NON-ARCHIMEDEAN MONOTONE FUNCTIONS

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Introduction.

In the sequel, \( K \) is a non-archimedean, non-trivially valued field, that is complete under the metric induced by the valuation. The residue class field of \( K \) is denoted by \( k \). \( X \) will always be a closed, non empty subset of \( K \) without isolated points (except in 2.2, if you want).

Since \( K \) admits no ordering in the usual sense it is not possible to define monotone functions \( X \rightarrow K \) just by taking over the classical definitions. Thus, our procedure will be to try and find statements for functions \( \mathbb{R} \rightarrow \mathbb{R} \) equivalent to monotony, and formulated in terms that are translatable to \( K \). This way we will obtain several definitions of "\( f : X \rightarrow K \) is monotone", that are, although not equivalent, closely related.

The connections between these various definitions and the properties of the non-archimedean monotone functions can be put together to form a little theory which is interesting in its own right, but of which the relations to the other parts of \( p \)-adic analysis are yet not very tight.

1. Monotone functions.

For a function \( f : \mathbb{R} \rightarrow \mathbb{R} \) the following conditions are equivalent:

(a) \( f \) is monotone (in the non-strict sense),
(b) If \( C \subset \mathbb{R} \) is convex then \( f^{-1}(C) \) is convex,
(c) If \( x \) is between \( y \), \( z \) then \( f(x) \) is between \( f(y) \) and \( f(z) \).

Also, the following conditions are equivalent:

(a) \( f \) is strictly monotone,
(b) \( f \) is injective. If \( C \subset \mathbb{R} \) is convex then \( f(C) \) is relatively convex in \( f(\mathbb{R}) \),
(c) If \( f(x) \) is between \( f(y) \) and \( f(z) \) then \( x \) is between \( y \) and \( z \).

Let \( x, y \in K \). Then the smallest ball that contains \( x, y \) is denoted by \([x, y]\). \( z \in K \) is between \( x \) and \( y \) if \( z \in [x, y] \). (If \( z \notin [x, y] \), we

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call \(x, y\) at the same side of \(z\). A subset \(C \subseteq K\) is called convex if \(x, y \in C\), \(z \in [x, y]\) implies \(z \in C\). Each convex subset of \(K\) can be written in at least one of the following forms
\[
\{x : |x - a| < r\}, \{x : |x - a| \leq r\}
\]
for some \(a \in K\), \(r \in (0, \infty)\).

Let \(Z \subseteq Y \subseteq K\). Then \(Z\) is called convex in \(Y\) if \(Z = C \cap Y\), where \(C\) is convex.

With all these definitions we have the following theorem.

**THEOREM 1.1.** Let \(f : X \to K\). Then the following conditions are equivalent:

1. If \(x, y, z \in X\), \(x\) is between \(y\) and \(z\) then \(f(x)\) is between \(f(y)\) and \(f(z)\),
2. If \(C \subseteq K\) is convex, then \(f^{-1}(C)\) is convex in \(X\).

We denote the collection of those \(f : X \to K\) satisfying (1) or (2) by \(M_b(X)\), i.e., \(f \in M_b(X)\) if, and only if, for each \(x, y, z \in X\),
\[
|x - y| \leq |y - z| \text{ implies } |f(x) - f(y)| \leq |f(y) - f(z)|.
\]

Isometries are in \(M_b\) (viz. exp), but also non-trivial locally constant functions (e.g., choose a center in each ball of radius \(r > 0\), and let \(f\) be the map assigning to \(x \in X\) the center of the ball of radius \(r\) to which \(x\) belongs. Then \(f \in M_b(X)\)).

**THEOREM 1.2.** Let \(f : X \to K\). Then the following conditions are equivalent

1'. If \(x, y, z \in X\), \(f(x)\) is between \(f(y)\) and \(f(z)\) then \(x\) is between \(y\) and \(z\),
2'. If \(C \subseteq X\) is convex in \(X\) then \(f(C)\) is convex in \(f(X)\). \(f\) is injective.

We denote the collection of those \(f : X \to K\) satisfying (1') or (2') by \(M_s(X)\), i.e., \(f \in M_s(X)\) if, and only if, for each \(x, y, z \in X\),
\[
|x - y| < |y - z| \text{ implies } |f(x) - f(y)| < |f(y) - f(z)|.
\]

The classical situations suggests the question as to wether \(M_s(X) \subseteq M_b(X)\) and also whether \(f \in M_b(X)\), \(f\) injective implies \(f \in M_s(X)\). In general, both statements are false, but we do have the following:

**THEOREM 1.3.** \(f \in M_s(X)\) implies \(f^{-1} \in M_b(f(X))\). \(f \in M_b(X)\), \(f\) injective implies \(f^{-1} \in M_s(f(X))\). If \(k\) is finite and \(X\) is convex, then an injective \(M_b\)-function is in \(M_s(X)\).
So we are led to define \( M_{ba}(X) := M_b(X) \cap M_s(X) \) as being the more or less natural translation of "the space of the strictly monotone functions".

The following theorem concerns continuity of monotone functions. For a function \( f : X \to K \), we define its oscillation function, \( \omega_f \), in the usual way:

\[
\omega_f(a) := \lim_{n \to \infty} \sup \{|f(x) - f(y)| ; |x - a| \leq \frac{1}{n}; |y - a| \leq \frac{1}{n}| \}
\]

\[
= \lim_{n \to \infty} \sup \{|f(x) - f(a)| ; |x - a| \leq \frac{1}{n}| (a \in X). \]

\( f \) is continuous at \( a \) if, and only if, \( \omega_f(a) = 0 \).

**Theorem 1.4.** - Let \( f \) be either in \( M_b(X) \) or in \( M_s(X) \). Then

(i) \( \omega_f(a) = \inf_{z \neq a} |f(z) - f(a)| \quad (a \in X) \)

(ii) \( f \) is bounded on compact subsets of \( X \),

(iii) For each \( a \in X \) we have the following alternative. Either \( f \) is continuous at \( a \), or for each sequence \( x_1, x_2, \ldots \) \((x_n \neq a)\) converging to \( a \), the sequence \( f(x_1), f(x_2), \ldots \) is bounded and has no convergent subsequence.

Let \( g \in M_b(X) \). If \( Y \subset X \) is spherically complete, then so is \( g(Y) \).

Let \( h \in M_s(X) \). If \( Z \subset h(X) \) is spherically complete, then so is \( h^{-1}(Z) \).

**Proof (sketch).** - If \( f \in M_b(X) \cup M_s(X) \), then:

\[
|x - y| < |y - z| \quad \text{implies} \quad |f(z) - f(y)| \leq |f(y) - f(x)| .
\]

So \( f \) is locally bounded, and (ii) follows. Of (i), only the \( \leq \) part is interesting. Choose \( z \neq a \). If \( |x - a| < |z - a| \), then

\[
|f(x) - f(a)| \leq |f(z) - f(a)| \quad \text{whence} \quad \omega_f(a) \leq |f(z) - f(a)| .
\]

Let \( \lim x_n = a \) \((x_n \neq a \text{ for all } n)\) and \( \lim f(x_n) = \alpha \). Let \( \lim y_n = a \). It suffices to show that \( \lim f(y_n) = \alpha \). Indeed, let \( \varepsilon > 0 \), and choose \( k \) such that \( |f(x_k) - \alpha| < \varepsilon \). Then \( |y_n - a| < |x_k - a| \) for large \( n \), so

\[
|y_n - x_m| < |x_k - x_m|
\]

for large \( m \) depending on \( m \). Hence \( |f(y_n) - f(x_n)| \leq |f(x_k) - f(x_m)| \), so \((m \to \infty)\) \( |f(y_n) - \alpha| \leq |f(x_k) - \alpha| < \varepsilon \), and we have (iii). The rest of the proof is straightforward.

**Corollary 1.5.** - Let \( f : X \to K \) be in \( M_b(X) \cup M_s(X) \).

(i) If \( K \) is a local field, then \( f \) is continuous,

(ii) If \(|K|\) is discrete, then \( f \in M_s(X) \Rightarrow f \) is a homeomorphism \( X \to f(X) \), and \( f \in M_b(X) \Rightarrow f \) is a closed map.

(iii) The graph of \( f \) is closed in \( K^2 \).

(iv) If \( f(X) \) has no isolated points, then \( f \) is continuous.
An $M_b$-function may be everywhere discontinuous on $K$ (even when $|K|$ is discrete).

**THEOREM 1.6.** - Let $B$ be the unit ball of $K$.

(i) If $K$ is a local field and $f \in M_b(B) \cup M_s(B)$, then $f$ has bounded difference quotients (i.e., there is $C > 0$ such that $|f(x) - f(y)| \leq C|x - y|$ for all $x \in B$). If, in addition, $f(B)$ is convex, then $f$ is a similarity (i.e., a scalar multiple of an isometry).

(ii) If $K$ has discrete valuation and $f \in M_b(B)$ is bounded, then $f$ has bounded difference quotients. If $f \in M_{bs}(B)$ and if $f(B)$ is convex, then $f$ is a similarity.

2. Monotone functions having a type.

In this section, we want to translate the usual classification of (strictly) monotone functions $\mathbb{R} \to \mathbb{R}$ into two types: the increasing and the decreasing functions. The equivalence relation in $\mathbb{R}^r$: $x \sim y$ if $x$ and $y$ are at the same side of $0$, yields $(-\infty, 0)$ and $(0, \infty)$ as equivalence classes. The relation $\sim$ is compatible with the canonical group homomorphism $\mathbb{R}^* \to \mathbb{R}^*/\mathbb{R}^+$, the latter group being $\{1, -1\}$. $\pi(x)$ (usually called $\text{sgn}(x)$) assigns $+1$ to every positive element and $-1$ to every negative element. A function $f : \mathbb{R} \to \mathbb{R}$ is strictly monotone if there exists $\sigma : \mathbb{R}^*/\mathbb{R}^+ \to \mathbb{R}^*/\mathbb{R}^+$ such that for all $x \neq y$,

$$\pi(f(x) - f(y)) = \sigma(\pi(x - y)).$$

If $\sigma$ is the identity then $f$ is called increasing; if $\sigma(1) = -1$, $\sigma(-1) = 1$, $f$ is called decreasing. Other maps $\sigma : [-1, 1] \to [-1, 1]$ can not occur (i.e., there is no $f$ such that, for all $x \neq y$,

$$\pi(f(x) - f(y)) = \sigma(\pi(x - y)).$$

This rather weird description of real monotone functions can be used in the non-archimedean case.

For $x, y \in K^*$ define $x \sim y$ if $x$, $y$ are at the same side of $0$. This means: $0 \notin [x, y]$, or $|x - y| > |y|$, or $|xy^{-1} - 1| < 1$. Thus $x \sim y$ if, and only if, $xy^{-1} \in K^+$ where

$$K^+ := \{x \in K ; |1 - x| < 1\}.$$

We call the elements of $K^+$ the positive element of $K$.

The relation $\sim$ is compatible with the canonical homomorphism of (multiplicative) groups

$$\pi : K^* \to K^*/K^+ = : \Sigma.$$

We call $\Sigma$ the group of signs and $\pi(x)$ the sign of an element $x \in K^*$ ( $x$ is
positive if, and only if, $\pi(x) = 1$).

If $K$ is a local field, we can make a group embedding $\rho : \Sigma \rightarrow K^*$ such that $\pi \circ \rho$ is the identity on $\Sigma$. For example, if $K = \mathbb{Q}_p$, $\delta$ is a primitive $(p - 1)$th root of unity, then

$$\pi\left(\sum_{n \geq 0} a_n p^n \right) = a_k p^k \quad (k \in \mathbb{Z}, \ a_k \neq 0)$$

(Here $a_n \in \{0, 1, \delta, \ldots, \delta^{p-2}\}$ for each $n$).

**Definition 2.1.** Let $\sigma : \Sigma \rightarrow \Sigma$ be any map. A function $f : X \rightarrow K$ is monotone of type $\sigma$ if, for all $x, y \in X$, $x \neq y$,

$$\sigma(f(x) - f(y)) = \sigma(\pi(x - y))$$

(i.e., if $x - y \in \alpha \in \Sigma$ then $f(x) - f(y) \in \sigma(\alpha)$).

We call $f$ of type $\beta \in \Sigma$ if $f$ is of type $\sigma$ where $\sigma$ is the multiplication with $\beta$, i.e.,

$$\frac{f(x) - f(y)}{x - y} \in \beta \quad (x, y \in X, \ x \neq y).$$

We call $f$ increasing if $f$ is of type $\sigma$ where $\sigma$ is the identity, i.e.,

$$\frac{f(x) - f(y)}{x - y}$$

is positive $(x \neq y)$.

Clearly, if $f$ is of type $\beta$, and if $b \in \beta$, then $b^{-1} f$ is increasing.

First, we look at increasing functions, then we discuss more general types $\sigma$.

Notice that increasing functions are isometries hence are in $N_{ba}(X)$. If $f$ is increasing then $f(x) = x + h(x)$, where $|h(x) - h(y)| < |x - y|$ $(x, y \in X, \ x \neq y)$.

Such $h$ we call pseudo-contractions.

**Lemma 2.2.** Let $X$ be an ultrametric space. Then the following are equivalent

(a) $X$ is spherically complete,

(b) Each pseudocontraction $X \rightarrow X$ has a (unique) fixed point.

**Proof (sketch).** $(a) \rightarrow (b)$. Let $\sigma : X \rightarrow X$ be a pseudocontraction. A convex set $C \subset X$ is called invariant if $\sigma(C) \subset C$. It is easily proved that the invarianaut convex subsets of $X$ form a nest. Let $C_0$ be the smallest invariant convex set. If $a \in C_0$ and $\sigma(a) \neq a$ then

$$B_0 := \{x \in X ; \ d(x, \sigma(a)) < d(a, \sigma(a))\}$$

is invariant, convex, and does not contain $a$. Hence $\sigma(a) = a$ for all $a \in C_0$, and $C_0$ is a singleton. $(b) \rightarrow (a)$. If $B_1 \neq B_2 \neq \ldots$ are balls in $X$ with $\bigcap B_n = \emptyset$, then choose $x_n \in B_n \setminus B_{n+1}$ $(n \in \mathbb{N})$. The map $\sigma : X \rightarrow X$ defined by $\sigma(x) = x_{n+1}$ $(x \in B_n \setminus B_{n+1})$ is a pseudocontraction without a fixed point.
COROLLARY 2.3. - Let \( X \) be convex, let \( K \) be spherically complete, and let \( f : X \to K \) be increasing. Then \( f(X) \) is convex. If \( f(X) \subset X \), then \( f \) is surjective.

Proof. - Let \( f(X) \subset X \). Choose \( a \in X \). Then \( x \mapsto f(x) + x + a \) is a pseudocontraction mapping \( X \) into \( X \), hence has a fixed point. So \( f(x) = a \) for some \( x \in X \).

If \( K \) is not spherically complete, we have always increasing \( f : K \to K \) that are not surjective. (Let \( h : K \to K \) be a pseudocontraction without a fixed point. Let \( f(x) = x - h(x) \) \( (x \in K) \), then \( 0 \notin \text{im} f \). The inverse \( f^{-1} : f(K) \to K \) can, of course, not be extended to an increasing function \( K \to K \).

THEOREM 2.4. - Let \( K \) be spherically complete, and let \( f : X \to K \) be increasing. Then \( f \) can be extended to an increasing function \( K \to K \).

Proof. - By Zorn's Lemma, it suffices to extend \( f \) to an increasing function on \( X \cup \{a\} \), where \( a \notin X \). We are done if we can find \( \alpha \in K \) such that, for all \( x \in X \),

\[
\left| \frac{\alpha - f(x)}{a - x} - 1 \right| < 1
\]

i.e., \( \alpha \in B_f(x) - (a - x) \) for all \( x \in X \). These balls form a nest.

Let us call a function \( f : X \to K \) positive if \( f(X) \subset K^+ \).

THEOREM 2.5.

(i) If \( f : X \to K \) is increasing then \( f' \) is positive,

(ii) If \( g : X \to K \) is a positive Baire class one function, then \( g \) has an increasing antiderivative,

(iii) If \( g : X \to K \) is continuous and positive, then \( g \) has a \( C^1 \)-antiderivative,

(iv) If \( f \in C^1(X) \) and \( f' \) is positive then \( f = j + h \) where \( j \) is increasing, and \( h \) is locally constant.

EXAMPLES.

1° The exponential function (defined on its natural convergence region) is increasing.

2° Let \( f \in C(\mathbb{Z}_p) \), and let \( e_0 = \xi_{\mathbb{Z}_p} \), for \( n \in \mathbb{N} \),

\[
e_n(x) = \begin{cases} 1 & \text{if } |x - n| < \frac{1}{n} \\ 0 & \text{elsewhere} \end{cases} \quad (x \in \mathbb{Z}_p).
\]

Then \( e_0, e_1, \ldots \) form an orthonormal base of \( C(\mathbb{Z}_p) \), so there exist \( \lambda_0, \lambda_1, \ldots \in \mathbb{Q}_p \) such that \( f = \sum_{n=0}^{\infty} \lambda_n e_n \), uniformly.
f is increasing if, and only if, for all \( n \in \mathbb{N} \),

\[ |\lambda_n - \lfloor n \rfloor| < \lfloor n \rfloor \]

(where, if \( n = a_0 + a_1 p + \ldots + a_k p^k \) \((a_i \in \{0, 1, \ldots, p-1\} \text{ for each } i, a_k \neq 0)\), then \( \lfloor n \rfloor = a_k p^k \)).

In other words, \( f = \sum \lambda_n e_n \in C(\mathbb{Z}_p) \) is increasing if, and only if, \( \lambda_n / \lfloor n \rfloor \) is positive for all \( n \in \mathbb{N} \).

Let \( \alpha, \beta \in \Sigma \). If the set theoretic sum \( \alpha + \beta := \{x + y ; x \in \alpha, y \in \beta\} \) does not contain 0 then \( \alpha + \beta \in \Sigma \), notation \( \alpha \oplus \beta \). It follows that \( \alpha \oplus \beta \) is defined if, and only if, \( \alpha \neq - \beta \).

If \( x, y \in \alpha \in \Sigma \) then \( |x| = |y| \). This defines \( |\alpha| \) in a natural way.

We have the following results.

**Theorem 2.6.** Let \( f : K \to K \) be monotone of type \( \sigma : \Sigma \to \Sigma \). Let \( \alpha, \beta \in \Sigma \),

(i) \( \sigma(- \alpha) = - \sigma(\alpha) \),

(ii) If \( \sigma(\alpha) \oplus \sigma(\beta) \) is defined then so is \( \alpha \oplus \beta \) and \( \sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta) \),

(iii) \( |\alpha| < |\beta| \) implies \( |\sigma(\alpha)| < |\sigma(\beta)| \),

(iv) If \( |\beta| = 1 \), \( \beta \) contains an element of the prime field of \( K \) then \( \sigma(\beta \alpha) = \beta \sigma(\alpha) \),

(v) \( f \in M_a(K) \),

(vi) \( f \) is either nowhere continuous or uniformly continuous.

**Theorem 2.7.** Let \( f : K \to K \) be monotone of type \( \sigma : \Sigma \to \Sigma \). Then the following conditions are equivalent,

(a) \( \sigma \) is injective,

(b) \( f \in M_b(X) \),

(c) If for some \( \alpha, \beta \in \Sigma \), \( \alpha \oplus \beta \) is defined, then so is \( \sigma(\alpha) \oplus \sigma(\beta) \),

(d) \( |\sigma(\alpha)| < |\sigma(\beta)| \) implies \( |\alpha| < |\beta| \) \((\alpha, \beta \in \Sigma)\).

**Corollary 2.8.** Let \( k \) be a prime field, and let \( f : K \to K \) be monotone of type \( \sigma : \Sigma \to \Sigma \). Then \( \sigma \) is injective.

(If \( K = \mathbb{Q}_p (\sqrt{-1}) \), \( p = 3 \mod 4 \), we can find an example of an \( f : K \to K \) monotone of type \( \sigma \), where \( \sigma \) is not injective).

**Example 2.9.** Let \( K = \mathbb{Q}_p \). Then

\( \{\sigma : \Sigma \to \Sigma : \text{there is } f : \mathbb{Q}_p \to \mathbb{Q}_p, f \text{ monotone of type } \sigma\} \)
consists of all \( \sigma : \Sigma \to \Sigma \) of the form
\[
\delta^p \mapsto \delta^p \delta^s(n) \lambda(n)
\]
where \( s : \mathbb{Z} \to \{0, 1, 2, \ldots, p - 2\} \) and \( \lambda : \mathbb{Z} \to \mathbb{Z} \) is strictly increasing.

3. Functions of bounded variation.

**Lemma 3.1.** Let \( f : X \to K \) have bounded difference quotients. Then \( f \) is a linear combination of two increasing functions.

**Proof.** Choose \( \lambda \in K \),
\[
|\lambda| > \sup \{|\frac{f(x) - f(y)}{x - y}| ; \ x \neq y\}.
\]
Then \( \lambda^{-1} f \) is a (pseudo-) contraction, so \( g(x) := -x + \lambda^{-1} f(x) \) \( (x \in X) \) is increasing. If \( h(x) := x \) \( (x \in X) \), then \( \lambda h + \lambda g = f \).

**Corollary 3.2.** Let \( X \) be the unit ball of a local field \( K \) and let \( f : X \to K \). Then the following are equivalent
\( (a) \ f \in BA(X) \) \( (\text{i.e.} \ \sup \{|\frac{f(x) - f(y)}{x - y}| ; \ x \neq y\} < \infty) \),
\( (b) \ f \) is a linear combination of two increasing functions,
\( (\gamma) \ f \in H_\mathbb{B}(X) \),
\( (\delta) \ f \in H_\mathbb{B}(X) \).

**Proof.** Use 1.6.

REFERENCES
