Abstract of the lecture

NON - ARCHIMEDEAN DIFFERENTIATION

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by

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§ 1. Introduction.

The subject is part of the so-called non-archimedean (or ultrametric) analysis. Roughly speaking, one may say that this is the analysis that one obtains when replacing in the "classical" analysis IR or C by a non-archimedean valued field K.

A non-archimedean valued field is a (commutative) field K, together with a map | | : K \to IR (the valuation) satisfying

- \[ |a| \geq 0 \text{, } |a| = 0 \text{ iff } a = 0 \]
- \[ |ab| = |a| \cdot |b| \]
- \[ |a+b| \leq \max(|a|,|b|) \text{ (the strong triangle inequality)} \]

for all a,b \in K.

We have the following remarks.

(1) Apart from IR or C, every complete valued field is non-archimedean.

(2) If K is a non-archimedean valued field and if L \supseteq K is an overfield of K then the valuation on K can be extended to a non-archimedean valuation on L.

(3) If K is a (non-archimedean) valued field then its completion \( \hat{K} \) (with respect to the metric \( (x,y) \leftrightarrow |x-y| \)) can, in a natural way.
way, be given the structure of a non-archimedean valued field.

In the sequel we exclude the so-called trivial valuation given by

\[ |x|' = \begin{cases} 
0 & \text{if } x = 0 \\
1 & \text{if } x \neq 0.
\end{cases} \]

The non-archimedean analysis has several branches, similar to the classical analysis. Thus we have non-archimedean functional analysis, harmonic analysis, theory of analytic functions in one or several variables, etc.

In this talk we consider a more elementary subject, namely infinitesimal calculus in K. More specifically, we want to see what remains of the so-called Fundamental Theorem of Calculus (in \( \mathbb{R} \)) that states that the operations of differentiation and integration are in some sense each others inverses.

§ 2. Differentiation in K. Let \( X \subseteq K \) be a subset without isolated points. A function \( f : X \rightarrow K \) is called differentiable if for all \( a \in X \)

\[ f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \]

exists. The proof of the well known rules (sum-, product-, chain-rule) can formally be taken over from the classical theory. Thus, a rational function is differentiable if it has no poles on \( X \). An analytic function \( x + \sum a_n x^n \) is differentiable on

\[ \{ x : |x| < (\lim \sqrt[n]{|a_n|})^{-1} \}. \]

Deviations from the classical theory appear when we look at the functions whose derivative vanishes everywhere. For example,
let \( \varepsilon > 0, \ a \in K \). Then \( B(a, \varepsilon) := \{x \in K : |x-a| < \varepsilon\} \) is an open-
and-closed subset of \( K \), hence \( \xi_{B(a, \varepsilon)} \), defined by
\[
\xi_{B(a, \varepsilon)}(x) := \begin{cases} 
1 & \text{if } x \in B(a, \varepsilon) \\
0 & \text{elsewhere}
\end{cases}
\]
is differentiable and \( \xi'_{B(a, \varepsilon)} = 0 \).

Locally constant functions all have derivative zero. On the other
hand they form a uniformly dense subset of \( C(X) \), the space of all
continuous functions: \( X \to K \).

Even worse: let \( \mathbb{Q}_p \) the field of the \( p \)-adic numbers and let
\( \mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x| \leq 1\} \). Then the function \( f : \mathbb{Z}_p \to \mathbb{Q}_p \) defined by
\[
f(\sum_{n} a_n p^n) = \sum_{n} a_n p^{2n} \quad (\sum_{n} a_n p^n \in \mathbb{Z}_p)
\]
satisfies \( |f(x)-f(y)| = |x-y|^2 \) for all \( x, y \in \mathbb{Z}_p \). So \( f' = 0 \) but \( f \)
is injective, hence not locally constant.

The above example shows also that a Mean Value Theorem is necessa-
ry absent in our theory.

Notice that the difficulties encountered above also appear when we
study differentiability of functions \( f : D \to \mathbb{R} \), where \( D \subset [0,1] \)
is the Cantor set. So it is the domain of \( f \) that is responsible
for the troubles rather than its range.

§ 3. Continuously differentiable functions.

If we follow naively the path of the classical analysis and define
\[
C^1(X) := \{f : X \to K, f \text{ is differentiable, } f' \text{ is continuous}\}
\]
then we run up against difficulties.

First of all, one can prove that \( C^1(\mathbb{Z}_p) \) (with the norm
$f \mapsto \max(\|f\|_\infty, \|f'\|_\infty)$ is not a Banach space. In fact one shows that for every pair of continuous functions $f, g : \mathbb{Z}_p \to \mathbb{Q}_p$ there exists a sequence $f_1, f_2, \ldots$ in $C^1(\mathbb{Z}_p)$ for which both $f_n \to f$ and $f'_n \to g$ uniformly.

What is worse, we have no local invertibility theorem for such $C^1$-functions.

In fact, let $f : \mathbb{Z}_p \to \mathbb{Q}_p$ be defined by

$$f(x) = \begin{cases} x-p^{2n} & \text{if } |x-p^n| < p^{-2n} \\ x & \text{elsewhere} \end{cases} (n \in \mathbb{N})$$

Then $f'(x) = 1$ for all $x \in \mathbb{Z}_p$. But $f(p^n) = f(p^{-p^{-2n}})$ for all $n \in \mathbb{N}$, so $f$ is not even locally injective at 0.

Therefore we are led to define:

Let $f : X \to K$. Put

$$\Phi f(x,y) := \frac{f(x) - f(y)}{x-y} (x, y \in X, x \neq y).$$

We say that $f \in C^1(X)$ if $\Phi f$ can continuously be extended to a function $\overline{\Phi} f : X \times X \to K$.

Then $BC^1(X) := \{f \in C^1(X) : f \text{ and } \Phi f \text{ are bounded}\}$ is a Banach space under $f \mapsto \|f\|_1 := \max(\|f\|_\infty, \|\Phi f\|_\infty)$.

Further, if $f \in C^1(X)$, $f'(a) \neq 0$ for some $a \in X$, then $f$ has a $C^1$-inverse, locally at $a$.

**Theorem.** Differentiation is a continuous surjection $BC^1(X) \xrightarrow{D} BC(X)$.

(here $BC(X)$ is the space of all bounded continuous functions with the supremum norm)

§ 4. "Integration".

Next, we want to define an "indefinite integral" $P : BC(X) \to BC^1(X)$.
(an analogue of \((\text{Pf})(x) := \int_0^x f(t)dt\) for real functions) such that \(\text{DP}\) is the identity on \(\text{BC}(X)\).

A natural try is first to find an analogue of the Lebesgue measure in \(K\). But this turns out to be a dead end road. For example if \(K = \mathbb{Q}_p\) there does not exist a nonzero translation invariant bounded additive \(\mathbb{Q}_p\)-valued function \(m\) defined on the compact open subsets of \(\mathbb{R}\). (By translation invariance \(|m(p^n x)| = p^n |m(x)| \to \infty\) if \(m(x) \neq 0\). For similar reasons it goes wrong for every local field \(K\).

Following the ideas of Dieudonné, Treiber, we define for \(f \in \text{BC}(X)\)
\[
(\text{Pf})(x) := \sum_{n=1}^{\infty} f(x_n)(x_{n+1} - x_n) \quad (x \in X)
\]
Here the \(x_n\) are defined as follows. For each \(n \in \mathbb{N}\) the equivalence relation \(\sim_n\) defined by \(x \sim_n y\) if \(|x-y| < \frac{1}{n}\) yields a partition of \(X\) into balls. Choose a center in each ball and let \(R_n\) be the set of these centers.
For each \(x \in X\) and \(n \in \mathbb{N}\), \(x_n\) is defined by \(x_n \in R_n\), \(|x_n - x| < \frac{1}{n}\).

**Theorem.** (A NON-ARCHIMEDEAN FORM OF THE FUNDAMENTAL THEOREM).
\(P\) is a linear isometry of \(\text{BC}(X)\) into \(\text{BC}^1(X)\). \(\text{DP}\) is the identity on \(\text{BC}(X)\), whereas \(\text{PD}\) is a projection of \(\text{BC}^1(X)\) onto a complement of \(\{f \in \text{BC}^1(X) : f' = 0\}\).

§ 5. Generalizations of the Fundamental Theorem.

We may ask whether there exists some form of the Fundamental Theorem for functions belonging to spaces, larger than \(\text{BC}(X), \text{BC}^1(X)\)
respectively. (For example, compare the classical theorem on $L^1$-functions versus absolutely continuous functions).

We have the following striking fact that has no counterpart in classical analysis. We say that $g : X \rightarrow K$ is of the first class of Baire if there exists a sequence $g_1', g_2', \ldots$ of continuous functions $X \rightarrow K$ such that $\lim g_n = g$ pointwise.

**THEOREM.** (a) Let $f : X \rightarrow K$ be differentiable. Then $f'$ is of the first class of Baire.

(b) Let $g : X \rightarrow K$ be of the first class of Baire. Then $g$ has an antiderivative.

Let $B^1_b(X)$ be the Banach space consisting of all bounded functions $X \rightarrow K$ of the first class of Baire with respect to the supremum norm. Let $BD(X)$ be the Banach space of all differentiable $f : X \rightarrow K$ for which both $f$ and $\phi f$ are bounded, with respect to the norm $f \rightarrow ||f||_\infty \vee ||\phi f||_\infty$. Then we have

**THEOREM.** Differentiation is a quotient map $BD(X) \xrightarrow{D} B^1_b(X)$.

If $K$ has discrete valuation then there exists a continuous linear $P : B^1_b(X) \rightarrow BD(X)$ for which $DP$ is the identity on $B^1_b(X)$.

**Notes.**

1. The construction of the above $P$ is awful and, contrary to § 4, $P$ does not resemble an indefinite integral in any way.

2. If the valuation of $K$ is dense the existence of such a $P$ is still an open question.
5.6. Restriction of the Fundamental Theorem.

In classical analysis, we have that if $f \in C^n$ then $x \mapsto \int_0^x f(t)\,dt$ is in $C^{n+1}$. In our situation we define for $f : X \to K$:

If $f \in C^2(X)$ if the function $\Phi_2 f$, defined by

$$\Phi_2 f(x,y,z) = \frac{\Phi_f(x,z) - \Phi_f(y,z)}{x-y} \quad (x,y,z \in X, x \neq y, y \neq z, x \neq z)$$

can continuously be extended to $\Phi_2 f : X^3 \to K$. Similarly, we define $C^3(X), C^4(X), \ldots$. Let $C^\infty(X) := \bigcap_{n=1}^\infty C^n(X)$.

The map $P$, defined in § 4, does not always map $C^1$-functions into $C^2$-functions. But we have (notations as in § 4)

**THEOREM.** Let the characteristic of $K$ be unequal to 2. Then the map $P_2$ defined via

$$P_2 f(x) := \sum_{n=1}^\infty \frac{f(x_n) (x_{n+1} - x_n)}{n} + \frac{1}{2} \sum_{n=1}^\infty \frac{f'(x_n) (x_{n+1} - x_n)^2}{n} \quad (x \in X)$$

maps $C^1(X)$ into $C^2(X)$ and $(Pf)' = f$ for all $f \in C^1(X)$.

Similarly, one can define antiderivation maps $P_n : C^{n-1}(X) \to C^n(X)$ (in case the characteristic of $K$ is unequal to $2, 3, \ldots, n$).

**OPEN QUESTION.** Let $K$ have characteristic 0. Does every $f \in C^\infty(X)$ have a $C^\infty$-antiderivative?

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Reference