§ 1. Introduction.

The subject is part of the so-called non-archimedean (or ultrametric) analysis. Roughly speaking, one may say that this is the analysis that one obtains when replacing in the "classical" analysis \( \mathbb{R} \) or \( \mathbb{C} \) by a non-archimedean valued field \( K \).

A non-archimedean valued field is a (commutative) field \( K \), together with a map \( | \cdot | : K \to \mathbb{R} \) (the valuation) satisfying
\[
|a| \geq 0 , \quad |a| = 0 \iff a = 0 \\
|ab| = |a| |b| \\
|a+b| \leq \max(|a|,|b|) \quad \text{(the strong triangle inequality)}
\]
for all \( a,b \in K \).

We have the following remarks.

(1) Apart from \( \mathbb{R} \) or \( \mathbb{C} \), every complete valued field is non-archimedean.

(2) If \( K \) is a non-archimedean valued field and if \( L \supset K \) is an overfield of \( K \) then the valuation on \( K \) can be extended to a non-archimedean valuation on \( L \).

(3) If \( K \) is a (non-archimedean) valued field then its completion \( \sim K \) (with respect to the metric \( (x,y) \mapsto |x-y| \)) can, in a natural way,
way, be given the structure of a non-archimedean valued field.

In the sequel we exclude the so-called trivial valuation given by

\[ |x|' = \begin{cases} 
0 & \text{if } x = 0 \\
1 & \text{if } x \neq 0.
\end{cases} \]

The non-archimedean analysis has several branches, similar to the classical analysis. Thus we have non-archimedean functional analysis, harmonic analysis, theory of analytic functions in one or several variables, etc.

In this talk we consider a more elementary subject, namely infinitesimal calculus in K. More specifically, we want to see what remains of the so-called Fundamental Theorem of Calculus (in \( \mathbb{R} \)) that states that the operations of differentiation and integration are in some sense each others inverses.

§ 2. Differentiation in K. Let \( X \subset K \) be a subset without isolated points. A function \( f : X \to K \) is called differentiable if for all \( a \in X \)

\[ f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \]

exists. The proof of the well known rules (sum-, product-, chain-rule) can formally be taken over from the classical theory. Thus, a rational function is differentiable if it has no poles on \( X \). An analytic function \( x + \sum a_n x^n \) is differentiable on \( \{x : |x| < (\lim \sqrt[n]{|a_n|})^{-1}\} \).

Deviations from the classical theory appear when we look at the functions whose derivative vanishes everywhere. For example,
let $\epsilon > 0$, $a \in K$. Then $B(a, \epsilon) := \{x \in K : |x-a| < \epsilon\}$ is an open-and-closed subset of $K$, hence $\xi_{B(a, \epsilon)}$, defined by

$$
\xi_{B(a, \epsilon)}(x) := \begin{cases} 
1 & \text{if } x \in B(a, \epsilon) \\
0 & \text{elsewhere}
\end{cases}
$$

is differentiable and $\xi_{B(a, \epsilon)}' = 0$.

Locally constant functions all have derivative zero. On the other hand they form a uniformly dense subset of $C(X)$, the space of all continuous functions: $X \to K$.

Even worse: let $\mathbb{Q}_p$ the field of the $p$-adic numbers and let $\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x| \leq 1\}$. Then the function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ defined by

$$f(\sum_{n=0}^{\infty} a_n p^n) = \sum_{n=0}^{\infty} a_n p^{2n} \quad (\sum_{n=0}^{\infty} a_n p^n \in \mathbb{Z}_p)$$

satisfies $|f(x) - f(y)| = |x-y|^2$ for all $x, y \in \mathbb{Z}_p$. So $f' = 0$ but $f$ is injective, hence not locally constant.

The above example shows also that a Mean Value Theorem is necessarily absent in our theory.

Notice that the difficulties encountered above also appear when we study differentiability of functions $f : D \to \mathbb{R}$, where $D \subset [0,1]$ is the Cantor set. So it is the domain of $f$ that is responsible for the troubles rather than its range.

§ 3. Continuously differentiable functions.

If we follow naively the path of the classical analysis and define

$$C^1(X) := \{f : X \to K, f \text{ is differentiable, } f' \text{ is continuous}\}$$

then we run up against difficulties.

First of all, one can prove that $C^1(\mathbb{Z}_p)$ (with the norm
$f \mapsto \max(\|f\|_\infty, \|f\|_\infty')$ is not a Banach space. In fact one shows that for every pair of continuous functions $f, g : Z_p \to D_p$, there exists a sequence $f_1, f_2, \ldots$ in $C^1(Z_p)$ for which both $f_n \to f$ and $f'_n \to g$ uniformly.

What is worse, we have no local invertibility theorem for such $C^1$-functions.

In fact, let $f : Z_p \to D_p$ be defined by

$$f(x) = \begin{cases} x^{-p^2n} & \text{if } |x^{-p^n}| < p^{-2} & (n \in \mathbb{N}) \\ x & \text{elsewhere} \end{cases}$$

Then $f'(x) = 1$ for all $x \in Z_p$. But $f(p^n) = f(p^{-p^2n})$ for all $n \in \mathbb{N}$, so $f$ is not even locally injective at 0.

Therefore we are led to define:

Let $f : X \to K$. Put

$$\Phi f(x,y) := \frac{f(x) - f(y)}{x-y} \quad (x,y \in X, x \neq y).$$

We say that $f \in C^1(X)$ if $\Phi f$ can continuously be extended to a function $\Phi f : X \times X \to K$.

Then $BC^1(X) := \{f \in C^1(X) : f$ and $\Phi f$ are bounded$\}$ is a Banach space under $f \mapsto \|f\|_1 := \max(\|f\|_\infty, \|\Phi f\|_\infty)$.

Further, if $f \in C^1(X)$, $f'(a) \neq 0$ for some $a \in X$, then $f$ has a $C^1$-inverse, locally at $a$.

Theorem. Differentiation is a continuous surjection $BC^1(X) \overset{D}{\to} BC(X)$.

(here $BC(X)$ is the space of all bounded continuous functions with the supremum norm)

§ 4. "Integration".

Next, we want to define an "indefinite integral" $P : BC(X) \to BC^1(X)$
(an analogue of \((Pf)(x) := \int_0^x f(t) \, dt\) for real functions) such that \(DP\) is the identity on \(BC(X)\).

A natural try is first to find an analogue of the Lebesgue measure in \(K\). But this turns out to be a dead end road. For example if \(K = \mathbb{Q}_p\), there does not exist a nonzero translation invariant bounded additive \(\mathbb{Q}_p\)-valued function \(m\) defined on the compact open subsets of \(\mathbb{Z}_p\). (By translation invariance \(\left| m(p^n \mathbb{Z}_p) \right| = p^n \left| m(\mathbb{Z}_p) \right| \to \infty\) if \(m(\mathbb{Z}_p) \neq 0\)). For similar reasons it goes wrong for every local field \(K\).

Following the ideas of Dieudonné, Treiber, we define for \(f \in BC(X)\)

\[
(Pf)(x) := \sum_{n=1}^{\infty} f(x_n) (x_{n+1} - x_n) \quad (x \in X)
\]

Here the \(x_n\) are defined as follows. For each \(n \in \mathbb{N}\) the equivalence relation \(\sim_n\) defined by \(x \sim_n y\) if \(|x-y| < \frac{1}{n}\) yields a partition of \(X\) into balls. Choose a center in each ball and let \(R_n\) be the set of these centers.

For each \(x \in X\) and \(n \in \mathbb{N}\), \(x_n\) is defined by \(x_n \in R_n\), \(|x_n - x| < \frac{1}{n}\).

Theorem. (A NON-ARCHIMEDEAN FORM OF THE FUNDAMENTAL THEOREM). \(P\) is a linear isometry of \(BC(X)\) into \(BC^1(X)\). \(DP\) is the identity on \(BC(X)\), whereas \(PD\) is a projection of \(BC^1(X)\) onto a complement of \(\{f \in BC^1(X) : f' = 0\}\).

§ 5. Generalizations of the Fundamental Theorem.

We may ask whether there exists some form of the Fundamental Theorem for functions belonging to spaces, larger than \(BC(X), BC^1(X)\).
respectively. (For example, compare the classical theorem on $L^1$-functions versus absolutely continuous functions).

We have the following striking fact that has no counterpart in classical analysis. We say that $g : X \to K$ is of the first class of Baire if there exists a sequence $g_1, g_2, \ldots$ of continuous functions $X \to K$ such that $\lim g_n = g$ pointwise.

**THEOREM.** (a) Let $f : X \to K$ be differentiable. Then $f'$ is of the first class of Baire.

(b) Let $g : X \to K$ be of the first class of Baire. Then $g$ has an antiderivative.

Let $B^1(X)$ be the Banach space consisting of all bounded functions $X \to K$ of the first class of Baire with respect to the supremum norm. Let $BD(X)$ be the Banach space of all differentiable $f : X \to K$ for which both $f$ and $\Phi f$ are bounded, with respect to the norm $f + ||f||_\infty + ||\Phi f||_\infty$. Then we have

**THEOREM.** Differentiation is a quotient map $BD(X) \overset{D}{\to} B^1(X)$.

If $K$ has discrete valuation then there exists a continuous linear $P : B^1(X) \to BD(X)$ for which $DP$ is the identity on $B^1(X)$.

**Notes.**

1. The construction of the above $P$ is awful and, contrary to § 4, $P$ does not resemble an indefinite integral in any way.

2. If the valuation of $K$ is dense the existence of such a $P$ is still an open question.
5.6. **Restriction of the Fundamental Theorem.**

In classical analysis, we have that if \( f \in C^n \) then \( x \mapsto \int_0^x f(t) \, dt \) is in \( C^{n+1} \). In our situation we define for \( f : X \to K \):

\( f \in C^2(X) \) if the function \( \phi_2 f \), defined by

\[
\phi_2 f(x,y,z) = \frac{\phi_1 f(x,z) - \phi_1 f(y,z)}{x-y} \quad (x,y,z \in X, x \neq y, y \neq z, x \neq z)
\]

can continuously be extended to \( \phi_2 f : X^3 \to K \). Similarly, we define \( C^3(X), C^4(X), \ldots \). Let \( C^\infty(X) := \bigcap_{n=1}^{\infty} C^n(X) \).

The map \( P \), defined in § 4, does not always map \( C^1 \)-functions into \( C^2 \)-functions. But we have (notations as in § 4)

**THEOREM.** Let the characteristic of \( K \) be unequal to 2. Then the map

\( P_2 \) defined via

\[
(P_2 f)(x) := \sum_{n=0}^{\infty} f_n (x_n - x_n) + \frac{1}{2} \sum_{n=0}^{\infty} f_n (x_n - x_n)^2 \quad (x \in X)
\]

maps \( C^1(X) \) into \( C^2(X) \) and \( (Pf)' = f \) for all \( f \in C^1(X) \).

Similarly, one can define antiderivation maps \( P_n : C^{n-1}(X) \to C^n(X) \) (in case the characteristic of \( K \) is unequal to \( 2,3,\ldots,n \)).

**OPEN QUESTION.** Let \( K \) have characteristic 0. Does every \( f \in C^\infty(X) \)

have a \( C^\infty \)-antiderivative?

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**Reference**