$ be an algebra over a complete non-archimedean valued field $K$. We say that $A$ has UBAT (a unique Banach algebra topology) if each two (non-archimedean) Banach algebra norms on $A$ are equivalent. Our problem is to find reasonable conditions on $A$ implying the UBAT property.

It is known [3] that in the non-archimedean case semi-simple algebras (even commutative fields) may fail to have UBAT. In fact, we have

1.1 EXAMPLE. Let $p$ be a prime. Let $\mathbb{C}_p$ be the completion (with respect to the natural valuation $|\cdot|$) of the algebraic closure of the field $\mathbb{Q}_p$ of the $p$-adic numbers. Then $(\mathbb{C}_p, |\cdot|)$ is a valued field and a $\mathbb{Q}_p$-Banach algebra. There exists a valuation $|\cdot|$ on $\mathbb{C}_p$, not equivalent to $|\cdot|$, for which $(\mathbb{C}_p, |\cdot|')$ is also a $\mathbb{Q}_p$-Banach algebra.
PROOF. It is well known that $\mathbb{C}_p$ is algebraically closed. Let $I$ be a maximal set of algebraically independent elements over $\mathbb{Q}_p$. Then $\mathbb{Q}_p \subset \mathbb{Q}_p(I) \subset \mathbb{C}_p$. If $x \in I$, and define $\sigma : \mathbb{Q}_p(I) \rightarrow \mathbb{Q}_p(I)$ by $\sigma(x) = px$ and $\sigma(y) = y$ for $y \in I, y \neq x$. Then $\sigma$ is an endomorphism $\mathbb{Q}_p(I) \rightarrow \mathbb{Q}_p(I)$ that can be extended to an endomorphism $\tilde{\sigma} : \mathbb{C}_p \rightarrow \mathbb{C}_p$. It is easy to see that $\tilde{\sigma}$ is also a $\mathbb{Q}_p$-algebra homomorphism. Define $|\cdot|$ via

$$ |x|' := |\tilde{\sigma}(x)| \quad (x \in \mathbb{C}_p). $$

Then $|\cdot|$ is not equivalent to $|\cdot|$ since $|x^n|' = |p|^n|x^n|$, so there is no $c > 0$ for which $|\cdot|' \geq c|\cdot|$. The rest is obvious. 

With 1.1 in mind it is rather surprising that we can prove that a $K$-Banach algebra whose norm is multiplicative and that is not a field has UBAT (see 4.7). Further results are:

- Tate algebras without nilpotents $\neq 0$ have UBAT. (4.4)
- $L(E)$ has UBAT if $E$ is a well-behaved Banach space. (5.4).

For background information on non-archimedean fields, Banach spaces and algebras we refer to [5].

In the sequel $K$ is a non-archimedean non-trivially valued complete field.

Instead of "$A$ is a $K$-Banach algebra with respect to the norms $\| \cdot \|_1'$ and $\| \cdot \|_2'$" we will sometimes use the expression "$(A, \| \cdot \|_1', \| \cdot \|_2)$ is bicomplete".

2. Algebras of functions.

Theorem 2.1 is more or less contained in [3].

Let $X$ be a nonempty set. For $f \in K^X$ set...
\[ \| f \|_\infty := \sup \{|f(x)| : x \in X\} \] (possibly \(\infty\)). A function algebra is a \(K\)-algebra that is, for some \(X\), (algebraically isomorphic to) a sub-algebra of \(X^X\). Without much effort we can prove

2.1 **Theorem.** Let \(F\) be a function algebra. Then

(i) \(F\) has UBAT

(ii) If \(\|\|\|\) is a Banach algebra norm on \(F\) then
\[ \|\|\| \geq \|\| \|\|_\infty \] .

**Proof.** Let \(\|\|\|\) be a Banach algebra norm on \(F\). Let \(a \in X\). The map \(f \mapsto f(a)\) \((f \in F)\) is a homomorphism: \(F \to K\), so by [5] it has norm \(\leq 1\):
\[ |f(a)| \leq \| f \|. \] It follows that \(\|\|\| \leq \|\| \|\|_\infty \). Now let \(\|\|\|_1\) and \(\|\|\|_2\) be two Banach algebra norms on \(F\). We prove that the identity:
\[ (F,\|\|\|_1) \to (F,\|\|\|_2) \] is continuous. Let \(f,f_1,f_2,\ldots \in F\) such that
\[ \| f_n \|_1 \to 0, \| f_n - f \|_2 \to 0. \] By the foregoing, \(\| f_n \|_\infty \to 0, \| f_n - f \|_\infty \to 0\), so \(f = 0\). Continuity follows after applying the closed graph theorem.

3. **The separating seminorm.**

(This is a non-archimedean version of [4], (2.5.1))

3.1 **Definition.** Let \(\|\|\|_1\) and \(\|\|\|_2\) be norms on a \(K\)-vector space \(E\).

The function \(\Delta : E \to \mathbb{R}\) defined by
\[ \Delta(s) := \inf \{\max(\| x \|_1, \| y \|_2) : x + y = s\} \quad (s \in E) \]

is called the separating seminorm of \(\|\|\|_1\) and \(\|\|\|_2\).

One easily checks that \(\Delta\) is the largest among the (non-archimedean) seminorms that are \(\leq \|\|\|_1\) and \(\leq \|\|\|_2\). As in [4] we have

3.2 **Lemma.** In case \(\|\|\|_1\) and \(\|\|\|_2\) are complete norms on \(E\) then:
\(\Delta\) is a norm \(\iff \|\|_1 \sim \|\|_2\).
3.3 LEMMA. Let A be a normed K-algebra with respect to \( \| \|_1 \) and \( \| \|_2 \), and let \( \Delta \) be its separating seminorm. Then Ker \( \Delta \) is a two-sided ideal that is closed with respect to both norms. In fact, we have for \( s, t \in A \):

\[
\Delta(st) \leq \Delta(s) \max(\|t\|_1, \|t\|_2)
\]

\[
\Delta(st) \leq \Delta(t) \max(\|s\|_1, \|s\|_2).
\]

The proofs of 3.2 and 3.3 are elementary and similar to the ones in [4], (2.5) and are omitted.

3.4 DEFINITION. For a linear subspace \( D \) of an algebra \( A \) that is normed by \( \| \|_1, \| \|_2 \) we set for \( d \in D \):

\[
\Delta_D(d) := \inf\{\max(\|x\|_1, \|y\|_2) : x, y \in D, x+y = d\}
\]

(\( \Delta_D \) is the separating seminorm of the restriction of \( \| \|_1 \) and \( \| \|_2 \) to \( D \)).

We have the following elementary facts concerning the behaviour of \( \Delta \) with respect to subalgebras and quotients:

3.5 LEMMA. With the notations as above we have

(i) \( \Delta_D \geq \Delta|D, \) so Ker \( \Delta_D \subset \) Ker \( \Delta \cap D \).

(ii) Let \( D \) be a left ideal, then for \( x \in A, t \in D \):

\[
\Delta_D(xt) \leq \Delta(x) \max(\|t\|_1, \|t\|_2)
\]

\[
\Delta_D(xt) \leq \max(\|x\|_1, \|x\|_2 \Delta_D t), \text{ so Ker } \Delta_D \text{ is a left ideal in } A, \text{ satisfying}
\]

(Ker \( \Delta \)) \cdot D \subset Ker \( \Delta_D \subset \) Ker \( \Delta \cap D \).

PROOF. (i) For \( t \in D \) we have \( \Delta(t) = \inf\{\max(\|x\|_1, \|y\|_2) : x+y = t, x, y \in A\} \leq \inf\{\max(\|x\|_1, \|y\|_2) : x+y = t, x, y \in D\} = \Delta_D(t) \).
(ii) $\Delta_D(xt) = \inf_{z \in D} \max(\|z\|_1, \|xt-z\|_2) \leq \inf_{y \in A} \max(\|yt\|_1, \|xt-yt\|_2)$
\[ \leq \inf_{y \in A} \max(\|y\|_1, \|x-y\|_2) \|t\|_2) \leq \Delta(x) \max(\|t\|_1, \|t\|_2). \]

Also, $\Delta_D(xt) = \inf_{z \in D} \max(\|z\|_1, \|xt-z\|_2) \leq \Delta(t)$.

\[ \leq \inf_{d \in D} \max(\|zd\|_1, \|xt-xd\|_2) \leq \max(\|x\|_1, \|x\|_2) \cdot \Delta_D(t). \]

3.6 LEMMA. Let $(A, \|\|_1, \|\|_2)$ be bicomplete. Suppose $D$ is a closed linear subspace with respect to both $\|\|_1$ and $\|\|_2$, and suppose that the quotient norms on $A/D$ are equivalent. Then $\ker \Delta \subset D$.

PROOF. Let $\Delta(x) = 0$ for some $x \in A$. Then there are $x_1, x_2, \ldots$ in $A$ such that $\|x-x_n\|_1 \to 0$, $\|x_n\|_2 \to 0$. Let $\pi : A \to A/D$ be the quotient map. Then $\lim \pi(x_n) = \pi(x)$ for the first quotient norm and $\lim \pi(x_n) = 0$ for the second one. Hence, $\pi(x) = 0$ i.e., $x \in D$.

A subset of a $K$-algebra $A$ is called universally closed if it is closed with respect to each Banach algebra topology on $A$. (In case $A$ has no Banach algebra topology then, by definition, each subset of $A$ is universally closed). Examples of universally closed sets are

(i) $\emptyset$, $A$, singletons, finite dimensional linear subspaces.

(ii) For each set $X \subset A$ its commutant $X' := \{y \in A : yx = xy \text{ for all } x \in X\}$, in particular, the center of $A$.

(iii) For each $X \subset A$ the left and right annihilator of $X$:
\[ ^1X := \{y \in A : yx = 0 \text{ for all } x \in X\} \]
\[ ^1X := \{y \in A : xy = 0 \text{ for all } x \in X\}. \]

(iv) For each idempotent $e$ of $A$ the left ideal $eA$, the right ideal $eA$, the subalgebra $eAe$. 

(v) Maximal modular left, right, two-sided ideals.

(vi) If A is unitary, the set of the non-invertible elements of A.

We proceed by stating some corollaries of the lemmas 3.5, and 3.6.

3.7 LEMMA. Let \((A, \| \cdot \|_1, \| \cdot \|_2)\) be bicomplete and let \(e\) be an idempotent in \(A\). Then
\[
\Delta|_{eA} \text{ is equivalent to } \Delta|_{eAe}
\]
\[
\Delta|_{Ae} \text{ is equivalent to } \Delta|_{AeA}
\]
\[
\Delta|_{eAe} \text{ is equivalent to } \Delta|_{eAe}.
\]

PROOF. For \(s \in eA\) we have \(\Delta|_{eA}(s) = \Delta|_{eA}(es) \leq (\text{bij 3.5}) \leq \max(\|e\|_1, \|e\|_2) \Delta(e) \leq \max(\|e\|_1, \|e\|_2) \Delta|_{eA}(s)\). The other proofs are similar. 

3.8 LEMMA. Let \((A, \| \cdot \|_1, \| \cdot \|_2)\) be bicomplete, and let \(I\) be a universally closed left ideal, that, as a K-algebra, has UBAT.

Then \(\text{Ker } \Delta \subset \frac{1}{\| \cdot \|} I\).

PROOF. \((I, \| \cdot \|_1, \| \cdot \|_2)\) is bicomplete, I has UBAT, so \(\| \cdot \|_1 \sim \| \cdot \|_2\) on I. Thus \(\Delta|_I\) is a norm: \(\text{Ker } \Delta|_I = \{0\}\). By 3.5, \((\text{Ker } \Delta) \cdot I = \{0\}\) so \(\text{Ker } \Delta \subset \frac{1}{\| \cdot \|} I\).

3.9 THEOREM. Let \(I\) be a universally closed two-sided ideal in a K-algebra \(A\). Suppose that \(I \cap \frac{1}{\| \cdot \|} \cap \frac{1}{\| \cdot \|} = \{0\}\) (this is true, for example, if for any two-sided ideal \(J\) in \(A\), \(J = \{0\}\) implies \(J = \{0\}\)). Then, if \(I, A/I\) have UBAT then so has \(A\).

PROOF. By 3.6 and 3.5, if \((A, \| \cdot \|_1, \| \cdot \|_2)\) is bicomplete then
\[
\text{Ker } \Delta \subset I \cap \frac{1}{\| \cdot \|} \cap \frac{1}{\| \cdot \|} = \{0\} : \Delta \text{ is a norm, so } \| \cdot \|_1 \sim \| \cdot \|_2\).
\]

3.10 THEOREM. ([3], (1.1)) Let \(A\) be a K-algebra. Suppose the intersection of the maximal modular left (right, two-sided) ideals
with finite codimension is zero. Then $A$ has UBAT.

PROOF. Maximal modular left (right, two-sided) ideals are universally closed. Now apply 3.6. $\square$

4. Topological zero divisors.

In this section we will show that in many cases, for a bicomplete algebra, the ideal $\text{Ker } A$ consists only of topological zero divisors. (Compare [4] (2.5.6)). We first consider unitary algebras.

Let $A$ be a $K$-Banach algebra with identity $1$. Set

$$A^i := \{ x \in A : x^{-1} \text{ exists} \}$$

Then $A^i$ is open.

Let us call $T(A) := A^i$.

An element $x \in A$ is called a strong two-sided topological zero divisor iff there exist $s_1, s_2, \ldots \in A$ such that $\inf_n \| s_n \| > 0$ and

$$\lim_{n \to \infty} s_n x = \lim_{n \to \infty} x s_n = 0.$$  

4.1 LEMMA. Let $A$ be a $K$-Banach algebra with unit. Then

$$x \in T(A) \setminus A^i \Rightarrow x \text{ is a strong two-sided topological zero divisor.}$$

Proof: Let $x \in T(A) \setminus A^i$. Then there are $x_n \in A^i$ such that $\lim_{n \to \infty} x_n = x$.

Then we claim that $\| x_n^{-1} \|$ is unbounded. Suppose namely that

$$\sup_n \| x_n^{-1} \| = M < \infty \text{ then for } n, m \in \mathbb{N}.$$  

$$\| x_n^{-1} x_m^{-1} \| = \| x_n^{-1} (x_n-x_m) x_m^{-1} \| \leq M^2 \| x_n - x_m \| , \text{ so } y := \lim_{n} x_n^{-1} \text{ exists.}$$

But then $xy = yx = 1 : x$ would be invertible, a contradiction.

By taking a suitable subsequence, assume $\lim_{n \to \infty} \| x_n^{-1} \| = \infty$.

There are $\lambda_n \in K, c_1, c_2 \in \mathbb{R}^+$ such that
Then \( \lim_{n \to \infty} |\lambda_n| = \infty \) and
\[
\frac{x x_n^{-1}}{\lambda_n} = \frac{(x-x_n) x_n^{-1}}{\lambda_n} + \frac{1}{\lambda_n} + 0 \quad \text{(if } n \to \infty\text{)}
\]
hence \( x s_n \to 0 \), where \( s_n := \frac{1}{\lambda_n} x_n^{-1} \).

Analogously,
\[
\frac{1}{\lambda_n} x_n^{-1} x_n = \frac{x_n^{-1}}{\lambda_n} x_n - \frac{1}{\lambda_n} + 0 \quad \text{(if } n \to \infty\text{)}
\]

Thus indeed, \( x \) is a strong two-sided topological zero divisor in the above sense.

For a bicomplete algebra with unit \((\mathcal{A}, \| \cdot \|_1, \| \cdot \|_2)\) let us define \( T^1(\mathcal{A}) \) (resp. \( T^2(\mathcal{A}) \)) to be the closure of \( \mathcal{A} \) with respect to \( \| \cdot \|_1 \) (resp. \( \| \cdot \|_2 \)). We have

4.2 LEMMA. Let \((\mathcal{A}, \| \cdot \|_1, \| \cdot \|_2)\) be a bicomplete algebra with unit. Then \( \ker \Delta \subset T^1(\mathcal{A}) \cap T^2(\mathcal{A}) \).

PROOF. Choose \( \lambda_1, \lambda_2, \ldots \in \mathbb{K} \) such that \( |\lambda_n| \geq n \) (\( n \in \mathbb{N} \)). Let \( \Delta(x) = 0 \) for some \( x \in \mathcal{A} \). Let \( n \in \mathbb{N} \). Then \( \Delta(\lambda_n x) = 0 \), so there is a sequence \( x_1, x_2, \ldots \) in \( \mathcal{A} \) such that \( \lim_{k \to \infty} \| x_k \|_1 = 0 \) and \( \lim_{k \to \infty} \| \lambda_n x - x_k \|_2 = 0 \). So \( 1-x_k \) is invertible for large \( k \). It follows that \( 1-\lambda_n x \in T^2(\mathcal{A}) \), hence so is \( \frac{1}{\lambda_n} x \). Now \( x = \lim_{n \to \infty} \frac{x - \lambda_n^{-1}}{n} \) (with respect to \( \| \cdot \|_2 \)), so \( x \in T^2(\mathcal{A}) \).

Similarly, \( x \in T^1(\mathcal{A}) \).

Thus we have the following alternative.

4.3 THEOREM. Let \((\mathcal{A}, \| \cdot \|_1, \| \cdot \|_2)\) be a bicomplete algebra with unit.

and with separating seminorm \( \Delta \). Then we have either (i) or (ii):
(i) Ker $\Delta = A$, $A = T_1(A) = T_2(A)$. If an element of $A$ is not invertible then it is a strong two-sided topological zero divisor with respect to both norms.

(ii) Ker $\Delta$ is a proper ideal. Ker $\Delta$ consists only of strong two-sided topological zero divisors with respect to both norms.

NOTE. In contrast to the classical theory, case (i) can occur. In fact the separating seminorm of $|\ |$ and $|\ |'$ in Example (1.1) must be zero.

An example of case (i) in which $A$ is not a field can easily be made. Let $A := \mathbb{C} \times \mathbb{C}$ with pointwise operations. Let

$$
\| (a_1, a_2) \| := \max(|a_1|, |a_2|)
$$

$$(a_1, a_2) \in A
$$

$$
\| (a_1, a_2) \|' := \max(|a_1|', |a_2|')
$$

Then $(A, \|\|, \|\|', \|\|')$ is a bicomplete $\mathbb{Q}_p$-algebra, is not a field.

The separating seminorm is zero. (Bij 3.5 (i), with $D := (0) \times \mathbb{C}_p$, we have $\Delta(0,1) = 0$. Similarly, $\Delta(1,0) = 0$ so $\Delta = 0$).

A Tate algebra is a quotient of $K\{X_1', \ldots, X_n\}$, where the latter is the algebra of formal power series in $X_1', \ldots, X_n'$ of which the coefficients tend to zero. (see [1] and [6]). We have the following application of 4.3.

4.4 THEOREM. Let $(A, \|\|, \|\|_1, \|\|_2)$ be a bicomplete Tate algebra with separating seminorm $\Delta$. Then Ker $\Delta$ consists of only nilpotent elements. In particular, a Tate algebra without nilpotents $\neq 0$ has UBAT.

PROOF. Since $A$ is noetherian ([3] 1.5) each ideal in $A$ is universally closed. Let $P$ be a prime ideal of $A$. Then $A/P$ is a noetherian Banach
algebra with respect to both quotient norms, (again denoted by \( \| \|_1 \) and \( \| \|_2 \)). Now A/P has maximal ideals of finite codimension ([6] (4.5)), so the separating seminorm of the norms on A/P is nonzero by 3.6. Theorem 4.2 (ii) tells us that its kernel consists only of topological zero divisors with respect to both norms.

On the other hand for any \( x \in A/P, x \neq 0 \) the map \( t \mapsto tx \) \( (t \in A/P) \) is a bijection of A/P onto the principal ideal I generated by x (A/P has no zero divisors). I is universally closed in A/P, the norms \( tx \mapsto \| t \|_1 \) and \( tx \mapsto \| tx \|_1 \) \( (t \in A/P) \) on I are complete, the latter is majorized by the first. By the open mapping theorem they are equivalent: there is a \( c > 0 \) such that \( \| tx \|_1 \geq c \| t \|_1 \) \( (t \in A/P) \). It follows that x is not a topological zero with respect to \( \| \|_1 \).

Combining the results of the two previous paragraphs we conclude that \( \| \|_1 \) and \( \| \|_2 \) induce equivalent quotient norms on A/P. By 3.6, Ker \( \Delta \) is contained in the intersection of all prime ideals of A, hence consists only of nilpotents.

Next we turn to K-algebras A without unit. Application of 4.3 to \( A_1 \) where \( A_1 \) is the usual unitary extension of A does not seem to lead to interesting results. We follow a different path.

An element \( x \) of a normed K-algebra A is called a two-sided topological zero divisor if there are sequences \( s_1, s_2, \ldots, t_1, t_2, \ldots \) such that \( \inf_n \| s_n \| > 0 \), \( \inf_n \| t_n \| > 0 \), \( \lim_n s_n x = \lim_n t_n x = 0 \).

We have the following analog of 4.3

4.5 THEOREM. Let \( (A, \| \|_1, \| \|_2) \) be a bicomplete K-algebra without a unit, and with separating seminorm \( \Delta \). Then we have either

(i) or (ii):

(i) Ker \( \Delta = A \). A has a one-sided unit.
(ii) Ker $\Delta$ consists only of two-sided topological zero divisors with respect to both norms.

PROOF. We prove that if we have not (ii) then we have (i). Hence suppose we have $s \in A$ for which $\Delta(s) = 0$ and such that $s$ is not a two-sided topological zero divisor with respect to both norms. Without loss, assume that the map $x \mapsto xs$ ($x \in A$) is a homeomorphism of $A$ onto $As$ with respect to $\| \|_1$. Now let $A^*_1$ be the usual unitary extension of $A$.

Define for $i = 1, 2$

$$\| (\lambda, x) \|_{i} := \max(\| \lambda \|_1, \| x \|_1)$$

$(\lambda \in K, x \in A)$

Then $(A^*_1, \| \|_1, \| \|_2)$ is bicomplete and, by 3.5, $\Delta_{A^*_1}(s) = 0$. Since $A$ is a maximal ideal in $A_1$ of codimension 1, $\text{Ker } \Delta_{A^*_1} \neq A_1$ (3.6). Hence by 4.3 (ii) there are $(\lambda_n, x_n) \in A_1$ such that $\lim_{n} s(\lambda_n, x_n) = \lim_{n} s_1(A_n, x_n) = 0$ in the sense of $\| \|_1$ and such that $c := \inf \| (\lambda_n, x_n) \|_1 > 0$. If for some subsequence $\lambda_1, \lambda_2, \ldots$ of $\lambda_1, \lambda_2, \ldots$ we had $\lim_{n} \mu_n = 0$ then $\| y_n \|_1 \rightarrow 0$, $\| y_n \|_1 \rightarrow 0$, $\| y_n \| \geq c$ for some subsequence $y_1, y_2, \ldots$ of $x_1, x_2, \ldots$, contradicting our assumption on $s$.

Hence we may assume $\inf \| \lambda_n \|_1 > 0$. From

$$\lim_{n} (\lambda_n s + x_n s) = 0 \quad (\text{in the sense of } \| \|_1)$$

we arrive at

$$\lim_{n} (s + \frac{x_n}{\lambda_n} s) = 0 \quad (\text{in the sense of } \| \|_1)$$

It follows that $s \in \overline{As}$ (here the closure if meant with respect to $\| \|_1$). But $As$ is closed, hence there is $e \in A$ for which $s = es$. For each $x \in A$ we have $(xe-x)s = 0$ and since $s$ is no left zero divisor, $xe-x = 0$.

We conclude that $e$ is a one-sided unit for $A$. We proceed to prove that $\Delta(e) = 0$ which will finish the proof. The algebra $eAe$ is universally
closed in \( A, e \) is a unit in \( eAe \) and \( s = es = ese \in eAe \). We have

\[ \Delta(s) = 0, \text{ so by 3.7, } \Delta_{eAe}(s) = 0. \text{ Since } s \text{ is not a left topological zero divisor in } A \text{ it is certainly not in } eAe. \text{ Applying 4.3 to } eAe \text{ we see that we are in case (i): } \Delta_{eAe} = 0. \text{ It follows that } \Delta(e) = 0. \]

In order to be able to conclude for certain algebras to be in case (i), we briefly look at \( K \)-algebras \( A \) without unit but having a one-sided unit \( e \), say \( xe = x \) for all \( x \in A \). Consider \( \downarrow = \{ y \in A : ey = 0 \} \).

It is perfectly easy to see from \( x = (x-e)x + ex (x \in A) \) that \( A = e^\perp \oplus eAe \). Since \( eA = eAe \) is an algebra with a two-sided unit, we have \( e^\perp \neq (0) \). \( e^\perp \) is a two-sided ideal for which \( Ae^\perp = (0) \). In particular all products in \( e^\perp \) are zero. Therefore:

4.6 COROLLARY. Let \( (A, \| \|_1, \| \|_2) \) be a bicomplete \( K \)-algebra without unit. Suppose one of the following conditions holds.

(i) \( A \) is commutative.

(ii) \( A \) has no one-sided unit.

(iii) For a two-sided ideal \( J \) in \( A \), \( J^2 = (0) \) implies \( J = (0) \).

(iv) \( \downarrow A = (0), A^\perp = (0) \).

Then \( \text{Ker } \Delta \) contains only two-sided topological zero divisors with respect to both norms.

An application:

4.7 THEOREM. Let \( (A, \| \|) \) be a Banach algebra whose norm is multiplicative. If \( A \) is not a (skew) field then \( A \) has UBAT.

PROOF. Let \( \| \|_1 \) be some Banach algebra norm on \( A \) and let \( \Delta \) be the separating seminorm of \( \| \|_1 \) and \( \| \|_1' \). Since \( \| \|_1 \) is multiplicative, \( A \) has no topological zero divisors with respect to \( \| \|_1 \), except 0. If \( A \) has no unit, apply 4.6 (use (iii) or (iv)) to arrive at \( \text{Ker } \Delta = (0) \).
If $A$ has a unit we may use 4.3: case (i) would imply that $A$ is a (skew) field which is forbidden and case (ii) leads again to $\text{Ker } A = \{0\}$.

5. The uniqueness of the norm topology of $L(E)$.

In this section $E$ is a $K$-Banach space, $L(E)$ is the $K$-algebra of all continuous linear operators $E \to E$, and $A$ is a $K$-Banach algebra.

Let $E$ be a (left) $A$-module with structure map $(a, \xi) \mapsto a\xi$ ($a \in A$, $\xi \in E$). We say that $E$ is 2-fold transitive if for each $\xi_1, \xi_2, \eta_1, \eta_2 \in E$, where $\xi_1, \xi_2$ are linearly independent, there is a $c \in A$ such that $a\xi_1 = \eta_1$, $a\xi_2 = \eta_2$.

By the density lemma of Jacobson we then have $n$-fold transitivity for each $n \in \mathbb{N}$ i.e., if $\eta_1, \ldots, \eta_n \in E$ are linearly independent and $\eta_1, \ldots, \eta_n \in E$ then there exists $a \in A$ such that $a\xi_i = \eta_i$ ($i = 1, \ldots, n$).

The following is essentially what remains of the proof of Johnsons theorem [2] in the non-archimedean case.

5.1 Lemma. Let $E$ be a 2-fold transitive $A$-module such that the maps $\xi \mapsto a\xi$ ($\xi \in E$) are continuous for each $a \in A$. (Or, equivalently, in the corresponding representation $a \mapsto T_a$ all the $T_a$ are in $L(E)$). Then there exists $M > 0$ such that

$$||a\xi|| \leq M||a|| ||\xi|| \quad (a \in A, \xi \in E)$$

Proof. By the uniform boundedness principle it suffices to show that the structure map $(a, \xi) \mapsto a\xi$ ($a \in A$, $\xi \in E$) is separately continuous i.e., we have to show that for each $\xi \in E$ the map $a \mapsto a\xi$ ($a \in A$) is continuous. By 2-fold transitivity (in fact, irreducibility) these maps are continuous either for all $\xi \in E$, $\xi \neq 0$ or for no such $\xi$.

First assume $\dim_K E = \infty$. We assume that $a \mapsto a\xi$ is continuous only in
case $\xi = 0$ and shall derive a contradiction. Choose independent
\(\xi_1, \xi_2, \ldots \in E\) such that \(1 \leq \|\xi_i\| \leq 2\) for all \(i\), and set
\[J_i := \{a \in A; a\xi_i = 0\} \quad (i = 1, 2, \ldots)\). Each \(J_i\) is a maximal modular left
ideal of \(A\) (if \(x\xi_i = \xi_i\) then \(x\) is an identity modulo \(J_i\)), hence closed
in \(A\). For each \(m \geq 2\) we have

\[\text{(*)} \quad A = (J_1 \cap J_2 \cap \ldots \cap J_{m-1}) + J_m\]

(By the \(m\)-fold transitivity there is \(x \in A\) such that \(x\xi_1 = x\xi_2 = \ldots = x\xi_m = 0\),
x\(\xi_m \neq 0\), hence \(x \in (J_1 \cap J_2 \cap \ldots \cap J_{m-1}\), \(x \neq J_m\). Now \(J_m\) is maximal and
(*\) follows). The addition map \((J_1 \cap \ldots \cap J_{m-1}) \times J_m \to A\) is continuous and
surjective hence open by Banach's open mapping theorem. So there is \(\gamma > 0\)
such that we can write each \(a \in A\) as \(b+c\) where \(b \in J_1 \cap \ldots \cap J_{m-1}\), \(c \in J_m\)
\(|b| \leq \gamma \|a\|, \|c\| \leq \gamma \|a\|\). With the help of this one can choose inductively
\(x_1, x_2, \ldots \in A\) such that for each \(n < N, n \geq 2\)

\[\|x_n\| \leq 2^{-n}; \quad x_n \in J_1 \cap \ldots \cap J_{n-1}; \quad \|x_n\xi_n\| \geq n + \sum_{i=2}^{n-1} x_i\xi_n\|\]

using also the discontinuity at 0 of \(x \mapsto x\xi_n\).

Set \(z := \sum x_i \in A\). Since for \(n \in N, n \geq 2\) we have \(\sum x_i \in J\) we get
\[\|z\xi_n\| = \|x_2 + \ldots + x_n\| \xi_n\| \geq \|x_n\xi_n\| - \sum_{i=2}^{n-1} x_i\xi_n\| \geq n.\]
Thus, \(\lim_{n \to \infty} \|z\xi_n\| = \infty\). But the sequence \(\xi_1, \xi_2, \ldots\) is bounded, so this
conflicts with the continuity of \(\xi \mapsto z\xi\) (\(\xi \in E\)).

If, finally, \(\dim_k E = \infty\) the map \(a \mapsto a\xi\) (\(a \in A\)) can be decomposed:

\[A + A/I \cong E\]

where \(A/I\) is equipped with the quotient norm and where \(I := \{x \in A; x\xi = 0\}\). It follows that \(a \mapsto a\xi\) (\(a \in A\)) is continuous.

5.2 THEOREM. Let \((B, \|\|_1, \|\|_2)\) be a bicomplete \(K\)-algebra, and
suppose \(E\) is a 2-fold transitive \(B\)-module such that the map
\( \xi \mapsto b\xi \ (\xi \in E) \) is continuous for each \( b \in B \). Set
\[
I_E := \{ x \in B : x\xi = 0 \text{ for all } \xi \in E \}.
\]

Then \( \text{Ker } \Delta \subseteq I_E \) where \( \Delta \) is the separating seminorm of \( \| \|_1 \) and \( \| \|_2 \).

**Proof.** Let \( b \notin I_E \). Then there is \( \xi \in E \) such that \( b\xi \neq 0 \).

Lemma 5.1 yields the existence of \( M > 0 \) such that
\[
\begin{align*}
\| x\xi \| &\leq M\| x \|_1 \| \xi \| \\
\| x\xi \| &\leq M\| x \|_2 \| \xi \|
\end{align*}
\]
\( (x \in B, \xi \in E) \).

The seminorm \( p : x \mapsto M^{-1} \| \xi \|^{-1} \| x\xi \| \ (x \in B) \) satisfies \( p \leq \| \|_1 \), \( p \leq \| \|_2 \), \( p(b) \neq 0 \). So \( 0 < p(b) \leq \Delta(b) \). It follows that \( \text{Ker } \Delta \subseteq I_E \).

5.3 **Corollary.** Let \( E \) have the property that for each independent \( \xi_1, \xi_2 \in E \) and \( \eta_1, \eta_2 \) there exists \( T \in L(E) \) such that \( T\xi_1 = \eta_1 \), \( T\xi_2 = \eta_2 \). Then \( L(E) \) has UBAT.

**Proof.** \( E \) is a 2-fold transitive \( L(E) \)-module under \( (T, \xi) \mapsto T\xi \ (T \in L(E), \xi \in E) \), satisfying the continuity condition of 5.2. \( I_E = \{ T \in L(E) : T\xi = 0 \text{ for all } \xi \in E \} = \{0\} \). Hence for each two Banach algebra norms the separating seminorm is a norm, so the norms are equivalent.

Finally we indicate a class of Banach spaces \( E \) for which \( L(E) \) has UBAT. For the notions used below see [5].

5.4 **Theorem.** Let \( E \) be a K-Banach space. Each of the following conditions implies that \( L(E) \) has a unique Banach algebra topology.

(i) \( K \) is spherically complete.

(ii) \( E \) has a base. (In particular, \( E \) is of countable type.)

(iii) \( E \) is the dual of some K-Banach space.

(iv) \( E \) is spherically complete.
PROOF. We shall first prove: if the elements of the dual $E'$ separate the points of $E$ then $l(E)$ has UBAT, which takes care of (i), (ii) and (iii). Let $\xi_1, \xi_2 \in E$ be independent and $\eta_1, \eta_2 \in E$. There are $f, g \in E'$ such that $f_i(\xi_j) = \delta_{ij}$ ($i, j \in \{1, 2\}$). The map

$$\xi \mapsto f_1(\xi)\eta_1 + f_2(\xi)\eta_2$$

is in $l(E)$ and sends $\xi_i$ into $\eta_i$ ($i = 1, 2$). Now apply 5.3.

Finally we prove (v). Let $\xi_1, \xi_2 \in E$ be independent and $\eta_1, \eta_2 \in E$. Let $A$ be the map $\lambda_1 \xi_1 + \lambda_2 \xi_2 \mapsto \lambda_1 \eta_1 + \lambda_2 \eta_2$ ($\lambda_1, \lambda_2 \in K$), $A : D + E$ where $D$ is the subspace of $E$ spanned by $\xi_1$ and $\xi_2$. By the spherical completeness of $E$, $A$ can be extended to an element of $l(E)$. Now apply 5.3.

PROBLEM: Do there exist $K$-Banach spaces $E$ for which $l(E)$ admits inequivalent Banach algebra norms?

REFERENCES


