EXTRAIT DE

"SYMPOSION DÉDIÉ À A.F. MONNA"

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0. Introduction

Our aim in this paper is to present reasonable definitions for a function \( f: K \to K \) to be "monotone", where \( K \) is a local field (i.e., a non-archimedean non-trivially valued field that is locally compact in the topology induced by the valuation). The future will tell us whether filling this gap in p-adic analysis has been of any use.

The fact that - until recently - the concept of "monotone function" has been absent in p-adic analysis is not surprising since a decent (partial) ordering in \( K \), (compatible with the algebraic and topological structure) is not available. Thus, in the sequel we will look for substitutes for "ordering" in \( K \) as a basis for our theory. One of them is the notion of "pseudo-ordering", introduced by A.F. Monna [1].

The theory can be build up in a more general setting, namely for functions \( f: X \to K \), where \( K \) is any complete non-archimedean field and \( X \subseteq K \), but restriction to local fields avoids a lot of technicalities and enlights the main track. For the extended theory, see [4].

The elementary facts about analysis in local fields can be found in [2].

Notations and definitions

FROM NOW ON IN THIS PAPER \( K \) IS A LOCAL FIELD WITH RESIDUE CLASS FIELD \( k \).
Set \( |K| : = \{|x| : x \in K\} \)
\( |K^*| : = |K\setminus\{0\}\) (the value group)

\( \pi\) : a (fixed) element of \( K^* \) such that \(|\pi| \) generates \(|K^*|\), \(|\pi| < 1 \).

For a prime \( p \) we denote by \( \mathbb{Q}_p \) the non-archimedean valued field of the \( p \)-adic numbers, by \( \mathbb{Z}_p \) its valuation ring \( \{x \in \mathbb{Q}_p : |x| \leq 1\} \).

The residue class field, \( \mathbb{Z}_p/p \mathbb{Z}_p \) of \( \mathbb{Q}_p \) is the field of \( p \) elements and is denoted by \( \mathbb{F}_p \).

The characteristic of a field \( L \) is denoted by \( \chi(L) \).

For a \( K \)-vector space \( E \) and a subset \( S \) of \( E \) we denote its \( K \)-linear span by \( \langle S \rangle \).

Let \( a \in K, r \in [0,\infty) \). The ball with center \( a \) and radius \( r \) is by definition \( \{x \in K : |x - a| \leq r\} \). It is easy to see that the intersection of a collection of balls is either empty or again a ball.

Let \( x,y \in K \). Then the smallest ball containing \( x \) and \( y \) is denoted by \( [x,y] \). A subset \( C \) of \( K \) is called convex if \( x,y \in C \) implies \( [x,y] \subseteq C \). Each ball is convex. A convex set \( \neq K, \neq \emptyset \) is a ball.

It follows that \( K \) is the only unbounded convex set in \( K \).

\textbf{FROM NOW ON X IS A CONVEX SUBSET OF K.}

\textbf{1. Two notions of monotony}

Interpreting the above notion of convexity also for the real numbers it is quite natural to introduce the following geometric expressions.

Let \( x,y,z \in K \). We say that \( z \) is between \( x \) and \( y \) if \( z \in [x,y] \).

If \( z \) is not between \( x \) and \( y \) we say that \( x,y \) are at the same side of \( z \). This yields more or less automatically the following
DEFINITION 1. Let \( f: X \to K \). We say that \( f \in M_b(X) \) (\( f \) respects "betweenness") if for all \( x, y, z \in X \)
\[ (*) \quad z \in [x,y] \to f(z) \in [f(x), f(y)]. \]
We say that \( f \in M_s(X) \) (\( f \) respects "sides") if for all \( x, y, z \in X \)
\[ (**) \quad z \notin [x,y] \to f(z) \notin [f(x), f(y)]. \]

REMARKS
1. If we, in the above definition, replace formally \( K \) by \( \mathbb{R} \) and \( X \) by an interval, we see that \( f \in M_b(X) \) just means that \( f \) is monotone and that \( f \in M_s(X) \) becomes "\( f \) is strictly monotone". (These facts can easily be proved). So for the time being we let our intuition be guided by this analogy:
The statements in 2 and 3 below are direct consequences of the definitions, and the proofs are left to the reader.

2. \( M_b(X) \) is closed under pointwise limits.
The constant functions are in \( M_b(X) \). If \( f \in M_b(X) \) and \( f(a) = f(b) \) then \( f \) is constant on \([a,b]\).
\( f \in M_b(X) \Rightarrow \) For each convex \( C \subseteq K \) the inverse image \( f^{-1}(C) \)
is convex. For each \( a,b \in K \) the map \( x \mapsto ax + b \) is in \( M_b(X) \).
\( f \in M_s(X) \Rightarrow f \) is injective.
If \( a,b \in K, a \neq 0 \) then \( x \mapsto ax + b \) is in \( M_s(X) \).
Each isometry \( X \to K \) is in \( M_{bs}(X) \) where
\[ M_{bs}(X) = M_b(X) \cap M_s(X). \]

3. Without harm we may replace in Definition 1 \((*)\) by \((*)'\) or \((*)''\) or \((*)'''\), where
\[ (*)' : |x-z| \leq |x-y| \to |f(x) - f(z)| \leq |f(x) - f(y)| \]
\[ (*)'' : |x-z| = |x-y| \to |f(x) - f(z)| = |f(x) - f(y)| \]
\[ (*)''' : |f(x) - f(z)| < |f(x) - f(y)| \to |x-z| < |x-y| \]
Similarly, we may replace \((**)\) by \((**)\)' or \((**)\)' or \((**)\)'', where

$$\text{\((**)\)' : } |x-z| < |x-y| \rightarrow |f(x) - f(z)| < |f(x) - f(y)|$$
$$\text{\((**)\)' : } |f(x) - f(z)| = |f(x) - f(y)| \rightarrow |x-z| = |x-y|$$
$$\text{\((**)\)' : } |f(x) - f(z)| < |f(x) - f(y)| \rightarrow |x-z| < |x-y|.$$ 

4. In the next section we will study \(M_b(X)\) and \(M_s(X)\). For example, 
the natural questions: \(M_s(X) \subseteq M_b(X)\)? \(f \in M_b(X)\), \(f\) injective \(\Rightarrow\) \(f \in M_s(X)\)? Notice that our definitions do not refer to any 
"type" of monotony (such as "increasing" and "decreasing" for 
real functions). In section 4 we will introduce such a concept. 
It will turn out that monotone functions having a "type" are 
\(M_{bs}\)-functions, but not conversely.

2. Properties of monotone functions

THEOREM 2.1. Let \(f \in M_b(X)\). If \(a, b, c \in X\), \(|a-b| < |a-c|\), \(f(a) \neq f(c)\) 
then \(|f(a) - f(b)| < |f(a) - f(c)|\). In particular, if 
\(f \in M_b(X)\), \(f\) is injective \(\Rightarrow\) \(f \in M_s(X)\).

PROOF. Without loss, assume \(X = [a,c]\). Since \(f \in M_b(X)\) we have 
\([f(a), f(c)] \subseteq f(X) \subseteq [f(a), f(c)]\), hence the diameter of \(f(X)\) 
equals \(M = |f(a) - f(c)|\). The ball \([f(a), f(c)]\) has a partition 
into \(n\) balls \(V_1, \ldots, V_n\) each having radius \(M/n\), where \(n\) is the 
number of elements of \(k\). The sets \(f^{-1}(V_i), \ldots, f^{-1}(V_n)\) form a 
partition of \(X\), each \(f^{-1}(V_i)\) is convex (since \(f \in M_b(X)\)), at least 
two of the \(f^{-1}(V_i)\) are non-empty (since \(a\) and \(c\) cannot both lie in 
the same \(f^{-1}(V_i)\)). It follows that the diameter of each \(f^{-1}(V_i)\) is 
strictly less than \(|a-c|\). (Otherwise \(f^{-1}(V_i) = X\) for some \(i\) and 
\(f^{-1}(V_j) = \emptyset\) for \(j \neq i\)). 
Consequently the partition \(f^{-1}(V_1), \ldots, f^{-1}(V_n)\) of \(X\) must be the 
partition of \(X\) into balls with radius \(|a-c||n|\).
Now if \(|a-b| < |a-c|\) then \(a, b \in f^{-1}(V_i)\) for some \(i\), so \(|f(a) - f(b)| \leq M|\pi| < |f(a) - f(c)|\).

**EXAMPLE.** Let \(p \neq 2\) and let \(f: \mathbb{Z}_p \to \mathbb{Q}_p\) "tear apart" \(\mathbb{Z}_p\) by sending \(k + p\mathbb{Z}_p\) into \(p^{-k} + p\mathbb{Z}_p\) \((k = 0, 1, 2, \ldots, p-1)\) via translations. Then one easily checks that \(f \in M_b(\mathbb{Z}_p) \setminus M_s(\mathbb{Z}_p)\).

Hence, it seems that \(M_{bs}\)-functions are the "translation" of the real strictly monotone functions (rather than \(M_s\)-functions).

In the sequel we often make use of the following observation. If \(f\) is either in \(M_b(X)\) or in \(M_s(X)\) then

\[
(*) \quad |x-z| < |x-y| \Rightarrow |f(x) - f(z)| \leq |f(x) - f(y)| \quad (x, y, z \in X)
\]

(Functions with property (*) are called weakly monotone in \([4]\)).

**LEMMA 2.2.** Let \(f \in M_b(X)\) or \(f \in M_s(X)\). Then, if \(Y \subset X\) is bounded then \(f(Y)\) is bounded.

**PROOF.** \(Y\) is precompact, so \(r: = \max\{|x-y|: x, y \in Y\}\) exists. We may assume \(r > 0\). The equivalence relation \(x \sim y\) iff \(|x-y| < r\) \((x, y \in Y)\) divides \(Y\) into finitely many classes \(Y_1, \ldots, Y_n\) where \(n \geq 2\). Choose \(a_i \in Y_i\) for each \(i\) and set \(M: = \max \{|f(a_i)|\}\). We prove \(|f| \leq M\) on \(Y\).

In fact, let \(x \in Y\). There is \(i\) \(\in\{1, \ldots, n\}\) such that \(|x-a_i| < r\).

For \(j \neq i\) we have \(|x-a_i| < r = |a_i - a_j|\), so \(|f(x) - f(a_i)| \leq |f(a_i) - f(a_j)| \leq M\). Hence \(|f(x)| \leq M\).

We have a "dual" statement which is only of interest in case \(X = K\):

**LEMMA 2.3.** Let \(f \in M_s(K)\) or \(f \in M_b(K)\). If \(f\) is not constant then for a bounded \(Z \subset K\) the inverse image \(f^{-1}(Z)\) is bounded.

**PROOF.** We prove: \(Z \subset K\) bounded, \(T: = f^{-1}(Z)\) is unbounded implies \(f\) is constant. In fact, let \(a, b \in K\). There are \(x_1, x_2, \ldots\) in \(T\) such that
(*) \( \max(|a|, |b|) < |x_1| < |x_2| < \ldots \)

The precompactness of \( \{f(x_1), f(x_2), \ldots\} \) implies convergence of a subsequence of \( f(x_1), f(x_2), \ldots \). Without loss, assume that \( \lim_{n \to \infty} f(x_n) \) exists. From (*) we obtain

\[
|a-b| < |x_1-a| < |x_2-x_1| < |x_3 - x_2| < \ldots ,
\]

so that for all \( n \in \mathbb{N} \)

\[
|f(a) - f(b)| \leq |f(x_{n+1}) - f(x_n)|.
\]

Hence,

\[
|f(a) - f(b)| \leq \lim_{n \to \infty} |f(x_{n+1}) - f(x_n)| = 0.
\]

It follows that \( f \) is constant.

2.1., 2.2. and 2.3. yield the continuity properties 2.4., 2.5. and 2.6.:

**THEOREM 2.4.** Let \( X \) be bounded with diameter \( r > 0 \) and let \( f \in M_b(X) \) or \( f \in M_s(X) \). Then \( f \) satisfies the Lipschitz-condition

\[
|f(x) - f(y)| \leq M|x-y| \quad (x, y \in X)
\]

where \( M = r^{-1}\sup\{|f(x) - f(y)|: x, y \in X\} < \infty \).

**PROOF.** By 2.2. \( f \) is bounded, so \( M < \infty \). Choose any \( a \in X \). We prove by induction on \( n \): \( P(n): "\)If \( |x-a| = |\pi|^n r \) then \( |f(x) - f(a)| \leq |\pi|^n r M \) \( (x \in X)" \) Clearly \( P(0) \) holds. Suppose \( P(n-1) \). Let \( x \in X \) such that \( |x-a| = |\pi|^n r \) and choose \( b \in X \) with \( |b-a| = |\pi|^{n-1} r \).

Then \( |x-a| < |b-a| \). If \( f(b) = f(a) \) then \( |f(x) - f(a)| \leq |f(b) - f(a)| = 0 \), so certainly \( |f(x) - f(a)| \leq |\pi|^n r M \). If \( f(a) \neq f(b) \) then either by Theorem 2.1. or since \( f \in M_s(X) \) we have \( |f(x) - f(a)| < |f(b) - f(a)| \leq \) (induction hypothesis) \( \leq |\pi|^{n-1} r M, \) so that \( |f(x) - f(a)| \leq |\pi|^n r M \).

**REMARK.** The map \( \Sigma_{n} p^n \to \Sigma_{n} p^{2n} \) is in \( M_{bs}(Q_p) \) and has unbounded difference quotients.
THEOREM 2.5. Let \( f \in M_b(X) \) or \( f \in M_s(X) \). Then \( f \) is continuous.

Further, if \( Y \subset X \) is closed then \( f(Y) \) is closed in \( K \).

In particular, an \( f \in M_s(X) \) is a homeomorphism \( X \xrightarrow{\sim} f(X) \).

PROOF. If \( X \) is bounded then everything follows from 2.1., so let \( X = K \). The continuity of \( f \) is clear (restrict \( f \) to bounded convex subsets). Let \( Y \subset K \) be closed and suppose \( a \in \overline{f(Y) \setminus f(Y)} \). There are \( a_1, a_2, \ldots \) in \( Y \) for which \( \lim_{n \to \infty} f(a_n) = a \). \( f \) is not constant, so by 2.3. the sequence \( a_1, a_2, \ldots \) is bounded, assume it converges, say \( a = \lim_{n \to \infty} a_n \). By continuity, \( f(a) = \lim_{n \to \infty} f(a_n) \). By 2.3. the sequence converges, say \( a = \lim a_n \). By continuity, \( f(a) = \lim f(a_n) = a \), a contradiction.

The last statement is now trivial.

THEOREM 2.6. Let \( f \in M_b(X) \) (or \( f \in M_s(X) \) for that matter). Then the following are equivalent.

1. \( f \) is a homeomorphism \( X \xrightarrow{\sim} f(X) \).
2. \( f \) is injective.
3. \( f \in M_s(X) \).
4. \( f(X) \) has no isolated points.

PROOF. The implications \( (\alpha) \Rightarrow (\beta) \Rightarrow (\gamma) \Rightarrow (\delta) \) are easy (2.1., continuity and injectivity of \( M_s \)-functions). We prove \( (\delta) \Rightarrow (\alpha) \), that is, if \( \lim_{n \to \infty} f(x_n) = f(a) \) then \( \lim_{n \to \infty} x_n = a \). Now since \( f(a) \) is not isolated we can find \( a_1, a_2, \ldots \) such that \( f(a_n) \neq f(a) \) for all \( n \), \( \lim_{n \to \infty} f(a_n) = f(a) \). By 2.3. we may assume that \( b = \lim_{n \to \infty} a_n \) exists. If suffices to prove that \( \lim_{n \to \infty} x_n = b \) (apply the result for \( x_n = a \) for all \( n \) and we find \( a = b \)). Let \( \epsilon > 0 \). There is \( k \in \mathbb{N} \) for which \( |a_k - b| < \epsilon \). For large \( n \), we have \( |f(x_n) - f(a)| < |f(a_k) - f(a)| \), hence for large \( n \) and \( m(n) \) we get \( |f(x_n) - f(a_m)| < |f(a_k) - f(a_m)| \) whence \( |x_n - a_m| \leq |a_k - a_m| \), so \( |x_n - b| = |a_k - b| < \epsilon \) for large \( n \).

THEOREM 2.7. Let \( f \in M_b(X) \) or \( f \in M_s(X) \) and assume that \( f(X) \) is convex. Then \( f \) is a scalar multiple of an isometry, or \( f \) is constant.
PROOF. We may assume that $f$ is not constant, so $f(X)$ is open, non-empty. Let $X = f(X)$ be bounded. By 2.6, $f$ is injective. It is clear that also $f^{-1} \in M_b(X) \cup M_s(X)$. Applying 2.4 to both $f$ and $f^{-1}$ we get (in both cases $M = 1$) for all $x,y,u,v \in X$ that $|f(x) - f(y)| \leq |x-y|$ and $|f^{-1}(u) - f^{-1}(v)| \leq |u-v|$. It follows that $f$ is an isometry. The general case is now easy. (If $X_f$ are bounded a transformation of the type $x \mapsto ax+b$ sends $f(X)$ into $X$; if $X = f(X) = K$, apply 2.7. to $X_n = \{ x \in K : |x| \leq n \} \ (n \in \mathbb{N})$.

The following theorem describes the functions "of bounded variation":

THEOREM 2.8. Let $X$ be bounded. Then $[M_s(X)] = [M_b(X)] = B\Delta(X)$, where $B\Delta(X)$ is the linear space of functions $f : X \to K$ having bounded difference quotients.

PROOF. By 2.4 we are done if we can prove that an $f \in B\Delta(X)$ is the sum of two $M_{bs}$-functions. Let $\lambda \in K$ such that $|f(x) - f(y)| < |\lambda||x-y|$ for all $x,y \in X$, $x \neq y$. Let $g(x) : = \lambda x$ $(x \in X)$ and let $h : = f - g$. Then $g, h$ are in $M_{bs}(X)$ (scalar multiples of isometries) and $f = g + h$.

3. Monotone sequences

We will not delve deeply into this subject, but content ourselves with presenting definitions and some facts indicating that these notions are not that bad.

DEFINITION 3.1. Let $x_1, x_2, \ldots$ be a sequence in $K$. It is called

$b$-monotone, if $k \leq l \leq m$ implies $x_l \in [x_k, x_m]$, $s$-monotone, if $l < k, m$ implies $x_l \notin [x_k, x_m]$.

THEOREM 3.2. A sequence $x_1, x_2, \ldots$ in $K$ is $s$-monotone if and only if $|x_1-x_2| > |x_2-x_3| > \ldots$

A sequence $x_1, x_2, \ldots$ in $K$ is $b$-monotone if and only if for each $k, m \in \mathbb{N}, k < m$: 
|x_m - x_k| = \max \{ |x_{i+1} - x_i| : k \leq i < m \}.

**PROOF.** Let x_1, x_2, ... be s-monotone, and let n \in \mathbb{N}. We have 
\[ x_n \notin [x_{n+1}, x_{n+2}] \] so
\[ |x_n - x_{n+1}| > |x_{n+1} - x_{n+2}|. \]
Conversely, if
\[ |x_1 - x_2| > |x_2 - x_3| > \ldots, \]
let 1 \leq k, m. Then
\[ |x_k - x_1| = |x_{k+1} - x_1| \]
and
\[ |x_m - x_k| = |x_{k+1} - x_k| \]
hence
\[ |x_k - x_1| > |x_{m} - x_k| \]
i.e.,
\[ x_1 \notin [x_k, x_m]. \]

Let x_1, x_2, ... be b-monotone, and let k, m \in \mathbb{N}, k < m, and k \leq i < m.

Then
\[ x_i \in [x_k, x_m] \]
and
\[ x_{i+1} \in [x_k, x_m], \]
so
\[ [x_i, x_{i+1}] \subset [x_k, x_m], \]
hence
\[ |x_{i+1} - x_i| \leq |x_m - x_k|. \]
The rest is obvious. To prove the converse, let k < 1 < m. Then
\[ |x_1 - x_k| = \max \{ |x_{i+1} - x_i| : k \leq i < 1 \} \leq \max \{ |x_{i+1} - x_i| : k \leq i < m \} = |x_m - x_k|. \]
Hence
\[ x_1 \in [x_k, x_m]. \]

**COROLLARY 3.3.** Each s-monotone sequence is b-monotone. An s-monotone sequence x_1, x_2, ... is convergent and \( x_n \neq x_m \) whenever \( n \neq m \). \( M_b \)-functions map b-monotone sequences into b-monotone sequences. \( M_s \)-functions map s-monotone sequences into s-monotone sequences.

A b-monotone sequence need not be convergent. In fact, if
\[ |x_1| < |x_2| < \ldots \] then x_1, x_2, ... is b-monotone. But we have

**THEOREM 3.4.** Let x_1, x_2, ... be b-monotone. Then either \[ \lim_{n \to \infty} |x_n| = \infty \]
or \[ \lim_{n \to \infty} x_n \]
exists.

**PROOF.** If not \[ \lim_{n \to \infty} |x_n| = \infty \] then the sequence has a bounded, hence a convergent subsequence, say, \( \lim_{i \to \infty} x_{n_i} = x \). We show that \( \lim_{n \to \infty} x_n = x \).

Let \( \epsilon > 0 \). Then
\[ |x - x_{n_k}| < \epsilon \] for \( k \geq k_0 \). If \( 1, m \geq n_{k_0} \) then if \( n_k \geq 1, m \) we have
\[ |x_{k} - x_{m}| \leq |x_{n_{k_0}} - x_{n_{k}}| \leq \max \{ |x - x_{n_{k_0}}|, |x - x_{n_{k}}| \} < \epsilon. \]

Hence x_1, x_2, ... is Cauchy, so it must converge, to x.

**THEOREM 3.5.** Each sequence in \( K \) has a b-monotone subsequence.
PROOF. We may assume that $x_1, x_2, \ldots$ is bounded, and that it has a convergent subsequence $y_1, y_2, \ldots$ where $y_n \neq y_m$ whenever $n \neq m$. Set $y = \lim_{n \to \infty} y_n$. Now it is easy to construct a subsequence $z_1, z_2, \ldots$ of $y_1, y_2, \ldots$ for which $|y-z_1| > |y-z_2| > |y-z_3| > \ldots$ Hence $|z_1-z_2| > |z_2-z_3| > \ldots$. The sequence $z_1, z_2, \ldots$ is s-monotone, hence $b$-monotone.

EXAMPLE. Let $x = \sum_{n=0}^{\infty} a_n p^n \in \mathbb{Z}_p$, and let $x_k = \sum_{n=0}^{k} a_n p^n$. Then $x_0, x_1, \ldots$ is a $b$-monotone sequence, converging to $x$.

4. Monotone functions of type $\sigma$

Following Monna [1] we introduce the concept of "sides of zero" in $\mathbb{K}$ (a generalization of the partition $\mathbb{R}^* = \mathbb{R}^+ \cup \mathbb{R}^-$). Let $K^* = K \setminus \{0\}$. Define

$$x \sim y \text{ if } x \text{ and } y \text{ are at the same side of } 0 \ (x, y \in K^*).$$

Then we have $x \sim y \iff 0 \notin [x,y] \iff |x-y| < |y| \iff |xy^{-1}| < 1 \iff xy^{-1} \in K^+$,

where

$$K^+ = \{x \in K : |1-x| < 1\}.$$

$K^+$ is a multiplicative open compact subgroup of $K^*$, called the group of the positive elements of $K$. We see that $\sim$ is the equivalence relation induced by the canonical group homomorphism, the "sign map":

$$\text{sgn}: K^* \to K^*/K^+.$$

The quotient group $\Sigma = K^*/K^+$ (comparable with $\{1, -1\}$ in the real case) is called the group of signs of elements of $K$, or the group of sides of zero of $K$. $\Sigma$ is an infinite group, whose elements are multiplicative cosets of $K^+$.

Let $\alpha \in \Sigma$ and let $x, y \in \alpha$. Then $|x-y| < |x|$, so in particular,
|x| = |y|. Therefore we may define the absolute value of a sign \( \alpha \in \Sigma \) as

\[ |\alpha| = |x| \quad (x \in \alpha) \]

The map \( \alpha \mapsto |\alpha| \) is a surjective group homomorphism \( \Sigma \to |K^*| \).

Its kernel, \( \{ \alpha \in \Sigma: |\alpha| = 1 \} \) is a multiplicative subgroup of \( \Sigma \), which is naturally isomorphic to the multiplicative group \( K^* \) under \( \alpha \mapsto \bar{\alpha} \) (where \( x \mapsto \overline{x} \) is the canonical map \( \{ x \in K: |x| < 1 \} \to k \)). Let us denote its inverse \( k^\ast \cong \{ \alpha \in \Sigma: |\alpha| = 1 \} \) by

\[ 1 \mapsto \alpha_1 \quad (1 \in K^*) \]

For each \( n \in \mathbb{Z} \), let \( \alpha_n^* = \operatorname{sgn}(\pi^n) \).

Now for each \( \alpha \in \Sigma \) there are unique \( n \in \mathbb{Z}, l \in k^* \) such that

\[ \alpha = \alpha_1^l \alpha_n \]

Further, we have

\[ \alpha_1^l \alpha_1^{l'} = \alpha_1^{l+l'} \quad (l,l' \in k^*) \]
\[ \alpha_n^a \alpha_m^a = \alpha_n^a \alpha_m^a \quad (n,m \in \mathbb{Z}). \]

It follows, that \( \Sigma \) is isomorphic to \( k^* \times \mathbb{Z} \) (or to \( k^* \times |k^*| \) for that matter). It is also possible to identify \( \Sigma \) with a subgroup of \( K^* \) but we shall not need it here.

In the sequel we will use addition of signs. For \( \alpha \in \Sigma \), let

\[ -\alpha = \{ -x: x \in \alpha \}. \]

Then clearly \( -\alpha \) is a sign and if \( \alpha = \alpha_1^l \alpha_n \ (l \in k^*, n \in \mathbb{Z}) \) then

\[ -\alpha = \alpha_{-1}^{-l} \alpha_n. \]

Let \( \alpha, \beta \in \Sigma \) Then \( \alpha + \beta = \{ x+y: x \in \alpha, y \in \beta \} \) is easily seen to be a ball, which turns out to be again a sign iff \( \alpha \neq -\beta \) (iff \( 0 \not\in \alpha + \beta \)). Therefore we define
\( \alpha \oplus \beta = \alpha + \beta \quad (\alpha, \beta \in \Sigma, \alpha \neq -\beta) \).

We have the following rules that are easy to prove:

RULES 4.1. (i) The operation \( \oplus \) is commutative, associative, distributive, whenever the occurring formulas are defined.

(ii) \( |\alpha \oplus \beta| = \max(|\alpha|, |\beta|) \quad (\alpha, \beta \in \Sigma, \alpha \neq -\beta) \)

(iii) \(|\alpha| < |\beta| \) if and only if \( \alpha \oplus \beta = \beta \quad (\alpha, \beta \in \Sigma) \)

(iv) Let \( n \in \mathbb{N}, 1 \leq n < \chi(k), \alpha \in \Sigma. \) Then \( \sum_{n} \alpha := \alpha \oplus \alpha \oplus \ldots \oplus \alpha(n \text{ times}) \) exists and equals \( n\alpha \).

Thus, we get for \( 1, 1' \in k^{*}, m, n \in \mathbb{Z} : \)

\[
\alpha_{1} \alpha_{m} \oplus \alpha_{1'} \alpha_{n} = \begin{cases} 
\alpha_{1} \alpha_{m} & \text{if } m < n \\
\alpha_{1} \alpha_{n} & \text{if } m > n \\
\alpha_{1} \alpha_{1'+1} \alpha_{n} & \text{if } m = n \text{ and } 1 + 1' \neq 0.
\end{cases}
\]

Next, we define a "pseudo-ordering" in \( K \). Let \( x, y \in K, \alpha \in \Sigma \). We say that \( x > y \) (\( x \) is greater than \( y \) in the sense of \( \alpha \)) in case \( x - y \in \alpha \). The following easy consequences obtain.

RULES 4.2. (i) Let \( x, y \in K \). Then if \( y \neq x \) there is exactly one \( \alpha \in \Sigma \) for which \( x > y \). If \( y = x \) then \( x > y \) for no \( \alpha \). ("\( K \) is totally pseudo-ordered").

(ii) If \( x > y, y > z \) for some \( x, y, z \in \mathbb{K}; \alpha, \beta \in \Sigma, \) and if \( \alpha \oplus \beta \) exists then \( x > z \). ("Transitivity").

(iii) If \( x > y \) for some \( x, y \in \mathbb{K}; \alpha \in \Sigma \) and if \( z \in \mathbb{K}, \)

then \( x + z > y + z \) ("Compatibility with addition").

(iv) If \( x, y, z \in \mathbb{K}; \alpha, \beta \in \Sigma, x > y \) and \( z > 0 \) then \( xz > yz \) ("Compatibility with multiplication").

We define:
DEFINITION 4.3. Let $\sigma: \Sigma \to \Sigma$ be an injection and $f: K \to K$. 

$f$ is called monotone of type $\sigma$ if for all $\alpha \in \Sigma$, $x, y \in K$ we have 

$x > y$ implies $f(x) > f(y)$.

$f$ is called increasing if $\sigma$ is the identity map.

REMARKS

1. One can extend easily the above definition for functions $f: X \to K$, where $X$ is a convex subset of $K$, $\sigma: \Sigma(X) \to \Sigma$.

Here $\Sigma(X)$ is the collection of signs that "occur in $X"$ i.e. 

$\{ \text{sgn}(x-y): x, y \in X, x \neq y \}$. We leave it to the reader to do this and, in case $X \neq K$, to show that there is $\beta \in \Sigma$ such that 

$$\Sigma(X) = \{ \alpha \in \Sigma: |\alpha| < |\beta| \}.$$

2. Notice that $f: K \to K$ is increasing if and only if the difference quotient 

$$\Phi_f(x, y) := \frac{f(x) - f(y)}{x - y} \quad (x \neq y)$$

is positive. In particular, $f$ is an isometry.

3. The requirement made in 4.3. that $\sigma$ be an injection is essential in case $k$ is not a prime field (see [4]).

4. In case $\chi(K) = 0$, the exponential function, defined by the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is increasing on its convergence region. 

If $f$ is an increasing function and $\beta \in \Sigma$ then for each $b \in \beta$ the function $bf$ is of type $\sigma$ where $\sigma$ is the multiplier $\alpha \mapsto \alpha \beta$.

THEOREM 4.4. (i) Let $f: K \to K$ be a monotone of type $\sigma: \Sigma \to \Sigma$. Then 

$$\sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta) \quad (\alpha, \beta \in \Sigma, \alpha \neq -\beta)$$

(ii) Let $\sigma: \Sigma \to \Sigma$ satisfy (*). Then there is a function $g: K \to K$, monotone of type $\sigma$. 

PROOF. (i) First we show that $\sigma(-\alpha) = -\sigma(\alpha)$ ($\alpha \in \Sigma$). In fact choose $x$, $y \in K$ such that $x - y \in \alpha$. Then $y - x \in -\alpha$, so $f(x) - f(y) \in \sigma(\alpha) \cap (-\sigma(-\alpha)) \neq \emptyset$, which implies $\sigma(\alpha) = -\sigma(-\alpha)$.

Now take $\alpha, \beta \in \Sigma$, $\alpha \neq -\beta$. Then by injectivity of $\sigma$, $\sigma(\alpha) \neq \sigma(-\beta) = -\sigma(\beta)$ so that $\sigma(\alpha) \oplus \sigma(\beta)$ exists. Now choose $x$, $y$, $z \in K$ such that $x - y \in \alpha$, $y - z \in \beta$. Then $f(x) - f(z) = f(x) - f(y) + f(y) - f(z) \in \sigma(\alpha) \oplus \sigma(\beta)$. Also, $x - z \in \alpha \oplus \beta$ so $f(x) - f(z) \in \sigma(\alpha \oplus \beta)$. As $\sigma(\alpha) \oplus \sigma(\beta)$ and $\sigma(\alpha \oplus \beta)$ have a non-empty intersection they are equal.

The proof of (ii) will be furnished by Lemma 4.5. and Lemma 4.6.:

**LEMMA 4.5.** Let $\sigma: \Sigma \to \Sigma$ satisfy $\sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta)$ ($\alpha, \beta \in \Sigma$, $\alpha \neq -\beta$). Then we have

(i) $\sigma(n\alpha) = n\sigma(\alpha)$ ($n \in \mathbb{N}$, $1 \leq n < \chi(k)$, $\alpha \in \Sigma$).

(ii) $\sigma(-\alpha) = -\sigma(\alpha)$ ($\alpha \in \Sigma$)

(iii) For all $\alpha, \beta \in \Sigma$: $|\alpha| < |\beta|$ if and only if $|\sigma(\alpha)| < |\sigma(\beta)|$.

(iv) $\lim_{|\alpha| \to 0} |\sigma(\alpha)| = 0$.

PROOF. (i) is a direct consequence of 4.1.(iv). To prove (ii), set $q := \chi(k)$. Then for $\beta \in \Sigma$, $(q-1)\beta = -\beta$, so, by (i), $\sigma(-\alpha) = \sigma(q-1)\alpha) = (q-1)\sigma(\alpha) = -\sigma(\alpha)$. Let $\alpha, \beta \in \Sigma$, $|\alpha| < |\beta|$. Then $\alpha \oplus \beta = \beta$ so $\sigma(\alpha) \oplus \sigma(\beta) = \sigma(\alpha \oplus \beta) = \sigma(\beta)$ whence $|\alpha\sigma(\alpha)| < |\sigma(\beta)|$. Conversely, suppose $|\alpha\sigma(\alpha)| < |\sigma(\beta)|$. Then clearly $\alpha \neq -\beta$ (otherwise $|\sigma(\alpha)| = |\sigma(-\beta)| = |-\sigma(\beta)| = |\sigma(\beta)|$), so $\alpha \oplus \beta$ exists. Now $\sigma(\beta) = \sigma(\alpha) \oplus \sigma(\beta) = \sigma(\alpha \oplus \beta)$. By injectivity of $\sigma$ we obtain $\beta = \alpha \oplus \beta$ whence $|\alpha| < |\beta|$. So we have (iii). (iv) follows from the fact that if $|\alpha_n| > |\alpha_2| > \ldots \to 0$ then $|\sigma(\alpha_n)| > |\sigma(\alpha_2)| > \ldots$, which last sequence tends to 0 due to the discreteness of the valuation.
LEMMA 4.6. (Extension theorem for monotone functions). Let
\( \phi \neq Y \subset K, f: Y \to K, \sigma: \Sigma \to \Sigma \) satisfy \( \sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta) \) (\( \alpha, \beta \in \Sigma, \alpha \neq -\beta \)). Suppose
\[ x > y \implies f(x) > f(y) \quad (x, y \in Y, \alpha \in \Sigma) \]
Then \( f \) can be extended to a function \( \overline{f}: K \to K \), monotone of type \( \sigma \).

PROOF. By Zorn's lemma it suffices to extend \( f \) to \( Y \cup \{a\} \) (\( a \not\in Y \)) such that \( f(x) - \overline{f}(a) \in \sigma(\text{sgn}(x - a)) \) and \( \overline{f}(a) - f(x) \in \sigma(\text{sgn}(a - x)) \) for all \( x \in Y \). By 4.5.(ii) it suffices to consider only the second case. Let
\[ B_x := f(x) + \sigma(\text{sgn}(a - x)) \quad (x \in Y) \]
Then each \( B_x \) is a ball having diameter \( |\pi| |\sigma(\text{sgn}(a - x))| \neq 0 \).
By the local compactness (in fact, spherical completeness) of \( K \) we are done if we can show that \( B_x \cap B_y \neq \emptyset \) whenever \( x, y \in Y, x \neq y \).

Set \( \alpha := \text{sgn}(a - x) \) and \( \beta := \text{sgn}(a - y) \), \( b \in \sigma(\alpha) \), \( c \in \sigma(\beta) \). We have to prove that \( |f(x) + b - (f(y) + c)| \leq |\pi| \max (|\sigma(\alpha)|, |\sigma(\beta)|) \).
Consider two cases.
1) \( \alpha = \beta \). Then \( a - x \) and \( a - y \) are in \( \alpha \), so \( |a - x - (a - y)| = |x - y| < |\alpha| \), hence \( |\text{sgn}(x - y)| < |\alpha| \). By 4.5.(iii) we have
\[ |\sigma(\text{sgn}(x - y))| < |\sigma(\alpha)|, \text{ so } |\text{sgn}(f(x) - f(y))| < |\sigma(\alpha)| \text{ whence } |f(x) - f(y)| < |\sigma(\alpha)|. \]
Also \( |b - c| < |\sigma(\alpha)| \) since both \( b \) and \( c \) are in \( \sigma(\alpha) \). Consequently \( |f(x) + b - (f(y) + c)| < |\sigma(\alpha)| \).
2) \( \alpha \neq \beta \). Then \( x - y = a - y - (a - x) \in \beta \oplus -\alpha \), so \( f(x) - f(y) \in \sigma(\beta \oplus -\alpha) \). Now \( b - c \in \sigma(\alpha) \oplus (\sigma(\beta) = \sigma(\alpha) \oplus -\sigma(\beta) = \sigma(\alpha \oplus -\beta) = -\sigma(\beta \oplus -\alpha) \). Therefore \( |f(x) - f(y) - (b - c)| < |\sigma(\beta \oplus -\alpha)| = \max (|\sigma(\beta)|, |\sigma(\alpha)|) \).

(The proof of 4.4.(ii): Choose \( Y := \{0\} \) and let \( f: Y \to K \) be defined via \( f(0): = 0 \). Extend \( f \) in the way of 4.6.).
COROLLARY 4.7. Let $f: K \to K$ be monotone of type $\sigma: \Sigma \to \Sigma$. Then $f \in M_{bs}(K)$ (see section 2). More than that: there exists a strictly increasing function $\phi: |K| \to |K|$, continuous at 0, $(\phi(0) = 0)$ such that $|f(x) - f(y)| = \phi(|x - y|)$. $(x,y \in K)$

PROOF. Let $x, y, u, v \in K$ and $x - y \in \alpha \in \Sigma$, $u - v \in \beta \in \Sigma$. Then we have by 4.5.(iii): $|x - y| < |u - v| \Rightarrow |\alpha| < |\beta| \Rightarrow |\sigma(\alpha)| < |\sigma(\beta)| \Rightarrow |f(x) - f(y)| < |f(u) - f(v)|$. The existence of $\phi$ is now clear. The continuity follows from 4.5.(iv).

REMARK. There exist isometries $K \to K$ that are monotone of type $\sigma$ for no $\sigma$ (see [4]).

THEOREM 4.8. Let $f: K \to K$ be monotone of type $\sigma: \Sigma \to \Sigma$. Then $\sigma$ is surjective if and only if $f$ is a bijection (in fact, $f$ is a nonzero scalar multiple of an isometry, by 2.7.).

PROOF. If $\sigma$ is surjective then $\sigma^{-1}$ exists and satisfies the condition of 4.6., so there is a $g: K \to K$, monotone of type $\sigma^{-1}$. Then $f \circ g$ is monotone of type 1, i.e., increasing. It suffices to show that an increasing $h: K \to K$ is surjective. Let $a \in K$ and consider the map $\psi: x \mapsto x - h(x) + a$ $(x \in K)$. Then $|\psi(x) - \psi(y)| \leq |\pi||x - y|$ $(x, y \in K)$. By the Banach contraction theorem, $\psi$ has a fixed point $t$. Then $h(t) = a$: $h$ is surjective. The converse is easy.

EXAMPLE. The monotone functions on $\mathbb{Q}_p$.

As we have seen in section 4, the group of signs of $\mathbb{Q}_p$ is isomorphic to $\mathbb{F}_p^* \times \mathbb{Z}$. Using this interpretation we can describe the sign function as follows. Let $x \in \mathbb{Q}_p$, $x = \sum_{n \geq k} a_n p^n$ be its standard expansion (i.e., $k \in \mathbb{Z}$, $a_n \in \{0,1,2,\ldots,p-1\}$, $a_k \neq 0$). Then $\text{sgn}(x) = (a_k, n) \in \mathbb{F}_p^* \times \mathbb{Z}$. 
Let $\sigma: \Sigma \to \Sigma$ be a type of some monotone function. By 4.5. we have $\sigma(1,n) = 1\sigma(1,n), ((1,n) \in \Sigma)$. Set

$$\sigma(1,n) = (s(n), \lambda(n)) \quad (n \in \mathbb{Z})$$

Since $n < m \iff |p^n| > |p^m| \iff |(1,n)| > |(1,m)| \iff |\sigma(1,n)| > |\sigma(1,m)|$

$\iff |p^\lambda(n)| > |p^\lambda(m)| \iff \lambda(n) < \lambda(m)$, we see that $\lambda: \mathbb{Z} \to \mathbb{Z}$ is strictly increasing. Thus, $\sigma$ has the form

$$(*) \quad (1,n) \mapsto (1s(n), \lambda(n))$$

where $s: \mathbb{Z} \to \mathbb{F}_p^*$ and $\lambda: \mathbb{Z} \to \mathbb{Z}$ is strictly increasing.

Conversely, if we have a map $\sigma$ satisfying (*) where $s: \mathbb{Z} \to \mathbb{F}_p^*$ and $\lambda: \mathbb{Z} \to \mathbb{Z}$ is strictly increasing the function

$$\Sigma a_n p^n \mapsto \Sigma a_n s(n)p^\lambda(n)$$

is monotone of type $\sigma$, as can easily be verified.

For a criterion in order that a continuous function $\mathbb{Z}_p \to \mathbb{Q}_p$ be increasing (expressed by means of its coordinates with respect to some orthonormal base), see [4].
Appendix

Differentiation

We briefly consider the relationship between monotony and differentiation. We refer to [4] for the proofs. Although even increasing functions may be nowhere differentiable there are some connections that are similar to those in the real case.

A function $g: K \to K$ is called positive if $g(K) \subseteq K^+$. A function $h: K \to K$ is of the first class of Baire if there exists a sequence $h_1, h_2, \ldots$ of continuous functions $K \to K$ that converges pointwise to $h$.

THEOREM. (i) Let $f: K \to K$ be increasing, differentiable. Then $f'$ is positive, of the first class of Baire.

(ii) A positive function of the first class of Baire has an increasing antiderivative.

THEOREM. Let $f: K \to K$ be continuously differentiable (which means here that $\lim_{x,y \to a} (x-y)^{-1}(f(x) - f(y))$ exists for $a \in K$), and suppose $f'(a) \neq 0$. Then there is a (convex) neighborhood $X$ of $a$ such that $f|X$ is monotone of type $\sigma$, where $\sigma$ is the map $\alpha \mapsto \text{sgn}(f'(a)) \alpha$.

THEOREM. Let $f: K \to K$ be monotone of type $\sigma$, differentiable. Then there are two cases.

I. $f'(a) = 0$ for some $a \in K$. Then $f' = 0$ everywhere and

$$\lim_{|\alpha| \to 0} \frac{\sigma(\alpha)}{|\alpha|} = 0.$$

II. $f'(a) \neq 0$ for some $a \in K$. Then $f' \neq 0$ everywhere. In fact, $f'$ has constant sign ($x \mapsto \text{sgn}(f'(x))$ is constant).

For small $|\alpha|$, $\frac{\sigma(\alpha)}{\alpha}$ is constant. $f'(a)^{-1}f$ is locally increasing.
REMARK. One can make an example of an everywhere differentiable \( f: K \to K \) with \( f' = 1 \) (so \( f' \) is positive) such that \( f \) is not even locally injective at 0. (\( f \) is, of course, not continuously differentiable).