Abstract of the lecture

NON - ARCHIMEDEAN DIFFERENTIATION

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by

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§ 1. Introduction.

The subject is part of the so-called non-archimedean (or ultrametric) analysis. Roughly speaking, one may say that this is the analysis that one obtains when replacing in the "classical" analysis IR or C by a non-archimedean valued field K.

A non-archimedean valued field is a (commutative) field K, together with a map | | : K → IR (the valuation) satisfying

\[ |a| ≥ 0 , \quad |a| = 0 \iff a = 0 \]

\[ |ab| = |a| |b| \]

\[ |a+b| ≤ \max(|a|,|b|) \quad \text{(the strong triangle inequality)} \]

for all a, b ∈ K.

We have the following remarks.

(1) Apart from IR or C, every complete valued field is non-archimedean.

(2) If K is a non-archimedean valued field and if L ⊃ K is an overfield of K then the valuation on K can be extended to a non-archimedean valuation on L.

(3) If K is a (non-archimedean) valued field then its completion \( \tilde{K} \) (with respect to the metric \( (x,y) \mapsto |x-y| \)) can, in a natural
way, be given the structure of a non-archimedean valued field. In the sequel we exclude the so-called trivial valuation given by

\[ |x|' = \begin{cases} 
0 & \text{if } x = 0 \\
1 & \text{if } x \neq 0.
\end{cases} \]

The non-archimedean analysis has several branches, similar to the classical analysis. Thus we have non-archimedean functional analysis, harmonic analysis, theory of analytic functions in one or several variables, etc.

In this talk we consider a more elementary subject, namely infinitesimal calculus in K. More specifically, we want to see what remains of the so-called Fundamental Theorem of Calculus (in \( \mathbb{R} \)) that states that the operations of differentiation and integration are in some sense each other's inverses.

§ 2. Differentiation in K. Let \( X \subset K \) be a subset without isolated points. A function \( f : X \to K \) is called differentiable if for all \( a \in X \)

\[ f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \]

exists. The proof of the well-known rules (sum-, product-, chain-rule) can formally be taken over from the classical theory. Thus, a rational function is differentiable if it has no poles on \( X \). An analytic function \( x + \sum a_n x^n \) is differentiable on \( \{x : |x| < (\lim \sqrt[n]{|a_n|})^{-1}\} \).

Deviations from the classical theory appear when we look at the functions whose derivative vanishes everywhere. For example,
let \( \epsilon > 0 \), \( a \in K \). Then \( B(a, \epsilon) := \{ x \in K : |x-a| < \epsilon \} \) is an open-and-closed subset of \( K \), hence \( \xi_{B(a, \epsilon)} \), defined by
\[
\xi_{B(a, \epsilon)}(x) := \begin{cases} 
1 & \text{if } x \in B(a, \epsilon) \\
0 & \text{elsewhere}
\end{cases}
\]
is differentiable and \( \xi_{B(a, \epsilon)}' = 0 \).

Locally constant functions all have derivative zero. On the other hand they form a uniformly dense subset of \( C(X) \), the space of all continuous functions: \( X \to K \).

Even worse: let \( \mathbb{Q}_p \) the field of the \( p \)-adic numbers and let
\( Z_p := \{ x \in \mathbb{Q}_p : |x| \leq 1 \} \). Then the function \( f : Z_p \to \mathbb{Q}_p \) defined by
\[
f(\sum_{n=0}^{\infty} a_n p^{2n}) = \sum_{n=0}^{\infty} a_n p^{2n} \quad (\sum_{n=0}^{\infty} a_n p^{2n} \in Z_p)
\]
satisfies \( |f(x) - f(y)| = |x-y|^2 \) for all \( x, y \in Z_p \). So \( f' = 0 \) but \( f \) is injective, hence not locally constant.

The above example shows also that a Mean Value Theorem is necessarily absent in our theory.

Notice that the difficulties encountered above also appear when we study differentiability of functions \( f : D \to \mathbb{R} \), where \( D \subset [0,1] \) is the Cantor set. So it is the domain of \( f \) that is responsible for the troubles rather than its range.

§ 3. Continuously differentiable functions.

If we follow naively the path of the classical analysis and define
\[
C^1(X) := \{ f : X \to K, f \text{ is differentiable, } f' \text{ is continuous} \}
\]
then we run up against difficulties.

First of all, one can prove that \( C^1(Z_p) \) (with the norm
max(\( \| f \|_\infty, \| f'\|_\infty ) \)) is not a Banach space. In fact one shows that for every pair of continuous functions \( f, g : \mathbb{Z}_p \to \mathbb{Q}_p \) there exists a sequence \( f_1, f_2, \ldots \) in \( C^1(\mathbb{Z}_p) \) for which both \( f_n \to f \) and \( f'_n \to g \) uniformly.

What is worse, we have no local invertibility theorem for such \( C^1 \)-functions.

In fact, let \( f : \mathbb{Z}_p \to \mathbb{Q}_p \) be defined by

\[
f(x) = \begin{cases} 
x-p^{-2n} & \text{if } |x-p^n| < p^{-2n} \\
x & \text{elsewhere}
\end{cases} \quad (n \in \mathbb{N})
\]

Then \( f'(x) = 1 \) for all \( x \in \mathbb{Z}_p \). But \( f(p^n) = f(p^n - p^{-2n}) \) for all \( n \in \mathbb{N} \), so \( f \) is not even locally injective at 0.

Therefore we are led to define:

Let \( f : X \to K \). Put

\[
\Phi f(x,y) := \frac{f(x)-f(y)}{x-y} \quad (x,y \in X, x \neq y).
\]

We say that \( f \in C^1(X) \) if \( \Phi f \) can continuously be extended to a function \( \Phi f : X \times X \to K \).

Then \( BC^1(X) := \{ f \in C^1(X) : f \text{ and } \Phi f \text{ are bounded} \} \) is a Banach space under \( f \mapsto \| f \|_1 := \max(\| f \|_\infty, \| \Phi f \|_\infty ) \).

Further, if \( f \in C^1(X) \), \( f'(a) \neq 0 \) for some \( a \in X \), then \( f \) has a \( C^1 \)-inverse, locally at \( a \).

**Theorem. Differentiation is a continuous surjection** \( BC^1(X) \overset{D}{\to} BC(X) \).

(here \( BC(X) \) is the space of all bounded continuous functions with the supremum norm)

§ 4. "Integration".

Next, we want to define an "indefinite integral" \( P : BC(X) \to BC^1(X) \).
(an analogue of \((Pf)(x) := \int_0^X f(t)dt\) for real functions) such that \(DP\) is the identity on \(BC(X)\).

A natural try is first to find an analogue of the Lebesgue measure in \(K\). But this turns out to be a dead end road. For example if \(K = \mathbb{Q}_p\) there does not exist a nonzero translation invariant bounded additive \(\mathbb{Q}_p\)-valued function \(m\) defined on the compact open subsets of \(\mathbb{Z}_p\). (By translation invariance \(|m(p^n\mathbb{Z}_p)| = p^n|m(\mathbb{Z}_p)| \to \infty\) if \(m(\mathbb{Z}_p) \neq 0\). For similar reasons it goes wrong for every local field \(K\).

Following the ideas of Dieudonné, Treiber, we define for \(f \in BC(X)\)

\[
(Pf)(x) := \sum_{n=1}^{\infty} f(x_n)(x_{n+1} - x_n) \quad (x \in X)
\]

Here the \(x_n\) are defined as follows. For each \(n \in \mathbb{N}\) the equivalence relation \(\sim\) defined by \(x \sim y\) if \(|x-y| < \frac{1}{n}\) yields a partition of \(X\) into balls. Choose a center in each ball and let \(R_n\) be the set of these centers.

For each \(x \in X\) and \(n \in \mathbb{N}\), \(x_n\) is defined by \(x_n \in R_n\), \(|x_n - x| < \frac{1}{n}\).

**Theorem.** (A NON-ARCHIMEDEAN FORM OF THE FUNDAMENTAL THEOREM).

\(P\) is a linear isometry of \(BC(X)\) into \(BC(X)\). \(DP\) is the identity on \(BC(X)\), whereas \(PD\) is a projection of \(BC(X)\) onto a complement of \(\{f \in BC(X) : f' = 0\}\).

§ 5. Generalizations of the Fundamental Theorem.

We may ask whether there exists some form of the Fundamental Theorem for functions belonging to spaces, larger than \(BC(X), BC^1(X)\)
respectively. (For example, compare the classical theorem on $L^1$-functions versus absolutely continuous functions).

We have the following striking fact that has no counterpart in classical analysis. We say that $g : X \to K$ is of the first class of Baire if there exists a sequence $g_1, g_2, \ldots$ of continuous functions $X \to K$ such that $\lim g_n = g$ pointwise.

**Theorem.** (a) Let $f : X \to K$ be differentiable. Then $f'$ is of the first class of Baire.

(b) Let $g : X \to K$ be of the first class of Baire. Then $g$ has an antiderivative.

Let $B^1(X)$ be the Banach space consisting of all bounded functions $X \to K$ of the first class of Baire with respect to the supremum norm.

Let $BD(X)$ be the Banach space of all differentiable $f : X \to K$ for which both $f$ and $\phi f$ are bounded, with respect to the norm $f + \|f\|_\infty + \|\phi f\|_\infty$. Then we have

**Theorem.** Differentiation is a quotient map $BD(X) \overset{D}{\to} B^1(X)$.

If $K$ has discrete valuation then there exists a continuous linear $P : B^1(X) \to BD(X)$ for which $DP$ is the identity on $B^1(X)$.

**Notes.**

1. The construction of the above $P$ is awful and, contrary to § 4, $P$ does not resemble an indefinite integral in any way.

2. If the valuation of $K$ is dense the existence of such a $P$ is still an open question.
5 6. **Restriction of the Fundamental Theorem.**

In classical analysis, we have that if \( f \in C^n \) then \( x \mapsto \int_0^x f(t) \, dt \) is in \( C^{n+1} \). In our situation we define for \( f : X \to K \):

\[ f \in C^2(X) \text{ if the function } \Phi_2 f, \text{ defined by} \]

\[ \Phi_2 f(x,y,z) = \frac{\Phi_1 f(x,z) - \Phi_1 f(y,z)}{x-y} \quad (x,y,z \in X, x \neq y, y \neq z, x \neq z) \]

can continuously be extended to \( \Phi_2 f : X^3 \to K \). Similarly, we define \( C^3(X), C^4(X), \ldots \). Let \( C^\infty(X) := \bigcap_{n=1}^\infty C^n(X) \).

The map \( P \), defined in § 4, does not always map \( C^1 \)-functions into \( C^2 \)-functions. But we have (notations as in § 4)

**THEOREM.** Let the characteristic of \( K \) be unequal to 2. Then the map \( P_2 \) defined via

\[ (P_2 f)(x) := \sum \frac{f(x)}{n!} (x_{n+1} - x_n) + \frac{1}{2!} \sum f'(x) (x_{n+1} - x_n)^2 \quad (x \in X) \]

maps \( C^1(X) \) into \( C^2(X) \) and \( (Pf)' = f \) for all \( f \in C^1(X) \).

Similarly, one can define antiderivation maps \( P_n : C^{n-1}(X) \to C^n(X) \) (in case the characteristic of \( K \) is unequal to 2, 3, ..., \( n \)).

**OPEN QUESTION.** Let \( K \) have characteristic 0. Does every \( f \in C^\infty(X) \) have a \( C^\infty \)-antiderivative?

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**Reference**