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NON-ARCHIMEDEAN MONOTONE FUNCTIONS

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Introduction.

In the sequel, K is a non-archimedean, non-trivially valued field, that is complete under the metric induced by the valuation. The residue class field of K is denoted by \( k \). \( X \) will always be a closed, non empty subset of \( K \) without isolated points (except in 2.2, if you want).

Since \( K \) admits no ordering in the usual sense it is not possible to define monotone functions \( X \to K \) just by taking over the classical definitions. Thus, our procedure will be to try and find statements for functions \( \mathbb{R} \to \mathbb{R} \) equivalent to monotony, and formulated in terms that are translatable to \( K \). This way we will obtain several definitions of "\( f : X \to K \) is monotone", that are, although not equivalent, closely related.

The connections between these various definitions and the properties of the non-archimedean monotone functions can be put together to form a little theory which is interesting in its own right, but of which the relations to the other parts of p-adic analysis are yet not very tight.

1. Monotone functions.

For a function \( f : \mathbb{R} \to \mathbb{R} \) the following conditions are equivalent:

(a) \( f \) is monotone (in the non-strict sense),
(b) If \( C \subset \mathbb{R} \) is convex then \( f^{-1}(C) \) is convex,
(c) If \( x \) is between \( y \), \( z \) then \( f(x) \) is between \( f(y) \) and \( f(z) \).

Also, the following conditions are equivalent:

(a) \( f \) is strictly monotone,
(b) \( f \) is injective. If \( C \subset \mathbb{R} \) is convex then \( f(C) \) is relatively convex in \( f(\mathbb{R}) \),
(c) If \( f(x) \) is between \( f(y) \) and \( f(z) \) then \( x \) is between \( y \) and \( z \).

Let \( x, y \in K \). Then the smallest ball that contains \( x, y \) is denoted by \([x, y]\). \( z \in K \) is between \( x \) and \( y \) if \( z \in [x, y] \). (If \( z \notin [x, y] \), we

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call $x$, $y$ at the same side of $z$). A subset $C \subseteq K$ is called convex if $x, y \in C$, $z \in [x, y]$ implies $z \in C$. Each convex subset of $K$ can be written in at least one of the following forms

$$\{x : |x - a| < r\}, \{x : |x - a| \leq r\}$$

for some $a \in K$, $r \in (0, \infty)$.

Let $Z \subseteq Y \subseteq K$. Then $Z$ is called convex in $Y$ if $Z = C \cap Y$, where $C$ is convex.

With all these definitions we have the following theorem.

**Theorem 1.1.** Let $f : X \rightarrow K$. Then the following conditions are equivalent:

1. If $x, y, z \in X$, $x$ is between $y$ and $z$ then $f(x)$ is between $f(y)$ and $f(z)$,

2. If $C \subseteq K$ is convex, then $f^{-1}(C)$ is convex in $X$.

We denote the collection of those $f : X \rightarrow K$ satisfying (1) or (2) by $M_b(X)$, i.e. $f \in M_b(X)$ if, and only if, for each $x, y, z \in X$,

$$|x - y| \leq |y - z| \implies |f(x) - f(y)| \leq |f(y) - f(z)|.$$

Isometries are in $M_b$ (viz. exp), but also non trivial locally constant functions (e.g., choose a center in each ball of radius $r > 0$, and let $f$ be the map assigning to $x \in X$ the center of the ball of radius $r$ to which $x$ belongs. Then $f \in M_b(X)$).

**Theorem 1.2.** Let $f : X \rightarrow K$. Then the following conditions are equivalent

1. If $x, y, z \in X$, $f(x)$ is between $f(y)$ and $f(z)$ then $x$ is between $y$ and $z$,

2. If $C \subseteq X$ is convex in $X$ then $f(C)$ is convex in $f(X)$. $f$ is injective.

We denote the collection of those $f : X \rightarrow K$ satisfying (1') or (2') by $M_s(X)$, i.e. $f \in M_s(X)$ if, and only if, for each $x, y, z \in X$,

$$|x - y| < |y - z| \implies |f(x) - f(y)| < |f(y) - f(z)|.$$

The classical situations suggests the question as to whether $M_s(X) \subseteq M_b(X)$ and also whether $f \in M_b(X)$, $f$ injective implies $f \in M_s(X)$. In general, both statements are false, but we do have the following:

**Theorem 1.3.** $f \in M_s(X)$ implies $f^{-1} \in M_b(f(X))$. $f \in M_b(X)$, $f$ injective implies $f^{-1} \in M_s(f(X))$. If $k$ is finite and $X$ is convex, then an injective $M_b$-function is in $M_s(X)$. 
So we are led to define \( M_{ba}(X) := M_b(X) \cap M_s(X) \) as being the more or less natural translation of "the space of the strictly monotone functions".

The following theorem concerns continuity of monotone functions. For a function \( f : X \to K \), we define its oscillation function, \( \omega_f \), in the usual way:

\[
\omega_f(a) := \lim_{n \to \infty} \sup \{|f(x) - f(y)| ; |x - a| \leq \frac{1}{n} ; |y - a| \leq \frac{1}{n}\}
\]

\[
= \lim_{n \to \infty} \sup \{|f(x) - f(a)| ; |x - a| \leq \frac{1}{n}\} \quad (a \in X).
\]

\( f \) is continuous at \( a \) if, and only if, \( \omega_f(a) = 0 \).

**Theorem 1.4.** Let \( f \) be either in \( M_b(X) \) or in \( M_s(X) \). Then

(i) \( \omega_f(a) = \inf_{x \neq a} |f(x) - f(a)| \quad (a \in X) \)

(ii) \( f \) is bounded on compact subsets of \( X \),

(iii) For each \( a \in X \) we have the following alternative. Either \( f \) is continuous at \( a \), or for each sequence \( x_1, x_2, \ldots \) \((x_n \neq a)\) converging to \( a \), the sequence \( f(x_1), f(x_2), \ldots \) is bounded and has no convergent subsequence.

Let \( g \in M_b(X) \). If \( Y \subset X \) is spherically complete, then so is \( g(Y) \).

Let \( h \in M_s(X) \). If \( Z \subset h(X) \) is spherically complete, then so is \( h^{-1}(Z) \).

**Proof (sketch).** If \( f \in M_b(X) \cup M_s(X) \), then:

\(|x - y| < |y - z|\) implies \(|f(x) - f(y)| \leq |f(y) - f(z)|\).

So \( f \) is locally bounded, and (ii) follows. Of (i), only the \( \leq \) part is interesting. Choose \( z \neq a \). If \(|x - a| < |z - a|\), then

\[|f(x) - f(a)| \leq |f(z) - f(a)|\] whence \( \omega_f(a) \leq |f(z) - f(a)|\).

Let \( \lim x_n = a \) \((x_n \neq a \text{ for all } n)\) and \( \lim f(x_n) = a \). Let \( \lim y_n = a \). It suffices to show that \( \lim f(y_n) = a \). Indeed, let \( \varepsilon > 0 \), and choose \( k \) such that \(|f(x_k) - a| < \varepsilon\). Then \(|y_n - a| < |x_k - a|\) for large \( n \), so

\[|y_n - x_m| < |x_k - x_m|
\]

for large \( m \) depending on \( m \). Hence \(|f(x_n) - f(x_m)| \leq |f(x_k) - f(x_m)|\), so \((m \to \infty)\) \(|f(y_n) - a| \leq |f(x_k) - a| < \varepsilon\), and we have (iii). The rest of the proof is straightforward.

**Corollary 1.5.** Let \( f : X \to K \) be in \( M_b(X) \cup M_s(X) \).

(i) If \( K \) is a local field, then \( f \) is continuous,

(ii) If \(|K|\) is discrete, then \( f \in M_b(X) \Rightarrow f \) is a homeomorphism \( X \sim f(X) \), and \( f \in M_s(X) \Rightarrow f \) is a closed map.

(iii) The graph of \( f \) is closed in \( K^2 \).

(iv) If \( f(X) \) has no isolated points, then \( f \) is continuous.
An $M_b$-function may be everywhere discontinuous on $K$ (even when $|K|$ is discrete).

**Theorem 1.6.** Let $B$ be the unit ball of $K$,

(i) If $K$ is a local field and $f \in M_b(B) \cup M_s(B)$, then $f$ has bounded difference quotients (i.e., there is $C > 0$ such that $|f(x) - f(y)| \leq C|x - y|$ for all $x \in B$). If, in addition, $f(B)$ is convex, then $f$ is a similarity (i.e., a scalar multiple of an isometry).

(ii) If $K$ has discrete valuation and $f \in M_b(B)$ is bounded, then $f$ has bounded difference quotients. If $f \in M_{bs}(B)$ and if $f(B)$ is convex, then $f$ is a similarity.

2. Monotone functions having a type.

In this section, we want to translate the usual classification of (strictly) monotone functions $\mathbb{R} \to \mathbb{R}$ into two types: the increasing and the decreasing functions. The equivalence relation in $\mathbb{R}^r$: $x \sim y$ if $x$ and $y$ are at the same side of 0, yields $(-\infty, 0)$ and $(0, \infty)$ as equivalence classes. The relation $\sim$ is compatible with the canonical group homomorphism $\mathbb{R}^* \to \mathbb{R}^*/\mathbb{R}^+_*$, the latter group being $\{1, -1\}$. $\pi(x)$ (usually called $\text{sgn}(x)$) assigns $+1$ to every positive element and $-1$ to every negative element. A function $f: \mathbb{R} \to \mathbb{R}$ is strictly monotone if there exists $\sigma: \mathbb{R}^*/\mathbb{R}^+_* \to \mathbb{R}^*/\mathbb{R}^+_*$ such that for all $x \neq y$

$$\pi(f(x) - f(y)) = \sigma(\pi(x - y)).$$

If $\sigma$ is the identity then $f$ is called increasing; if $\sigma(1) = 1$, $\sigma(-1) = 1$, $f$ is called decreasing. Other maps $\sigma: [-1, 1] \to [-1, 1]$ can not occur (i.e., there is no $f$ such that, for all $x \neq y$,

$$\pi(f(x) - f(y)) = \sigma(\pi(x - y)).$$

This rather weird description of real monotone functions can be used in the non-archimedean case.

For $x, y \in K^*$ define $x \sim y$ if $x, y$ are at the same side of 0. This means: $0 \not\in [x, y]$, or $|x - y| > |y|$, or $|xy^{-1} - 1| < 1$. Thus $x \sim y$ if, and only if, $xy^{-1} \in K^+$ where

$$K^+ := \{x \in K; |1 - x| < 1\}.$$

We call the elements of $K^+$ the **positive** element of $K$.

The relation $\sim$ is compatible with the canonical homomorphism of (multiplicative) groups

$$\pi: K^* \to K^*/K^+_* =: \Sigma.$$

We call $\Sigma$ the **group of signs** and $\pi(x)$ the **sign** of an element $x \in K^*$ (i.e.,


positive if, and only if, $\pi(x) = 1$.

If $K$ is a local field, we can make a group embedding $\rho : \Sigma \rightarrow K^\ast$ such that $\pi \circ \rho$ is the identity on $\Sigma$. For example, if $K = \mathbb{Q}_p$, $\delta$ is a primitive $(p - 1)$th root of unity, then

$$\pi\left(\sum_{n \geq 0} a_n p^n\right) = a_k p^k \quad (k \in \mathbb{Z}, \ a_k \neq 0)$$

(Here $a_n \in \{0, 1, \delta, \ldots, \delta^{p-2}\}$ for each $n$).

**DEFINITION 2.1.** - Let $\sigma : \Sigma \rightarrow \Sigma$ be any map. A function $f : X \rightarrow K$ is monotone of type $\sigma$ if, for all $x, y \in X$, $x \neq y$,

$$\pi(f(x) - f(y)) = \sigma(\pi(x - y))$$

(i.e., if $x - y \in \sigma$ then $f(x) - f(y) \in \sigma(\sigma)$).

We call $f$ of type $\beta \in \Sigma$ if $f$ is of type $\sigma$ where $\sigma$ is the multiplication with $\beta$, i.e.,

$$\frac{f(x) - f(y)}{x - y} \in \beta \quad (x, y \in X, x \neq y).$$

We call $f$ increasing if $f$ is of type $\sigma$ where $\sigma$ is the identity, i.e.,

$$\frac{f(x) - f(y)}{x - y}$$

is positive $(x \neq y)$.

Clearly, if $f$ is of type $\beta$, and if $b \in \beta$, then $b^{-1} f$ is increasing. First, we look at increasing functions, then we discuss more general types $\sigma$.

Notice that increasing functions are isometries hence are in $K_{bs}(X)$. If $f$ is increasing then $f(x) = x + h(x)$, where $|h(x) - h(y)| < |x - y|$ $(x, y \in X, x \neq y)$.

Such $h$ we call pseudo-contractions.

**LEMMA 2.2.** - Let $X$ be an ultrametric space. Then the following are equivalent

(a) $X$ is spherically complete,

(b) Each pseudocontraction $X \rightarrow X$ has a (unique) fixed point.

**Proof (sketch).** - (a) $\Rightarrow$ (b). Let $\sigma : X \rightarrow X$ be a pseudocontraction. A convex set $C \subset X$ is called invariant if $\sigma(C) \subset C$. It is easily proved that the invariant convex subsets of $X$ form a nest. Let $C_0$ be the smallest invariant convex set. If $a \in C_0$ and $\sigma(a) \neq a$ then

$$B_0 := \{x \in X ; \ d(x, \sigma(a)) < d(a, \sigma(a))\}$$

is invariant, convex, and does not contain $a$. Hence $\sigma(a) = a$ for all $a \in C_0$, and $C_0$ is a singleton. (b) $\Rightarrow$ (a). If $B_1 \neq B_2 \neq \ldots$ are balls in $X$ with $\cap B_n = \emptyset$ then choose $x_n \in B_n \setminus B_{n+1}$ $(n \in \mathbb{N})$. The map $\sigma : X \rightarrow X$ defined by

$$\sigma(x) = x_{n+1} \quad (x \in B_n \setminus B_{n+1})$$

is a pseudocontraction without a fixed point.
COROLLARY 2.3. - Let \( X \) be convex, let \( K \) be spherically complete, and let \( f : X \to K \) be increasing. Then \( f(X) \) is convex. If \( f(X) \subseteq X \), then \( f \) is surjective.

Proof. - Let \( f(X) \subseteq X \). Choose \( a \in X \). Then \( x \mapsto f(x) + x + a \) is a pseudocontraction mapping \( X \) into \( X \), hence has a fixed point. So \( f(x) = a \) for some \( x \in X \).

If \( K \) is not spherically complete, we have always increasing \( f : K \to K \) that are not surjective. (Let \( h : K \to K \) be a pseudocontraction without a fixed point. Let \( f(x) = x - h(x) \) \((x \in K)\), then \( 0 \notin \im f \). The inverse \( f^{-1} : f(K) \to K \) can, of course, not be extended to an increasing function \( K \to K \).

THEOREM 2.4. - Let \( K \) be spherically complete, and let \( f : X \to K \) be increasing. Then \( f \) can be extended to an increasing function \( K \to K \).

Proof. - By Zorn's Lemma, it suffices to extend \( f \) to an increasing function on \( X \cup \{a\} \), where \( a \notin X \). We are done if we can find \( a \in K \) such that, for all \( x \in X \),

\[
\frac{a - f(x)}{a - x} - 1 < 1
\]

i. e. \( a \in B_{f(x)-(a-x)}(|a-x|) \) for all \( x \in X \). These balls form a nest.

Let us call a function \( f : X \to K \) positive if \( f(X) \subseteq K^+ \).

THEOREM 2.5.

(i) If \( f : X \to K \) is increasing then \( f' \) is positive,

(ii) If \( g : X \to K \) is a positive Baire class one function, then \( g \) has an increasing antiderivative,

(iii) If \( g : X \to K \) is continuous and positive, then \( g \) has a \( C^1 \)-antiderivative,

(iv) If \( f \in C^1(X) \) and \( f' \) is positive then \( f = j + h \) where \( j \) is increasing, and \( h \) is locally constant.

EXAMPLES.

1° The exponential function (defined on its natural convergence region) is increasing.

2° Let \( f \in C(\mathbb{Z}_p) \), and let \( e_0 = \xi_{\mathbb{Z}_p} \), for \( n \in \mathbb{N}_0 \),

\[
e_n(x) = \begin{cases} 1 & \text{if } |x - n| < \frac{1}{n} \\ 0 & \text{elsewhere} \end{cases} \quad (x \in \mathbb{Z}_p).
\]

Then \( e_0, e_1, \ldots \) form an orthonormal base of \( C(\mathbb{Z}_p) \), so there exist \( \lambda_0, \lambda_1, \ldots \in \mathbb{Q}_p \) such that \( f = \sum_{n=0}^{\infty} \lambda_n e_n \), uniformly.
f is increasing if, and only if, for all \( n \in \mathbb{N} \),

\[
|\lambda_n - \{n\}| < \{n\}
\]

(where, if \( n = a_0 + a_1 p + \ldots + a_k p^k \) (\( a_i \in \{0, 1, \ldots, p - 1\} \) for each \( i \), \( a_k \neq 0 \)), then \( \{n\}_1 = a_k p^k \)).

In other words, \( f = \sum \lambda_n \in C(\mathbb{Z}_p) \) is increasing if, and only if, \( \lambda_n/\{n\} \) is positive for all \( n \in \mathbb{N} \).

Let \( \alpha, \beta \in \Sigma \). If the set theoretic sum \( \alpha + \beta := \{x + y ; x \in \alpha, y \in \beta\} \) does not contain \( 0 \) then \( \alpha + \beta \in \Sigma \), notation \( \alpha \oplus \beta \). It follows that \( \alpha \oplus \beta \) is defined if, and only if, \( \alpha \neq - \beta \).

If \( x, y \in \alpha \in \Sigma \) then \( |x| = |y| \). This defines \(|\alpha|\) in a natural way.

We have the following results.

**THEOREM 2.6.** Let \( f : K \to K \) be monotone of type \( \sigma : \Sigma \to \Sigma \). Let \( \alpha, \beta \in \Sigma \),

(i) \( \sigma(-\alpha) = -\sigma(\alpha) \),

(ii) If \( \sigma(\alpha) \oplus \sigma(\beta) \) is defined then so is \( \sigma(\alpha) \oplus \sigma(\beta) \) and \( \sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta) \),

(iii) \( |\alpha| < |\beta| \) implies \( |\sigma(\alpha)| < |\sigma(\beta)| \),

(iv) If \( |\beta| = 1 \), \( \beta \) contains an element of the prime field of \( K \) then \( \sigma(\beta) = \beta \sigma(\alpha) \),

(v) \( f \in M_a(K) \),

(vi) \( f \) is either nowhere continuous or uniformly continuous.

**THEOREM 2.7.** Let \( f : K \to K \) be monotone of type \( \sigma : \Sigma \to \Sigma \). Then the following conditions are equivalent,

(a) \( \sigma \) is injective,

(b) \( f \in M_b(X) \),

(c) If for some \( \alpha, \beta \in \Sigma \), \( \alpha \oplus \beta \) is defined, then so is \( \sigma(\alpha) \oplus \sigma(\beta) \),

(d) \( |\sigma(\alpha)| < |\sigma(\beta)| \) implies \( |\alpha| < |\beta| \) (\( \alpha, \beta \in \Sigma \)).

**COROLLARY 2.8.** Let \( K \) be a prime field, and let \( f : K \to K \) be monotone of type \( \sigma : \Sigma \to \Sigma \). Then \( \sigma \) is injective.

(If \( K = Q_p(N^{-1}) \), \( p = 3 \mod 4 \), we can find an example of an \( f : K \to K \) monotone of type \( \sigma \), where \( \sigma \) is not injective).

**EXAMPLE 2.9.** Let \( K = Q_p \). Then

\( \{\sigma : \Sigma \to \Sigma : \text{there is } f : Q_p \to Q_p , \ f \text{ monotone of type } \sigma\} \)
consists of all \( \sigma: \Sigma \to \Sigma \) of the form
\[
\delta_{p}^{i} \gamma_{n} \to \delta_{p}^{i} \gamma_{n} \lambda(n)
\]
where \( s: \mathbb{Z} \to \{0, 1, 2, \ldots, p - 2\} \) and \( \lambda: \mathbb{Z} \to \mathbb{Z} \) is strictly increasing.

3. Functions of bounded variation.

**Lemma 3.1.** - Let \( f: X \to K \) have bounded difference quotients. Then \( f \) is a linear combination of two increasing functions.

**Proof.** - Choose \( \lambda \in K \),
\[
|\lambda| > \sup \{|f(x) - f(y)|; x \neq y\}.
\]
Then \( \lambda^{-1} f \) is a (pseudo-) contraction, so \( g(x) := -x + \lambda^{-1} f(x) \) (\( x \in X \)) is increasing. If \( h(x) := x \) (\( x \in X \)), then \( \lambda h + \lambda g = f \).

**Corollary 3.2.** - Let \( X \) be the unit ball of a local field \( K \) and let \( f: X \to K \). Then the following are equivalent

(a) \( f \in BA(X) \) (i.e., \( \sup \{|f(x) - f(y)|; x \neq y\} < \infty \)),
(b) \( f \) is a linear combination of two increasing functions,
(c) \( f \in [\mathcal{B}_{g}(X)] \),
(d) \( f \in [\mathcal{B}_{b}(X)] \).

**Proof.** - Use 1.6.

**References**
