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NON-ARCHIMEDEAN MONOTONE FUNCTIONS

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Introduction.

In the sequel, K is a non-archimedean, non-trivially valued field, that is complete under the metric induced by the valuation. The residue class field of K is denoted by k. X will always be a closed, non empty subset of K without isolated points (except in 2.2, if you want).

Since K admits no ordering in the usual sense it is not possible to define monotone functions $X \to K$ just by taking over the classical definitions. Thus, our procedure will be to try and find statements for functions $\mathbb{R} \to \mathbb{R}$ equivalent to monotony, and formulated in terms that are translatable to K. This way we will obtain several definitions of " $f : X \to K$ is monotone", that are, although not equivalent, closely related.

The connections between these various definitions and the properties of the non-archimedean monotone functions can be put together to form a little theory which is interesting in its own right, but of which the relations to the other parts of p-adic analysis are yet not very tight.

1. Monotone functions.

For a function $f : \mathbb{R} \to \mathbb{R}$ the following conditions are equivalent :

(α) f <u>is monotone (in the non-strict sense)</u>,

(β) <u>If</u> $C \subset \mathbb{R}$ <u>is convex then</u> $f^{-1}(C)$ <u>is convex</u>,

(γ) <u>If</u> x <u>is between</u> y , z <u>then</u> $f(x)$ <u>is between</u> $f(y)$ <u>and</u> $f(z)$.

Also, the following conditions are equivalent :

(a) f <u>is strictly monotone</u>,

(b) f <u>is injective. If</u> $C \subset \mathbb{R}$ <u>is convex then</u> $f(C)$ <u>is relatively convex in</u> $f(\mathbb{R})$,

(c) <u>If</u> $f(x)$ <u>is between</u> $f(y)$ <u>and</u> $f(z)$ <u>then</u> x <u>is between</u> y <u>and</u> z .

Let $x , y \in K$. Then the smallest ball that contains x , y is denoted by $[x , y]$. $z \in K$ is <u>between</u> x and y if $z \in [x , y]$. (If $z \notin [x , y]$, we

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call x , y at <u>the same side of</u> z). A subset $C \subset K$ is called <u>convex</u> if $x , y \in C$, $z \in [x , y]$ implies $z \in C$. Each convex subset of K can be written in at least one of the following forms

$$\{x : |x - a| < r\} , \quad \{x : |x - a| \leq r\}$$

for some $a \in K$, $r \in (0 , \infty)$.

Let $Z \subset Y \subset K$. Then Z is called <u>convex in</u> Y <u>if</u> $Z = C \cap Y$, <u>where</u> C <u>is convex</u>.

With all these definitions we have the following theorem.

THEOREM 1.1. - <u>Let</u> $f : X \to K$. <u>Then the following conditions are equivalent</u> :

(1) <u>If</u> $x , y , z \in X$, x <u>is between</u> y <u>and</u> z <u>then</u> $f(x)$ <u>is between</u> $f(y)$ <u>and</u> $f(z)$,

(2) <u>If</u> $C \subset K$ <u>is convex, then</u> $f^{-1}(C)$ <u>is convex in</u> X .

We denote the collection of those $f : X \to K$ satisfying (1) or (2) by $M_b(X)$, i. e. $f \in M_b(X)$ if, and only if, for each $x , y , z \in X$,

$$|x - y| \leq |y - z| \quad \text{implies} \quad |f(x) - f(y)| \leq |f(y) - f(z)| .$$

Isometries are in M_b (viz. exp), but also non trivial locally constant functions (e. g., choose a center in each ball of radius $r > 0$, and let f be the map assigning to $x \in X$ the center of the ball of radius r to which x belongs. Then $f \in M_b(X)$).

THEOREM 1.2. - <u>Let</u> $f : X \to K$. <u>Then the following conditions are equivalent</u>

(1') <u>If</u> $x , y , z \in X$, $f(x)$ <u>is between</u> $f(y)$ <u>and</u> $f(z)$ <u>then</u> x <u>is between</u> y <u>and</u> z ,

(2') <u>If</u> $C \subset X$ <u>is convex in</u> X <u>then</u> $f(C)$ <u>is convex in</u> $f(X)$. f <u>is injective</u>.

We denote the collection of those $f : X \to K$ satisfying (1') or (2') by $M_s(X)$, i. e. $f \in M_s(X)$ if, and only if, for each $x , y , z \in X$.

$$|x - y| < |y - z| \quad \text{implies} \quad |f(x) - f(y)| < |f(y) - f(z)| .$$

The classical situations suggests the question as to wether $M_s(X) \subset M_b(X)$ and also wether $f \in M_b(X)$, f injective implies $f \in M_s(X)$. In general, both statements are false, but we do have the following :

THEOREM 1.3. - $f \in M_s(X)$ <u>implies</u> $f^{-1} \in M_b(f(X))$. $f \in M_b(X)$, f <u>injective implies</u> $f^{-1} \in M_s(f(X))$. <u>If</u> k <u>is finite and</u> X <u>is convex, then an injective</u> M_b<u>-function is in</u> $M_s(X)$.

So we are led to define $M_{bs}(X) := M_b(X) \cap M_s(X)$ as being the more or less natural translation of "the space of the strictly monotone functions".

The following theorem concerns continuity of monotone functions. For a function $f : X \to K$, we define its oscillation function, ω_f, in the usual way :

$$\omega_f(a) := \lim_{n\to\infty} \sup\{|f(x) - f(y)| \ ; \ |x - a| \leq \frac{1}{n} \ ; \ |y - a| \leq \frac{1}{n}\}$$
$$= \lim_{n\to\infty} \sup\{|f(x) - f(a)| \ ; \ |x - a| \leq \frac{1}{n}\} \quad (a \in X) .$$

f is continuous at a if, and only if, $\omega_f(a) = 0$.

THEOREM 1.4. - Let f be either in $M_b(X)$ or in $M_s(X)$. Then

(i) $\omega_f(a) = \inf_{z\neq a}|f(z) - f(a)| \quad (a \in X)$

(ii) f is bounded on compact subsets of X,

(iii) For each $a \in X$ we have the following alternative. Either f is continuous at a, or for each sequence $x_1, x_2, \ldots$ $(x_n \neq a)$ converging to a, the sequence $f(x_1), f(x_2), \ldots$ is bounded and has no convergent subsequence.

Let $g \in M_b(X)$. If $Y \subset X$ is spherically complete, then so is $g(Y)$.

Let $h \in M_s(X)$. If $Z \subset h(X)$ is spherically complete, then so is $h^{-1}(Z)$.

Proof (sketch). - If $f \in M_b(X) \cup M_s(X)$, then :

$$|x - y| < |y - z| \text{ implies } |f(x) - f(y)| \leq |f(y) - f(z)| .$$

So f is locally bounded, and (ii) follows. Of (i), only the $\leq$ part is interesting. Choose $z \neq a$. If $|x - a| < |z - a|$, then

$$|f(x) - f(a)| \leq |f(z) - f(a)| \text{ whence } \omega_f(a) \leq |f(z) - f(a)| .$$

Let $\lim x_n = a$ ($x_n \neq a$ for all n) and $\lim f(x_n) = \alpha$. Let $\lim y_n = a$. It suffices to show that $\lim f(y_n) = \alpha$. Indeed, let $\varepsilon > 0$, and choose k such that $|f(x_k) - \alpha| < \varepsilon$. Then $|y_n - a| < |x_k - a|$ for large n, so

$$|y_n - x_m| < |x_k - x_m|$$

for large m depending on m. Hence $|f(y_n) - f(x_m)| \leq |f(x_k) - f(x_m)|$, so $(m \to \infty)$ $|f(y_n) - \alpha| \leq |f(x_k) - \alpha| < \varepsilon$, and we have (iii). The rest of the proof is straightforward.

COROLLARY 1.5. - Let $f : X \to K$ be in $M_b(X) \cup M_s(X)$.

(i) If K is a local field, then f is continuous,

(ii) If $|K|$ is discrete, then $f \in M_s(X) \Rightarrow f$ is a homeomorphism $X \sim f(X)$, and $f \in M_b(X) \Rightarrow f$ is a closed map.

(iii) The graph of f is closed in K^2,

(iv) If $f(X)$ has no isolated points, then f is continuous.

An M_b-function may be everywhere discontinuous on K (even when $|K|$ is discrete).

THEOREM 1.6. - *Let* B *be the unit ball of* K ,

(i) *If* K *is a local field and* $f \in M_b(B) \cup M_s(B)$, *then* f *has bounded difference quotients* (*i. e. there is* $C > 0$ *such that* $|f(x) - f(y)| \leq C|x - y|$ *for all* $x \in B$). *If, in addition,* $f(B)$ *is convex, then* f *is a similarity* (*i. e., a scalar multiple of an isometry*).

(ii) *If* K *has discrete valuation and* $f \in M_s(B)$ *is bounded, then* f *has bounded difference quotients. If* $f \in M_{bs}(B)$ *and if* $f(B)$ *is convex, then* f *is a similarity.*

2. Monotone functions having a type.

In this section, we want to translate the usual classification of (strictly) monotone functions $\mathbb{R} \to \mathbb{R}$ into two types : the increasing and the decreasing functions. The equivalence relation in $\mathbb{R}^*$: $x \sim y$ if x and y are at the same side of 0 , yields $(-\infty , 0)$ and $(0 , \infty)$ as equivalence classes. The relation $\sim$ is compatible with the canonical group homomorphism $\mathbb{R}^* \xrightarrow{\pi} \mathbb{R}^*/\mathbb{R}^+$, the latter group being $\{1 , -1\}$. $\pi(x)$ (usually called $\mathrm{sgn}(x)$) assigns $+1$ to every positive element and -1 to every negative element. A function $f : \mathbb{R} \to \mathbb{R}$ is strictly monotone if there exists $\sigma : \mathbb{R}^*/\mathbb{R}^+ \to \mathbb{R}^*/\mathbb{R}^+$ such that for all $x \neq y$

$$\pi(f(x) - f(y)) = \sigma(\pi(x - y)) .$$

If σ is the identity then f is called increasing ; if $\sigma(1) = -1$, $\sigma(-1) = 1$, f is called decreasing. Other maps $\sigma : \{-1 , 1\} \to \{-1 , 1\}$ can not occur (i. e., there is no f such that, for all $x \neq y$,

$$\pi(f(x) - f(y)) = \sigma(\pi(x - y))).$$

This rather weird description of real monotone functions can be used in the non-archimedean case.

For $x , y \in K^*$ define $x \sim y$ if x , y are at the same side of 0 . This means : $0 \notin [x , y]$, or $|x - y| > |y|$, or $|xy^{-1} - 1| < 1$. Thus $x \sim y$ if, and only if, $xy^{-1} \in K^+$ where

$$K^+ := \{x \in K \; ; \; |1 - x| < 1\} .$$

We call the elements of K^+ the *positive* element of K .

The relation $\sim$ is compatible with the canonical homomorphism of (multiplicative) groups

$$\pi : K^* \to K^*/K^+ =: \Sigma .$$

We call Σ *the group of signs* and $\pi(x)$ the *sign* of an element $x \in K^*$ (x is positive if, and only if, $\pi(x) = 1$).

If K is a local field, we can make a group embedding $\rho : \Sigma \hookrightarrow K^*$ such that $\pi \circ \rho$ is the identity on Σ. For example, if $K = \mathbb{Q}_p$, δ is a primitive $(p-1)^{th}$ root of unity, then

$$\pi(\Sigma_{n\geq k} a_n p^n) = a_k p^k \qquad (k \in \mathbb{Z}, \quad a_k \neq 0)$$

(Here $a_n \in \{0, 1, \delta, \ldots, \delta^{p-2}\}$ for each n).

DEFINITION 2.1. - Let $\sigma : \Sigma \longrightarrow \Sigma$ be any map. A function $f : X \longrightarrow K$ is monotone of type σ if, for all $x, y \in X$, $x \neq y$,

$$\pi(f(x) - f(y)) = \sigma(\pi(x - y))$$

(i. e., if $x - y \in \alpha \in \Sigma$ then $f(x) - f(y) \in \sigma(\alpha)$).

We call f of type $\beta \in \Sigma$ if f is of type σ where σ is the multiplication with β, i. e.

$$\frac{f(x) - f(y)}{x - y} \in \beta \qquad (x, y \in X, \quad x \neq y) .$$

We call f increasing if f is of type σ where σ is the identity, i. e.,

$$\frac{f(x) - f(y)}{x - y} \text{ is positive } (x \neq y) .$$

Clearly, if f is of type β, and if $b \in \beta$, then $b^{-1} f$ is increasing. First, we look at increasing functions, then we discuss more general types σ. Notice that increasing functions are isometries hence are in $M_{bs}(X)$. If f is increasing then $f(x) = x + h(x)$, where $|h(x) - h(y)| < |x - y|$ $(x,y \in X, x \neq y)$. Such h we call pseudo-contractions.

LEMMA 2.2. - Let X be an ultrametric space. Then the following are equivalent

(α) X is spherically complete,

(β) Each pseudocontraction $X \longrightarrow X$ has a (unique) fixed point.

Proof (sketch). - $(\alpha) \longrightarrow (\beta)$. Let $\sigma : X \longrightarrow X$ be a pseudocontraction. A convex set $C \subset X$ is called invariant if $\sigma(C) \subset C$. It is easily proved that the invariant convex subsets of X form a nest. Let C_0 be the smallest invariant convex set. If $a \in C_0$ and $\sigma(a) \neq a$ then

$$B_0 := \{x \in X ; \quad d(x, \sigma(a)) < d(a, \sigma(a))\}$$

is invariant, convex, and does not contain a. Hence $\sigma(a) = a$ for all $a \in C_0$, and C_0 is a singleton. $(\beta) \longrightarrow (\alpha)$. If $B_1 \supsetneq B_2 \supsetneq \ldots$ are balls in X with $\bigcap B_n = \emptyset$ then choose $x_n \in B_n \setminus B_{n+1}$ $(n \in \mathbb{N})$. The map $\sigma : X \longrightarrow X$ defined by

$$\sigma(x) = x_{n+1} \qquad (x \in B_n \setminus B_{n+1})$$

is a pseudocontraction without a fixed point.

COROLLARY 2.3. - Let X be convex, let K be spherically complete, and let $f : X \to K$ be increasing. Then $f(X)$ is convex. If $f(X) \subset X$, then f is surjective.

Proof. - Let $f(X) \subset X$. Choose $\alpha \in X$. Then $x \longmapsto -f(x) + x + \alpha$ is a pseudocontraction mapping X into X, hence has a fixed point. So $f(x) = \alpha$ for some $x \in X$.

If K is not spherically complete, we have always increasing $f : K \to K$ that are not surjective. (Let $h : K \to K$ be a pseudocontraction without a fixed point Let $f(x) = x - h(x)$ $(x \in K)$, then $0 \notin \operatorname{im} f$). The inverse $f^{-1} : f(K) \to K$ can, of course, not be extended to an increasing function $K \to K$.

THEOREM 2.4. - Let K be spherically complete, and let $f : X \to K$ be increasing. Then f can be extended to an increasing function $K \to K$.

Proof. - By Zorn's Lemma, it suffices to extend f to an increasing function on $X \cup \{a\}$, where $a \notin X$. We are done if we can find $\alpha \in K$ such that, for all $x \in X$,

$$\left|\frac{\alpha - f(x)}{a - x} - 1\right| < 1$$

i. e. $\alpha \in B_{f(x)-(a-x)}(|a - x|^{-})$ for all $x \in X$. These balls form a nest.

Let us call a function $f : X \to K$ positive if $f(X) \subset K^{+}$.

THEOREM 2.5.

(i) If $f : X \to K$ is increasing then f' is positive,

(ii) If $g : X \to K$ is a positive Baire class one function, then g has an increasing antiderivative,

(iii) If $g : X \to K$ is continuous and positive, then g has a C^1-antiderivative,

(iv) If $f \in C^1(X)$ and f' is positive then $f = j + h$ where j is increasing, and h is locally constant.

EXAMPLES.

1° The exponential function (defined on its natural convergence region) is increasing.

2° Let $f \in C(\mathbb{Z}_p)$, and let $e_0 = \xi_{\mathbb{Z}_p}$, for $n \in \mathbb{N}$,

$$e_n(x) = \begin{cases} 1 & \text{if } |x - n| < \frac{1}{n} \\ 0 & \text{elsewhere} \end{cases} \qquad (x \in \mathbb{Z}_p) .$$

Then $e_0, e_1, \ldots$ form an orthonormal base of $C(\mathbb{Z}_p)$, so there exist $\lambda_0, \lambda_1, \ldots \in \mathbb{Q}_p$ such that $f = \sum_{n=0}^{\infty} \lambda_n e_n$, uniformly.

f _is increasing if_, _and only if_, _for all_ $n \in \underset{\sim}{N}$,

$$|\lambda_n - \{n\}| < \{n\}$$

(where, if $n = a_0 + a_1 p + \dots + a_k p^k$ $(a_i \in \{0, 1, \dots, p-1\}$ for each i , $a_k \neq 0)$, then $\{n\}_1 = a_k p^k$).

In other words, $f = \sum \lambda_n e_n \in C(\underset{\sim}{Z}_p)$ is increasing if, and only if, $\lambda_n/\{n\}$ is positive for all $n \in \underset{\sim}{N}$.

Let α , $\beta \in \Sigma$. If the set theoretic sum $\alpha + \beta := \{x + y ; x \in \alpha , y \in \beta\}$ does not contain 0 then $\alpha + \beta \in \Sigma$, notation $\alpha \oplus \beta$. It follows that $\alpha \oplus \beta$ is defined if, and only if, $\alpha \neq - \beta$.

If x , $y \in \alpha \in \Sigma$ then $|x| = |y|$. This defines $|\alpha|$ in a natural way.

We have the following results.

THEOREM 2.6. - _Let_ $f : K \to K$ _be monotone of type_ $\sigma : \Sigma \to \Sigma$. _Let_ $\alpha, \beta \in \Sigma$,

(i) $\sigma(-\alpha) = -\sigma(\alpha)$,

(ii) _If_ $\sigma(\alpha) \oplus \sigma(\beta)$ _is defined then so is_ $\alpha \oplus \beta$ _and_ $\sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta)$,

(iii) $|\alpha| < |\beta|$ _implies_ $|\sigma(\alpha)| < |\sigma(\beta)|$,

(iv) _If_ $|\beta| = 1$, β _contains an element of the prime field of_ K _then_ $\sigma(\beta\alpha) = \beta\sigma(\alpha)$,

(v) $f \in M_s(K)$,

(vi) f _is either nowhere continuous or uniformly continuous_.

THEOREM 2.7. - _Let_ $f : K \to K$ _be monotone of type_ $\sigma : \Sigma \to \Sigma$. _Then the following conditions are equivalent_,

(α) σ _is injective_,

(β) $f \in M_b(X)$,

(γ) _If for some_ α , $\beta \in \Sigma$, $\alpha \oplus \beta$ _is defined_, _then so is_ $\sigma(\alpha) \oplus \sigma(\beta)$,

(δ) $|\sigma(\alpha)| < |\sigma(\beta)|$ _implies_ $|\alpha| < |\beta|$ $(\alpha , \beta \in \Sigma)$.

COROLLARY 2.8. - _Let_ k _be a prime field_, _and let_ $f : K \to K$ _be monotone of type_ $\sigma : \Sigma \to \Sigma$. _Then_ σ _is injective_.

(If $K = \underset{\sim}{Q}_p(\sqrt{-1})$, $p = 3$ mod 4 , we can find an example of an $f : K \to K$ monotone of type σ , where σ is not injective).

EXAMPLE 2.9. - Let $K = \underset{\sim}{Q}_p$. Then

$$\{\sigma : \Sigma \to \Sigma : \text{there is } f : \underset{\sim}{Q}_p \to \underset{\sim}{Q}_p , \ f \text{ monotone of type } \sigma\}$$

consists of all $\sigma : \Sigma \to \Sigma$ of the form

$$\delta^i p^n \longmapsto \delta^i \delta^{s(n)} p^{\lambda(n)}$$

where $s : \mathbb{Z} \to \{0, 1, 2, \ldots, p-2\}$ and $\lambda : \mathbb{Z} \to \mathbb{Z}$ is strictly increasing.

3. Functions of bounded variation.

LEMMA 3.1. – *Let* $f : X \to K$ *have bounded difference quotients. Then* f *is a linear combination of two increasing functions.*

Proof. – Choose $\lambda \in K$,

$$|\lambda| > \sup\{|\frac{f(x) - f(y)}{x - y}| \ ; \ x \neq y\} .$$

Then $\lambda^{-1} f$ is a (pseudo-) contraction, so $g(x) := -x + \lambda^{-1} f(x)$ $(x \in X)$ is increasing. If $h(x) := x$ $(x \in X)$, then $\lambda h + \lambda g = f$.

COROLLARY 3.2. – *Let* X *be the unit ball of a local field* K *and let* $f : X \to K$. *Then the following are equivalent*

(α) $f \in B\Delta(X)$ (i. e. $\sup\{|\frac{f(x) - f(y)}{x - y}| \ ; \ x \neq y\} < \infty$),

(β) f *is a linear conbination of two increasing functions,*

(γ) $f \in [\![M_s(X)]\!]$,

(δ) $f \in [\![I_b(X)]\!]$.

Proof. – Use 1.6.

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