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Abstract of the lecture

NON - ARCHIMEDEAN DIFFERENTIATION

held on Tuesday, June 5, 1979 at the "VI Jornadas de Matemáticas Hispano-Lusas" organized by the University of SANTANDER,

by

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§ 1. Introduction.

The subject is part of the so-called non-archimedean (or ultrametric) analysis. Roughly speaking, one may say that this is the analysis that one obtains when replacing in the "classical" analysis \( \mathbb{R} \) or \( \mathbb{C} \) by a non-archimedean valued field \( K \).

A **non-archimedean valued field** is a (commutative) field \( K \), together with a map \( | | : K \rightarrow \mathbb{R} \) (the valuation) satisfying

\[
|a| \geq 0 , \quad |a| = 0 \iff a = 0 \\
|ab| = |a| |b| \\
|a+b| \leq \max(|a|,|b|) \quad \text{(the strong triangle inequality)}
\]

for all \( a,b \in K \).

We have the following remarks.

(1) Apart from \( \mathbb{R} \) or \( \mathbb{C} \), every complete valued field is non-archimedean.

(2) If \( K \) is a non-archimedean valued field and if \( L \supset K \) is an overfield of \( K \) then the valuation on \( K \) can be extended to a non-archimedean valuation on \( L \).

(3) If \( K \) is a (non-archimedean) valued field then its completion \( \sim K \) (with respect to the metric \( (x,y) \mapsto |x-y| \)) can, in a natural...
way, be given the structure of a non-archimedean valued field.

In the sequel we exclude the so-called trivial valuation given by

$$|x|' = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0. \end{cases}$$

The non-archimedean analysis has several branches, similar to the classical analysis. Thus we have non-archimedean functional analysis, harmonic analysis, theory of analytic functions in one or several variables, etc.

In this talk we consider a more elementary subject, namely infinitesimal calculus in $K$. More specifically, we want to see what remains of the so-called Fundamental Theorem of Calculus (in $\mathbb{R}$) that states that the operations of differentiation and integration are in some sense each others inverses.

§ 2. Differentiation in $K$. Let $X \subset K$ be a subset without isolated points. A function $f : X \to K$ is called differentiable if for all $a \in X$

$$f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. The proof of the well known rules (sum-, product-, chain-rule) can formally be taken over from the classical theory. Thus, a rational function is differentiable if it has no poles on $X$. An analytic function $x + \sum a_n x^n$ is differentiable on $\{x : |x| < (\lim \sqrt[n]{|a_n|})^{-1}\}$.

Deviations from the classical theory appear when we look at the functions whose derivative vanishes everywhere. For example,
let $\varepsilon > 0$, $a \in K$. Then $B(a, \varepsilon) := \{x \in K : |x-a| < \varepsilon\}$ is an open-
and-closed subset of $K$, hence $\xi_{B(a, \varepsilon)}$, defined by

$$
\xi_{B(a, \varepsilon)}(x) := \begin{cases} 
1 & \text{if } x \in B(a, \varepsilon) \\
0 & \text{elsewhere}
\end{cases}
$$

is differentiable and $\xi_{B(a, \varepsilon)}' = 0$.

Locally constant functions all have derivative zero. On the other hand they form a uniformly dense subset of $C(X)$, the space of all continuous functions: $X \to K$.

Even worse: let $\mathbb{Q}_p$ the field of the $p$-adic numbers and let $\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x| \leq 1\}$. Then the function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ defined by

$$f(\sum_{n} \frac{a_n}{p^n}) = \sum_{n} \frac{a_n 2^n}{p^n}$$

for $(\sum_{n} \frac{a_n}{p^n} \in \mathbb{Z}_p)$ satisfies $|f(x)-f(y)| = |x-y|^2$ for all $x, y \in \mathbb{Z}_p$. So $f' = 0$ but $f$ is injective, hence not locally constant.

The above example shows also that a Mean Value Theorem is necessarily absent in our theory.

Notice that the difficulties encountered above also appear when we study differentiability of functions $f : X \to \mathbb{R}$, where $X \subset [0,1]$ is the Cantor set. So it is the domain of $f$ that is responsible for the troubles rather than its range.

§ 3. **Continuously differentiable functions.**

If we follow naively the path of the classical analysis and define

$$C^1(X) := \{f : X \to K, f \text{ is differentiable, } f' \text{ is continuous}\}$$

then we run up against difficulties.

First of all, one can prove that $C^1(\mathbb{Z}_p)$ (with the norm
max(\|f\|_\infty, \|f'\|_\infty) is not a Banach space. In fact one shows that for every pair of continuous functions \(f, g : \mathbb{Z}_p \to \mathbb{Q}_p\) there exists a sequence \(f_1, f_2, \ldots\) in \(C^1(\mathbb{Z}_p)\) for which both \(f_n \to f\) and \(f'_n \to g\) uniformly.

What is worse, we have no local invertibility theorem for such \(C^1\)-functions.

In fact, let \(f : \mathbb{Z}_p \to \mathbb{Q}_p\) be defined by
\[
f(x) = \begin{cases} 
  x-p^{2n} & \text{if } |x-p^n| < p^{-2n} \\
  x & \text{elsewhere}
\end{cases} \quad (n \in \mathbb{N})
\]

Then \(f'(x) = 1\) for all \(x \in \mathbb{Z}_p\). But \(f(p^n) = f(2p^n)\) for all \(n \in \mathbb{N}\), so \(f\) is not even locally injective at 0.

Therefore we are led to define:

Let \(f : X \to K\). Put
\[
\Phi f(x,y) := \frac{f(x)-f(y)}{x-y} \quad (x,y \in X, x \neq y).
\]

We say that \(f \in C^1(X)\) if \(\Phi f\) can continuously be extended to a function \(\widetilde{\Phi} f : X \times X \to K\).

Then \(BC^1(X) := \{f \in C^1(X) : f\) and \(\Phi f\) are bounded\} is a Banach space under \(f \mapsto \|f\|_1 := \max(\|f\|_\infty, \|\Phi f\|_\infty)\).

Further, if \(f \in C^1(X)\), \(f'(a) \neq 0\) for some \(a \in X\), then \(f\) has a \(C^1\)-inverse, locally at \(a\).

Theorem. Differentiation is a continuous surjection \(BC^1(X) \overset{D}{\to} BC(X)\).

(here \(BC(X)\) is the space of all bounded continuous functions with the supremum norm)

§ 4. "Integration".

Next, we want to define an "indefinite integral" \(P : BC(X) \to BC^1(X)\).
(an analogue of \((Pf)(x) := \int_0^X f(t)dt\) for real functions) such that \(DP\) is the identity on \(BC(X)\).

A natural try is first to find an analogue of the Lebesgue measure in \(K\). But this turns out to be a dead end road. For example if \(K = \mathbb{Q}_p\) there does not exist a nonzero translation invariant bounded additive \(\mathbb{Q}_p\)-valued function \(m\) defined on the compact open subsets of \(\mathbb{A}_p\). (By translation invariance \(|m(p^n\mathbb{A}_p)| = p^n|m(\mathbb{A}_p)| \to \infty\) if \(m(\mathbb{A}_p) \neq 0\). For similar reasons it goes wrong for every local field \(K\).

Following the ideas of Dieudonné, Treiber, we define for \(f \in BC(X)\)

\[
(Pf)(x) := \sum_{n=1}^{\infty} f(x_n) (x_{n+1} - x_n) \quad (x \in X)
\]

Here the \(x_n\) are defined as follows. For each \(n \in \mathbb{N}\) the equivalence relation \(\sim_n\) defined by \(x \sim_n y\) if \(|x-y| < \frac{1}{n}\) yields a partition of \(X\) into balls. Choose a center in each ball and let \(R_n\) be the set of these centers.

For each \(x \in X\) and \(n \in \mathbb{N}\), \(x_n\) is defined by \(x_n \in R_n, |x_n - x| < \frac{1}{n}\).

**Theorem.** (A NON-ARCHIMEDEAN FORM OF THE FUNDAMENTAL THEOREM).

\(P\) is a linear isometry of \(BC(X)\) into \(BC^1(X)\). \(DP\) is the identity on \(BC(X)\), whereas \(PD\) is a projection of \(BC^1(X)\) onto a complement of \(\{f \in BC^1(X) : f' = 0\}\).

§ 5. Generalizations of the Fundamental Theorem.

We may ask whether there exists some form of the Fundamental Theorem for functions belonging to spaces, larger than \(BC(X), BC^1(X)\)
respectively. (For example, compare the classical theorem on \( L^1 \)-functions versus absolutely continuous functions).

We have the following striking fact that has no counterpart in classical analysis. We say that \( g : X \to K \) is of the first class of Baire if there exists a sequence \( g_1, g_2, \ldots \) of continuous functions \( X \to K \) such that \( \lim g_n = g \) pointwise.

**THEOREM.** (a) Let \( f : X \to K \) be differentiable. Then \( f' \) is of the first class of Baire.

(b) Let \( g : X \to K \) be of the first class of Baire. Then \( g \) has an antiderivative.

Let \( B^1_b(X) \) be the Banach space consisting of all bounded functions \( X \to K \) of the first class of Baire with respect to the supremum norm. Let \( BD(X) \) be the Banach space of all differentiable \( f : X \to K \) for which both \( f \) and \( \phi f \) are bounded, with respect to the norm \( \| f \|_\infty \vee \| \phi f \|_\infty \). Then we have

**THEOREM.** Differentiation is a quotient map \( BD(X) \xrightarrow{D} B^1_b(X) \).

If \( K \) has discrete valuation then there exists a continuous linear \( P : B^1_b(X) \to BD(X) \) for which \( DP \) is the identity on \( B^1_b(X) \).

**Notes.**

1. The construction of the above \( P \) is awful and, contrary to § 4, \( P \) does not resemble an indefinite integral in any way.

2. If the valuation of \( K \) is dense the existence of such a \( P \) is still an open question.
§ 6. Restriction of the Fundamental Theorem.

In classical analysis, we have that if $f \in C^n$ then $x \mapsto \int_0^x f(t) \, dt$ is in $C^{n+1}$. In our situation we define for $f : X \to K$:

If $f \in C^2(X)$ if the function $\phi_2 f$, defined by

$$\phi_2 f(x,y,z) = \frac{\phi_1 f(x,z) - \phi_1 f(y,z)}{x-y} \quad (x,y,z \in X, \, x \neq y, \, y \neq z, \, x \neq z)$$

can continuously be extended to $\phi_2 f : X^3 \to K$. Similarly, we define $C^3(X), C^4(X), \ldots$. Let $C^\infty(X) := \bigcap_{n=1}^\infty C^n(X)$.

The map $P$, defined in § 4, does not always map $C^1$-functions into $C^2$-functions. But we have (notations as in § 4)

**THEOREM.** Let the characteristic of $K$ be unequal to 2. Then the map $P_2$ defined via

$$(P_2 f)(x) := \sum_{n} \left( \frac{x_{n+1} - x_n}{n+1} \right) + \frac{1}{2} \sum_{n} \left( \frac{x_{n+1} - x_n}{n+1} \right)^2 \quad (x \in X)$$

maps $C^1(X)$ into $C^2(X)$ and $(Pf)' = f$ for all $f \in C^1(X)$.

Similarly, one can define antiderivation maps $P_n : C^{n-1}(X) \to C^n(X)$ (in case the characteristic of $K$ is unequal to 2,3,...,n).

**OPEN QUESTION.** Let $K$ have characteristic 0. Does every $f \in C^\infty(X)$ have a $C^\infty$-antiderivative?

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Reference