EXTRAIT DE

"SYMPOSION DÉDIÉ À A.F. MONNA"

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P-adic monotone functions

by

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0. Introduction

Our aim in this paper is to present reasonable definitions for a function $f: K \to K$ to be "monotone", where K is a local field (i.e., a non-archimedean non-trivially valued field that is locally compact in the topology induced by the valuation). The future will tell us whether filling this gap in p-adic analysis has been of any use.

The fact that - until recently - the concept of "monotone function" has been absent in p-adic analysis is not surprising since a decent (partial) ordering in K, (compatible with the algebraic and topological structure) is not available. Thus, in the sequel we will look for substitutes for "ordering" in K as a basis for our theory. One of them is the notion of "pseudo-ordering", introduced by A.F. Monna [1].

The theory can be build up in a more general setting, namely for functions $f: X \to K$, where K is any complete non-archimedean field and $X \subset K$, but restriction to local fields avoids a lot of technicalities and enlights the main track. For the extended theory, see [4].

The elementary facts about analysis in local fields can be found in [2].

Notations and definitions

FROM NOW ON IN THIS PAPER K IS A LOCAL FIELD WITH RESIDUE CLASS FIELD k.

Set $|K|: = \{|x|: x \in K\}$

$|K^*|: = |K| \setminus \{0\}$ (the value group)

π: a (fixed) element of K^* such that $|\pi|$ generates $|K^*|$, $|\pi| < 1$.

For a prime p we denote by $\mathbb{Q}_p$ the non-archimedean valued field of the p-adic numbers, by $\mathbb{Z}_p$ its valuation ring $\{x \in \mathbb{Q}_p: |x| \leq 1\}$. The residue class field, $\mathbb{Z}_p/p\,\mathbb{Z}_p$ of $\mathbb{Q}_p$ is the field of p elements and is denoted by $\mathbb{F}_p$.

The characteristic of a field L is denoted by $\chi(L)$.

For a K-vector space E and a subset S of E we denote its K-linear span by $[\![S]\!]$.

Let $a \in K$, $r \in [0,\infty)$. The ball with center a and radius r is by definition $\{x \in K: |x - a| \leq r\}$. It is easy to see that the intersection of a collection of balls is either empty or again a ball. Let $x,y \in K$. Then the smallest ball containing x and y is denoted by $[x,y]$. A subset C of K is called convex if $x,y \in C$ implies $[x,y] \subset C$. Each ball is convex. A convex set $\neq K$, $\neq \emptyset$ is a ball. It follows that K is the only unbounded convex set in K.

FROM NOW ON X IS A CONVEX SUBSET OF K.

1. Two notions of monotony

Interpreting the above notion of convexity also for the real numbers it is quite natural to introduce the following geometric expressions.

Let $x,y,z \in K$. We say that z is between x and y if $z \in [x,y]$. If z is not between x and y we say that x,y are at the same side of z. This yields more or less automatically the following

DEFINITION 1. *Let* $f: X \to K$. *We say that* $f \in M_b(X)$ (f *respects "betweenness"*) *if for all* $x, y, z \in X$

(*) $z \in [x,y] \to f(z) \in [f(x), f(y)]$.

We say that $f \in M_s(X)$ (f *respects "sides"*) *if for all* $x, y, z \in X$

(**) $z \notin [x,y] \to f(z) \notin [f(x), f(y)]$.

REMARKS

1. If we, in the above definition, replace formally K by $\mathbb{R}$ and X by an interval, we see that $f \in M_b(X)$ just means that f is monotone and that $f \in M_s(X)$ becomes "f is strictly monotone". (These facts can easily be proved). So for the time being we let our intuition be guided by this analogy:
   The statements in 2 and 3 below are direct consequences of the definitons, and the proofs are left to the reader.
2. $M_b(X)$ is closed under pointwise limits.
   The constant functions are in $M_b(X)$. If $f \in M_b(X)$ and $f(a) = f(b)$ then f is constant on $[a,b]$.
   $f \in M_b(X) \Leftrightarrow$ For each convex $C \subset K$ the inverse image $f^{-1}(C)$ is convex. For each $a,b \in K$ the map $x \mapsto ax + b$ is in $M_b(X)$.
   $f \in M_s(X) \Rightarrow f$ is injective.
   If $a,b \in K$, $a \neq 0$ then $x \mapsto ax + b$ is in $M_s(X)$.
   Each isometry $X \to K$ is in $M_{bs}(X)$ where

$$M_{bs}(X) := M_b(X) \cap M_s(X).$$

3. Without harm we may replace in Definition 1 (*) by (*)' or (*)'' or (*)''', where

$$(*)' \quad : |x-z| \leqslant |x-y| \to |f(x) - f(z)| \leqslant |f(x) - f(y)|$$

$$(*)'' \quad : |x-z| = |x-y| \to |f(x) - f(z)| = |f(x) - f(y)|$$

$$(*)''' : |f(x) - f(z)| < |f(x) - f(y)| \to |x-z| < |x-y|$$

Similarly, we may replace (**) by (**)' or (**)'' or (**)''', where

$$(**)' : |x-z| < |x-y| \to |f(x) - f(z)| < |f(x) - f(y)|$$
$$(**)'' : |f(x) - f(z)| = |f(x) - f(y)| \to |x-z| = |x-y|$$
$$(**)''': |f(x) - f(z)| \leqslant |f(x) - f(y)| \to |x-z| \leqslant |x-y|.$$

4. In the next section we will study $M_b(X)$ and $M_s(X)$. For example, the natural questions: $M_s(X) \subset M_b(X)$? $f \in M_b(X)$, f injective $\Rightarrow$ $f \in M_s(X)$? Notice that our definitions do not refer to any "type" of monotony (such as "increasing" and "decreasing" for real functions). In section 4 we will introduce such a concept. It will turn out that monotone functions having a "type" are M_{bs}-functions, but not conversely.

2. Properties of monotone functions

THEOREM 2.1. Let $f \in M_b(X)$. If $a,b,c \in X$, $|a-b| < |a-c|$, $f(a) \neq f(c)$ then $|f(a) - f(b)| < |f(a) - f(c)|$. In particular, if $f \in M_b(X)$, f is injective then $f \in M_s(X)$.

PROOF. Without loss, assume $X = [a,c]$. Since $f \in M_b(X)$ we have $\{f(a), f(c)\} \subset f(X) \subset [f(a), f(c)]$, hence the diameter of $f(X)$ equals $M: = |f(a) - f(c)|$. The ball $[f(a), f(c)]$ has a partition into n balls $V_1,\ldots,V_n$ each having radius $M|\pi|$, where n is the number of elements of k. The sets $f^{-1}(V_1),\ldots,f^{-1}(V_n)$ form a partition of X, each $f^{-1}(V_i)$ is convex (since $f \in M_b(X)$), at least two of the $f^{-1}(V_i)$ are non-empty (since a and c cannot both lie in the same $f^{-1}(V_i)$). It follows that the diameter of each $f^{-1}(V_i)$ is strictly less than $|a-c|$. (Otherwise $f^{-1}(V_i) = X$ for some i and $f^{-1}(V_j) = \emptyset$ for $j \neq i$).
Consequently the partition $f^{-1}(V_1),\ldots,f^{-1}(V_n)$ of X must be the partition of X into balls with radius $|a-c||\pi|$.

Now if $|a-b| < |a-c|$ then $a,b \in f^{-1}(V_i)$ for some i, so $|f(a) - f(b)| \leqslant M|\pi| < |f(a) - f(c)|$.

EXAMPLE. Let $p \neq 2$ and let $f: \mathbb{Z}_p \to \mathbb{Q}_p$ "tear apart" $\mathbb{Z}_p$ by sending $k + p\mathbb{Z}_p$ into $p^{-k} + p\mathbb{Z}_p$ $(k = 0,1,2,..,p-1)$ via translations. Then one easily checks that $f \in M_s(\mathbb{Z}_p)\backslash M_b(\mathbb{Z}_p)$.

Hence, it seems that M_{bs}-functions are the "translation" of the real strictly monotone functions (rather than M_s-functions).

In the sequel we often make use of the following observation. If f is either in $M_b(X)$ or in $M_s(X)$ then

$$(*) \quad |x-z| < |x-y| \to |f(x) - f(z)| \leqslant |f(x) - f(y)| \quad (x,y,z \in X)$$

(Functions with property (*) are called weakly monotone in [4]).

LEMMA 2.2. <u>Let</u> $f \in M_b(X)$ <u>or</u> $f \in M_s(X)$. <u>Then, if</u> $Y \subset X$ <u>is bounded then</u> $f(Y)$ <u>is bounded</u>.

PROOF. Y is precompact, so $r := \max\{|x-y| : x,y \in Y\}$ exists. We may assume $r > 0$. The equivalence relation $x \sim y$ iff $|x-y| < r$ $(x,y \in Y)$ divides Y into finitely many classes $Y_1,\ldots,Y_n$ where $n \geqslant 2$. Choose $a_i \in Y_i$ for each i and set $M := \max_i |f(a_i)|$. We prove $|f| \leqslant M$ on Y. In fact, let $x \in Y$. There is $i \in \{1,\ldots,n\}$ such that $|x-a_i| < r$. For $j \neq i$ we have $|x-a_i| < r = |a_i-a_j|$, so $|f(x) - f(a_i)| \leqslant |f(a_i) - f(a_j)| \leqslant M$. Hence $|f(x)| \leqslant M$.

We have a "dual" statement which is only of interest in case $X = K$:

LEMMA 2.3. <u>Let</u> $f \in M_s(K)$ <u>or</u> $f \in M_b(K)$. <u>If</u> f <u>is not constant then for a bounded</u> $Z \subset K$ <u>the inverse image</u> $f^{-1}(Z)$ <u>is bounded</u>.

PROOF. We prove: $Z \subset K$ bounded, $T := f^{-1}(Z)$ is unbounded implies f is constant. In fact, let $a,b \in K$. There are $x_1, x_2, \ldots$ in T such that

$$(*)\ \max(|a|, |b|) < |x_1| < |x_2| < \dots$$

The precompactness of $\{f(x_1), f(x_2), \dots\}$ implies convergence of a subsequence of $f(x_1), f(x_2), \dots$ Without loss, assume that $\lim_{n\to\infty} f(x_n)$ exists. From (*) we obtain

$$|a-b| < |x_1-a| < |x_2-x_1| < |x_3 - x_2| < \dots ,$$

so that for all $n \in \mathbb{N}$

$$|f(a) - f(b)| \leq |f(x_{n+1}) - f(x_n)|.$$

Hence,

$$|f(a) - f(b)| \leq \lim_{n\to\infty} |f(x_{n+1}) - f(x_n)| = 0.$$

It follows that f is constant.

2.1., 2.2. and 2.3. yield the continuity properties 2.4., 2.5. and 2.6.:

THEOREM 2.4. _Let_ X _be bounded with diameter_ $r > 0$ _and let_ $f \in M_b(X)$ _or_ $f \in M_s(X)$. _Then_ f _satisfies the Lipschitz-condition_

$$|f(x) - f(y)| \leq M|x-y| \qquad (x,y \in X)$$

where $M: = r^{-1}\sup\{|f(x) - f(y)|: x,y \in X\} < \infty$.

PROOF. By 2.2. f is bounded, so $M < \infty$. Choose any $a \in X$. We prove by induction on n: $P(n)$:"If $|x-a| = |\pi|^n r$ then $|f(x) - f(a)| \leq |\pi|^n rM$. $(x \in X)$" Clearly $P(0)$ holds. Suppose $P(n-1)$. Let $x \in X$ such that $|x-a| = |\pi|^n r$ and choose $b \in X$ with $|b-a| = |\pi|^{n-1}r$. Then $|x-a| < |b-a|$. If $f(b) = f(a)$ then $|f(x) - f(a)| \leq |f(b) - f(a)| = 0$, so certainly $|f(x) - f(a)| \leq |\pi|^n rM$. If $f(a) \neq f(b)$ then either by Theorem 2.1. or since $f \in M_s(X)$ we have $|f(x) - f(a)| < |f(b) - f(a)| \leq$ (induction hypothesis) $\leq |\pi|^{n-1}rM$, so that $|f(x) - f(a)| \leq |\pi|^n rM$.

REMARK. The map $\Sigma a_n p^n \mapsto \Sigma a_n p^{2n}$ is in $M_{bs}(\mathbb{Q}_p)$ and has unbounded difference quotients.

THEOREM 2.5. *Let* $f \in M_b(X)$ *or* $f \in M_s(X)$. *Then* f *is continuous. Further, if* $Y \subset X$ *is closed then* $f(Y)$ *is closed in* K. *In particular, an* $f \in M_s(X)$ *is a homeomorphism* $X \stackrel{\sim}{\to} f(X)$.

PROOF. If X is bounded then everything follows from 2.4., so let $X = K$. The continuity of f is clear (restrict f to bounded convex subsets). Let $Y \subset K$ be closed and suppose $\alpha \in \overline{f(Y)} \setminus f(Y)$. There are $a_1, a_2, \ldots$ in Y for which $\lim_{n\to\infty} f(a_n) = \alpha$. f is not constant, so by 2.3. the sequence $a_1, a_2, \ldots$ is bounded, assume it converges, say $a := \lim_{n\to\infty} a_n$. By continuity, $f(a) = \lim_{n\to\infty} f(a_n) = \alpha$, a contradiction. The last statement is now trivial.

THEOREM 2.6. *Let* $f \in M_b(X)$ (*or* $f \in M_s(X)$ *for that matter*). *Then the following are equivalent*.

(α) f *is a homeomorphism* $X \stackrel{\sim}{\to} f(X)$.

(β) f *is injective*.

(γ) $f \in M_s(X)$.

(δ) $f(X)$ *has no isolated points*.

PROOF. The implications $(\alpha) \Rightarrow (\beta) \Rightarrow (\gamma) \Rightarrow (\delta)$ are easy (2.1., continuity and injectivity of M_s-functions). We prove $(\delta) \Rightarrow (\alpha)$, that is, if $\lim_{n\to\infty} f(x_n) = f(a)$ then $\lim_{n\to\infty} x_n = a$. Now since $f(a)$ is not isolated we can find $a_1, a_2, \ldots$ such that $f(a_n) \neq f(a)$ for all n, $\lim_{n\to\infty} f(a_n) = f(a)$. By 2.3. we may assume that $b := \lim_{n\to\infty} a_n$ exists. If suffices to prove that $\lim x_n = b$ (apply the result for $x_n := a$ for all n and we find $a = b$). Let $\varepsilon > 0$. There is $k \in \mathbb{N}$ for which $|a_k - b| < \varepsilon$. For large n we have $|f(x_n) - f(a)| < |f(a_k) - f(a)|$, hence for large n and $m(n)$ we get $|f(x_n) - f(a_m)| < |f(a_k) - f(a_m)|$ whence $|x_n - a_m| \leqslant |a_k - a_m|$, so $|x_n - b| \leqslant |a_k - b| < \varepsilon$ for large n.

THEOREM 2.7. *Let* $f \in M_b(X)$ *or* $f \in M_s(X)$ *and assume that* $f(X)$ *is convex. Then* f *is a scalar multiple of an isometry*, *or* f *is constant*.

PROOF. We may assume that f is not constant, so $f(X)$ is open, non-empty. Let $X = f(X)$ be bounded. By 2.6. f is injective. It is clear that also $f^{-1} \in M_b(X) \cup M_s(X)$. Applying 2.4 to both f and f^{-1} we get (in both cases $M = 1$) for all $x,y,u,v \in X$ that $|f(x) - f(y)| \leqslant |x-y|$ and $|f^{-1}(u) - f^{-1}(v)| \leqslant |u-v|$. It follows that f is an isometry. The general case is now easy. (If X, $f(X)$ are bounded a transformation of the type $x \to ax+b$ sends $f(X)$ into X; if $X = f(X) = K$, apply 2.7. to $X_n: = \{x \in K: |x| \leqslant n\}$ $(n \in \mathbb{N})$).

The following theorem describes the functions "of bounded variation":

THEOREM 2.8. Let X be bounded. Then $[\![M_s(X)]\!] = [\![M_b(X)]\!] = B\Delta(X)$, where $B\Delta(X)$ is the linear space of functions $f: X \to K$ having bounded difference quotients.

PROOF. By 2.4 we are done if we can prove that an $f \in B\Delta(X)$ is the sum of two M_{bs}-functions. Let $\lambda \in K$ such that $|f(x) - f(y)| < |\lambda||x-y|$ for all $x,y \in X$, $x \neq y$. Let $g(x): = \lambda x$ $(x \in X)$ and let $h: = f - g$. Then g, h are in $M_{bs}(X)$ (scalar multiples of isometries) and $f = g + h$.

3. Monotone sequences

We will not delve deeply into this subject, but content ourselves with presenting definitions and some facts indicating that these notions are not that bad.

DEFINITION 3.1. Let $x_1, x_2, \ldots$ be a sequence in K. It is called b-monotone, if $k \leqslant l \leqslant m$ implies $x_l \in [x_k, x_m]$, s-monotone, if $l < k, m$ implies $x_l \notin [x_k, x_m]$.

THEOREM 3.2. A sequence $x_1, x_2, \ldots$ in K is s-monotone if and only if $|x_1-x_2| > |x_2-x_3| > \ldots$

A sequence $x_1, x_2, \ldots$ in K is b-monotone if and only if for each $k, m \in \mathbb{N}$, $k < m$:

$$|x_m - x_k| = \max\{|x_{i+1} - x_i|: k \leqslant i < m\}.$$

PROOF. Let $x_1, x_2, \ldots$ be s-monotone, and let $n \in \mathbb{N}$. We have $x_n \notin [x_{n+1}, x_{n+2}]$ so $|x_n - x_{n+1}| > |x_{n+1} - x_{n+2}|$. Conversely, if $|x_1-x_2| > |x_2-x_3| > \ldots$, let $l < k,m$. Then $|x_k - x_l| = |x_{l+1} - x_l|$ and $|x_m - x_k| = |x_{k+1} - x_k|$ hence $|x_k - x_l| > |x_m - x_k|$ i.e., $x_l \notin [x_k, x_m]$. Let $x_1, x_2, \ldots$ be b-monotone, and let $k, m \in \mathbb{N}$, $k < m$, and $k \leqslant i < m$. Then $x_i \in [x_k, x_m]$ and $x_{i+1} \in [x_k, x_m]$, so $[x_i, x_{i+1}] \subset [x_k, x_m]$, hence $|x_{i+1} - x_i| \leqslant |x_m - x_k|$. The rest is obvious. To prove the converse, let $k < l < m$. Then $|x_l - x_k| = \max\{|x_{i+1} - x_i|: k \leqslant i < l\} \leqslant \max\{|x_{i+1} - x_i|: k \leqslant i < m\} = |x_m - x_k|$. Hence $x_l \in [x_k, x_m]$.

COROLLARY 3.3. <u>Each</u> s-<u>monotone sequence is</u> b-<u>monotone. An</u> s-<u>monotone sequence</u> $x_1, x_2, \ldots$ <u>is convergent and</u> $x_n \neq x_m$ <u>whenever</u> $n \neq m$. M_b-<u>functions map</u> b-<u>monotone sequences into</u> b-<u>monotone sequences</u>. M_s-<u>functions map</u> s-<u>monotone sequences into</u> s-<u>monotone sequences</u>.

A b-monotone sequence need not be convergent. In fact, if $|x_1| < |x_2| < \ldots$ then $x_1, x_2, \ldots$ is b-monotone. But we have

THEOREM 3.4. <u>Let</u> $x_1, x_2, \ldots$ <u>be</u> b-<u>monotone. Then either</u> $\lim_{n\to\infty} |x_n| = \infty$ <u>or</u> $\lim_{n\to\infty} x_n$ <u>exists</u>.

PROOF. If not $\lim_{n\to\infty} |x_n| = \infty$ then the sequence has a bounded, hence a convergent subsequence, say, $\lim_{i\to\infty} x_{n_i} = x$. We show that $\lim_{n\to\infty} x_n = x$. Let $\varepsilon > 0$. Then $|x - x_{n_k}| < \varepsilon$ for $k \geqslant k_0$. If $l, m \geqslant n_{k_0}$ then if $n_k \geqslant l,m$ we have $|x_l - x_m| \leqslant |x_{n_{k_0}} - x_{n_k}| \leqslant \max(|x - x_{n_{k_0}}|, |x - x_{n_k}|) < \varepsilon$. Hence $x_1, x_2, \ldots$ is Cauchy, so it must converge, to x.

THEOREM 3.5. <u>Each sequence in</u> K <u>has a</u> b-<u>monotone subsequence</u>.

PROOF. We may assume that $x_1, x_2, \ldots$ is bounded, and that it has a convergent subsequence $y_1, y_2, \ldots$ where $y_n \neq y_m$ whenever $n \neq m$. Set $y := \lim_{n\to\infty} y_n$. Now it is easy to construct a subsequence $z_1, z_2, \ldots$ of $y_1, y_2, \ldots$ for which $|y-z_1| > |y-z_2| > |y-z_3| > \ldots$ Hence $|z_1-z_2| > |z_2-z_3| > \ldots$ The sequence $z_1, z_2, \ldots$ is s-monotone, hence b-monotone.

EXAMPLE. Let $x = \sum_{n=0}^{\infty} a_n p^n \in \mathbb{Z}_p$, and let $x_k := \sum_{n=0}^{k} a_n p^n$. Then $x_0, x_1, \ldots$ is a b-monotone sequence, converging to x.

4. Monotone functions of type σ

Following Monna [1] we introduce the concept of "sides of zero" in K (a generalization of the partition $\mathbb{R}^* = \mathbb{R}^+ \cup \mathbb{R}^-$). Let $K^* := K\setminus\{0\}$. Define

$$x \sim y \text{ if } x \text{ and } y \text{ are at the same side of } 0 \quad (x, y \in K^*).$$

Then we have $x \sim y \Leftrightarrow 0 \notin [x,y] \Leftrightarrow |x-y| < |y| \Leftrightarrow |xy^{-1}-1| < 1 \Leftrightarrow xy^{-1} \in K^+$, where

$$K^+ := \{x \in K: |1-x| < 1\}.$$

K^+ is a multiplicative open compact subgroup of K^*, called the group of the positive elements of K. We see that $\sim$ is the equivalence relation induced by the canonical group homomorphism, the "sign map":

$$\mathrm{sgn}: K^* \to K^*/K^+.$$

The quotient group $\Sigma := K^*/K^+$ (comparable with $\{1, -1\}$ in the real case) is called the group of signs of elements of K, or the group of sides of zero of K. Σ is an infinite group, whose elements are multiplicative cosets of K^+.

Let $\alpha \in \Sigma$ and let $x,y \in \alpha$. Then $|x-y| < |x|$, so in particular,

$|x| = |y|$. Therefore we may define the <u>absolute value of a sign</u> $\alpha \in \Sigma$ as

$$|\alpha| := |x| \qquad (x \in \alpha)$$

The map $\alpha \mapsto |\alpha|$ is a surjective group homomorphism $\Sigma \to |K^*|$.
Its kernel, $\{\alpha \in \Sigma: |\alpha| = 1\}$ is a multiplicative subgroup of Σ, which is naturally isomorphic to the multiplicative group k^* under $\alpha \mapsto \overline{\alpha}$ (where $x \mapsto \overline{x}$ is the canonical map $\{x \in K: |x| \leqslant 1\} \to k$).
Let us denote its inverse $k^* \overset{\sim}{\mapsto} \{\alpha \in \Sigma: |\alpha| = 1\}$ by

$$l \mapsto \alpha_l \qquad (l \in k^*).$$

For each $n \in \mathbb{Z}$, let $\alpha_n := \mathrm{sgn}(\pi^n)$.
Now for each $\alpha \in \Sigma$ there are unique $n \in \mathbb{Z}$, $l \in k^*$ such that

$$\alpha = \alpha_l \alpha_n.$$

Further, we have

$$\alpha_l \alpha_{l'} = \alpha_{ll'} \qquad (l,l' \in k^*)$$
$$\alpha_n \alpha_m = \alpha_{n+m} \qquad (n,m \in \mathbb{Z}).$$

It follows, that Σ <u>is isomorphic to</u> $k^* \times \mathbb{Z}$ (or to $k^* \times |K^*|$ for that matter). It is also possible to identify Σ with a subgroup of K^* but we shall not need it here.
In the sequel we will use <u>addition</u> of signs. For $\alpha \in \Sigma$, let

$$-\alpha := \{-x: x \in \alpha\}.$$

Then clearly $-\alpha$ is a sign and if $\alpha = \alpha_l \alpha_n$ $(l \in k^*, n \in \mathbb{Z})$ then $-\alpha = \alpha_{-l}\alpha_n$.
Let $\alpha, \beta \in \Sigma$. Then $\alpha+\beta := \{x+y: x \in \alpha, y \in \beta\}$ is easily seen to be a ball, which turns out to be again a sign iff $\alpha \neq -\beta$ (iff $0 \notin \alpha+\beta$).
Therefore we define

$$\alpha \oplus \beta := \alpha+\beta \quad (\alpha,\beta \in \Sigma,\ \alpha \neq -\beta).$$

We have the following rules that are easy to prove:

RULES 4.1.(i) The operation $\oplus$ is commutative, associative, distributive, whenever the occurring formulas are defined.

(ii) $|\alpha \oplus \beta| = \max(|\alpha|, |\beta|) \quad (\alpha, \beta \in \Sigma, \alpha \neq -\beta)$

(iii) $|\alpha| < |\beta|$ if and only if $\alpha \oplus \beta = \beta \quad (\alpha, \beta \in \Sigma)$

(iv) Let $n \in \mathbb{N}$, $1 \leqslant n < \chi(k)$, $\alpha \in \Sigma$. Then $\oplus_n \alpha (:= \alpha \oplus \alpha \oplus \ldots \oplus \alpha$ (n times)) exists and equals $n\alpha$.

Thus, we get for $l, l' \in k^*$, $m, n \in \mathbb{Z}$:

$$\alpha_l\alpha_m \oplus \alpha_{l'}\alpha_n = \begin{cases} \alpha_l\alpha_m & \text{if } m < n \\ \alpha_{l'}\alpha_n & \text{if } m > n \\ \alpha_{l+l'}\alpha_n & \text{if } m = n \text{ and } l + l' \neq 0. \end{cases}$$

Next, we define a "pseudo-ordering" in K. Let $x,y \in K$, $\alpha \in \Sigma$. We say that $x \underset{\alpha}{>} y$ (x is greater than y in the sense of α) in case $x - y \in \alpha$. The following easy consequences obtain.

RULES 4.2. (i) Let $x, y \in K$. Then if $y \neq x$ there is exactly one $\alpha \in \Sigma$ for which $x \underset{\alpha}{>} y$. If $y = x$ then $x \underset{\alpha}{>} y$ for no α. ("K is totally pseudo-ordered").

(ii) If $x \underset{\alpha}{>} y$, $y \underset{\beta}{>} z$ for some $x, y, z \in K$; $\alpha, \beta \in \Sigma$, and if $\alpha \oplus \beta$ exists then $x \underset{\alpha\oplus\beta}{>} z$. ("Transitivity").

(iii) If $x \underset{\alpha}{>} y$ for some $x, y \in K$; $\alpha \in \Sigma$ and if $z \in K$, then $x + z \underset{\alpha}{>} y + z$ ("Compatibility with addition").

(iv) If $x, y, z \in K$; $\alpha, \beta \in \Sigma$, $x \underset{\alpha}{>} y$ and $z \underset{\beta}{>} 0$ then $xz \underset{\alpha\beta}{>} yz$ ("Compatibility with multiplication").

We define:

DEFINITION 4.3. Let $\sigma: \Sigma \to \Sigma$ be an injection and $f: K \to K$. f is called monotone of type σ if for all $\alpha \in \Sigma$, $x, y \in K$ we have

$$x \underset{\alpha}{>} y \text{ implies } f(x) \underset{\sigma(\alpha)}{>} f(y).$$

f is called increasing if σ is the identity map.

REMARKS

1. One can extend easily the above definition for functions $f: X \to K$, where X is a convex subset of K, $\sigma: \Sigma(X) \to \Sigma$. Here $\Sigma(X)$ is the collection of signs that "occur in X" i.e. $\{\mathrm{sgn}(x-y): x, y \in X, x \neq y\}$. We leave it to the reader to do this and, in case $X \neq K$, to show that there is $\beta \in \Sigma$ such that

$$\Sigma(X) = \{\alpha \in \Sigma: |\alpha| < |\beta|\}.$$

2. Notice that $f: K \to K$ is increasing if and only if the difference quotient

$$\Phi f(x, y) := \frac{f(x) - f(y)}{x - y} \qquad (x \neq y)$$

is positive. In particular, f is an isometry.

3. The requirement made in 4.3. that σ be an injection is essential in case k is not a prime field (see [4]).

4. In case $\chi(K) = 0$, the exponential function, defined by the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is increasing on its convergence region. If f is an increasing function and $\beta \in \Sigma$ then for each $b \in \beta$ the function bf is of type σ where σ is the multiplier $\alpha \mapsto \alpha\beta$.

THEOREM 4.4. (i) Let $f: K \to K$ be a monotone of type $\sigma: \Sigma \mapsto \Sigma$. Then

(*) $\sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta) \qquad (\alpha, \beta \in \Sigma, \alpha \neq -\beta)$

(ii) Let $\sigma: \Sigma \to \Sigma$ satisfy (*). Then there is a function $g: K \to K$, monotone of type σ.

PROOF. (i) First we show that $\sigma(-\alpha) = -\sigma(\alpha)$ $(\alpha \in \Sigma)$. In fact choose $x, y \in K$ such that $x - y \in \alpha$. Then $y - x \in -\alpha$, so $f(x) - f(y) \in \sigma(\alpha) \cap (-\sigma(-\alpha)) \neq \emptyset$, which implies $\sigma(\alpha) = -\sigma(-\alpha)$. Now take $\alpha, \beta \in \Sigma$, $\alpha \neq -\beta$. Then by injectivity of σ, $\sigma(\alpha) \neq \sigma(-\beta) = -\sigma(\beta)$ so that $\sigma(\alpha) \oplus \sigma(\beta)$ exists. Now choose $x, y, z \in K$ such that $x - y \in \alpha$, $y - z \in \beta$. Then $f(x) - f(z) = f(x) - f(y) + f(y) - f(z) \in \sigma(\alpha) \oplus \sigma(\beta)$. Also, $x - z \in \alpha \oplus \beta$ so $f(x) - f(z) \in \sigma(\alpha \oplus \beta)$. As $\sigma(\alpha) \oplus \sigma(\beta)$ and $\sigma(\alpha \oplus \beta)$ have a non-empty intersection they are equal.

The proof of (ii) will be furnished by Lemma 4.5. and Lemma 4.6.:

LEMMA 4.5. *Let* $\sigma: \Sigma \to \Sigma$ *satisfy* $\sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta)$ $(\alpha, \beta \in \Sigma, \alpha \neq -\beta)$. *Then we have*

(i) $\sigma(n\alpha) = n\sigma(\alpha)$ $\quad (n \in \mathbb{N}, 1 \leqslant n < \chi(k), \alpha \in \Sigma)$.

(ii) $\sigma(-\alpha) = -\sigma(\alpha)$ $\quad (\alpha \in \Sigma)$

(iii) *For all* $\alpha, \beta \in \Sigma$: $|\alpha| < |\beta|$ *if and only if* $|\sigma(\alpha)| < |\sigma(\beta)|$.

(iv) $\lim\limits_{|\alpha| \to 0} |\sigma(\alpha)| = 0$.

PROOF. (i) is a direct consequence of 4.1.(iv). To prove (ii), set $q := \chi(k)$. Then for $\beta \in \Sigma$, $(q-1)\beta = -\beta$, so, by (i), $\sigma(-\alpha) = \sigma((q-1)\alpha) = (q-1)\sigma(\alpha) = -\sigma(\alpha)$. Let $\alpha, \beta \in \Sigma$, $|\alpha| < |\beta|$. Then $\alpha \oplus \beta = \beta$ so $\sigma(\alpha) \oplus \sigma(\beta) = \sigma(\alpha \oplus \beta) = \sigma(\beta)$ whence $|\sigma(\alpha)| < |\sigma(\beta)|$. Conversely, suppose $|\sigma(\alpha)| < |\sigma(\beta)|$. Then clearly $\alpha \neq -\beta$ (otherwise $|\sigma(\alpha)| = |\sigma(-\beta)| = |-\sigma(\beta)| = |\sigma(\beta)|$), so $\alpha \oplus \beta$ exists. Now $\sigma(\beta) = \sigma(\alpha) \oplus \sigma(\beta) = \sigma(\alpha \oplus \beta)$. By injectivity of σ we obtain $\beta = \alpha \oplus \beta$ whence $|\alpha| < |\beta|$. So we have (iii). (iv) follows from the fact that if $|\alpha_1| > |\alpha_2| > \ldots \to 0$ then $|\sigma(\alpha_1)| > |\sigma(\alpha_2)| > \ldots$, which last sequence tends to 0 due to the discreteness of the valuation.

LEMMA 4.6. (Extension theorem for monotone functions). Let $\emptyset \neq Y \subset K$, $f\colon Y \to K$, $\sigma\colon \Sigma \to \Sigma$ satisfy $\sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta)$ $(\alpha, \beta \in \Sigma, \alpha \neq -\beta)$. Suppose

$$x \underset{\alpha}{>} y \text{ implies } f(x) \underset{\sigma(\alpha)}{>} f(y) \qquad (x,y \in Y,\ \alpha \in \Sigma)$$

Then f can be extended to a function $\overline{f}\colon K \to K$, monotone of type σ.

PROOF. By Zorn's lemma it suffices to extend f to $Y \cup \{a\}$ $(a \notin Y)$ such that $f(x) - \overline{f}(a) \in \sigma(\operatorname{sgn}(x - a))$ and $\overline{f}(a) - f(x) \in \sigma(\operatorname{sgn}(a - x))$ for all $x \in Y$. By 4.5.(ii) it suffices to consider only the second case. Let

$$B_x := f(x) + \sigma(\operatorname{sgn}(a - x)) \qquad (x \in Y)$$

Then each B_x is a ball having diameter $|\pi|\ |\sigma(\operatorname{sgn}(a - x))| \neq 0$. By the local compactness (in fact, spherical completeness) of K we are done if we can show that $B_x \cap B_y \neq \emptyset$ whenever $x, y \in Y$, $x \neq y$.

Set $\alpha := \operatorname{sgn}(a - x)$ and $\beta := \operatorname{sgn}(a - y)$, $b \in \sigma(\alpha)$, $c \in \sigma(\beta)$. We have to prove that $|f(x) + b - (f(y) + c)| \leqslant |\pi| \max(|\sigma(\alpha)|, |\sigma(\beta)|)$. Consider two cases.

1) $\alpha = \beta$. Then $a - x$ and $a - y$ are in α, so $|a - x - (a - y)| = |x - y| < |\alpha|$, hence $|\operatorname{sgn}(x - y)| < |\alpha|$. By 4.5.(iii) we have $|\sigma(\operatorname{sgn}(x - y))| < |\sigma(\alpha)|$, so $|\operatorname{sgn}(f(x) - f(y))| < |\sigma(\alpha)|$ whence $|f(x) - f(y)| < |\sigma(\alpha)|$. Also $|b - c| < |\sigma(\alpha)|$ since both b and c are in $\sigma(\alpha)$. Consequently $|f(x) + b - (f(y) + c)| < |\sigma(\alpha)|$.

2) $\alpha \neq \beta$. Then $x - y = a - y - (a - x) \in \beta \oplus -\alpha$, so $f(x) - f(y) \in \sigma(\beta \oplus -\alpha)$. Now $b - c \in \sigma(\alpha) \oplus (-\sigma(\beta)) = \sigma(\alpha) \oplus \sigma(-\beta) = \sigma(\alpha \oplus -\beta) = -\sigma(\beta \oplus -\alpha)$. Therefore $|f(x) - f(y) - (b - c)| < |\sigma(\beta \oplus -\alpha)| = \max(|\sigma(\beta)|, |\sigma(\alpha)|)$.

(The proof of 4.4.(ii): Choose $Y := \{0\}$ and let $f\colon Y \to K$ be defined via $f(0) := 0$. Extend f in the way of 4.6.).

COROLLARY 4.7. Let $f\colon K \to K$ be monotone of type $\sigma\colon \Sigma \to \Sigma$. Then $f \in M_{bs}(K)$ (see section 2). More than that: there exists a strictly increasing function $\phi\colon |K| \to |K|$, continuous at 0, $(\phi(0) = 0)$ such that

$$|f(x) - f(y)| = \phi(|x - y|). \quad (x,y \in K)$$

PROOF. Let $x, y, u, v \in K$ and $x - y \in \alpha \in \Sigma$, $u - v \in \beta \in \Sigma$. Then we have by 4.5.(iii): $|x - y| < |u - v| \Leftrightarrow |\alpha| < |\beta| \Leftrightarrow |\sigma(\alpha)| < |\sigma(\beta)| \Leftrightarrow |f(x) - f(y)| < |f(u) - f(v)|$. The existence of ϕ is now clear. The continuity follows from 4.5.(iv).

REMARK. There exist isometries $K \to K$ that are monotone of type σ for no σ (see [4]).

THEOREM 4.8. Let $f\colon K \to K$ be monotone of type $\sigma\colon \Sigma \to \Sigma$. Then σ is surjective if and only if f is a bijection (in fact, f is a nonzero scalar multiple of an isometry, by 2.7.).

PROOF. If σ is surjective then σ^{-1} exists and satisfies the condition of 4.6., so there is a $g\colon K \to K$, monotone of type σ^{-1}. Then $f \circ g$ is monotone of type 1, i.e., increasing. It suffices to show that an increasing $h\colon K \mapsto K$ is surjective. Let $a \in K$ and consider the map $\psi\colon x \mapsto x - h(x) + a$ $(x \in K)$. Then $|\psi(x) - \psi(y)| \leq |\pi||x - y|$ $(x, y \in K)$. By the Banach contraction theorem, ψ has a fixed point t. Then $h(t) = a$: h is surjective. The converse is easy.

EXAMPLE. The monotone functions on $\mathbb{Q}_p$.

As we have seen in section 4, the group of signs of $\mathbb{Q}_p$ is isomorphic to $\mathbb{F}_p^* \times \mathbb{Z}$. Using this interpretation we can describe the sign function as follows. Let $x \in \mathbb{Q}_p$, $x = \sum_{n \geq k} a_n p^n$ be its standard expansion (i.e., $k \in \mathbb{Z}$, $a_n \in \{0,1,2,\ldots,p-1\}$, $a_k \neq 0$). Then

$$\operatorname{sgn}(x) = (a_k, n) \in \mathbb{F}_p^* \times \mathbb{Z}.$$

Let $\sigma: \Sigma \to \Sigma$ be a type of some monotone function. By 4.5. we have $\sigma(1,n) = 1\sigma(1,n), ((1,n) \in \Sigma)$. Set

$$\sigma(1,n) = (s(n), \lambda(n)) \quad (n \in \mathbb{Z})$$

Since $n < m \Leftrightarrow |p^n| > |p^m| \Leftrightarrow |(1,n)| > |(1,m)| \Leftrightarrow |\sigma(1,n)| > |\sigma(1,m)| \Leftrightarrow |p^{\lambda(n)}| > |p^{\lambda(m)}| \Leftrightarrow \lambda(n) < \lambda(m)$, we see that $\lambda: \mathbb{Z} \to \mathbb{Z}$ is strictly increasing. Thus, σ has the form

$$(*) \ (1,n) \mapsto (1s(n), \lambda(n))$$

where $s: \mathbb{Z} \to \mathbb{F}_p^*$ and $\lambda: \mathbb{Z} \to \mathbb{Z}$ is strictly increasing.

Conversely, if we have a map σ satisfying $(*)$ where $s: \mathbb{Z} \to \mathbb{F}_p^*$ and $\lambda: \mathbb{Z} \to \mathbb{Z}$ is strictly increasing the function

$$\sum_n a_n p^n \mapsto \sum_n a_n s(n) p^{\lambda(n)}$$

is monotone of type σ, as can easily be verified.

For a criterion in order that a continuous function $\mathbb{Z}_p \to \mathbb{Q}_p$ be increasing (expressed by means of its coordinates with respect to some orthonormal base), see [4].

Appendix

Differentiation

We briefly consider the relationship between monotony and differentiation. We refer to [4] for the proofs. Although even increasing functions may be nowhere differentiable there are some connections that are similar to those in the real case.

A function $g: K \to K$ is called <u>positive</u> if $g(K) \subset K^+$.
A function $h: K \to K$ is of the <u>first class of Baire</u> if there exists a sequence $h_1, h_2, \ldots$ of continuous functions $K \to K$ that converges pointwise to h.

THEOREM. (i) <u>Let</u> $f: K \to K$ <u>be increasing, differentiable. Then</u> f' <u>is positive, of the first class of Baire</u>.
(ii) <u>A positive function of the first class of Baire has an increasing antiderivative</u>.

THEOREM. <u>Let</u> $f: K \to K$ <u>be continuously differentiable (which means here that</u> $\lim_{x,y\to a} (x-y)^{-1}(f(x) - f(y))$ <u>exists for</u> $a \in K$), <u>and suppose</u> $f'(a) \neq 0$. <u>Then there is a</u> (convex) <u>neighborhood</u> X <u>of</u> a <u>such that</u> $f|X$ <u>is monotone of type</u> σ, <u>where</u> σ <u>is the map</u> $\alpha \mapsto \mathrm{sgn}(f'(a)).\alpha$.

THEOREM. <u>Let</u> $f: K \to K$ <u>be monotone of type</u> σ, <u>differentiable. Then there are two cases</u>.

I. $f'(a) = 0$ <u>for some</u> $a \in K$. <u>Then</u> $f' = 0$ <u>everywhere and</u> $\lim_{|\alpha|\to 0} \frac{|\sigma(\alpha)|}{|\alpha|} = 0$.

II. $f'(a) \neq 0$ <u>for some</u> $a \in K$. <u>Then</u> $f' \neq 0$ <u>everywhere</u>. <u>In fact</u>, f' <u>has constant sign</u> ($x \mapsto \mathrm{sgn}(f'(x))$ <u>is constant</u>). <u>For small</u> $|\alpha|$, $\frac{\sigma(\alpha)}{\alpha}$ <u>is constant</u>. $f'(a)^{-1}f$ <u>is locally increasing</u>.

REMARK. One can make an example of an everywhere differentiable $f: K \to K$ with $f' = 1$ (so f' is positive) such that f is not even locally injective at 0. (f is, of course, not continuously differentiable).