NON-ARCHIMEDEAN MONOTONE FUNCTIONS

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Introduction.

In the sequel, $K$ is a non-archimedean, non-trivially valued field, that is complete under the metric induced by the valuation. The residue class field of $K$ is denoted by $k$. $X$ will always be a closed, non empty subset of $K$ without isolated points (except in 2.2, if you want).

Since $K$ admits no ordering in the usual sense it is not possible to define monotone functions $X \to K$ just by taking over the classical definitions. Thus, our procedure will be to try and find statements for functions $\mathbb{R} \to \mathbb{R}$ equivalent to monotony, and formulated in terms that are translatable to $K$. This way we will obtain several definitions of "$f : X \to K$ is monotone", that are, although not equivalent, closely related.

The connections between these various definitions and the properties of the non-archimedean monotone functions can be put together to form a little theory which is interesting in its own right, but of which the relations to the other parts of $p$-adic analysis are yet not very tight.

1. Monotone functions.

For a function $f : \mathbb{R} \to \mathbb{R}$ the following conditions are equivalent:

(a) $f$ is monotone (in the non-strict sense),

(b) If $C \subset \mathbb{R}$ is convex then $f^{-1}(C)$ is convex,

(c) If $x$ is between $y$, $z$ then $f(x)$ is between $f(y)$ and $f(z)$.

Also, the following conditions are equivalent:

(a) $f$ is strictly monotone,

(b) $f$ is injective. If $C \subset \mathbb{R}$ is convex then $f(C)$ is relatively convex in $f(\mathbb{R})$,

(c) If $f(x)$ is between $f(y)$ and $f(z)$ then $x$ is between $y$ and $z$.

Let $x, y \in K$. Then the smallest ball that contains $x, y$ is denoted by $[x, y]$. $z \in K$ is between $x$ and $y$ if $z \in [x, y]$. (If $z \notin [x, y]$, we

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call \( x, y \) at the same side of \( z \). A subset \( C \subset K \) is called convex if \( x, y \in C, z \in [x, y] \) implies \( z \in C \). Each convex subset of \( K \) can be written in at least one of the following forms

\[
\{x : |x - a| < r\}, \{x : |x - a| \leq r\}
\]

for some \( a \in K, r \in (0, \infty) \).

Let \( Z \subset Y \subset K \). Then \( Z \) is called convex in \( Y \) if \( Z = C \cap Y \), where \( C \) is convex.

With all these definitions we have the following theorem.

**Theorem 1.1.** Let \( f : X \to K \). Then the following conditions are equivalent:

1. If \( x, y, z \in X \), \( x \) is between \( y \) and \( z \) then \( f(x) \) is between \( f(y) \) and \( f(z) \),

2. If \( C \subset K \) is convex, then \( f^{-1}(C) \) is convex in \( X \).

We denote the collection of those \( f : X \to K \) satisfying (1) or (2) by \( M_b(X) \), i.e. \( f \in M_b(X) \) if, and only if, for each \( x, y, z \in X \),

\[
|x - y| \leq |y - z| \implies |f(x) - f(y)| \leq |f(y) - f(z)|.
\]

Isometries are in \( M_b \) (viz. exp), but also non trivial locally constant functions (e.g., choose a center in each ball of radius \( r > 0 \), and let \( f \) be the map assigning to \( x \in X \) the center of the ball of radius \( r \) to which \( x \) belongs. Then \( f \in M_b(X) \)).

**Theorem 1.2.** Let \( f : X \to K \). Then the following conditions are equivalent

1. If \( x, y, z \in X \), \( f(x) \) is between \( f(y) \) and \( f(z) \) then \( x \) is between \( y \) and \( z \),

2. If \( C \subset X \) is convex in \( X \) then \( f(C) \) is convex in \( f(X) \). \( f \) is injective.

We denote the collection of those \( f : X \to K \) satisfying (1') or (2') by \( M_s(X) \), i.e. \( f \in M_s(X) \) if, and only if, for each \( x, y, z \in X \),

\[
|x - y| < |y - z| \implies |f(x) - f(y)| < |f(y) - f(z)|.
\]

The classical situations suggests the question as to wether \( M_s(X) \subset M_b(X) \) and also wether \( f \in M_b(X) \), \( f \) injective implies \( f \in M_s(X) \). In general, both statements are false, but we do have the following:

**Theorem 1.3.** \( f \in M_s(X) \) implies \( f^{-1} \in M_b(f(X)) \). \( f \in M_b(X) \), \( f \) injective implies \( f^{-1} \in M_s(f(X)) \). If \( k \) is finite and \( X \) is convex, then an injective \( M_b \)-function is in \( M_s(X) \).
So we are led to define \( M_{ba}(X) := M_b(X) \cap M_s(X) \) as being the more or less natural translation of "the space of the strictly monotone functions".

The following theorem concerns continuity of monotone functions. For a function \( f : X \to K \), we define its oscillation function, \( \omega_f \), in the usual way:

\[
\omega_f(a) := \lim_{n \to \infty} \sup \{|f(x) - f(y)| : |x - a| \leq \frac{1}{n}; \ |y - a| \leq \frac{1}{n}\}
\]

\[
= \lim_{n \to \infty} \sup \{|f(x) - f(a)| : |x - a| \leq \frac{1}{n}\} \quad (a \in X).
\]

\( f \) is continuous at \( a \) if, and only if, \( \omega_f(a) = 0 \).

**Theorem 1.4.** Let \( f \) be either in \( M_b(X) \) or in \( M_s(X) \). Then

(i) \( \omega_f(a) = \inf_{z \neq a} |f(z) - f(a)| \quad (a \in X) \)

(ii) \( f \) is bounded on compact subsets of \( X \).

(iii) For each \( a \in X \) we have the following alternative. Either \( f \) is continuous at \( a \), or for each sequence \( x_1, x_2, \ldots \) \((x_n \neq a)\) converging to \( a \), the sequence \( f(x_1), f(x_2), \ldots \) is bounded and has no convergent subsequence.

Let \( g \in M_b(X) \). If \( Y \subset X \) is spherically complete, then so is \( g(Y) \).

Let \( h \in M_s(X) \). If \( Z \subset h(X) \) is spherically complete, then so is \( h^{-1}(Z) \).

**Proof (sketch).** If \( f \in M_b(X) \cup M_s(X) \), then:

\[
|x - y| < |y - a| \quad \text{implies} \quad |f(x) - f(y)| \leq |f(y) - f(z)|.
\]

So \( f \) is locally bounded, and (ii) follows. Of (i), only the \( \leq \) part is interesting. Choose \( z \neq a \). If \( |x - a| < |z - a| \), then

\[
|f(x) - f(a)| \leq |f(z) - f(a)| \quad \text{whence} \quad \omega_f(a) \leq |f(z) - f(a)|.
\]

Let \( \lim_{n \to \infty} x_n = a \) \((x_n \neq a \text{ for all } n)\) and \( \lim f(x_n) = \alpha \). Let \( \lim_{n \to \infty} x_n = a \). It suffices to show that \( \lim f(y_n) = \alpha \). Indeed, let \( \varepsilon > 0 \), and choose \( k \) such that \( |f(x_k) - \alpha| < \varepsilon \). Then \( |y_n - a| < |x_k - a| \) for large \( n \), so

\[
|y_n - x_m| < |x_k - x_m|
\]

for large \( m \) depending on \( m \). Hence \( |f(y_n) - f(x_n)| \leq |f(x_k) - f(x_m)| \), so

\[
(m \to \infty) \quad |f(y_n) - \alpha| \leq |f(x_k) - \alpha| < \varepsilon \quad \text{and we have (iii).} \]

The rest of the proof is straightforward.

**Corollary 1.5.** Let \( f : X \to K \) be in \( M_b(X) \cup M_s(X) \).

(i) If \( K \) is a local field, then \( f \) is continuous.

(ii) If \( |K| \) is discrete, then \( f \in M_s(X) \Rightarrow f \) is a homeomorphism \( X \to f(X) \), and \( f \in M_b(X) \Rightarrow f \) is a closed map.

(iii) The graph of \( f \) is closed in \( K^2 \).

(iv) If \( f(X) \) has no isolated points, then \( f \) is continuous.
An $M_b$-function may be everywhere discontinuous on $K$ (even when $|K|$ is discrete).

**THEOREM 1.6.** Let $B$ be the unit ball of $K$,

(i) If $K$ is a local field and $f \in M_b(B) \cup M_s(B)$, then $f$ has bounded difference quotients (i.e., there is $C > 0$ such that $|f(x) - f(y)| \leq C|x - y|$ for all $x \in B$). If, in addition, $f(B)$ is convex, then $f$ is a similarity (i.e., a scalar multiple of an isometry).

(ii) If $K$ has discrete valuation and $f \in M_b(B)$ is bounded, then $f$ has bounded difference quotients. If $f \in M_{bs}(B)$ and if $f(B)$ is convex, then $f$ is a similarity.

2. **Monotone functions having a type.**

In this section, we want to translate the usual classification of (strictly) monotone functions $\mathbb{R} \to \mathbb{R}$ into two types: the increasing and the decreasing functions. The equivalence relation in $\mathbb{R}^r$: $x \sim y$ if $x$ and $y$ are at the same side of 0, yields $(-\infty, 0)$ and $(0, \infty)$ as equivalence classes. The relation $\sim$ is compatible with the canonical group homomorphism $\mathbb{R}^* \xrightarrow{\pi} \mathbb{R}^*/\mathbb{R}^+$, the latter group being $\{1, -1\}$. $\pi(x)$ (usually called $\text{sgn}(x)$) assigns +1 to every positive element and -1 to every negative element. A function $f: \mathbb{R} \to \mathbb{R}$ is strictly monotone if there exists $\sigma: \mathbb{R}^*/\mathbb{R}^+ \to \mathbb{R}^*/\mathbb{R}^+$ such that for all $x \neq y$

$$\pi(f(x) - f(y)) = \sigma(\pi(x - y)).$$

If $\sigma$ is the identity then $f$ is called increasing; if $\sigma(1) = -1$, $\sigma(-1) = 1$, $f$ is called decreasing. Other maps $\sigma: [-1, 1] \to [-1, 1]$ cannot occur (i.e., there is no $f$ such that, for all $x \neq y$,

$$\pi(f(x) - f(y)) = \sigma(\pi(x - y)).$$

This rather weird description of real monotone functions can be used in the nonarchimedean case.

For $x, y \in K^*$ define $x \sim y$ if $x, y$ are at the same side of 0. This means: $0 \notin [x, y]$, or $|x - y| > |y|$, or $|xy^{-1} - 1| < 1$. Thus $x \sim y$ if, and only if, $xy^{-1} \in K^+$ where

$$K^+ := \{x \in K; |1 - x| < 1\}.$$

We call the elements of $K^+$ the positive element of $K$.

The relation $\sim$ is compatible with the canonical homomorphism of (multiplicative) groups

$$\pi: K^* \to K^*/K^+ =: \Sigma.$$

We call $\Sigma$ the group of signs and $\pi(x)$ the sign of an element $x \in K^*$ ( $x$ is
positive if, and only if, \( \pi(x) = 1 \).

If \( K \) is a local field, we can make a group embedding \( \rho : \Sigma \hookrightarrow K^\times \) such that \( \pi \circ \rho \) is the identity on \( \Sigma \). For example, if \( K = \mathbb{Q}_p \), \( \delta \) is a primitive \((p - 1)\)th root of unity, then

\[
\pi\left( \sum_{n \geq 1} a_n p^n \right) = a_k p^k \quad (k \in \mathbb{Z}, \ a_k \neq 0)
\]

(Here \( a_n \in \{0, 1, \delta, \ldots, \delta^{p-2}\} \) for each \( n \)).

**Definition 2.1.** Let \( \sigma : \Sigma \to \Sigma \) be any map. A function \( f : X \to K \) is monotone of type \( \sigma \) if, for all \( x, y \in X \), \( x \neq y \),

\[
\pi(f(x) - f(y)) = \sigma(\pi(x - y))
\]

(i.e., if \( x - y \in \alpha \in \Sigma \) then \( f(x) - f(y) \in \sigma(\alpha) \)).

We call \( f \) of type \( \beta \in \Sigma \) if \( f \) is of type \( \alpha \) where \( \sigma \) is the multiplication with \( \beta \), i.e.,

\[
\frac{f(x) - f(y)}{x - y} \in \beta \quad (x, y \in X, x \neq y).
\]

We call \( f \) increasing if \( f \) is of type \( \sigma \) where \( \sigma \) is the identity, i.e., \( \frac{f(x) - f(y)}{x - y} \) is positive \((x \neq y)\).

Clearly, if \( f \) is of type \( \beta \), and if \( b \in \beta \), then \( b^{-1} f \) is increasing.

First, we look at increasing functions, then we discuss more general types \( \sigma \). Notice that increasing functions are isometries hence are in \( \mathbb{N}_{bas}(X) \). If \( f \) is increasing then \( f(x) = x + h(x) \), where \( |h(x) - h(y)| < |x - y| \) \((x, y \in X, x \neq y)\).

Such \( h \) we call pseudo-contractions.

**Lemma 2.2.** Let \( X \) be an ultrametric space. Then the following are equivalent

(a) \( X \) is spherically complete,

(b) Each pseudocontraction \( X \to X \) has a (unique) fixed point.

**Proof (sketch).** (a) \( \to \) (b). Let \( \sigma : X \to X \) be a pseudocontraction. A convex set \( C \subseteq X \) is called invariant if \( \sigma(C) \subseteq C \). It is easily proved that the invariant convex subsets of \( X \) form a nest. Let \( C_0 \) be the smallest invariant convex set. If \( a \in C_0 \) and \( \sigma(a) \neq a \) then

\[
B_0 := \{ x \in X ; \ d(x, \sigma(a)) < d(a, \sigma(a)) \}
\]

is invariant, convex, and does not contain \( a \). Hence \( \sigma(a) = a \) for all \( a \in C_0 \), and \( C_0 \) is a singleton. (b) \( \to \) (a). If \( B_1 \neq B_2 \neq \ldots \) are balls in \( X \) with \( \cap B_n = \emptyset \) then choose \( x_n \in B_n \setminus B_{n+1} \) \((n \in \mathbb{N})\). The map \( \sigma : X \to X \) defined by

\[
\sigma(x) = x_{n+1} \quad (x \in B_n \setminus B_{n+1})
\]

is a pseudocontraction without a fixed point.
COROLLARY 2.3. - Let \( X \) be convex, let \( K \) be spherically complete, and let \( f : X \to K \) be increasing. Then \( f(X) \) is convex. If \( f(X) \subset X \), then \( f \) is surjective.

Proof. - Let \( f(X) \subset X \). Choose \( a \in X \). Then \( x \mapsto x + f(x) \) is a pseudo-contraction mapping \( X \) into \( X \), hence has a fixed point. So \( f(x) = a \) for some \( x \in X \).

If \( K \) is not spherically complete, we have always increasing \( f : K \to K \) that are not surjective. (Let \( h : K \to K \) be a pseudocontraction without a fixed point. Let \( f(x) = x - h(x) \) \((x \in K)\), then \( 0 \not\in f \).) The inverse \( f^{-1} : f(K) \to K \) can, of course, not be extended to an increasing function \( K \to K \).

THEOREM 2.4. - Let \( K \) be spherically complete, and let \( f : X \to K \) be increasing. Then \( f \) can be extended to an increasing function \( K \to K \).

Proof. - By Zorn's Lemma, it suffices to extend \( f \) to an increasing function on \( X \cup \{a\} \), where \( a \not\in X \). We are done if we can find \( a \in K \) such that, for all \( x \in X \),

\[
\left| \frac{a - f(x)}{a - x} - 1 \right| < 1
\]

i.e. \( a \in \text{Ball}(x, (a - x)^{-1}) \) for all \( x \in X \). These balls form a nest.

Let us call a function \( f : X \to K \) positive if \( f(X) \subset K^+ \).

THEOREM 2.5.

(i) If \( f : X \to K \) is increasing then \( f' \) is positive,

(ii) If \( g : X \to K \) is a positive Baire class one function, then \( g \) has an increasing antiderivative,

(iii) If \( g : X \to K \) is continuous and positive, then \( g \) has a \( C^1 \)-antiderivative,

(iv) If \( f \in C^1(X) \) and \( f' \) is positive then \( f = j + h \) where \( j \) is increasing, and \( h \) is locally constant.

EXAMPLES.

1° The exponential function (defined on its natural convergence region) is increasing.

2° Let \( f \in C(\mathbb{Z}_p) \), and let \( e_0 = \xi_{\mathbb{Z}_p} \), for \( n \in \mathbb{N} \),

\[
e_n(x) = \begin{cases} 1 & \text{if } |x - n| < \frac{1}{n} \\ 0 & \text{elsewhere} \end{cases} \quad (x \in \mathbb{Z}_p).
\]

Then \( e_0, e_1, \ldots \) form an orthonormal base of \( C(\mathbb{Z}_p) \), so there exist \( \lambda_0, \lambda_1, \ldots \in \mathbb{Q}_p \) such that \( f = \sum_{n=0}^{\infty} \lambda_n e_n \), uniformly.
If $f$ is increasing if, and only if, for all $n \in \mathbb{N}$,
\[ |\lambda_n - [n]| < [n] \]
(where, if $n = a_0 + a_1 p + \ldots + a_k p^k$ ($a_i \in \{0, 1, \ldots, p - 1\}$ for each $i$, $a_k \neq 0$), then \(\{n\}_l = a_k p^k\)).

In other words, $f = \sum \lambda_n \varepsilon_n \in C(\mathbb{Z}_p)$ is increasing if, and only if, $\lambda_n/\{n\}$ is positive for all $n \in \mathbb{N}$.

Let $\alpha, \beta \in \Sigma$. If the set theoretic sum $\alpha + \beta := \{x + y ; x \in \alpha, y \in \beta\}$ does not contain 0 then $\alpha + \beta \in \Sigma$, notation $\alpha \oplus \beta$. It follows that $\alpha \oplus \beta$ is defined if, and only if, $\alpha \neq -\beta$.

If $x, y \in \alpha \in \Sigma$ then $|x| = |y|$. This defines $|\alpha|$ in a natural way.

We have the following results.

**THEOREM 2.6.** - Let $f : K \rightarrow K$ be monotone of type $\sigma : \Sigma \rightarrow \Sigma$. Let $\alpha, \beta \in \Sigma$,

(i) $\sigma(-\alpha) = -\sigma(\alpha)$,

(ii) If $\sigma(\alpha) \oplus \sigma(\beta)$ is defined then so is $\alpha \oplus \beta$ and $\sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta)$,

(iii) $|\alpha| < |\beta|$ implies $|\sigma(\alpha)| < |\sigma(\beta)|$,

(iv) If $|\beta| = 1$, $\beta$ contains an element of the prime field of $K$ then $\sigma(\beta \alpha) = \beta \sigma(\alpha)$,

(v) $f \in M_{\mathbb{S}}(K)$,

(vi) $f$ is either nowhere continuous or uniformly continuous.

**THEOREM 2.7.** - Let $f : K \rightarrow K$ be monotone of type $\sigma : \Sigma \rightarrow \Sigma$. Then the following conditions are equivalent,

(a) $\sigma$ is injective,

(b) $f \in M_{\mathbb{S}}(X)$,

(c) If for some $\alpha, \beta \in \Sigma$, $\alpha \oplus \beta$ is defined, then so is $\sigma(\alpha) \oplus \sigma(\beta)$,

(d) $|\sigma(\alpha)| < |\sigma(\beta)|$ implies $|\alpha| < |\beta|$ ($\alpha, \beta \in \Sigma$).

**COROLLARY 2.8.** - Let $K$ be a prime field, and let $f : K \rightarrow K$ be monotone of type $\sigma : \Sigma \rightarrow \Sigma$. Then $\sigma$ is injective.

(If $K = \mathbb{Q}_p(\sqrt{-1})$, $p = 3 \mod 4$, we can find an example of an $f : K \rightarrow K$ monotone of type $\sigma$, where $\sigma$ is not injective).

**EXAMPLE 2.9.** - Let $K = \mathbb{Q}_p$. Then

$\{\sigma : \Sigma \rightarrow \Sigma : \text{there is } f : \mathbb{Q}_p \rightarrow \mathbb{Q}_p, f \text{ monotone of type } \sigma\}$
consists of all $\sigma: \Sigma \rightarrow \Sigma$ of the form

$$y^i p^j \rightarrow z^i \delta^a(p) \lambda(n)$$

where $s: \mathbb{Z} \rightarrow \{0, 1, 2, \ldots, p - 2\}$ and $\lambda: \mathbb{Z} \rightarrow \mathbb{Z}$ is strictly increasing.

3. Functions of bounded variation.

**Lemma 3.1.** Let $f: X \rightarrow K$ have bounded difference quotients. Then $f$ is a linear combination of two increasing functions.

**Proof.** Choose $\lambda \in K$,

$$|\lambda| > \sup\{|f(x) - f(y)| ; x \neq y\}.$$

Then $\lambda^{-1} f$ is a (pseudo-) contraction, so $g(x) := -x + \lambda^{-1} f(x)$ ($x \in X$) is increasing. If $h(x) := x$ ($x \in X$), then $\lambda h + \lambda g = f$.

**Corollary 3.2.** Let $X$ be the unit ball of a local field $K$ and let $f: X \rightarrow K$. Then the following are equivalent

1. $f \in BA(X)$ (i.e., $\sup\{|f(x) - f(y)| ; x \neq y\} < \infty$),
2. $f$ is a linear combination of two increasing functions,
3. $f \in [I_b(X)]$,
4. $f \in [l_b(X)]$.

**Proof.** Use 1.6.

**References**
