Introduction.

In the sequel, \( K \) is a non-archimedean, non-trivially valued field, that is complete under the metric induced by the valuation. The residue class field of \( K \) is denoted by \( k \). \( X \) will always be a closed, non empty subset of \( K \) without isolated points (except in 2.2, if you want).

Since \( K \) admits no ordering in the usual sense it is not possible to define monotone functions \( X \to K \) just by taking over the classical definitions. Thus, our procedure will be to try and find statements for functions \( \mathbb{R} \to \mathbb{R} \) equivalent to monotony, and formulated in terms that are translatable to \( K \). This way we will obtain several definitions of "\( f : X \to K \) is monotone", that are, although not equivalent, closely related.

The connections between these various definitions and the properties of the non-archimedean monotone functions can be put together to form a little theory which is interesting in its own right, but of which the relations to the other parts of \( p \)-adic analysis are yet not very tight.

1. Monotone functions.

For a function \( f : \mathbb{R} \to \mathbb{R} \) the following conditions are equivalent:

(a) \( f \) is monotone (in the non-strict sense),

(b) If \( C \subseteq \mathbb{R} \) is convex then \( f^{-1}(C) \) is convex,

(c) If \( x \) is between \( y \), \( z \) then \( f(x) \) is between \( f(y) \) and \( f(z) \).

Also, the following conditions are equivalent:

(a) \( f \) is strictly monotone,

(b) \( f \) is injective. If \( C \subseteq \mathbb{R} \) is convex then \( f(C) \) is relatively convex in \( f(\mathbb{R}) \),

(c) If \( f(x) \) is between \( f(y) \) and \( f(z) \) then \( x \) is between \( y \) and \( z \).

Let \( x, y \in K \). Then the smallest ball that contains \( x, y \) is denoted by \([x, y]\). \( z \in K \) is between \( x \) and \( y \) if \( z \in [x, y] \). (If \( z \notin [x, y] \), we

(*) Texte reçu le 12 mars 1979.
call $x$, $y$ at the same side of $z$). A subset $C \subset K$ is called convex if $x, y \in C$, $z \in [x, y]$ implies $z \in C$. Each convex subset of $K$ can be written in at least one of the following forms

$$\{x : |x - a| < r\}, \{x : |x - a| \leq r\}$$

for some $a \in K$, $r \in (0, \infty)$.

Let $Z \subset Y \subset K$. Then $Z$ is called convex in $Y$ if $Z = C \cap Y$, where $C$ is convex.

With all these definitions we have the following theorem.

**Theorem 1.1.** Let $f : X \rightarrow K$. Then the following conditions are equivalent:

1. If $x, y, z \in X$, $x$ is between $y$ and $z$ then $f(x)$ is between $f(y)$ and $f(z)$,

2. If $C \subset K$ is convex, then $f^{-1}(C)$ is convex in $X$.

We denote the collection of those $f : X \rightarrow K$ satisfying (1) or (2) by $M_b(X)$, i.e. $f \in M_b(X)$ if, and only if, for each $x, y, z \in X$,

$$|x - y| \leq |y - z| \quad \text{implies} \quad |f(x) - f(y)| \leq |f(y) - f(z)|.$$

Isometries are in $M_b$ (viz. exp), but also non trivial locally constant functions (e.g., choose a center in each ball of radius $r > 0$, and let $f$ be the map assigning to $x \in X$ the center of the ball of radius $r$ to which $x$ belongs. Then $f \in M_b(X)$).

**Theorem 1.2.** Let $f : X \rightarrow K$. Then the following conditions are equivalent:

1. If $x, y, z \in X$, $f(x)$ is between $f(y)$ and $f(z)$ then $x$ is between $y$ and $z$,

2. If $C \subset X$ is convex in $X$ then $f(C)$ is convex in $f(X)$. $f$ is injective.

We denote the collection of those $f : X \rightarrow K$ satisfying (1') or (2') by $M_s(X)$, i.e. $f \in M_s(X)$ if, and only if, for each $x, y, z \in X$,

$$|x - y| < |y - z| \quad \text{implies} \quad |f(x) - f(y)| < |f(y) - f(z)|.$$

The classical situations suggests the question as to wether $M_s(X) \subset M_b(X)$ and also wether $f \in M_b(X)$, $f$ injective implies $f \in M_s(X)$. In general, both statements are false, but we do have the following:

**Theorem 1.3.** $f \in M_s(X)$ implies $f^{-1} \in M_b(f(X))$. $f \in M_b(X)$, $f$ injective implies $f^{-1} \in M_s(f(X))$. If $k$ is finite and $X$ is convex, then an injective $M_b$-function is in $M_s(X)$. 
So we are led to define $M_{ba}(X) := M_b(X) \cap M_s(X)$ as being the more or less natural translation of "the space of the strictly monotone functions".

The following theorem concerns continuity of monotone functions. For a function $f : X \to K$, we define its oscillation function, $\omega_f$, in the usual way:

$$\omega_f(a) := \lim_{n \to \infty} \sup \{|f(x) - f(y)| : |x - a| \leq \frac{1}{n}; |y - a| \leq \frac{1}{n}\}$$

$$= \lim_{n \to \infty} \sup \{|f(x) - f(a)| : |x - a| \leq \frac{1}{n}\} \quad (a \in X) .$$

$f$ is continuous at $a$ if, and only if, $\omega_f(a) = 0$.

**Theorem 1.4.** - Let $f$ be either in $M_b(X)$ or in $M_s(X)$. Then

(i) $\omega_f(a) = \inf_{z \not= a} |f(z) - f(a)| \quad (a \in X)$

(ii) $f$ is bounded on compact subsets of $X$,

(iii) For each $a \in X$ we have the following alternative. Either $f$ is continuous at $a$, or for each sequence $x_1, x_2, \ldots$ ($x_n \not= a$) converging to $a$, the sequence $f(x_1), f(x_2), \ldots$ is bounded and has no convergent subsequence.

Let $g \in M_b(X)$. If $Y \subset X$ is spherically complete, then so is $g(Y)$.

Let $h \in M_s(X)$. If $Z \subset h(x)$ is spherically complete, then so is $h^{-1}(z)$.

**Proof (sketch).** - If $f \in M_b(X) \cup M_s(X)$, then:

$$|x - y| < |y - z| \implies |f(x) - f(y)| \leq |f(y) - f(z)| .$$

So $f$ is locally bounded, and (ii) follows. Of (i), only the $\leq$ part is interesting. Choose $z \not= a$. If $|x - a| < |z - a|$, then

$$|f(x) - f(a)| \leq |f(z) - f(a)| \text{ whence } \omega_f(a) \leq |f(z) - f(a)| .$$

Let $\lim x_n = a$ ($x_n \not= a$ for all $n$) and $\lim f(x_n) = \alpha$. Let $\lim y_n = a$. It suffices to show that $\lim f(y_n) = \alpha$. Indeed, let $\varepsilon > 0$, and choose $k$ such that $|f(x_k) - \alpha| < \varepsilon$. Then $|y_n - a| < |x_k - a|$ for large $n$, so

$$|y_n - x_m| < |x_k - x_m|$$

for large $n$ depending on $m$. Hence $|f(y_n) - f(x_n)| \leq |f(x_k) - f(x_n)|$, so

$$(m \to \infty) |f(y_n) - \alpha| \leq |f(x_k) - \alpha| < \varepsilon ,$$

and we have (iii). The rest of the proof is straightforward.

**Corollary 1.5.** - Let $f : X \to K$ be in $M_b(X) \cup M_s(X)$.

(i) If $K$ is a local field, then $f$ is continuous,

(ii) If $|K|$ is discrete, then $f \in M_b(X) \Rightarrow f$ is a homeomorphism $X \sim f(X)$, and $f \in M_s(X) \Rightarrow f$ is a closed map.

(iii) The graph of $f$ is closed in $K^2$.

(iv) If $f(X)$ has no isolated points, then $f$ is continuous.
An \(M_b\)-function may be everywhere discontinuous on \(K\) (even when \(|K|\) is discrete).

**Theorem 1.6.** Let \(B\) be the unit ball of \(K\),

(i) If \(K\) is a local field and \(f \in M_b(B) \cup M_s(B)\), then \(f\) has bounded difference quotients (i.e., there is \(C > 0\) such that \(|f(x) - f(y)| \leq C|x - y|\) for all \(x \in B\)). If, in addition, \(f(B)\) is convex, then \(f\) is a similarity (i.e., a scalar multiple of an isometry).

(ii) If \(K\) has discrete valuation and \(f \in M_b(B)\) is bounded, then \(f\) has bounded difference quotients. If \(f \in M_{bs}(B)\) and if \(f(B)\) is convex, then \(f\) is a similarity.

2. Monotone functions having a type.

In this section, we want to translate the usual classification of (strictly) monotone functions \(\mathbb{R} \to \mathbb{R}\) into two types: the increasing and the decreasing functions. The equivalence relation in \(\mathbb{R}\), \(x \sim y\) if \(x\) and \(y\) are at the same side of \(0\), yields \((-\infty, 0)\) and \((0, \infty)\) as equivalence classes. The relation \(\sim\) is compatible with the canonical group homomorphism \(\mathbb{R}^* \to \mathbb{R}^*/\mathbb{R}^+\), the latter group being \(\{1, -1\}\). \(\pi(x)\) (usually called \(\operatorname{sgn}(x)\)) assigns \(+1\) to every positive element and \(-1\) to every negative element. A function \(f : \mathbb{R} \to \mathbb{R}\) is strictly monotone if there exists \(\sigma : \mathbb{R}^*/\mathbb{R}^+ \to \mathbb{R}^*/\mathbb{R}^+\) such that for all \(x \neq y\)

\[
\pi(f(x) - f(y)) = \sigma(\pi(x - y)).
\]

If \(\sigma\) is the identity then \(f\) is called increasing; if \(\sigma(1) = -1\), \(\sigma(-1) = 1\), \(f\) is called decreasing. Other maps \(\sigma : \{-1, 1\} \to \{-1, 1\}\) cannot occur (i.e., there is no \(f\) such that, for all \(x \neq y\),

\[
\pi(f(x) - f(y)) = \sigma(\pi(x - y)).
\]

This rather weird description of real monotone functions can be used in the non-archimedean case.

For \(x, y \in K^*\) define \(x \sim y\) if \(x, y\) are at the same side of \(0\). This means: \(0 \notin [x, y]\), or \(|x - y| > |y|\), or \(|xy^{-1} - 1| < 1\). Thus \(x \sim y\) if, and only if, \(xy^{-1} \in K^+\) where

\[
K^+ := \{x \in K ; \ |1-x| < 1\}.
\]

We call the elements of \(K^+\) the positive element of \(K\).

The relation \(\sim\) is compatible with the canonical homomorphism of (multiplicative) groups

\[
\pi : K^* \to K^*/K^+ =: \Sigma.
\]

We call \(\Sigma\) the group of signs and \(\pi(x)\) the sign of an element \(x \in K^*\) (\(x\) is
positive if, and only if, \( \eta(x) = 1 \).

If \( K \) is a local field, we can make a group embedding \( \rho : \Sigma \rightarrow K^* \) such that \( \eta \circ \rho \) is the identity on \( \Sigma \). For example, if \( K = \mathbb{Q}_p \), \( \delta \) is a primitive \((p - 1)\)th root of unity, then

\[
\eta(\sum_{n \geq k} a_n p^n) = a_k p^k \quad (k \in \mathbb{Z}, a_k \neq 0)
\]

(Here \( a_n \in \{0, 1, \delta, \ldots, \delta^{p-2}\} \) for each \( n \)).

**DEFINITION 2.1.** Let \( \sigma : \Sigma \rightarrow \Sigma \) be any map. A function \( f : X \rightarrow K \) is monotone of type \( \sigma \) if, for all \( x, y \in X, x \neq y \),

\[
\eta(f(x) - f(y)) = \sigma(\eta(x - y))
\]

(i.e., if \( x - y \in \alpha \in \Sigma \) then \( f(x) - f(y) \in \sigma(\alpha) \)).

We call \( f \) of type \( \beta \in \Sigma \) if \( f \) is of type \( \sigma \) where \( \sigma \) is the multiplication with \( \beta \), i.e.,

\[
\frac{f(x) - f(y)}{x - y} \in \beta \quad (x, y \in X, x \neq y).
\]

We call \( f \) increasing if \( f \) is of type \( \sigma \) where \( \sigma \) is the identity, i.e.,

\[
\frac{f(x) - f(y)}{x - y} \text{ is positive} \quad (x \neq y).
\]

Clearly, if \( f \) is of type \( \beta \), and if \( b \in \beta \), then \( b^{-1} f \) is increasing.

First, we look at increasing functions, then we discuss more general types \( \sigma \).

Notice that increasing functions are isometries hence are in \( M_{bs}(X) \). If \( f \) is increasing then \( f(x) = x + h(x) \), where \( |h(x) - h(y)| < |x - y| \) \((x, y \in X, x \neq y)\).

Such \( h \) we call pseudo-contractions.

**LEMMA 2.2.** Let \( X \) be an ultrametric space. Then the following are equivalent

(a) \( X \) is spherically complete,

(b) Each pseudocontraction \( X \rightarrow X \) has a (unique) fixed point.

**Proof (sketch).** (a) \( \rightarrow \) (b). Let \( \sigma : X \rightarrow X \) be a pseudocontraction. A convex set \( C \subset X \) is called invariant if \( \sigma(C) \subset C \). It is easily proved that the invariant convex subsets of \( X \) form a nest. Let \( C_0 \) be the smallest invariant convex set. If \( a \in C_0 \) and \( \sigma(a) \neq a \) then

\[
B_0 := \{ x \in X ; \, d(x, \sigma(a)) < d(a, \sigma(a)) \}
\]

is invariant, convex, and does not contain \( a \). Hence \( \sigma(a) = a \) for all \( a \in C_0 \), and \( C_0 \) is a singleton. (b) \( \rightarrow \) (a). If \( B_1 \neq B_2 \neq \ldots \) are balls in \( X \) with \( \bigcap B_n = \emptyset \) then choose \( x_n \in B_n \setminus B_{n+1} \) \((n \in \mathbb{N})\). The map \( \sigma : X \rightarrow X \) defined by

\[
\sigma(x) = x_{n+1} \quad (x \in B_n \setminus B_{n+1})
\]

is a pseudocontraction without a fixed point.
COROLLARY 2.3. - Let $X$ be convex, let $K$ be spherically complete, and let $f : X \to K$ be increasing. Then $f(X)$ is convex. If $f(X) \subseteq X$, then $f$ is surjective.

Proof. - Let $f(X) \subseteq X$. Choose $\alpha \in X$. Then $x \mapsto -f(x) + x + \alpha$ is a pseudo-contraction mapping $X$ into $X$, hence has a fixed point. So $f(x) = \alpha$ for some $x \in X$.

If $K$ is not spherically complete, we have always increasing $f : K \to K$ that are not surjective. (Let $h : K \to K$ be a pseudocontraction without a fixed point. Let $f(x) = x - h(x)$ ($x \in K$), then $0 \not\in \text{im } f$). The inverse $f^{-1} : f(K) \to K$ can, of course, not be extended to an increasing function $K \to K$.

THEOREM 2.4. - Let $K$ be spherically complete, and let $f : X \to K$ be increasing. Then $f$ can be extended to an increasing function $K \to K$.

Proof. - By Zorn's Lemma, it suffices to extend $f$ to an increasing function on $X \cup \{a\}$, where $a \not\in X$. We are done if we can find $\alpha \in K$ such that, for all $x \in X$,

$$|\frac{\alpha - f(x)}{a - x} - 1| < 1$$

i.e., $\alpha \in B_{f(x)}(a-x)(|a-x|^{-})$ for all $x \in X$. These balls form a nest.

Let us call a function $f : X \to K$ positive if $f(X) \subseteq K^+$.

THEOREM 2.5.

(i) If $f : X \to K$ is increasing then $f'$ is positive,

(ii) If $g : X \to K$ is a positive Baire class one function, then $g$ has an increasing antiderivative,

(iii) If $g : X \to K$ is continuous and positive, then $g$ has a $C^1$-antiderivative,

(iv) If $f \in C^1(X)$ and $f'$ is positive then $f = j + h$ where $j$ is increasing, and $h$ is locally constant.

EXAMPLES.

1° The exponential function (defined on its natural convergence region) is increasing.

2° Let $f \in C(\mathbb{Z}_p)$, and let $e_0 = 1_{\mathbb{Z}_p}$, for $n \in \mathbb{N}$,

$$e_n(x) = \begin{cases} 1 & \text{if } |x - n| < \frac{1}{n} \\ 0 & \text{elsewhere} \end{cases} \quad (x \in \mathbb{Z}_p).$$

Then $e_0, e_1, \ldots$ form an orthonormal base of $C(\mathbb{Z}_p)$, so there exist $\lambda_0, \lambda_1, \ldots \in \mathbb{Q}_p$ such that $f = \sum_{n=0}^{\infty} \lambda_n e_n$, uniformly.
\( f \) is increasing if, and only if, for all \( n \in \mathbb{N} \),
\[
|\lambda_n - \{n\}| < \{n\}
\]
(where, if \( n = a_0 + a_1 p + \ldots + a_k p^k \) (\( a_i \in \{0, 1, \ldots, p - 1\} \) for each \( i \), \( a_k \neq 0 \)), then \( \{n\} \downarrow = a_k p^k \).

In other words, \( f = \sum \lambda_n \in \mathcal{C}(\mathbb{Z}_p) \) is increasing if, and only if, \( \lambda_n/\{n\} \) is positive for all \( n \in \mathbb{N} \).

Let \( \alpha, \beta \in \Sigma \). If the set theoretic sum \( \alpha + \beta := \{x + y : x \in \alpha, y \in \beta\} \) does not contain 0 then \( \alpha + \beta \in \Sigma \), notation \( \alpha \oplus \beta \). It follows that \( \alpha \oplus \beta \) is defined if, and only if, \( \alpha \neq -\beta \).

If \( x, y \in \alpha, \beta \in \Sigma \) then \(|x| = |y|\). This defines \(|\alpha|\) in a natural way.
We have the following results.

**THEOREM 2.6.** - Let \( f : \mathbb{K} \to \mathbb{K} \) be monotone of type \( \sigma : \Sigma \to \Sigma \). Let \( \alpha, \beta \in \Sigma \),

(i) \( \sigma(-\alpha) = -\sigma(\alpha) \),

(ii) If \( \sigma(\alpha) \oplus \sigma(\beta) \) is defined then so is \( \alpha \oplus \beta \) and \( \sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta) \),

(iii) \(|\alpha| < |\beta| \) implies \(|\sigma(\alpha)| < |\sigma(\beta)| \),

(iv) If \( |\beta| = 1 \), \( \beta \) contains an element of the prime field of \( \mathbb{K} \) then \( \sigma(\beta) = \beta \sigma(\alpha) \),

(v) \( f \in \mathcal{M}_{sa}(\mathbb{K}) \),

(vi) \( f \) is either nowhere continuous or uniformly continuous.

**THEOREM 2.7.** - Let \( f : \mathbb{K} \to \mathbb{K} \) be monotone of type \( \sigma : \Sigma \to \Sigma \). Then the following conditions are equivalent,

(a) \( \sigma \) is injective,

(b) \( f \in \mathcal{M}_b(X) \),

(c) If for some \( \alpha, \beta \in \Sigma \), \( \alpha \oplus \beta \) is defined, then so is \( \sigma(\alpha) \oplus \sigma(\beta) \),

(d) \(|\sigma(\alpha)| < |\sigma(\beta)| \) implies \(|\alpha| < |\beta| \) (\( \alpha, \beta \in \Sigma \)).

**COROLLARY 2.8.** - Let \( \mathbb{K} \) be a prime field, and let \( f : \mathbb{K} \to \mathbb{K} \) be monotone of type \( \sigma : \Sigma \to \Sigma \). Then \( \sigma \) is injective.

(If \( \mathbb{K} = \mathbb{Q}_p^p(\sqrt{-1}) \), \( p = 3 \mod 4 \), we can find an example of an \( f : \mathbb{K} \to \mathbb{K} \) monotone of type \( \sigma \), where \( \sigma \) is not injective).

**EXAMPLE 2.9.** - Let \( \mathbb{K} = \mathbb{Q}_p \). Then \[\{\sigma : \Sigma \to \Sigma : \text{there is } f : \mathbb{Q}_p \to \mathbb{Q}_p, \ f \text{ monotone of type } \sigma\}\]
consists of all $\sigma : \Sigma \to \Sigma$ of the form

\[ \sigma^n : p \to \sigma^i \delta^a(n) \rho \lambda(n) \]

where $s : \mathbb{Z} \to \{0, 1, 2, \ldots, p - 2\}$ and $\lambda : \mathbb{Z} \to \mathbb{Z}$ is strictly increasing.

3. Functions of bounded variation.

**Lemma 3.1.** Let $f : X \to K$ have bounded difference quotients. Then $f$ is a linear combination of two increasing functions.

**Proof.** Choose $\lambda \in K$,

\[ |\lambda| > \sup\left| \frac{f(x) - f(y)}{x - y} \right| ; x \neq y \]

Then $\lambda^{-1} f$ is a (pseudo-) contraction, so $g(x) := -x + \lambda^{-1} f(x) (x \in X)$ is increasing. If $h(x) := x$ $(x \in X)$, then $\lambda h + \lambda g = f$.

**Corollary 3.2.** Let $X$ be the unit ball of a local field $K$ and let $f : X \to K$. Then the following are equivalent

(a) $f \in BA(X)$ (i.e., $\sup\left| \frac{f(x) - f(y)}{x - y} \right| ; x \neq y < \infty$),

(b) $f$ is a linear combination of two increasing functions,

(c) $f \in \mathcal{B}_0(X)$,

(d) $f \in \mathcal{B}_0(X)$.

**Proof.** Use 1.6.

**References**
