NON-ARCHIMEDEAN MONOTONE FUNCTIONS

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Introduction.

In the sequel, $K$ is a non-archimedean, non-trivially valued field, that is complete under the metric induced by the valuation. The residue class field of $K$ is denoted by $k$. $X$ will always be a closed, non empty subset of $K$ without isolated points (except in 2.2, if you want).

Since $K$ admits no ordering in the usual sense it is not possible to define monotone functions $X \rightarrow K$ just by taking over the classical definitions. Thus, our procedure will be to try and find statements for functions $\mathbb{R} \rightarrow \mathbb{R}$ equivalent to monotony, and formulated in terms that are translatable to $K$. This way we will obtain several definitions of "$f : X \rightarrow K$ is monotone", that are, although not equivalent, closely related.

The connections between these various definitions and the properties of the non-archimedean monotone functions can be put together to form a little theory which is interesting in its own right, but of which the relations to the other parts of $p$-adic analysis are yet not very tight.

1. Monotone functions.

For a function $f : \mathbb{R} \rightarrow \mathbb{R}$ the following conditions are equivalent:

(a) $f$ is monotone (in the non-strict sense),

(b) If $C \subset \mathbb{R}$ is convex then $f^{-1}(C)$ is convex,

(c) If $x$ is between $y$ , $z$ then $f(x)$ is between $f(y)$ and $f(z)$.

Also, the following conditions are equivalent:

(a) $f$ is strictly monotone,

(b) $f$ is injective. If $C \subset \mathbb{R}$ is convex then $f(C)$ is relatively convex in $f(\mathbb{R})$,

(c) If $f(x)$ is between $f(y)$ and $f(z)$ then $x$ is between $y$ and $z$.

Let $x , y \in K$. Then the smallest ball that contains $x , y$ is denoted by $[x , y]$. $z \in K$ is between $x$ and $y$ if $z \in [x , y]$. (If $z \notin [x , y]$, we...
call \( x, y \) at the same side of \( z \). A subset \( C \subseteq K \) is called convex if \( x, y \in C \), \( z \in [x, y] \) implies \( z \in C \). Each convex subset of \( K \) can be written in at least one of the following forms
\[
\{ x : |x - a| < r \}, \{ x : |x - a| \leq r \}
\]
for some \( a \in K \), \( r \in (0, \infty) \).

Let \( Z \subseteq Y \subseteq K \). Then \( Z \) is called convex in \( Y \) if \( Z = C \cap Y \), where \( C \) is convex.

With all these definitions we have the following theorem.

**Theorem 1.1.** - Let \( f : X \to K \). Then the following conditions are equivalent:

1. If \( x, y, z \in X \), \( x \) is between \( y \) and \( z \) then \( f(x) \) is between \( f(y) \) and \( f(z) \),

2. If \( C \subseteq K \) is convex, then \( f^{-1}(C) \) is convex in \( X \).

We denote the collection of those \( f : X \to K \) satisfying (1) or (2) by \( M_b(X) \), i.e. \( f \in M_b(X) \) if, and only if, for each \( x, y, z \in X \),
\[
|x - y| \leq |y - z| \text{ implies } |f(x) - f(y)| \leq |f(y) - f(z)|.
\]

Isometries are in \( M_b \) (viz. exp), but also non trivial locally constant functions (e.g., choose a center in each ball of radius \( r > 0 \), and let \( f \) be the map assigning to \( x \in X \) the center of the ball of radius \( r \) to which \( x \) belongs. Then \( f \in M_b(X) \)).

**Theorem 1.2.** - Let \( f : X \to K \). Then the following conditions are equivalent:

1. If \( x, y, z \in X \), \( f(x) \) is between \( f(y) \) and \( f(z) \) then \( x \) is between \( y \) and \( z \),

2. If \( C \subseteq X \) is convex in \( X \) then \( f(C) \) is convex in \( f(X) \). \( f \) is injective.

We denote the collection of those \( f : X \to K \) satisfying (1') or (2') by \( M_s(X) \), i.e. \( f \in M_s(X) \) if, and only if, for each \( x, y, z \in X \),
\[
|x - y| < |y - z| \text{ implies } |f(x) - f(y)| < |f(y) - f(z)|.
\]

The classical situations suggests the question as to weather \( M_s(X) \subseteq M_b(X) \) and also whether \( f \in M_b(X) \), \( f \) injective implies \( f \in M_s(X) \). In general, both statements are false, but we do have the following:

**Theorem 1.3.** - \( f \in M_s(X) \) implies \( f^{-1} \in M_b(f(X)) \). \( f \in M_b(X) \), \( f \) injective implies \( f^{-1} \in M_s(f(X)) \). If \( k \) is finite and \( X \) is convex, then an injective \( M_b \)-function is in \( M_s(X) \).
So we are led to define \( M_{b_0}(X) := M_b(X) \cap M_g(X) \) as being the more or less natural translation of "the space of the strictly monotone functions".

The following theorem concerns continuity of monotone functions. For a function \( f : X \to K \), we define its oscillation function, \( \omega_f \), in the usual way:

\[
\omega_f(a) := \lim_{n \to \infty} \sup \{|f(x) - f(y)| : |x - a| \leq \frac{1}{n}, |y - a| \leq \frac{1}{n}\}
\]

\[
= \lim_{n \to \infty} \sup \{|f(x) - f(a)| : |x - a| \leq \frac{1}{n}\} \quad (a \in X).
\]

\( f \) is continuous at \( a \) if, and only if, \( \omega_f(a) = 0 \).

**THEOREM 1.4.** Let \( f \) be either in \( M_b(X) \) or in \( M_g(X) \). Then

(i) \( \omega_f(a) = \inf_{y \neq a} |f(y) - f(a)| \quad (a \in X) \)

(ii) \( f \) is bounded on compact subsets of \( X \).

(iii) For each \( a \in X \) we have the following alternative. Either \( f \) is continuous at \( a \), or for each sequence \( x_1, x_2, \ldots \) \((x_n \neq a)\) converging to \( a \), the sequence \( f(x_1), f(x_2), \ldots \) is bounded and has no convergent subsequence.

Let \( g \in M_b(X) \). If \( Y \subset X \) is spherically complete, then so is \( g(Y) \).

Let \( h \in M_g(X) \). If \( Z \subset h(X) \) is spherically complete, then so is \( h^{-1}(Z) \).

**Proof (sketch).** If \( f \in M_b(X) \cup M_g(X) \), then:

\[
|x - y| < |y - z| \text{ implies } |f(x) - f(y)| \leq |f(y) - f(z)|.
\]

So \( f \) is locally bounded, and (ii) follows. Of (i), only the \( \leq \) part is interesting. Choose \( z \neq a \). If \( |x - a| < |z - a| \), then

\[
|f(x) - f(a)| \leq |f(z) - f(a)| \text{ whence } \omega_f(a) \leq |f(z) - f(a)|.
\]

Let \( \lim_{n \to \infty} x_n = a \) \((x_n \neq a \text{ for all } n)\) and \( \lim f(x_n) = \alpha \). Let \( \lim_{n \to \infty} y_n = a \). It suffices to show that \( \lim f(y_n) = \alpha \). Indeed, let \( \epsilon > 0 \), and choose \( k \) such that \( |f(x_k) - \alpha| < \epsilon \). Then \( |y_n - a| < |x_k - a| \) for large \( n \), so

\[
|y_n - x_m| < |x_k - x_m|
\]

for large \( m \) depending on \( m \). Hence \( |f(y_n) - f(x_n)| \leq |f(x_k) - f(x_m)| \), so

\[
(m \to \infty) \quad |f(y_n) - \alpha| \leq |f(x_k) - \alpha| < \epsilon,
\]

and we have (iii). The rest of the proof is straightforward.

**COROLLARY 1.5.** Let \( f : X \to K \) be in \( M_b(X) \cup M_g(X) \).

(i) If \( K \) is a local field, then \( f \) is continuous.

(ii) If \( |K| \) is discrete, then \( f \in M_b(X) \Rightarrow f \) is a homeomorphism \( X \sim f(X) \), and \( f \in M_g(X) \Rightarrow f \) is a closed map.

(iii) The graph of \( f \) is closed in \( K^2 \).

(iv) If \( f(X) \) has no isolated points, then \( f \) is continuous.
An \(M_b\)-function may be everywhere discontinuous on \(K\) (even when \(|K|\) is discrete).

**THEOREM 1.6.** Let \(B\) be the unit ball of \(K\),

(i) If \(K\) is a local field and \(f \in M_b(B) \cup M_s(B)\), then \(f\) has bounded difference quotients (i.e., there is \(C > 0\) such that \(|f(x) - f(y)| \leq C|x - y|\) for all \(x \in B\)). If, in addition, \(f(B)\) is convex, then \(f\) is a similarity (i.e., a scalar multiple of an isometry).

(ii) If \(K\) has discrete valuation and \(f \in M_b(B)\) is bounded, then \(f\) has bounded difference quotients. If \(f \in M_{bs}(B)\) and if \(f(B)\) is convex, then \(f\) is a similarity.

2. **Monotone functions having a type.**

In this section, we want to translate the usual classification of (strictly) monotone functions \(\mathbb{R} \to \mathbb{R}\) into two types: the increasing and the decreasing functions. The equivalence relation in \(\mathbb{R}^r\): \(x \sim y\) if \(x\) and \(y\) are at the same side of 0, yields \((-\infty, 0)\) and \((0, +\infty)\) as equivalence classes. The relation \(\sim\) is compatible with the canonical group homomorphism \(\mathbb{R}^+ \to \mathbb{R}_+^*/\mathbb{R}_+\), the latter group being \(\{1, -1\}\). \(\pi(x)\) (usually called \(\text{sgn}(x)\)) assigns +1 to every positive element and -1 to every negative element. A function \(f: \mathbb{R} \to \mathbb{R}\) is strictly monotone if there exists \(\sigma: \mathbb{R}^*/\mathbb{R}_+^+ \to \mathbb{R}^+_*/\mathbb{R}_+\) such that for all \(x \neq y\)

\[
\pi(f(x) - f(y)) = \sigma(\pi(x - y))
\]

If \(\sigma\) is the identity then \(f\) is called increasing; if \(\sigma(1) = -1\), \(\sigma(-1) = 1\), \(f\) is called decreasing. Other maps \(\sigma: \{-1, 1\} \to \{-1, 1\}\) cannot occur (i.e., there is no \(f\) such that, for all \(x \neq y\),

\[
\pi(f(x) - f(y)) = \sigma(\pi(x - y))
\]

This rather weird description of real monotone functions can be used in the non-archimedean case.

For \(x, y \in K^\#\) define \(x \sim y\) if \(x, y\) are at the same side of 0. This means: \(0 \notin [x, y]\), or \(|x - y| > |y|\), or \(|xy^{-1} - 1| < 1\). Thus \(x \sim y\) if, and only if, \(xy^{-1} \in K^+\) where

\[
K^+ := \{x \in K; |1 - x| < 1\}.
\]

We call the elements of \(K^+\) the positive element of \(K\).

The relation \(\sim\) is compatible with the canonical homomorphism of (multiplicative) groups

\[
\pi: K^\# \to K^*/K^+_+ = : \Sigma
\]

We call \(\Sigma\) the group of signs and \(\pi(x)\) the sign of an element \(x \in K^\#\) (\(x\) is
positive if, and only if, $\pi(x) = 1$).

If $K$ is a local field, we can make a group embedding $\rho : \Sigma \hookrightarrow K^\times$ such that $\pi \circ \rho$ is the identity on $\Sigma$. For example, if $K = \mathbb{Q}_p$, $\delta$ is a primitive $(p - 1)$th root of unity, then

$$\pi(\sum_{n \geq 0} a_n p^n) = a_k p^k \quad (k \in \mathbb{Z}, \ a_k \neq 0)$$

(Here $a_n \in \{0, 1, \delta, \ldots, \delta^{p-2}\}$ for each $n$).

**Definition 2.1.** Let $\sigma : \Sigma \rightarrow \Sigma$ be any map. A function $f : X \rightarrow K$ is monotone of type $\sigma$ if, for all $x, y \in X$, $x \neq y$,

$$\pi(f(x) - f(y)) = \sigma(\pi(x - y))$$

(i.e., if $x - y \in \sigma \in \Sigma$ then $f(x) - f(y) \in \sigma(a)$).

We call $f$ of type $\beta \in \Sigma$ if $f$ is of type $\sigma$ where $\sigma$ is the multiplication with $\beta$, i.e.,

$$\frac{f(x) - f(y)}{x - y} \in \beta \quad (x, y \in X, x \neq y).$$

We call $f$ increasing if $f$ is of type $\sigma$ where $\sigma$ is the identity, i.e.,

$$\frac{f(x) - f(y)}{x - y}$$

is positive ($x \neq y$).

Clearly, if $f$ is of type $\beta$, and if $b \in \beta$, then $b^{-1}f$ is increasing.

First, we look at increasing functions, then we discuss more general types $\sigma$.

Notice that increasing functions are isometries hence are in $\text{Ker}(\sigma)$. If $f$ is increasing then $f(x) = x + h(x)$, where $|h(x) - h(y)| < |x - y|$ ($\forall x, y \in X, x \neq y$).

Such $h$ we call pseudo-contractive.

**Lemma 2.2.** Let $X$ be an ultrametric space. Then the following are equivalent

(a) $X$ is spherically complete,

(b) Each pseudocontraction $X \rightarrow X$ has a (unique) fixed point.

**Proof (sketch).** (a) $\rightarrow$ (b). Let $\sigma : X \rightarrow X$ be a pseudocontraction. A convex set $C \subseteq X$ is called invariant if $\sigma(C) \subseteq C$. It is easily proved that the invariant convex subsets of $X$ form a nest. Let $C_0$ be the smallest invariant convex set. If $a \in C_0$ and $\sigma(a) \neq a$ then

$$B_0 := \{x \in X ; \ d(x, \sigma(a)) < d(a, \sigma(a))\}$$

is invariant, convex, and does not contain $a$. Hence $\sigma(a) = a$ for all $a \in C_0$, and $C_0$ is a singleton. (b) $\rightarrow$ (a). If $B_1 \neq B_2 \neq \ldots$ are balls in $X$ with

$\cap B_n = \emptyset$ then choose $x_n \in B_n \setminus B_{n+1}$ ($n \in \mathbb{N}$). The map $\sigma : X \rightarrow X$ defined by

$$\sigma(x) = x_{n+1} \quad (x \in B_n \setminus B_{n+1})$$

is a pseudocontraction without a fixed point.
COROLLARY 2.3. - Let X be convex, let K be spherically complete, and let 
\( f : X \to K \) be increasing. Then \( f(X) \) is convex. If \( f(X) \subset X \), then \( f \) is surjective.

Proof. - Let \( f(X) \subset X \). Choose \( \alpha \in X \). Then \( x \mapsto f(x) + x + \alpha \) is a pseudo-contraction mapping \( X \) into \( X \), hence has a fixed point. So \( f(x) = \alpha \) for some \( x \in X \).

If \( K \) is not spherically complete, we have always increasing \( f : K \to K \) that are not surjective. (Let \( h : K \to K \) be a pseudocontraction without a fixed point. Let \( f(x) = x - h(x) \) \( (x \in K) \), then \( 0 \not\in f') \). The inverse \( f^{-1} : f(K) \to K \) can, of course, not be extended to an increasing function \( K \to K \).

THEOREM 2.4. - Let \( K \) be spherically complete, and let \( f : X \to K \) be increasing. Then \( f \) can be extended to an increasing function \( K \to K \).

Proof. - By Zorn's Lemma, it suffices to extend \( f \) to an increasing function on \( X \cup \{a\} \), where \( a \not\in X \). We are done if we can find \( \alpha \in K \) such that, for all \( x \in X \),

\[
\left| \frac{\alpha - f(x)}{a - x} - 1 \right| < 1
\]

i.e. \( \alpha \in B_{f(x)}(a-x)(|a-x|) \) for all \( x \in X \). These balls form a nest.

Let us call a function \( f : X \to K \) positive if \( f(X) \subset K^+ \).

THEOREM 2.5.

(i) If \( f : X \to K \) is increasing then \( f' \) is positive,
(ii) If \( g : X \to K \) is a positive Baire class one function, then \( g \) has an increasing antiderivative,
(iii) If \( g : X \to K \) is continuous and positive, then \( g \) has a \( C^1 \)-antiderivative,
(iv) If \( f \in C^1(X) \) and \( f' \) is positive then \( f = j + h \) where \( j \) is increasing, and \( h \) is locally constant.

EXAMPLES.

1° The exponential function (defined on its natural convergence region) is increasing.

2° Let \( f \in C(\mathbb{Z}_p) \), and let \( e_0 = \xi_{\mathbb{Z}_p} \), for \( n \in \mathbb{N} \),

\[
e_n(x) = \begin{cases} 1 & \text{if } |x - n| < \frac{1}{n} \\ 0 & \text{elsewhere} \end{cases} (x \in \mathbb{Z}_p).
\]

Then \( e_0, e_1, \ldots \) form an orthonormal base of \( C(\mathbb{Z}_p) \), so there exist \( \lambda_0, \lambda_1, \ldots \in \mathbb{Q}_p \) such that \( f = \sum_{n=0}^{\infty} \lambda_n e_n \), uniformly.
\( f \) is increasing if, and only if, for all \( n \in \mathbb{N} \),
\[
|\lambda_n - [n]| < [n]
\]
(where, if \( n = a_0 + a_1 p + \ldots + a_k p^k \) (\( a_i \in \{0, 1, \ldots, p - 1\} \) for each \( i \), \( a_k \neq 0 \)), then \( \{n\}_1 = a_k p^k \).

In other words, \( f = \sum \lambda_n \in C(\mathbb{Z}_p) \) is increasing if, and only if, \( \lambda_n/[n] \) is positive for all \( n \in \mathbb{N} \).

Let \( \alpha, \beta \in \Sigma \). If the set theoretic sum \( \alpha + \beta := \{x + y; x \in \alpha, y \in \beta\} \) does not contain \( 0 \) then \( \alpha + \beta \in \Sigma \), notation \( \alpha \otimes \beta \). It follows that \( \alpha \otimes \beta \) is defined if, and only if, \( \alpha \neq -\beta \).

If \( x, y \in \alpha \in \Sigma \) then \( |x| = |y| \). This defines \( |\alpha| \) in a natural way.

We have the following results.

**Theorem 2.6.** Let \( f : K \rightarrow K \) be monotone of type \( \sigma : \Sigma \rightarrow \Sigma \). Let \( \alpha, \beta \in \Sigma \),

(i) \( \sigma(- \alpha) = -\sigma(\alpha) \),

(ii) If \( \sigma(\alpha) \otimes \sigma(\beta) \) is defined then so is \( \alpha \otimes \beta \) and \( \sigma(\alpha \otimes \beta) = \sigma(\alpha) \otimes \sigma(\beta) \),

(iii) \( |\alpha| < |\beta| \) implies \( |\sigma(\alpha)| < |\sigma(\beta)| \),

(iv) If \( |\beta| = 1 \), \( \beta \) contains an element of the prime field of \( K \) then \( \sigma(\beta \alpha) = \beta \sigma(\alpha) \),

(v) \( f \in H_b(K) \),

(vi) \( f \) is either nowhere continuous or uniformly continuous.

**Theorem 2.7.** Let \( f : K \rightarrow K \) be monotone of type \( \sigma : \Sigma \rightarrow \Sigma \). Then the following conditions are equivalent,

(a) \( \sigma \) is injective,

(b) \( f \in H_b(X) \),

(c) If for some \( \alpha, \beta \in \Sigma \), \( \alpha \otimes \beta \) is defined, then so is \( \sigma(\alpha) \otimes \sigma(\beta) \),

(d) \( |\sigma(\alpha)| < |\sigma(\beta)| \) implies \( |\alpha| < |\beta| \) (\( \alpha, \beta \in \Sigma \)).

**Corollary 2.8.** Let \( k \) be a prime field, and let \( f : K \rightarrow K \) be monotone of type \( \sigma : \Sigma \rightarrow \Sigma \). Then \( \sigma \) is injective.

(If \( K = \mathbb{Q}((\sqrt{-1}) \), \( p \equiv 3 \) mod 4, we can find an example of an \( f : K \rightarrow K \) monotone of type \( \sigma \), where \( \sigma \) is not injective).

**Example 2.9.** Let \( K = \mathbb{Q}_p \). Then

\[ \{\sigma : \Sigma \rightarrow \Sigma : \text{there is } f : \mathbb{Q}_p \rightarrow \mathbb{Q}_p, \text{ } f \text{ monotone of type } \sigma\} \]
consists of all \( \sigma : \Sigma \to \Sigma \) of the form
\[
r^i p^n \to s^i \delta^s(n) p\lambda(n)
\]
where \( s : \mathbb{Z} \to \{0, 1, 2, \ldots, p - 2\} \) and \( \lambda : \mathbb{Z} \to \mathbb{Z} \) is strictly increasing.

3. Functions of bounded variation.

**Lemma 3.1.** Let \( f : X \to K \) have bounded difference quotients. Then \( f \) is a linear combination of two increasing functions.

**Proof.** Choose \( \lambda \in K \),
\[|\lambda| > \sup \left\{ \frac{|f(x) - f(y)|}{x - y} ; x \neq y \right\} .\]

Then \( \lambda^{-1} f \) is a (pseudo-) contraction, so \( g(x) := -x + \lambda^{-1} f(x) \) \( (x \in X) \) is increasing. If \( h(x) := x \) \( (x \in X) \), then \( \lambda h + \lambda g = f \).

**Corollary 3.2.** Let \( X \) be the unit ball of a local field \( K \) and let \( f : X \to K \). Then the following are equivalent

\((\alpha)\) \( f \in \text{BA}(X) \) (i.e., \( \sup \left\{ \frac{|f(x) - f(y)|}{x - y} ; x \neq y \right\} < \infty \)),

\((\beta)\) \( f \) is a linear combination of two increasing functions,

\((\gamma)\) \( f \in \text{wBA}(X) \),

\((\delta)\) \( f \in \text{BA}(X) \).

**Proof.** Use 1.6.

**References**
