Abstract of the lecture

NON - ARCHIMEDEAN DIFFERENTIATION

held on Tuesday, June 5, 1979 at the "VI Jornadas de Matemáticas Hispano-Lusas" organized by the University of SANTANDER,

by

W.H. Schikhof

§ 1. Introduction.

The subject is part of the so-called non-archimedean (or ultrametric) analysis. Roughly speaking, one may say that this is the analysis that one obtains when replacing in the "classical" analysis $\mathbb{R}$ or $\mathbb{C}$ by a non-archimedean valued field K.

A non-archimedean valued field is a (commutative) field K, together with a map $|\ |: K \to \mathbb{R}$ (the valuation) satisfying

$$|a| \geq 0 \quad , \quad |a| = 0 \text{ iff } a = 0$$

$$|ab| = |a|\ |b|$$

$$|a+b| \leq \max(|a|,|b|) \quad \text{(the strong triangle inequality)}$$

for all $a,b \in K$.

We have the following remarks.

(1) Apart from $\mathbb{R}$ or $\mathbb{C}$, every complete valued field is non-archimedean.

(2) If K is a non-archimedean valued field and if $L \supset K$ is an overfield of K then the valuation on K can be extended to a non-archimedean valuation on L.

(3) If K is a (non-archimedean) valued field then its completion $\tilde{K}$ (with respect to the metric $(x,y) \mapsto |x-y|$ can, in a natural

way, be given the structure of a non-archimedean valued field.

In the sequel we exclude the so-called trivial valuation given by

$$|x|' = \begin{cases} 0 \text{ if } x = 0 \\ 1 \text{ if } x \neq 0. \end{cases}$$

The non-archimedean analysis has several branches, similar to the classical analysis. Thus we have non-archimedean functional analysis, harmonic analysis, theory of analytic functions in one or several variables, etc.

In this talk we consider a more elementary subject, namely infinitesimal calculus in K. More specifically, we want to see what remains of the so-called Fundamental Theorem of Calculus (in $\mathbb{R}$) that states that the operations of differentiation and integration are in some sense each others inverses.

§ 2. Differentiation in K. Let $X \subset K$ be a subset without isolated points. A function $f : X \to K$ is called differentiable if for all $a \in X$

$$f'(a) := \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

exists. The proof of the well known rules (sum-, product-, chain-rule) can formally be taken over from the classical theory. Thus, a rational function is differentiable if it has no poles on X. An analytic function $x \to \Sigma a_n x^n$ is differentiable on $\{x : |x| < (\overline{\lim} \sqrt[n]{|a_n|})^{-1}\}$.

Deviations from the classical theory appear when we look at the functions whose derivative vanishes everywhere For example, let $\varepsilon > 0$, $a \in K$. Then $B(a,\varepsilon) := \{x \in K : |x-a| < \varepsilon\}$ is an open-and-closed subset of K, hence $\xi_{B(a,\varepsilon)}$, defined by

$$\xi_{B(a,\varepsilon)}(x) := \begin{cases} 1 & \text{if } x \in B(a,\varepsilon) \\ 0 & \text{elsewhere} \end{cases}$$

is differentiable and $\xi'_{B(a,\varepsilon)} = 0$.

Locally constant functions all have derivative zero. On the other hand they form a uniformly dense subset of $C(X)$, the space of all continuous functions: $X \to K$.

Even worse: let $\mathbb{Q}_p$ the field of the p-adic numbers and let $\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x| \leq 1\}$. Then the function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ defined by

$$f(\Sigma a_n p^n) = \Sigma a_n p^{2n} \qquad (\Sigma a_n p^n \in \mathbb{Z}_p)$$

satisfies $|f(x)-f(y)| = |x-y|^2$ for all $x,y \in \mathbb{Z}_p$. So $f' = 0$ but f is injective, hence not locally constant.

The above example shows also that a Mean Value Theorem is necessarily absent in our theory.

Notice that the difficulties encountered above also appear when we study differentiability of functions $f : \mathbb{D} \to \mathbb{R}$, where $\mathbb{D} \subset [0,1]$ is the Cantor set. So it is the domain of f that is responsable for the troubles rather than its range.

§ 3. Continuously differentiable functions.

If we follow naively the path of the classical analysis and define

$$C^1(X) := \{f : X \to K,\ f \text{ is differentiable, } f' \text{ is continuous}\}$$

then we run up against difficulties.

First of all, one can prove that $C^1(\mathbb{Z}_p)$ (with the norm

$f \mapsto \max(||f||_\infty, ||f'||_\infty))$ is not a Banach space. In fact one shows that for every pair of continuous functions $f,g : \mathbb{Z}_p \to \mathbb{Q}_p$ there exists a sequence $f_1, f_2, \ldots$ in $C^1(\mathbb{Z}_p)$ for which both $f_n \to f$ and $f'_n \to g$ uniformly.

What is worse, we have no local invertibility theorem for such C^1-functions.

In fact, let $f : \mathbb{Z}_p \to \mathbb{Q}_p$ be defined by

$$f(x) = \begin{cases} x-p^{2n} & \text{if } |x-p^n| < p^{-2n} \quad (n \in \mathbb{N}) \\ x & \text{elsewhere} \end{cases}$$

Then $f'(x) = 1$ for all $x \in \mathbb{Z}_p$. But $f(p^n) = f(p^n - p^{2n})$ for all $n \in \mathbb{N}$, so f is not even locally injective at 0.

Therefore we are led to define:

Let $f : X \to K$. Put

$$\Phi f(x,y) := \frac{f(x)-f(y)}{x-y} \qquad (x,y \in X,\ x \neq y).$$

We say that $f \in C^1(X)$ if Φf can continuously be extended to a function $\overline{\Phi} f : X \times X \to K$.

Then $BC^1(X) := \{f \in C^1(X) : f \text{ and } \Phi f \text{ are bounded}\}$ is a Banach space under $f \mapsto ||f||_1 := \max(||f||_\infty, ||\Phi(f)||_\infty)$.

Further, if $f \in C^1(X)$, $f'(a) \neq 0$ for some $a \in X$, then f has a C^1-inverse, locally at a.

<u>Theorem</u>. <u>Differentiation is a continuous surjection</u> $BC^1(X) \xrightarrow{D} BC(X)$.

(here $BC(X)$ is the space of all bounded continuous functions with the supremum norm)

§ 4. <u>"Integration"</u>.

Next, we want to define an "indefinite integral" $P : BC(X) \to BC^1(X)$

(an analogue of $(Pf)(x) := {}_0\int^x f(t)dt$ for real functions) such that DP is the identily on BC(X).

A natural try is first to find an analogue of the Lebesgue measure in K. But this turns out to be a dead end road. For example if $K = \mathbb{Q}_p$ there does not exist a nonzero translation invariant bounded additive $\mathbb{Q}_p$-valued function m defined on the compact open subsets of $\mathbb{Z}_p$. (By translation invariance $|m(p^n\mathbb{Z}_p)| = p^n|m(\mathbb{Z}_p)| \to \infty$ if $m(\mathbb{Z}_p) \neq 0$). For similar reasons it goes wrong for every local field K.

Following the ideas of Dieudonné, Treiber, we define for $f \in BC(X)$

$$(Pf)(x) := \sum_{n=1}^{\infty} f(x_n)(x_{n+1}-x_n) \qquad (x \in X)$$

Here the x_n are defined as follows. For each $n \in \mathbb{N}$ the equivalence relation $\underset{n}{\sim}$ defined by $x \underset{n}{\sim} y$ if $|x-y| < \frac{1}{n}$ yields a partition of X into balls. Choose a center in each ball and let R_n be the set of these centers.

For each $x \in X$ and $n \in \mathbb{N}$, x_n is defined by $x_n \in R_n$, $|x_n-x| < \frac{1}{n}$.

Theorem. (A NON-ARCHIMEDEAN FORM OF THE FUNDAMENTAL THEOREM).

P *is a linear isometry of* BC(X) *into* $BC^1(X)$. DP *is the identity on* BC(X), *whereas* PD *is a projection of* $BC^1(X)$ *onto a complement of* $\{f \in BC^1(X) : f' = 0\}$.

§ 5. Generalizations of the Fundamental Theorem.

We may ask whether there exists some form of the Fundamental Theorem for functions belonging to spaces, larger than BC(X), $BC^1(X)$

respectively. (For example, compare the classical theorem on L^1-functions versus absolutely continuous functions).

We have the following striking fact that has no counterpart in classical analysis. We say that $g : X \to K$ is of _the first class of_ Baire if there exists a sequence $g_1, g_2, \ldots$ of continuous functions $X \to K$ such that $\lim g_n = g$ pointwise.

THEOREM. (a) _Let_ $f : X \to K$ _be differentiable. Then_ f' _is of the first class of Baire._

(b) _Let_ $g : X \to K$ _be of the first class of Baire. Then_ g _has an antiderivative._

Let $B\mathcal{B}^1(X)$ be the Banach space consisting of all bounded functions $X \to K$ of the first class of Baire with respect to the supremum norm. Let $BD(X)$ be the Banach space of all differentiable $f : X \to K$ for which both f and Φf are bounded, with respect to the norm $f \to ||f||_\infty \vee ||\Phi f||_\infty$. Then we have

THEOREM. _Differentiation is a quotient map_ $BD(X) \overset{D}{\to} B\mathcal{B}^1(X)$.

_If_ K _has discrete valuation then there exists a continuous linear_ $P : B\mathcal{B}^1(X) \to \mathcal{B}D(X)$ _for which_ DP _is the identity on_ $B\mathcal{B}^1(X)$.

Notes.

1. The construction of the above P is awful and, contrary to § 4, P does not resemble an indefinite integral in any way.
2. If the valuation of K is dense the existence of such a P is still an open question.

§ 6. Restriction of the Fundamental Theorem.

In classical analysis, we have that if $f \in C^n$ then $x \mapsto {}_0\int^x f(t)\,dt$ is in C^{n+1}. In our situation we define for $f : X \to K$: $f \in C^2(X)$ if the function $\Phi_2 f$, defined by

$$\Phi_2 f(x,y,z) = \frac{\Phi_1 f(x,z) - \Phi_1 f(y,z)}{x-y} \qquad (x,y,z \in X\ ,\ x \neq y,\ y \neq z,\ x \neq z)$$

can continuously be extended to $\overline{\Phi}_2 f : X^3 \to K$. Similarly, we define $C^3(X), C^4(X), \ldots$. Let $C^\infty(X) := \bigcap_{n=1}^{\infty} C^n(X)$.

The map P, defined in § 4, does not always map C^1-functions into C^2-functions. But we have (notations as in § 4)

THEOREM. Let the characteristic of K be unequal to 2. Then the map P_2 defined via

$$(P_2 f)(x) := \Sigma f(x_n)(x_{n+1}-x_n) + \tfrac{1}{2}\Sigma f'(x_n)(x_{n+1}-x_n)^2 \qquad (x \in X)$$

maps $C^1(X)$ into $C^2(X)$ and $(Pf)' = f$ for all $f \in C^1(X)$.

Similarly, one can define antiderivation maps $P_n : C^{n-1}(X) \to C^n(X)$ (in case the characteristic of K is unequal to $2,3,\ldots,n$).

OPEN QUESTION. Let K have characteristic 0. Does every $f \in C^\infty(X)$ have a C^∞-antiderivative ?

W. Schikhof

Reference

Schikhof, W. : Non-archimedean calculus. (Lecture Notes). Report 7812, Mathematisch Instituut, Nijmegen, The Netherlands.