Abstract of the lecture

NON - ARCHIMEDEAN DIFFERENTIATION

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by

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§ 1. Introduction.

The subject is part of the so-called non-archimedean (or ultrametric) analysis. Roughly speaking, one may say that this is the analysis that one obtains when replacing in the "classical" analysis IR or C by a non-archimedean valued field K.

A non-archimedean valued field is a (commutative) field K, together with a map | | : K → IR (the valuation) satisfying

\[ |a| \geq 0, \quad |a| = 0 \text{ iff } a = 0 \]
\[ |ab| = |a| |b| \]
\[ |a+b| \leq \max(|a|,|b|) \quad \text{(the strong triangle inequality)} \]

for all \( a,b \in K \).

We have the following remarks.

(1) Apart from IR or C, every complete valued field is non-archimedean.

(2) If \( K \) is a non-archimedean valued field and if \( L \supset K \) is an overfield of \( K \) then the valuation on \( K \) can be extended to a non-archimedean valuation on \( L \).

(3) If \( K \) is a (non-archimedean) valued field then its completion \( \hat{K} \) (with respect to the metric \( (x,y) \mapsto |x-y| \)) can, in a natural...
way, be given the structure of a non-archimedean valued field. In the sequel we exclude the so-called trivial valuation given by

$$ |x|' = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0. \end{cases} $$

The non-archimedean analysis has several branches, similar to the classical analysis. Thus we have non-archimedean functional analysis, harmonic analysis, theory of analytic functions in one or several variables, etc.

In this talk we consider a more elementary subject, namely infinitesimal calculus in $K$. More specifically, we want to see what remains of the so-called Fundamental Theorem of Calculus (in $\mathbb{R}$) that states that the operations of differentiation and integration are in some sense each others inverses.

§ 2. Differentiation in $K$. Let $X \subset K$ be a subset without isolated points. A function $f : X \to K$ is called differentiable if for all $a \in X$

$$ f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a} $$

exists. The proof of the well known rules (sum-, product-, chain-rule) can formally be taken over from the classical theory. Thus, a rational function is differentiable if it has no poles on $X$. An analytic function $x + \sum a_n x^n$ is differentiable on

$$ \{ x : |x| < (\lim_{n \to \infty} \sqrt[n]{|a_n|})^{-1} \}. $$

Deviations from the classical theory appear when we look at the functions whose derivative vanishes everywhere. For example,
Let $\varepsilon > 0$, $a \in K$. Then $B(a, \varepsilon) := \{x \in K : |x-a| < \varepsilon\}$ is an open-and-closed subset of $K$, hence $\xi_{B(a, \varepsilon)}$, defined by

$$
\xi_{B(a, \varepsilon)}(x) := \begin{cases} 
1 & \text{if } x \in B(a, \varepsilon) \\
0 & \text{elsewhere}
\end{cases}
$$

is differentiable and $\xi'_{B(a, \varepsilon)} = 0$.

Locally constant functions all have derivative zero. On the other hand they form a uniformly dense subset of $C(X)$, the space of all continuous functions: $X \to K$.

Even worse: let $\mathbb{Q}_p$ the field of the $p$-adic numbers and let $\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x| \leq 1\}$. Then the function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ defined by

$$
f(p_n a^n) = p_n a_n 2^{n} \quad (p_n a_n \in \mathbb{Z}_p)
$$

satisfies $|f(x) - f(y)| = |x-y|^2$ for all $x, y \in \mathbb{Z}_p$. So $f' = 0$ but $f$ is injective, hence not locally constant.

The above example shows also that a Mean Value Theorem is necessarily absent in our theory.

Notice that the difficulties encountered above also appear when we study differentiability of functions $f : D \to \mathbb{R}$, where $D \subset [0,1]$ is the Cantor set. So it is the domain of $f$ that is responsible for the troubles rather than its range.

\section*{§ 3. Continuously differentiable functions.}

If we follow naively the path of the classical analysis and define

$$
C^1(X) := \{f : X \to K, f \text{ is differentiable, } f' \text{ is continuous}\}
$$

then we run up against difficulties.

First of all, one can prove that $C^1(\mathbb{Z}_p)$ (with the norm
f \mapsto \max(\|f\|_\infty, \|f'\|_\infty) \) is not a Banach space. In fact one shows that for every pair of continuous functions \( f, g : \mathbb{Z}_p \to \mathbb{Q}_p \) there exists a sequence \( f_1, f_2, \ldots \) in \( C^1(\mathbb{Z}_p) \) for which both \( f_n \to f \) and \( f'_n \to g \) uniformly.

What is worse, we have no local invertibility theorem for such \( C^1 \) functions.

In fact, let \( f : \mathbb{Z}_p \to \mathbb{Q}_p \) be defined by

\[
f(x) = \begin{cases} \frac{2^n}{p^n} & \text{if } |x-p^n| < p^{-2n} \quad (n \in \mathbb{N}) \\ x & \text{elsewhere} \end{cases}
\]

Then \( f'(x) = 1 \) for all \( x \in \mathbb{Z}_p \). But \( f(p^n) = f(p^n - 2^n) \) for all \( n \in \mathbb{N} \), so \( f \) is not even locally injective at 0.

Therefore we are led to define:

Let \( f : X \to K \). Put

\[
\Phi f(x, y) := \frac{f(x) - f(y)}{x - y} \quad (x, y \in X, x \neq y).
\]

We say that \( f \in C^1(X) \) if \( \Phi f \) can continuously be extended to a function \( \Phi f : X \times X \to K \).

Then \( BC^1(X) := \{ f \in C^1(X) : f \) and \( \Phi f \) are bounded \) is a Banach space under \( f \mapsto \| f \|_1 := \max(\| f \|_{\infty}, \| \Phi f \|_{\infty}) \).

Further, if \( f \in C^1(X), f'(a) \neq 0 \) for some \( a \in X \), then \( f \) has a \( C^1 \)-inverse, locally at \( a \).

Theorem. Differentiation is a continuous surjection \( BC^1(X) \xrightarrow{D} BC(X) \).

(here \( BC(X) \) is the space of all bounded continuous functions with the supremum norm)

§ 4. "Integration".

Next, we want to define an "indefinite integral" \( \mathcal{P} : BC(X) \to BC^1(X) \)
(an analogue of $\langle Pf \rangle(x) := \int_0^x f(t)dt$ for real functions) such that DP is the identity on BC(X).

A natural try is first to find an analogue of the Lebesgue measure in $K$. But this turns out to be a dead end road. For example if $K = \mathbb{Q}_p$ there does not exist a nonzero translation invariant bounded additive $\mathbb{Q}_p$-valued function $m$ defined on the compact open subsets of $\mathbb{Z}_p$. (By translation invariance $|m(p^n) - m) = p^n|m(\mathbb{Z}_p)| \to \infty$ if $m(\mathbb{Z}_p) \neq 0$.) For similar reasons it goes wrong for every local field $K$.

Following the ideas of Dieudonné, Treiber, we define for $f \in BC(X)$

$$\langle Pf \rangle(x) := \sum_{n=1}^{\infty} f(x_n) (x_{n+1} - x_n) \quad (x \in X)$$

Here the $x_n$ are defined as follows. For each $n \in \mathbb{N}$ the equivalence relation $\sim_n$ defined by $x \sim_n y$ if $|x-y| < \frac{1}{n}$ yields a partition of $X$ into balls. Choose a center in each ball and let $R_n$ be the set of these centers.

For each $x \in X$ and $n \in \mathbb{N}$, $x_n$ is defined by $x_n \in R_n$, $|x_n - x| < \frac{1}{n}$.

**Theorem.** (A NON-ARCHIMEDEAN FORM OF THE FUNDAMENTAL THEOREM).

$P$ is a linear isometry of $BC(X)$ into $BC^1(X)$. DP is the identity on $BC(X)$, whereas PD is a projection of $BC^1(X)$ onto a complement of $\{f \in BC^1(X) : f' = 0\}$.

§ 5. Generalizations of the Fundamental Theorem.

We may ask whether there exists some form of the Fundamental Theorem for functions belonging to spaces, larger than $BC(X)$, $BC^1(X)$.
respectively. (For example, compare the classical theorem on $L^1$-functions versus absolutely continuous functions).

We have the following striking fact that has no counterpart in classical analysis. We say that $g : X \to K$ is of the first class of Baire if there exists a sequence $g_1, g_2, \ldots$ of continuous functions $X \to K$ such that $\lim g_n = g$ pointwise.

**THEOREM.**

(a) Let $f : X \to K$ be differentiable. Then $f'$ is of the first class of Baire.

(b) Let $g : X \to K$ be of the first class of Baire. Then $g$ has an antiderivative.

Let $B^1_b(X)$ be the Banach space consisting of all bounded functions $X \to K$ of the first class of Baire with respect to the supremum norm. Let $BD(X)$ be the Banach space of all differentiable $f : X \to K$ for which both $f$ and $\Phi f$ are bounded, with respect to the norm $f \to ||f||_\infty \vee ||\Phi f||_\infty$. Then we have

**THEOREM.** Differentiation is a quotient map $BD(X) \overset{D}{\to} B^1_b(X)$.

If $K$ has discrete valuation then there exists a continuous linear $P : B^1_b(X) \to BD(X)$ for which $DP$ is the identity on $B^1_b(X)$.

**Notes.**

1. The construction of the above $P$ is awful and, contrary to § 4, $P$ does not resemble an indefinite integral in any way.

2. If the valuation of $K$ is dense the existence of such a $P$ is still an open question.
5 6. **Restriction of the Fundamental Theorem.**

In classical analysis, we have that if \( f \in C^n \) then \( x \mapsto \int_0^x f(t) \, dt \) is in \( C^{n+1} \). In our situation we define for \( f : X \to K \):

\( f \in C^2(X) \) if the function \( \phi_2 f \), defined by

\[
\phi_2 f(x,y,z) = \frac{\phi_1 f(x,z) - \phi_1 f(y,z)}{x - y} \quad (x,y,z \in X, \ x \neq y, \ y \neq z, \ x \neq z)
\]

can continuously be extended to \( \phi_2 f : X^3 \to K \). Similarly, we define \( C^3(X), C^4(X), \ldots \). Let \( C^\infty(X) := \bigcap_{n=1}^{\infty} C^n(X) \).

The map \( P \), defined in § 4, does not always map \( C^1 \)-functions into \( C^2 \)-functions. But we have (notations as in § 4)

**THEOREM.** Let the characteristic of \( K \) be unequal to 2. Then the map \( P_2 \) defined via

\[
(P_2 f)(x) := \sum_{n=0}^{\infty} f_n(x_n) (x_{n+1} - x_n) + \frac{1}{2} \sum_{n=0}^{\infty} f'_n(x_n) (x_{n+1} - x_n)^2 \quad (x \in X)
\]

maps \( C^1(X) \) into \( C^2(X) \) and \( (P f)' = f \) for all \( f \in C^1(X) \).

Similarly, one can define antiderivation maps \( P_n : C^{n-1}(X) \to C^n(X) \) (in case the characteristic of \( K \) is unequal to \( 2, 3, \ldots, n \)).

**OPEN QUESTION.** Let \( K \) have characteristic 0. Does every \( f \in C^\infty(X) \) have a \( C^\infty \)-antiderivative?

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**Reference**