NON - ARCHIMEDEAN DIFFERENTIATION

held on Tuesday, June 5, 1979 at the "VI Jornadas de Matemáticas Hispano-Lusas" organized by the University of SANTANDER,

by

W.H. Schikhof

§ 1. Introduction.

The subject is part of the so-called non-archimedean (or ultrametric) analysis. Roughly speaking, one may say that this is the analysis that one obtains when replacing in the "classical" analysis \( \mathbb{R} \) or \( \mathbb{C} \) by a non-archimedean valued field \( K \).

A non-archimedean valued field is a (commutative) field \( K \), together with a map \( | \cdot | : K \rightarrow \mathbb{R} \) (the valuation) satisfying

\[
|a| \geq 0, \quad |a| = 0 \text{ iff } a = 0
\]

\[
|ab| = |a| |b|
\]

\[
|a+b| \leq \max(|a|,|b|) \quad \text{(the strong triangle inequality)}
\]

for all \( a,b \in K \).

We have the following remarks.

(1) Apart from \( \mathbb{R} \) or \( \mathbb{C} \), every complete valued field is non-archimedean.

(2) If \( K \) is a non-archimedean valued field and if \( L \supset K \) is an overfield of \( K \) then the valuation on \( K \) can be extended to a non-archimedean valuation on \( L \).

(3) If \( K \) is a (non-archimedean) valued field then its completion \( \tilde{K} \) (with respect to the metric \( (x,y) \mapsto |x-y| \)) can, in a natural
way, be given the structure of a non-archimedean valued field.

In the sequel we exclude the so-called trivial valuation given by

$$|x|' = \begin{cases} 
0 & \text{if } x = 0 \\
1 & \text{if } x \neq 0.
\end{cases}$$

The non-archimedean analysis has several branches, similar to the classical analysis. Thus we have non-archimedean functional analysis, harmonic analysis, theory of analytic functions in one or several variables, etc.

In this talk we consider a more elementary subject, namely infinitesimal calculus in $K$. More specifically, we want to see what remains of the so-called Fundamental Theorem of Calculus (in $\mathbb{R}$) that states that the operations of differentiation and integration are in some sense each others inverses.

§ 2. Differentiation in $K$. Let $X \subset K$ be a subset without isolated points. A function $f : X \to K$ is called differentiable if for all $a \in X$

$$f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. The proof of the well known rules (sum-, product-, chain-rule) can formally be taken over from the classical theory. Thus, a rational function is differentiable if it has no poles on $X$. An analytic function $x + \sum_{n} a_{n} x^{n}$ is differentiable on

$$\{x : |x| < \left( \lim_{n} \sqrt[n]{|a_{n}|} \right)^{-1} \}.$$  

Deviations from the classical theory appear when we look at the functions whose derivative vanishes everywhere. For example,
let $\varepsilon > 0$, $a \in K$. Then $B(a, \varepsilon) := \{x \in K : |x-a| < \varepsilon\}$ is an open-
and-closed subset of $K$, hence $\xi_{B(a, \varepsilon)}$, defined by

$$\xi_{B(a, \varepsilon)}(x) := \begin{cases} 1 & \text{if } x \in B(a, \varepsilon) \\ 0 & \text{elsewhere} \end{cases}$$

is differentiable and $\xi'_{B(a, \varepsilon)} = 0$.

Locally constant functions all have derivative zero. On the other
hand they form a uniformly dense subset of $C(X)$, the space of all
continuous functions: $X \to K$.

Even worse: let $\mathbb{Q}_p$ the field of the $p$-adic numbers and let
$
\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x| \leq 1\}.
$

Then the function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ defined by

$$f(\sum_n a_n p^n) = \sum_n a_n p^{2n} \quad (\sum_n a_n p^n \in \mathbb{Z}_p)$$

satisfies $|f(x)-f(y)| = |x-y|^2$ for all $x,y \in \mathbb{Z}_p$. So $f' = 0$ but $f$
is injective, hence not locally constant.

The above example shows also that a Mean Value Theorem is necessa­

rily absent in our theory.

Notice that the difficulties encountered above also appear when we
study differentiability of functions $f : \mathbb{D} \to \mathbb{R}$, where $\mathbb{D} \subset [0,1]$ is the Cantor set. So it is the domain of $f$ that is responsible
for the troubles rather than its range.

§ 3. Continuously differentiable functions.

If we follow naively the path of the classical analysis and define

$$C^1(X) := \{f : X \to K, f \text{ is differentiable, } f' \text{ is continuous}\}$$

then we run up against difficulties.

First of all, one can prove that $C^1(\mathbb{Z}_p)$ (with the norm
\( f \mapsto \max(|f|_\infty, |f'|_\infty) \) is not a Banach space. In fact one shows that for every pair of continuous functions \( f, g : \mathbb{Z}_p \to \mathbb{Q}_p \) there exists a sequence \( f_1, f_2, \ldots \) in \( C^1(\mathbb{Z}_p) \) for which both \( f_n \to f \) and \( f'_n \to g \) uniformly. 

What is worse, we have no local invertibility theorem for such \( C^1 \)-functions. 

In fact, let \( f : \mathbb{Z}_p \to \mathbb{Q}_p \) be defined by

\[
f(x) = \begin{cases}x-p^{2n} & \text{if } |x-p^n| < p^{-2n} \\ x & \text{elsewhere}
\end{cases} \quad (n \in \mathbb{N})
\]

Then \( f'(x) = 1 \) for all \( x \in \mathbb{Z}_p \). But \( f(p^n) = f(p^n - p^{2n}) \) for all \( n \in \mathbb{N} \), so \( f \) is not even locally injective at 0. 

Therefore we are led to define:

Let \( f : X \to K \). Put

\[
\Phi f(x,y) := \frac{f(x)-f(y)}{x-y} \quad (x,y \in X, x \neq y).
\]

We say that \( f \in C^1(X) \) if \( \Phi f \) can continuously be extended to a function \( \Phi f : X \times X \to K \).

Then \( BC^1(X) := \{ f \in C^1(X) : f \text{ and } \Phi f \text{ are bounded} \} \) is a Banach space under \( f \mapsto ||f|| := \max(||f||_\infty, ||\Phi f||_\infty) \).

Further, if \( f \in C^1(X) \), \( f'(a) \neq 0 \) for some \( a \in X \), then \( f \) has a \( C^1 \)-inverse, locally at \( a \).

**Theorem.** Differentiation is a continuous surjection \( BC^1(X) \xrightarrow{D} BC(X) \).

(here \( BC(X) \) is the space of all bounded continuous functions with the supremum norm)

§ 4. **Integration**.

Next, we want to define an "indefinite integral" \( P : BC(X) \to BC^1(X) \).
(an analogue of \( (Pf)(x) := \int_0^x f(t) \, dt \) for real functions) such that \( DP \) is the identity on \( BC(X) \).

A natural try is first to find an analogue of the Lebesgue measure in \( K \). But this turns out to be a dead end road. For example if \( K = \mathcal{O}_p \) there does not exist a nonzero translation invariant bounded additive \( \mathcal{O}_p \)-valued function \( m \) defined on the compact open subsets of \( \mathcal{A} \). (By translation invariance

\[
|m(p^n\mathcal{A}_p)| = p^n|m(\mathcal{A}_p)| \to \infty \text{ if } m(\mathcal{A}_p) \neq 0.
\]

For similar reasons it goes wrong for every local field \( K \).

Following the ideas of Dieudonné, Treiber, we define for \( f \in BC(X) \)

\[
(Pf)(x) := \sum_{n=1}^{\infty} f(x_n)(x_{n+1} - x_n) \quad (x \in X)
\]

Here the \( x_n \) are defined as follows. For each \( n \in \mathbb{N} \) the equivalence relation \( \sim_n \) defined by \( x \sim_n y \) if \( |x-y| < \frac{1}{n} \) yields a partition of \( X \) into balls. Choose a center in each ball and let \( R_n \) be the set of these centers.

For each \( x \in X \) and \( n \in \mathbb{N} \), \( x_n \) is defined by \( x_n \in R_n \), \( |x_n - x| < \frac{1}{n} \).

**Theorem.** (A NON-ARCHIMEDEAN FORM OF THE FUNDAMENTAL THEOREM). \( P \) is a linear isometry of \( BC(X) \) into \( BC^1(X) \). \( DP \) is the identity on \( BC(X) \), whereas \( PD \) is a projection of \( BC^1(X) \) onto a complement of \( \{ f \in BC^1(X) : f' = 0 \} \).

§ 5. Generalizations of the Fundamental Theorem.

We may ask whether there exists some form of the Fundamental Theorem for functions belonging to spaces, larger than \( BC(X), BC^1(X) \)
respectively. (For example, compare the classical theorem on $L^1$-functions versus absolutely continuous functions).

We have the following striking fact that has no counterpart in classical analysis. We say that $g : X \to K$ is of the first class of Baire if there exists a sequence $g_1, g_2, \ldots$ of continuous functions $X \to K$ such that $\lim g_n = g$ pointwise.

**THEOREM.** (a) Let $f : X \to K$ be differentiable. Then $f'$ is of the first class of Baire.

(b) Let $g : X \to K$ be of the first class of Baire. Then $g$ has an antiderivative.

Let $B^1(B(X))$ be the Banach space consisting of all bounded functions $X \to K$ of the first class of Baire with respect to the supremum norm. Let $BD(X)$ be the Banach space of all differentiable $f : X \to K$ for which both $f$ and $\Phi f$ are bounded, with respect to the norm $f + |f|_\infty \vee |\Phi f|_\infty$. Then we have

**THEOREM.** Differentiation is a quotient map $BD(X) \xrightarrow{D} B^1(B(X))$.

If $K$ has discrete valuation then there exists a continuous linear $P : B^1(B(X)) \to BD(X)$ for which $DP$ is the identity on $B^1(B(X))$.

**Notes.**

1. The construction of the above $P$ is awful and, contrary to § 4, $P$ does not resemble an indefinite integral in any way.

2. If the valuation of $K$ is dense the existence of such a $P$ is still an open question.
§ 6. **Restriction of the Fundamental Theorem.**

In classical analysis, we have that if $f \in C^n$ then $\int_0^x f(t) \, dt$ is in $C^{n+1}$. In our situation we define for $f : X \to K$:

- $f \in C^2(X)$ if the function $\phi^2 f$, defined by

$$
\phi^2 f(x,y,z) = \frac{\phi_1 f(x,z) - \phi_1 f(y,z)}{x-y} \quad (x,y,z \in X, x \neq y, y \neq z, x \neq z)
$$

- can continuously be extended to $\overline{\phi^2 f} : X^3 \to K$. Similarly, we define $C^3(X), C^4(X), \ldots$. Let $C^\infty(X) := \cap_{n=1}^\infty C^n(X)$.

The map $P$, defined in § 4, does not always map $C^1$-functions into $C^2$-functions. But we have (notations as in § 4)

**THEOREM.** Let the characteristic of $K$ be unequal to 2. Then the map $P_2$ defined via

$$
(P_2 f)(x) := \sum_{n=1}^\infty (x_n + 1 - x_n)^2 \sum_{n=1}^\infty \sum_{n=1}^\infty f(x_n) (x_{n+1} - x_n)^2 \quad (x \in X)
$$

maps $C^1(X)$ into $C^2(X)$ and $(Pf)' = f$ for all $f \in C^1(X)$.

Similarly, one can define antiderivation maps $P_n : C^{n-1}(X) \to C^n(X)$ (in case the characteristic of $K$ is unequal to $2, 3, \ldots, n$).

**OPEN QUESTION.** Let $K$ have characteristic 0. Does every $f \in C^\infty(X)$ have a $C^\infty$-antiderivative?

W. Schikhof

---

**Reference**