Abstract of the lecture

NON - ARCHIMEDEAN DIFFERENTIATION

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by

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§ 1. Introduction.

The subject is part of the so-called non-archimedean (or ultrametric) analysis. Roughly speaking, one may say that this is the analysis that one obtains when replacing in the "classical" analysis $\mathbb{R}$ or $\mathbb{C}$ by a non-archimedean valued field $K$.

A non-archimedean valued field is a (commutative) field $K$, together with a map $|\cdot| : K \to \mathbb{R}$ (the valuation) satisfying

\begin{align*}
|a| &\geq 0 \ , \ |a| = 0 \text{ iff } a = 0 \\
|ab| &= |a| \cdot |b| \\
|a+b| \leq \max(|a|,|b|) &\text{ (the strong triangle inequality)}
\end{align*}

for all $a,b \in K$.

We have the following remarks.

(1) Apart from $\mathbb{R}$ or $\mathbb{C}$, every complete valued field is non-archimedean.

(2) If $K$ is a non-archimedean valued field and if $L \supset K$ is an overfield of $K$ then the valuation on $K$ can be extended to a non-archimedean valuation on $L$.

(3) If $K$ is a (non-archimedean) valued field then its completion $\overline{K}$ (with respect to the metric $(x,y) \mapsto |x-y|$) can, in a natural...
way, be given the structure of a non-archimedean valued field.

In the sequel we exclude the so-called trivial valuation given by

\[ |x|' = \begin{cases} 
0 & \text{if } x = 0 \\
1 & \text{if } x \neq 0.
\end{cases} \]

The non-archimedean analysis has several branches, similar to the classical analysis. Thus we have non-archimedean functional analysis, harmonic analysis, theory of analytic functions in one or several variables, etc.

In this talk we consider a more elementary subject, namely infinitesimal calculus in $K$. More specifically, we want to see what remains of the so-called Fundamental Theorem of Calculus (in $\mathbb{R}$) that states that the operations of differentiation and integration are in some sense each others inverses.

§ 2. Differentiation in $K$. Let $X \subseteq K$ be a subset without isolated points. A function $f : X \to K$ is called differentiable if for all $a \in X$

\[ f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}, \]

exists. The proof of the well known rules (sum-, product-, chain-rule) can formally be taken over from the classical theory. Thus, a rational function is differentiable if it has no poles on $X$. An analytic function $x + \sum a_n x^n$ is differentiable on

\[ \{x : |x| < (\lim_{n \to \infty} \sqrt[n]{|a_n|})^{-1} \}. \]

Deviations from the classical theory appear when we look at the functions whose derivative vanishes everywhere. For example,
let $\varepsilon > 0$, $a \in K$. Then $B(a,\varepsilon) := \{x \in K : |x-a| < \varepsilon\}$ is an open-and-closed subset of $K$, hence $\xi_{B(a,\varepsilon)}$, defined by
\[
\xi_{B(a,\varepsilon)}(x) := \begin{cases} 
1 & \text{if } x \in B(a,\varepsilon) \\
0 & \text{elsewhere}
\end{cases}
\]
is differentiable and $\xi'_{B(a,\varepsilon)} = 0$.

Locally constant functions all have derivative zero. On the other hand they form a uniformly dense subset of $C(X)$, the space of all continuous functions: $X \to K$.

Even worse: let $\mathbb{Q}_p$ the field of the $p$-adic numbers and let $z_p := \{x \in \mathbb{Q}_p : |x| \leq 1\}$. Then the function $f : z_p \to \mathbb{Q}_p$ defined by
\[
f(\sum_{n} a_n p^n) = \sum_{n} a_n p^{2n} \quad (\sum_{n} a_n p^n \in z_p)
\]
satisfies $|f(x)-f(y)| = |x-y|^2$ for all $x,y \in z_p$. So $f' = 0$ but $f$ is injective, hence not locally constant.

The above example shows also that a Mean Value Theorem is necessarily absent in our theory.

Notice that the difficulties encountered above also appear when we study differentiability of functions $f : D \to \mathbb{R}$, where $D \subseteq [0,1]$ is the Cantor set. So it is the domain of $f$ that is responsible for the troubles rather than its range.

§ 3. Continuously differentiable functions.

If we follow naively the path of the classical analysis and define
\[
C^1(X) := \{f : X \to K, f \text{ is differentiable, } f' \text{ is continuous}\}
\]
then we run up against difficulties.

First of all, one can prove that $C^1(z_p)$ (with the norm
$f \mapsto \max(\|f\|_{\infty}, \|f^1\|_{\infty})$ is not a Banach space. In fact one shows that for every pair of continuous functions $f, g : \mathbb{Z}_p \to \mathbb{Q}_p$ there exists a sequence $f_1, f_2, \ldots$ in $C^1(\mathbb{Z}_p)$ for which both $f_n \to f$ and $f'_n \to g$ uniformly.

What is worse, we have no local invertibility theorem for such $C^1$-functions.

In fact, let $f : \mathbb{Z}_p \to \mathbb{Q}_p$ be defined by

$$f(x) = \begin{cases} x^{-p^{2n}} & \text{if } |x - p^n| < p^{-2n} \\ x & \text{elsewhere} \end{cases} \quad (n \in \mathbb{N})$$

Then $f'(x) = 1$ for all $x \in \mathbb{Z}_p$. But $f(p^n) = f(p^{n-2n})$ for all $n \in \mathbb{N}$, so $f$ is not even locally injective at 0.

Therefore we are led to define:

Let $f : X \to K$. Put

$$\Phi f(x,y) := \frac{f(x) - f(y)}{x-y} \quad (x,y \in X, x \neq y).$$

We say that $f \in C^1(X)$ if $\Phi f$ can continuously be extended to a function $\overline{\Phi f} : X \times X \to K$.

Then $BC^1(X) := \{f \in C^1(X) : f$ and $\Phi f$ are bounded\}$ is a Banach space under $f \mapsto \|f\|_1 := \max(\|f\|_{\infty}, \|\Phi(f)\|_{\infty})$.

Further, if $f \in C^1(X)$, $f'(a) \neq 0$ for some $a \in X$, then $f$ has a $C^1$-inverse, locally at $a$.

**Theorem.** Differentiation is a continuous surjection $BC^1(X) \overset{D}{\to} BC(X)$.

(here $BC(X)$ is the space of all bounded continuous functions with the supremum norm)

§ 4. "Integration".

Next, we want to define an "indefinite integral" $P : BC(X) \to BC^1(X)$
(an analogue of \( (Pf)(x) := \int_0^x f(t)\,dt \) for real functions) such that \( DP \) is the identity on \( BC(X) \).

A natural try is first to find an analogue of the Lebesgue measure in \( K \). But this turns out to be a dead end road. For example if \( K = \mathbb{Q}_p \) there does not exist a nonzero translation invariant bounded additive \( \mathbb{Q}_p \)-valued function \( m \) defined on the compact open subsets of \( \mathcal{D}_p \). (By translation invariance \( |m(p^n\mathcal{D}_p)| = p^n|m(\mathcal{D}_p)| \to \infty \) if \( m(\mathcal{D}_p) \neq 0 \). For similar reasons it goes wrong for every local field \( K \).

Following the ideas of Dieudonné, Treiber, we define for \( f \in BC(X) \)

\[
(Pf)(x) := \sum_{n=1}^{\infty} f(x_n)(x_{n+1} - x_n) \quad (x \in X)
\]

Here the \( x_n \) are defined as follows. For each \( n \in \mathbb{N} \) the equivalence relation \( \sim_n \) defined by \( x \sim_n y \) if \( |x-y| < \frac{1}{n} \) yields a partition of \( X \) into balls. Choose a center in each ball and let \( R_n \) be the set of these centers.

For each \( x \in X \) and \( n \in \mathbb{N} \), \( x_n \) is defined by \( x_n \in R_n, |x_n - x| < \frac{1}{n} \).

Theorem. (A NON-ARCHIMEDEAN FORM OF THE FUNDAMENTAL THEOREM).

\( P \) is a linear isometry of \( BC(X) \) into \( BC^1(X) \). \( DP \) is the identity on \( BC(X) \), whereas \( PD \) is a projection of \( BC^1(X) \) onto a complement of \( \{f \in BC^1(X) : f' = 0\} \).

§ 5. Generalizations of the Fundamental Theorem.

We may ask whether there exists some form of the Fundamental Theorem for functions belonging to spaces, larger than \( BC(X), BC^1(X) \)
respectively. (For example, compare the classical theorem on $L^1$-functions versus absolutely continuous functions).

We have the following striking fact that has no counterpart in classical analysis. We say that $g : X \to K$ is of the first class of Baire if there exists a sequence $g_1, g_2, \ldots$ of continuous functions $X \to K$ such that $\lim g_n = g$ pointwise.

**THEOREM.** (a) Let $f : X \to K$ be differentiable. Then $f'$ is of the first class of Baire.

(b) Let $g : X \to K$ be of the first class of Baire. Then $g$ has an antiderivative.

Let $B^1(X)$ be the Banach space consisting of all bounded functions $X \to K$ of the first class of Baire with respect to the supremum norm. Let $BD(X)$ be the Banach space of all differentiable $f : X \to K$ for which both $f$ and $\phi f$ are bounded, with respect to the norm $f + \|f\|_\infty + \|\phi f\|_\infty$. Then we have

**THEOREM.** Differentiation is a quotient map $BD(X) \xrightarrow{\phi} B^1(X)$.

If $K$ has discrete valuation then there exists a continuous linear $P : B^1(X) \to BD(X)$ for which $DP$ is the identity on $B^1(X)$.

**Notes.**

1. The construction of the above $P$ is awful and, contrary to § 4, $P$ does not resemble an indefinite integral in any way.

2. If the valuation of $K$ is dense the existence of such a $P$ is still an open question.
§ 6. **Restriction of the Fundamental Theorem.**

In classical analysis, we have that if $f \in C^n$ then $x \mapsto \int_0^x f(t) \, dt$ is in $C^{n+1}$. In our situation we define for $f : X \to K$:

If $f \in C^2(X)$ if the function $\phi^2 f$, defined by

$$\phi^2_f(x,y,z) = \frac{\phi^1_f(x,z) - \phi^1_f(y,z)}{x-y} \quad (x,y,z \in X, \ x\neq y, \ y\neq z, \ x\neq z)$$

can continuously be extended to $\phi^2_f : X^3 \to K$. Similarly, we define $C^3(X), C^4(X), \ldots$. Let $C^\infty(X) := \bigcap_{n=1}^\infty C^n(X)$.

The map $P$, defined in § 4, does not always map $C^1$-functions into $C^2$-functions. But we have (notations as in § 4)

**THEOREM.** Let the characteristic of $K$ be unequal to 2. Then the map $P_2$ defined via

$$(P_2 f)(x) := \sum_{n} (x_{n+1} - x_n) \phi^1_f(x_n) (x_{n+1} - x_n)^2 \quad (x \in X)$$

maps $C^1(X)$ into $C^2(X)$ and $(Pf)' = f$ for all $f \in C^1(X)$.

Similarly, one can define antiderivation maps $P_n : C^{n-1}(X) \to C^n(X)$ (in case the characteristic of $K$ is unequal to 2,3,\ldots,n).

**OPEN QUESTION.** Let $K$ have characteristic 0. Does every $f \in C^\infty(X)$ have a $C^\infty$-antiderivative?

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**Reference**