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These lecture notes (actually, the lectures never took place) form the result of some investigation by the author in order to get more insight in the behavior of differentiable functions: $K + K$, where $K$ is a non-archimedean valued field. These notes do not assume any knowledge about these functions, so some theorems are not new; wherever possible a reference to the existing literature has been given at the end of each chapter. (My apologies to the authors whom I may have overlooked.) In [6] the reader may find any background information on non-archimedean functional analysis that is needed here.

In Chapter 1 some elementary properties of differentiable functions are given and some examples, showing that a mean value theorem is necessarily absent in our theory and indicating that it does not seem to be wise to define a $C^1$-function as a differentiable $f$ for which $f'$ is continuous. Another choice (the continuity of the difference quotient) turns out to be much better.

Chapter 2 contains some pathology. First, the non-archimedean version of the well known classical theorem stating that the collection of the functions that are somewhere differentiable form a set of the first category (in the sense of Baire) in the space of the continuous functions (2.2). Somewhat more surprising is the existence of nowhere differentiable isometries (2.6). This result has no counterpart in real analysis (functions $f : [0,1] \to \mathbb{R}$ satisfying $|f(x)-f(y)| \leq |x-y|$ for all $x,y$ are of bounded variation and hence differentiable almost everywhere).
In Chapter 3 the differentiation map D is studied (defined on the set BD of all bounded differentiable functions). A striking fact is that im D, the set of all functions having an antiderivative, equals the set $B^1$ of the Baire class one functions (3.10). The kernel of D, the solution set of the differential equation $y' = 0$, turns out to be a closed subset of BD (with respect to some natural topology on BD) containing the locally constant functions as a dense subset (3.3).

Chapter 4 deals with translation of classical theorems about the behaviour of differentiable functions with respect to null sets ($K$ is supposed to be locally compact). There are no surprises here.

In Chapter 5 again the map D is studied but now restricted to $BC^1$: the space of the $C^1$-functions for which $f$ and its difference quotients are bounded. Results, similar to those in chapter 3, are obtained. Moreover a continuous linear antiderivation map $P : C \to C^1$ is defined. It has the form

$$(Pf)(x) = \sum_{n} f(x_n)(x_{n+1} - x_n) \quad (f \in C, \; x \in K)$$

(5.4). (Here $x_1, x_2, \ldots$ is a sequence converging to $x$, defined in some standard way).

In Chapter 6 an orthonormal base is constructed in $C^1(X)$ where $X \subset K$ is compact, and relations between $f$ and its coefficients are studied. Some Fourier theory turns out to be possible on $\{ f \in C^1(X) : f' = 0 \}$, where $X$ is the unit disc of a local field with characteristic zero.

In Chapter 7 two natural definitions of "uniform differentiability" are discussed. (For example uniform continuity of the difference quotient.)
Chapter 8 deals with the definition of $C^n$-functions ($f \in C^n$ if the difference quotient of order $n$ is continuous as a function of $n+1$ variables). The reader will experience the fact that proving innocent looking statements such as "$f \in C^n$ implies $f' \in C^{n-1}$" needs a lot more machinery than one might expect. In fact most of the results have analogues in the theory of real functions that are completely trivial.

Chapter 9 is devoted to the local invertibility theorem and composition for $C^n$-functions. (Again, the real counterpart is trivial.)

Chapter 10 deals with the space $C^n$ consisting of those functions having a Taylor expansion up to order $n$ with a continuous remainder term ($f \in C^n$ if there exist functions $D_i^kf$ ($1 \leq i \leq n$) and a continuous $R_n$ such that $f(x) = f(y)+(x-y)D_1f(y)+\ldots+(x-y)^nR_n(x,y)$ for all $x,y$). Then $C^n \subset C^n$ (as has been shown in Chapter 8) and also $C^1 = C^1$, $C^2 = C^2$. Example 8.8. bis shows that $C^3(X) \neq C^3(X)$ for some pathological subset $X$ of $\mathbb{Z}_p$. If $X$ is sufficiently nice, however, one has $C^n(X) = C^n(X)$ (e.g. open sets are nice).

In Chapter 11 a continuous linear antiderivation map $P_n : C^{n-1} \to C^n$ is constructed. (The map $P$ of Chapter 5 does not work for $n \geq 1$). It has the form

$$ (P_n f)(x) = \sum_{k=1}^{n} \frac{1}{i!} (x_{k+1} - x_k)^i D_{i-1}f(x_k) \quad (f \in C^{n-1}, \ x \in K) $$

(11.2). (Here: $j! D_j f = f^{(j)}$ for all $j$ and if the characteristic of $K$ is $p \neq 0$ we assume $n < p$). For $n \geq p$ differentiation $C^n \to C^{n-1}$ is not surjective, hence there is no antiderivation: $C^{n-1} \to C^n$.

Chapter 12 finally deals with properties of $C^\infty$-functions ($C^\infty = \cap_n C^n$).
Among other things, a non-archimedean version of Borel's theorem (12.12) is proved. (Given \( \lambda_n \in K \) there exists a \( C^\infty \)-function \( f \) for which \( D_n f(0) = \lambda_n \) for each \( n \)). A question that keeps annoying the author: does every \( C^\infty \)-function have a \( C^\infty \)-antiderivative? (See below)

As has been said before, the fact that one has to work hard in order to obtain trivial looking statements is rather surprising. On the other hand, the main results of the theory are valid for \( f : X \to K \) where the only assumption on \( X \) is that it has no isolated points. [Of course, if we had restricted ourselves from the beginning to the case where \( X \) is open and the characteristic of \( K \) is zero, then it would have shortened these notes considerably.]

As far as generalizations are concerned, a lot of the theory can be translated without much effort to functions \( f : K \to E \), where \( E \) is a Banach space over \( K \). Less obvious is a theory for functions \( K^n \to K^m \) although it seems clear how one should define \( C^k \) in this case.

Preface to the second edition. A few minor errors have been corrected. In the bibliography I have added some recent works. ([16] is an elaboration of [5], [17] is the book form of [6]). The reader who is not familiar to non-archimedean analysis is advised to consult [5] (or [16]), [15] or [18] for a better understanding of the basic concepts.

The answer to the above problem concerning \( C^\infty \)-functions is now known. In fact, if the characteristic of the base field is 0 each \( C^\infty \)-function has a \( C^\infty \)-antiderivative. A proof appears in [18].

Copies of these notes can be ordered by writing to the address on the cover.

April 1982

W. Schikhof
O. NOTATIONS, PRELIMINARIES

By \( \mathbb{N} \), \( \mathbb{Z} \), \( \mathbb{Q} \), \( \mathbb{R} \), \( \mathbb{C} \) we denote the set of the natural numbers, integers, rational numbers, real numbers, complex numbers respectively.

For \( a, b \in \mathbb{R} \) we denote \( \max(a, b) \) by \( a \vee b \) and \( \min(a, b) \) by \( a \wedge b \).

\( K \) will always be a non-archimedean nontrivially valued field that is complete with respect to the metric induced by the valuation.

For any prime number \( p \), \( \mathbb{Q}_p \) is the (valued) field of the \( p \)-adic numbers, \( \mathbb{Z}_p = \{ a \in \mathbb{Q}_p : |a|_p \leq 1 \} \)

Let \( S \) be a metric space, \( Y \subseteq S \), \( a \in S \). The distance of \( Y \) and \( a \), \( d(a,Y) \) is

\[
\inf \{ d(a,y) : y \in Y \}.
\]

\( \overline{Y} \) is the closure of \( Y \), \( Y^0 \) is the interior of \( Y \), \( Y^c \) the complement of \( Y \).

For \( f : S \to K \) we denote its restriction to \( Y \) by \( f|_Y \). \( Y \) is called clopen if \( Y \) is both closed and open.

Let \( E \) be a \( K \)-vector space. A seminorm is a map \( q : E \to \mathbb{R} \) satisfying

\[
q(x) \geq 0,
q(\lambda x) = |\lambda| q(x),
q(x+y) \leq \max(q(x), q(y))
\]

for all \( x, y \in E \), \( \lambda \in K \). \( q \) is called a norm if \( q(x) = 0 \) implies \( x = 0 \).

A set \( \Gamma \) of seminorms is called a separating set if for each \( x \neq 0 \) there is a \( q \in \Gamma \) with \( q(x) \neq 0 \). A topology on \( E \) is called a locally convex topology if it is induced by a separating set of seminorms. A closed linear subspace \( D \) of a locally convex space \( E \) is called complemented if
there exists a closed linear subspace $F \subset E$ such that $E = D + F$, $D \cap F = \{0\}$.

A normed space is a pair $(E, ||\cdot||)$, where $E$ is a $K$-vector space and $||\cdot||$ is a norm. A Banach space is a complete normed space.

Let $0 < a < 1$. A subset $\{e_i : i \in I\}$ of a Banach space is called an $a$-orthogonal base if for each $x \in E$ there exist $\xi_i \in K$ ($i \in I$) such that

$$x = \sum_{i \in I} \xi_i e_i \quad \text{and} \quad \alpha \sup_{i} ||\xi_i x_i|| \leq ||x||.$$

An orthonormal base is a 1-orthogonal base $\{e_i : i \in I\}$ such that $||e_i|| = 1$ for each $i$.

In the sequel we shall need frequently the following

**Lemma 0.1.** Let $X \subset K$ and let $\{U_i : i \in I\}$ be a covering of $X$, where each $U_i$ is clopen in $X$. Then for $r > 0$ there is a disjoint refinement $X = \bigcup_{i \in J} B_j$, such that each $B_j$ is a ball $\{x \in X : |x-a| \leq \rho\}$ with radius $< r$.

**Proof:** Choose $r = x_1 > x_2 > \ldots$, $\lim x_n = 0$. Let $\Omega$ be the collection of balls of the form $B_a(\rho) = \{x \in X : |x-a| \leq \rho\}$ such that $\rho \in \{x_1, x_2, \ldots\}$ and $B_a(\rho) \subset U_i$ for some $i$. Then $\Omega$ is a covering of $X$ which is a refinement of the covering by the $U_i$. Let $\Omega_1$ be the set of $B \in \Omega$ with radius $x_1$. If $\Omega_1, \Omega_2, \ldots, \Omega_{n-1}$ are defined, let $\Omega_n$ be the set of balls in $\Omega$ of radius $x_n$ that are not contained in an element of $\bigcup_{k=1}^{n-1} \Omega_k$. Now $\bigcup_{k=1}^{n-1} \Omega_k$ is a disjoint subcovering of $\Omega$.

Let $X \subset K$. Then by $BC(X)$ (BUC(X)) we mean the space of the bounded (uniformly) continuous functions $X \to K$, with the sup norm.
1. ELEMENTARY DEFINITIONS

We want to study differentiability of functions $f : X \to K$ where $X$ is a nonempty subset of $K$, without isolated points. The definition is classical.

DEFINITION 1.1. Let $a \in X \subseteq K$. A function $f : X \to K$ is called differentiable at $a$ if

$$f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x-a}$$

exists. $f'(a)$ is called the derivative of $f$ at $a$. $f$ is called differentiable (on $X$) if $f$ is differentiable at every point of $X$. Then $f' : X \to K$ is called the derivative of $f$ and $f$ is called an antiderivative of $f'$.

The well-known rules for differentiation of sums, products, quotients and compositions of differentiable functions are valid in our theory. Polynomial functions are differentiable. A differentiable function is continuous.

Let $\mathcal{A}$ be the class of the nonempty subsets of $K$ that do not have isolated points. Then every nonempty open set is in $\mathcal{A}$; $\mathcal{A}$ is closed under arbitrary unions. If $X \subseteq Y \subseteq K$ and $X$ is dense in $Y$, then $X \in \mathcal{A}$ iff $Y \in \mathcal{A}$. In particular, $X \in \mathcal{A}$ implies $\overline{X} \in \mathcal{A}$. If $X \in \mathcal{A}$ and $f : X \to K$ is a continuous injection, then $f(X) \in \mathcal{A}$. If $X \in \mathcal{A}$ then $X$ is infinite.

FROM NOW ON $X$ WILL BE ALWAYS A NONEMPTY SUBSET OF $K$, WITHOUT ISOLATED POINTS.
LEMA 1.2. Let \( f : X \to K \) be continuous. Then there exists a sequence
\[ f_1, f_2, \ldots \]
of locally constant functions: \( X \to K \) with
\[ \lim f_n = f \text{ uniformly.} \]

**Proof:** Let \( n \in \mathbb{N} \). Define \( x \sim y \) if \( |f(x) - f(y)| < \frac{1}{n} \). Then \( \sim \) is an equivalence relation on \( X \) with clopen equivalence classes \( U_i \) \( (i \in I) \). Choose \( a_i \in U_i \) for each \( i \in I \). For each \( x \) there is precisely one \( i \) with \( x \in U_i \).

Define \( f_n : X \to K \) via
\[ f_n(x) = f(a_i) \quad (x \in X). \]
Then \( f_n \) is locally constant for each \( n \), \( |f_n(x) - f(x)| < \frac{1}{n} \) for all \( x \in X \), so \( \lim f_n = f \) uniformly.

Thus Lemma 1.2 shows that \( \{ f : X \to K : f \text{ differentiable and } f' = 0 \} \)
is uniformly dense in the space of all continuous functions: \( X \to K \). A
natural question that can be raised is the following. Does \( f' = 0 \) everywhere imply local constantness of \( f \)? The answer is "no", as we can see from the following interesting example.

**Example 1.3.** There exists an injective \( g : \mathbb{Z}_p \to \mathbb{Q}_p \) with \( g' = 0 \).

**Proof:** Let \( g : \mathbb{Z}_p \to \mathbb{Q}_p \) be defined as follows. For \( x \in \mathbb{Z}_p \) set
\[ g(x) = \sum_{i=0}^{\infty} a_i p^i. \]
If \( x, y \in \mathbb{Z}_p \), \( |x - y| = p^{-k} \) for some \( k \in \{0, 1, 2, \ldots\} \) then \( |g(x) - g(y)| = p^{-k}. \)
Thus, \( g \) is injective and \( g' = 0 \). Moreover \( g \) has a remarkable property,
namely \( \lim_{x \to y} (x-y)^{-n} (g(x) - g(y)) = 0 \) for each \( n \in \mathbb{N} \). Also, \( g \) satisfies a
Lipschitz-condition of order \( n \) for each \( n \in \mathbb{N} \), since for each \( n \) there
exists \( C > 0 \) such that for all \( x, y \in \mathbb{Z}_p \) : \( |g(x) - g(y)| \leq C|x - y|^n \). We leave
the proofs to the reader.
Note. If the characteristic of $K$ equals $p \neq 0$ we have another example of an injective map with zero derivative, that is different in nature from the one given above: The map $x \mapsto x^p$ is a field homomorphism $\sigma : K \to K$, $\sigma' = 0$. (If every element of $K$ has a $p$th root then $\sigma$ is even an automorphism of $K$.)

The above examples show also that a mean value theorem is lacking in our theory. Indeed: $f' = 0$ and $f(x) - f(y) = f'(\xi)(x-y)$ for some $x, y, \xi \in X$ implies $f(x) = f(y)$. This fact (no mean value theorem) will also cause troubles if we want to define $C^1$-functions. We may be tempted to follow naively the path of the "classical" theory and say that a differentiable $f : X \to K$ is in $C^1$ if $f'$ is continuous. But it will turn out that this definition has great disadvantages.

First of all, if $X$ is compact the space $C^1(X)$ with the norm 

$$f \mapsto \max(|f|_\infty, |f'|_\infty)$$

is not a Banach space. By lemma 1.2 there exists a sequence $f_1, f_2, \ldots$ of locally constant functions such that $\lim_{n \to \infty} f_n(x) = x$ uniformly. Since $f'_n = 0$ for all $n$, the sequence $f_1, f_2, \ldots$ is Cauchy, but not convergent, since $0 = \lim_{n \to \infty} f'_n(x) \neq 1$.

Secondly, a $C^1$-function defined in the way of above, may fail to be locally invertible in points where $f'$ is non zero. In fact we have

**EXAMPLE 1.4.** There exists a differentiable $f : \mathbb{Z}_p \to \mathbb{Z}_p$ with $f' = 1$ everywhere, and null sequences $x_1, x'_2, \ldots; y_1, y'_2, \ldots$ such that $x_n \neq y_n$ but $f(x_n) = f(y_n)$ for all $n$.

**Proof.** For $n \in \mathbb{N}$, let $B_n = \{ x \in \mathbb{Z}_p : |x-p^n| < p^{-2n} \}$. Then $x \in B_n$ implies $\overline{|x|} = p^{-n}$, so the $B_n$ are disjoint clopen sets. For $x \in \mathbb{Z}_p$ let
\[ g(x) = \begin{cases} 
 p^{2n} & \text{if } x \in B_n \\
 0 & \text{otherwise.} 
\end{cases} \]

Clearly \( g'(x) = 0 \) if \( x \neq 0 \). If \( y \neq 0 \), then \( |y^{-1}g(y)| \) is either 0 (if \( y \) is in no \( B_n \)) or \( p^{-n} \) (if \( y \in B_n \)). Hence \( g'(0) = \lim_{y \to 0} y^{-1}g(y) = 0 \). Now define \( f : \mathbb{Z}_p \to \mathbb{Q}_p \) by
\[
f(x) = x - g(x) \quad (x \in \mathbb{Z}_p).
\]

Then \( f' = 1 \). For \( n \in \mathbb{N} \), set \( x_n = p^n \) and \( y_n = p^n - p^{2n} \). Then \( x_n \neq y_n \) for all \( n \), \( \lim x_n = \lim y_n = 0 \), but \( f(x_n) = p^n - p^n = p^{n-2n} \) and \( f(y_n) = p^n - p^{2n} - g(p^n - p^{2n}) = p^{n-2n} \). Thus \( f(x_n) = f(y_n) \) for all \( n \).

So we are prompted to try another definition of a \( C^1 \)-function (1.7, 1.10, 5.1 will show that this definition is more satisfactory).

**DEFINITION 1.5.** \( f : X \to \mathbb{K} \) is called continuously differentiable (\( f \in C^1(X) \)) if for each \( a \in X \)
\[
\lim_{(x,y) \to (a,a), x \neq y} (x-y)^{-1} (f(x) - f(y))
\]
exists.

Observe that for a real valued function, defined on an interval, the continuity of \( f' \) guaranties the existence of \( \lim_{(x,y) \to (a,a), x \neq y} (x-y)^{-1} (f(x) - f(y)) \), since by the mean value theorem \( (x-y)^{-1} (f(x) - f(y)) = f'(\xi) \) for some \( \xi \) between \( x \) and \( y \).

The reader may verify that for a differentiable \( f : X \to \mathbb{K} \) to be \( C^1 \), continuity of \( f' \) is necessary, but not sufficient.

Let us define \( \nabla^2 X = X \times X \setminus \Delta \). For \( f : X \to \mathbb{K} \), define \( \phi_1 f : \nabla^2 X \to \mathbb{K} \) by
\[
\phi_1 f(x,y) = (x-y)^{-1} (f(x) - f(y)) \quad ((x,y) \in \nabla^2 X)
\]
Since $X$ does not have isolated points, $\nabla^2 X$ is dense in $X \times X$, so $f_1^*f$ has at most one continuous extension $\Phi_1 f : X \times X \to K$. The following lemma is a simple consequence of the definitions.

**Lemma 1.6.** Let $f : X \to K$. Then the following are equivalent.

(a) $f$ is continuously differentiable.

(b) $\Phi_1 f$ can be extended to a continuous function $\Phi_1 f : X \times X \to K$.

(c) There exists a continuous function $R : X \times X \to K$ such that

$$f(x) = f(y) + (x-y)R(x,y).$$

**Proof:** Obvious.

We now investigate local invertibility of $C^1$-functions.

Let $Y \subset K$. A map $g : Y \to K$ is called an *isometry* if $|g(x)-g(y)| = |x-y|$ for all $x,y \in Y$. We call $h : Y \to K$ a *similarity* if there exists $\alpha \in K$, $\alpha \neq 0$ such that $|h(x)-h(y)| = |\alpha| |x-y|$ for all $x,y \in Y$. In other words, a similarity is a non-zero scalar multiple of an isometry.

**Lemma 1.7.** Let $f : X \to K$ be in $C^1$ and let $f'(a) \neq 0$ for some $a \in X$. Then there is a neighborhood $U$ of $a$ such that $f|U \cap X$ is a similarity.

**Proof:** Since $f \in C^1$ there is a $\delta > 0$ such that for all $x,y \in X$ with $|x-a| < \delta$, $|y-a| < \delta$ and $x \neq y$:

$$|f(x)-f(y)| \leq \frac{1}{\delta} |f'(a)|,$$

hence $|(x-y)^{-1}(f(x)-f(y))| = |f'(a)|$; for all $x,y \in X$ with $|x-a| < \delta$, $|y-a| < \delta$ we have $|f(x)-f(y)| = |f'(a)| |x-y|.$

**Lemma 1.8.** Let $f : X \to K$ be an injective $C^1$-function and let $f'(x) \neq 0$ for all $x \in X$. Suppose its inverse $g : f(X) \to K$ is continuous. Then $g$ is also $C^1$ and $g'(f(x)) = (f'(x))^{-1}$ for all $x \in X$.

**Proof:** Let $z,t \in f(X)$ with $z \neq t$. Then $f_1^*g(z,t) = (z-t)^{-1}(g(z)-g(t)) = (f(g(z))-f(g(t)))^{-1}(g(z)-g(t)) = [f_1^*f(g(z),g(t))]^{-1}$. Since $\Phi_1 f \neq 0$
everywhere, the map \((z,t) \mapsto [\Phi_1 f(g(z), g(t))]^{-1}\), defined on \(f(X) \times f(X)\) is a continuous extension of \(\Phi_1 g_\circ g \in C^1\).

It may happen that the inverse of an injective \(C^1\)-function is not continuous. Choose \(B \subset Z^p\) as in example 1.4. Then let \(A = \bigcup_{n=1}^{\infty} (p^{-n} + B_n)\), \(B = Z^p \setminus \bigcup_{n=1}^{\infty} B_n\). Then \(X := A \cup B\) is closed, does not have isolated points.

Define \(f : X \to Z^p\) by

\[
\begin{align*}
    f(x) &= -p^{-n} + x & \text{if } x \in p^{-n} + B_n \text{ for some } n \\
    f(x) &= x & \text{if } x \in B
\end{align*}
\]

Then clearly \(f \in C^1\), \(f\) is injective, but its inverse is not continuous since \(f(X) = Z^p\) is compact and \(X\) is unbounded. However, we do not encounter this difficulty in case \(X\) is an open subset of \(K\). (See 1.10).

**Lemma 1.9.** (Newton approximation) Let \(B = \{x \in K : |x-a| < r\}\) be a ball in \(K\). Let \(f : B \to K\) and suppose there is an \(a \in K\) such that

\[
\sup \{|\Phi_1 f(x,y) - a| : x,y \in B, x \neq y\} < |a|
\]

Then \(f(B)\) is a ball in \(K\) with radius \(|a|r\) and \(f\) is a similarity.

**Proof:** Clearly we have for all \(x,y \in B\) (\(x\neq y\)): \(|\Phi_1 f(x,y) - a| = |a|\), so

\[
|f(x) - f(y)| = |a| \cdot |x-y|.\]

Thus \(f\) is similarity and \(f(B) \subset \{z \in K : |z-f(a)| < |a|r\}\).

Let \(c \in K\) with \(|c-f(a)| \leq |a|r\). We are done if we can produce an \(x \in B\) with \(f(x) = c\). Define \(\psi : B \to K\) via

\[
\psi(x) = x - a^{-1}(f(x) - c) \quad (x \in B)
\]

Then for \(x \in B\):

\[
|\psi(x) - a| \leq \max(|x-a|, |a^{-1}| \cdot |f(x) - c|) \leq \max(r, |a^{-1}| \cdot |f(a) - f(a)|, |a^{-1}| \cdot |f(a) - c|) \leq \max(r, |a^{-1}| \cdot |a|r, |a^{-1}| \cdot |a|r) = r.
\]

Thus \(\psi\) maps \(B\) into \(B\). Now, if \(x, y \in B\) and \(x \neq y\), then

\[
|\psi(x) - \psi(y)| = |x-y - a^{-1}(f(x) - f(y))| = |x-y| \cdot |1 - a^{-1}\Phi_1 f(x,y)| = |a^{-1}| \cdot |x-y| \cdot |a - \Phi_1 f(x,y)| \leq k |x-y| \text{ with } 0 < k < 1.
\]
By the Banach contraction theorem, $\psi$ has a fixed point $x$. Then $f(x) = c$.

**THEOREM 1.10.** Let $X$ be an open subset of $K$ and let $f : X \to K$ be in $C^1$, and let $f'(x) \neq 0$ for every $x \in X$. Then

(i) $X$ is a disjoint union of balls $B_{a_i}(r_i)$ ($i \in I$) such that for each $i \in I$ the restriction $f|_{B_{a_i}(r_i)}$ is a similarity and $f(B_{a_i}(r_i)) = B_{f(a_i)}(|f'(a_i)| r_i)$.

(ii) If $f$ is injective then its inverse is in $C^1$.

**Proof:** (i) is a corollary of 1.9 (use 0.1). (ii) is a simple consequence of 1.9 and 1.8.

**Note.** Almost everything in this Chapter can be found in [1], [3], [5], [6] or [11].
2. NOWHERE DIFFERENTIABLE FUNCTIONS

Following the methods of classical analysis we can prove the existence of nowhere differentiable continuous functions, using the Baire category theorem. (Corollary 2.2).

**Lemma 2.1.** Let $n \in \mathbb{N}$, let

$$E_n = \{ f \in BC(X) : \text{there is an } a \in X \text{ such that for all } x \in X : \left| f(x) - f(a) \right| \leq n|x-a| \}$$

Then $E_n$ is nowhere dense in $BC(X)$.

**Proof:** Let $A \subset BC(X)$ be a ball with radius $\rho \in |X|$. We are done if we can show the existence of $g \in A$ such that the distance between $g$ and $E_n$ is positive. By 1.2, $A$ contains a locally constant function $f$. By 0.1, $X$ can be covered by disjoint balls $S_i = \{ x \in X : |x-a_i| < r_i \}$ such that $r_i < \frac{1}{2}n^{-1}\rho$ for all $i$, and such that $f$ is constant on each $S_i$. For each $i$, let $\tau_i$ be the diameter of $S_i$. Then $0 < \tau_i \leq r_i$. Choose, for each $i$, an element $c_i \in X$ such that $\frac{1}{n}\tau_i^{-1}\rho \leq |c_i| = \tau_i^{-1}\rho$. (If the valuation of $X$ is discrete, then $\tau_i \in |X|$, hence $\tau_i^{-1}\rho \in |X|$ and we can choose a $c_i \in X$ with $|c_i| = \tau_i^{-1}\rho$. If the valuation of $X$ is dense it is obvious that we can choose $c_i$ with the required property). Define the function $g : X + K$ as follows. For $x \in S_i$, let

$$g(x) = f(x) + c_i(x-a_i).$$

Then $g$ is continuous, and, for $x \in S_i$:

$$|g(x) - f(x)| = |c_i| \leq \frac{1}{n}\tau_i^{-1}\rho \tau_i = \rho,$$

hence $|g-f|_\infty \leq \rho$, so $g \in A$. Now we will show that $d(g, E_n) \geq \frac{1}{4}\rho$.

Let $h \in E_n$. Then there is an $a \in X$ such that $|h(x) - h(a)| \leq n|x-a|$ for all $x \in X$.

Now $a \in S_i$ for some $i$, choose $b \in S_i$ with $|b-a| \geq \frac{1}{2}\tau_i$. Then
\[ |h(b)-h(a)| \leq \pi^n |b-a| \leq \pi^n \tau_1 < h_0, \] whereas \[ |g(b)-g(a)| = (\text{since } \tau \text{ is constant on } S^1) = |c_i (b-a_i)-c_i (a-a_i)| = |c_i| |b-a| > \frac{h_0}{\tau_1} \frac{h_0}{\tau_1} = h_0. \]

Hence \[ ||g-h||_\infty \geq \max(|g(b)-h(b)|, |g(a)-h(a)|) \geq |g(b)-g(a)-h(b)+h(a)| \geq \frac{h_0}{2}. \]

**COROLLARY 2.2.** The collection of all those \( f \in BC(X) \) that are somewhere differentiable is of first category in \( BC(X) \). The nowhere differentiable functions in \( BC(X) \) form a dense subset of \( BC(X) \), of second category.

**Proof:** Let \( f \in BC(X) \) be differentiable in \( a \in X \). Then there is \( \delta > 0 \) such that \( |x-a| < \delta \) implies \( |f(x)-f(a)| < (|f'|(a)+1) |x-a| \). If \( |x-a| > \delta \) then \( |f(x)-f(a)| \leq ||f||_\infty \cdot \delta \cdot \delta^{-1} \leq \delta^{-1} ||f||_\infty |x-a| \). Hence there is \( M > 0 \), such that for all \( x \in X \):

\[ |f(x)-f(a)| \leq M |x-a|, \]

implying \( f \in E_n \) for some \( n \). Therefore, the set of all \( f \in BC(X) \) that are somewhere differentiable is contained in \( \bigcup E_n \), hence of first category by the previous theorem. The second statement follows after applying the Baire category theorem.

The nowhere differentiable functions that possibly can be constructed with the help of 2.1 and 2.2 (choose \( f_1 \notin E_1, f_2 \notin E_2, \ldots \) with \( ||f_{n+1}-f_n||_\infty < 2^{-n} \). Then \( \lim f_n \) is nowhere differentiable) have unbounded difference quotients at every \( a \in X \).

One may ask: do there exist nowhere differentiable functions that have bounded difference quotients?

In real analysis, an \( f : [0,1] \to \mathbb{R} \) such that there exists \( M > 0 \) such that for all \( x,y \in X \) : \( |f(x)-f(y)| \leq M |x-y| \) is of bounded variation and, by a theorem of Lebesgue, differentiable almost everywhere. The situation is radically different in n.a. analysis, as we will show now. (2.4, 2.6).
Lemma 2.3. Let $X = \{ x \in K : |x| \leq 1 \}$. Then there exists a nowhere differentiable $f : X \to X$, that has bounded difference quotients.

Proof: Let $\rho \in K$ such that $0 < |\rho| < 1$ and put $\pi = \rho^2$. Let $J = \{ x \in K : |x| \leq |\pi| \}$. Then $J$ is an ideal in the ring $X$. Let $R$ be a full set of representatives in $X$ of $X/J$. Then for every $x \in X$ there is a unique sequence $r_1, r_2, \ldots$ $(r_i \in R$ for all $i)$ such that

\[(*) \quad x = \sum_{i=0}^{\infty} r_i \pi^i\]

To see this, we observe that there is $r_0 \in R$ such that $x - r_0 \in J$. Hence $x - r_0 = \pi x_1$, where $x_1 \in X$. There exists $r_1 \in R$ such that $x_1 - r_1 \in J$. Hence $x_1 - r_1 = \pi x_2$, where $x_2 \in X$. Thus $x = x_0 + \pi r_1 + \pi^2 x_2$. By induction we see that $x$ has a representation $(*)$. If $\sum_{i=0}^{\infty} r_i \pi^i = \sum_{i=0}^{\infty} r'_i \pi^i$ $(r_i, r'_i \in R$ for all $i)$, then it follows that $r_0 - r'_0 \in J$, hence $r_0 = r'_0$. Dividing by $\pi$ we get $\sum_{i=1}^{\infty} r_i \pi^{i-1} = \sum_{i=1}^{\infty} r'_i \pi^{i-1}$, whence $r_1 - r'_1 \in J$, $r_1 = r'_1$. Again, an induction process shows that the representation $(*)$ is unique.

The map $\alpha \mapsto \alpha^2$ $(\alpha \in X/J)$ induces a map $\sigma : R \to R$ satisfying $x^2 = \sigma(x)$ for all $x \in R$. (Here we denote $y \mod J$ by $\overline{y}$.)

We define $f : X \to X$ via

\[ f(\sum_{n=0}^{\infty} r_n \pi^n) = \sum_{n=0}^{\infty} \sigma(r_n) \pi^n \]

$f$ has bounded difference quotients: Let $x = \sum_{n=0}^{\infty} r_n \pi^n$, $y = \sum_{n=0}^{\infty} s_n \pi^n$, where $r_n, s_n \in R$ for each $n$, and let $x \neq y$. Then there is $k \in \{0, 1, 2, \ldots \}$ such that $r_i = s_i$ for $i < k$ and $r_k \neq s_k$. We have

\[ |(r_k - s_k) \pi^k| > |\pi| |\pi|^k = |\pi|^{k+1} \]

whereas for $i > 1$

\[ |(r_{k+i} - s_{k+i}) \pi^{k+i}| \leq |\pi|^{k+i} \leq |\pi|^{k+1} \]

Thus $|x - y| = |\sum_{i=k+1}^{\infty} (r_i - s_i) \pi^i| = \max(\{|(r_k - s_k) \pi^k|, |\sum_{i=k+1}^{\infty} (r_i - s_i) \pi^i|\})$

$= |(r_k - s_k) \pi^k| > |\pi|^{k+1}$. 

Further, \(|f(x)-f(y)| = |\Sigma (\sigma(r_n) - \sigma(s_n)) p^n| \leq \max_{n \geq k} |\sigma(r_n) - \sigma(s_n)| \leq |u|^n \leq |u|^k - 1.\]

Therefore \(|\phi_1 f(x,y) = |(x-y)^{-1}(f(x)-f(y))| < \frac{1}{|\pi|}.\]

Next we show that \(f\) is nowhere differentiable. Let \(a \in X\), \(a = \Sigma r_k x^n_{n=0}\) with \(r_n \in R\). For each \(n \in \{0,1,2,\ldots\}\), choose \(u_n, v_n \in R\) such that
\(|u_n - r_n| = 1\), \(|v_n - r_n| = \rho\). (This is possible: there exist \(\alpha, \beta \in X\) with \(|\alpha - r_n| = 1\), \(|\beta - r_n| = \rho\). There exist \(u_n, v_n \in R\) with \(u_n - \alpha \in J\), \(v_n - \beta \in J\) i.e. \(|u_n - \alpha| < |\pi| < 1\) and \(|v_n - \beta| \leq |\pi| < |\rho|\). So \(|u_n - r_n| = \max(|u_n - \alpha|, |\alpha - r_n|) = 1\) and \(|v_n - r_n| = \max(|v_n - \beta|, |\beta - r_n|) = \rho\). For \(n \in \{0,1,2,\ldots\}\) put
\[a_n = r_0 + r_1 + r_2 + \ldots + r_{n-1} p + u_n p + r_n + 1 p + \ldots\]
\[a'_n = r_0 + r_1 + r_2 + \ldots + r_{n-1} p + v_n p + r_n + 1 p + \ldots\]

Then \(\lim_{n \to \infty} a_n = a\), \(\lim_{n \to \infty} a'_n = a\) and \(a_n \neq a\), \(a'_n \neq a\) for all \(n\).

Now
\[\phi_1 f(a_n, a) = (a_n - a)^{-1}(f(a_n) - f(a)) = (u_n - r_n)^{-1}(\sigma(u_n) - \sigma(r_n))\]
\[\phi_1 f(a'_n, a) = (a'_n - a)^{-1}(f(a'_n) - f(a)) = (v_n - r_n)^{-1}(\sigma(v_n) - \sigma(r_n))\]

We have
\[\phi_1 f(a_n, a) - \phi_1 f(a'_n, a) = |(u_n - r_n)^{-1}(v_n - r_n)^{-1}| - (u_n - r_n)(\sigma(u_n) - \sigma(r_n)) - (v_n - r_n)(\sigma(v_n) - \sigma(r_n))|\]

We see that \(|(u_n - r_n)^{-1}(v_n - r_n)^{-1}| = |\rho|^{-1}\) and \((u_n - r_n)(\sigma(u_n) - \sigma(r_n)) - (u_n - v_n)(\sigma(u_n) - \sigma(v_n)) = (\sigma(u_n) - \sigma(v_n))(u_n - v_n) - (u_n - r_n)(v_n - r_n) = (v_n - u_n)(u_n - r_n)(v_n - r_n) = (u_n - v_n)(u_n - r_n)(v_n - r_n).\]

Since \(|(v_n - r_n)(u_n - r_n)(u_n - v_n)| = |\rho| > |\pi|\), we find
\[|(v_n - r_n)(\sigma(u_n) - \sigma(r_n)) - (u_n - r_n)(\sigma(v_n) - \sigma(r_n))| = |\rho|\]

Hence, for all \(n\) we find
\[\phi_1 f(a_n, a) - \phi_1 f(a'_n, a) = 1\]

so \(f\) is not differentiable at \(a\).
COROLLARY 2.4. Let \( X = \{ x \in \mathbb{K} : |x| < 1 \} \), and let \( f : X \to \mathbb{K} \) be a bounded uniformly continuous function. Then for each \( \varepsilon > 0 \) there exists a nowhere differentiable \( g : X \to \mathbb{K} \), such that \( g \) has bounded difference quotients and such that \( ||f-g||_\infty < \varepsilon \).

Proof: Let \( D = \{ f \in \text{BUC}(X) : f \text{ differentiable, } f \text{ has bounded difference quotients} \} \). It suffices to show that \( D \) is uniformly dense in \( \text{BUC}(X) \).

[In fact, if \( h \) is a nowhere differentiable function with bounded difference quotients, then there is \( d \in D \) such that \( ||(f-h)-d||_\infty < \varepsilon \). Choose \( g = h+d \).]

To show that \( D \) is dense, let \( \varepsilon > 0 \) and \( f \in \text{BUC}(X) \). There is \( \delta > 0 \) such that \( |x-y| < \delta \) implies \( |f(x) - f(y)| < \varepsilon \). Then cover \( X \) with balls of radius \( \delta \), say \( X = \bigcup B_i \). Choose \( a_i \in B_i \) for each \( i \) and define \( d : X \to \mathbb{K} \) as follows.

\[
\text{If } x \in B_i, \text{ then } d(x) = f(a_i).
\]

Then clearly \( ||d-f||_\infty < \varepsilon \). If \( x \neq y \) then \( x \in B_i, y \in B_j \) for some \( i \neq j \), hence \( |(x-y)^{-1}(d(x) - d(y))| < \delta^{-1}||f(a_i) - f(a_j)|| \leq \delta^{-1}||f||_\infty \).

So \( d \) has bounded difference quotients; \( d \) is locally constant, hence differentiable.

Finally, we show that there exist nowhere differentiable isometries.

We need the following funny lemma.

LEMMA 2.5. Let \( f : X \to \mathbb{K} \) have bounded difference quotients. Then \( f \) is a linear combination of two isometries. More precisely, there exist \( \mu \in \mathbb{K} \) and an isometry \( g : X \to \mathbb{K} \) such that \( f(x) = \mu x + \mu g(x) \) (\( x \in X \)).

Proof: Choose \( \mu \in \mathbb{K} \) such that for all \( x,y \in \mathbb{K} \), \( |f(x) - f(y)| < |\mu| |x-y| \).

Then \( |f(x) - \mu x + (f(y) - \mu y)| = \max(|f(x) - f(y)|, |\mu| |x-y|) = |\mu| |x-y| \). Thus, the map
g : x \mapsto \mu^{-1}f(x) - x is an isometry, and \( f(x) = \mu x + \mu g(x) \) for all \( x \in X \).

**COROLLARY 2.6.** There exists a nowhere differentiable isometry of 
\[ \{ x \in K : |x| < 1 \} \quad \text{into itself.} \]

**Proof:** Let \( f \) be as in lemma 2.3. By 2.5, \( f(x) = \mu x + \mu g(x) \) for some isometry \( g; \mu \in K \). Since \( x \mapsto x \) is differentiable everywhere, \( g \) is nowhere differentiable. There is a constant \( c \) such that \( g + c \) maps into the unit disc of \( K \).

**Notes.**

1) It is not difficult to extend the results of this chapter to the case where \( X \) is an open subset of \( K \). In fact, let 
\[ f : \{ x \in K : |x| < 1 \} \rightarrow \{ x \in K : |x| < 1 \} \]
be nowhere differentiable, having bounded difference quotients. We can extend this \( f \) "periodically" to a function \( \overline{f} : K \rightarrow \{ x \in K : |x| < 1 \} \). (\( K \) is a disjoint union of balls \( B_{i} \) with radius 1, choose a center \( a_{i} \) in \( B_{i} \) and define \( \overline{f}(x) = f(x-a_{i}) \) for \( x \in B_{i} \).) Then \( \overline{f} \) is nowhere differentiable, \( \overline{f} \) has bounded difference quotients. If \( X \subseteq K \) is open, then \( \overline{f}|X \) is nowhere differentiable, \( \overline{f}|X \) has bounded difference quotients. The proof of corollary 2.4 works also if \( X \) is an open set. We get:

Let \( X \subseteq K \) be an open set, and \( f : X \rightarrow K \) be a bounded uniformly continuous function. Then for each \( \varepsilon > 0 \) there exists a nowhere differentiable \( g : X \rightarrow K \) such that \( g \) has bounded difference quotients and such that \( ||f-g||_{\infty} < \varepsilon \). Also, there exists a nowhere differentiable isometry \( X \rightarrow K \).

2) We may get examples of nowhere differentiable isometries by another method inspired by the fact that \( z \mapsto \overline{z} \) (\( z \in \mathbb{C} \)) is nowhere differentiable.
If \( L \subseteq K \) is a closed nontrivially valued subfield of \( K \) and if \( \sigma : K \to K \) is a continuous \( L \)-automorphism, then \( \sigma \) is an isometry. If \( \sigma \) is not the identity, then \( \sigma \) is nowhere differentiable (if \( \sigma \) were differentiable then \( |\sigma'| = 1 \), and \( \sigma(x)\sigma(y) = \sigma(xy) \) implies \( \sigma'(x)\sigma(y) = y\sigma'(xy) \) hence \( \sigma'(0)\sigma(y) = y\sigma'(0) \) for all \( y \) i.e. \( \sigma(y) = y \) for all \( y \).
3. DIFFERENTIABILITY AS SUCH

Let \( D(X) \) be the \( K \)-algebra of all differentiable functions : \( X \to K \), and let \( N(X) = \{ f \in D(X) : f' = 0 \} \). Then \( N(X) \) is a subalgebra of \( D(X) \).

For each \( a \in X \) and for each compact set \( C \subset X \), let

\[
||f||_{a,C} = |f(a)| + \sup_{x \in C \setminus \{ a \}} |(x-a)^{-1}(f(x)-f(a))| \quad (f \in D(X))
\]

Then \( || \cdot ||_{a,C} \) is a seminorm on \( D(X) \). (Since \( f \) is differentiable there exists \( \delta > 0 \) such that if \( 0 < |x-a| < \delta \), \( |(x-a)^{-1}(f(x)-f(a))| \leq \max(|f'(a)|, 1) \).

If \( |x-a| > \delta \), \( x \in C \), then \( \sup_{x \in C}(x-a)^{-1}|f(x)-f(a)| \leq \delta^{-1} \max(|f(a)|, \sup_{x \in C}|f(x)|) \).

Thus \( ||f||_{a,C} < \infty \) for all \( a \in X \), all compact \( C \subset X \). The rest is easy.)

The \( || \cdot ||_{a,C} \) define a locally convex topology on \( D(X) \). Unless otherwise stated, we assume \( D(X) \) to be equipped with this topology.

To define the same topology we also could have started with seminorms \( || \cdot ||_{a,C} \), where

\[
||f||_{a,C} = \sup_{x \in C} |f(x)| + \sup_{x \in C \setminus \{ a \}} |(x-a)^{-1}(f(x)-f(a))| \quad (f \in D(X))
\]

Indeed, we have \( \omega > ||f||_{a,C} > ||f||_{a,C} \) for all \( f \in D(X) \), \( a \in X \) and non-empty compact \( C \subset X \). Also, for each \( x \in C \) we have, \( |f(x)| \leq |f(x)-f(a)| + |f(a)| \leq ||f||_{a,C} |x-a| + |f(a)| \leq ||f||_{a,C} \max(1, d(a, C)) \).

Hence, \( ||f||_{a,C} \) is majorized by a multiple of \( ||f||_{a,C} \).

Let \( BD(X) \) denote the \( K \)-algebra of all bounded differentiable functions : \( X \to K \), and let \( BN(X) = \{ f \in BD(X) : f' = 0 \} \). Then \( BN(X) \) is a subalgebra of \( BD(X) \). For each \( a \in X \), let

\[
||f||_a = ||f||_\infty + \sup_{x \neq a} |(x-a)^{-1}(f(x)-f(a))| \quad (f \in BD(X))
\]
Then \(|\| \cdot \|_a\) is a norm on \(BD(X)\) and the norms \(|\| \cdot \|_a\) \((a \in X)\) define a locally convex topology on \(BD(X)\). Unless otherwise stated, we assume \(BD(X)\) to be equipped with this topology. It is clear from the definitions that this topology is stronger than the topology of \(D(X)\), restricted to \(BD(X)\). It is also clear that \(D(X) = BD(X)\) in case \(X\) is a compact subset of \(K\). The (semi)norms, defined so far, are meaningful for all functions \(f : X \to K\). (If we admit \(-\infty\) as a value.)

**Theorem 3.1.** The spaces \(D(X)\) and \(BD(X)\) are complete. \(N(X)\) is closed in \(D(X)\), \(BN(X)\) is closed in \(BD(X)\). For each \(a \in X\) the map \(f \mapsto f'(a)\) (\(f \in D(X)\)) is a continuous linear function, and its restriction to \(BD(X)\) is continuous.

**Proof:** For a compact set \(C\) containing \(a\) as a non-isolated point, we have \(|f'(a)| \leq |f|_{a,C} \) (\(< |f|_a\) in case \(f\) is bounded) for all \(f \in D(X)\). Hence \(f \mapsto f'(a)\) is continuous both as a map: \(D(X) \to K\) and as a map: \(BD(X) \to K\). Then \(N(X)\), being an intersection of kernels of continuous maps, is closed in \(D(X)\), and also \(BN(X)\) is closed in \(BD(X)\).

In order to show that \(BD(X)\) is complete, let \((f_\lambda)\) be a Cauchy net in \(BD(X)\). Then \(\lim_\lambda f_\lambda = f\) uniformly for some bounded, continuous function \(f\). For each \(a \in X\), \(|f_\lambda'(a) - f'_\mu(a)| \leq |f_\lambda - f_\mu|_a\), hence there is a function \(g : X \to K\) such that \(\lim_\lambda f_\lambda' = g\) pointwise. Now (with \(\Phi_\lambda\) as in (1.6)) for all indices \(\lambda, \mu\), and for all \(x \in X, x \neq a\)

\[
|\Phi_\lambda f(x,a) - \Phi_\mu f(x,a)| \leq |\Phi_\lambda f(x,a) - \Phi_\mu f(x,a)| + |f_\lambda - f_\mu|_a,
\]
whence

\[
|\Phi_\lambda f(x,a) - \Phi_\mu f(x,a)| \leq \lim_\mu |f_\lambda - f_\mu|_a
\]
It follows that \(\lim_\lambda |f - f_\lambda|_a = 0\). We finally show that \(f\) is differentiable at \(a \in X\). Let \(\varepsilon > 0\). Choose \(\lambda_0\) such that for all \(\lambda \geq \lambda_0\): \(|f - f_\lambda|_a < \varepsilon, |f_\lambda'(a) - g(a)| < \varepsilon\). Then choose any \(\lambda \geq \lambda_0\). We see that for \(x \in X, x \neq a\)
\[ |\phi_1 f(x,a) - g(a)| \leq |f-f_\lambda|_a + |\phi_1 f_\lambda(x,a) - f'_\lambda(a)| + |f'_\lambda(a) - g(a)|. \]

There is \( \delta > 0 \) such that if \( 0 < |x-a| < \delta \) then \( |\phi_1 f_\lambda(x,a) - f'_\lambda(a)| < \varepsilon \).

It follows that if \( 0 < |x-a| < \delta \) then \( |\phi_1 f(x,a) - g(a)| < \varepsilon \). Hence \( f \) is differentiable and \( f' = g \).

To show that \( D(X) \) is complete we can follow a similar reasoning. Let \( (f_\lambda) \) be a Cauchy net in \( D(X) \) and let \( C \subset X \) be a compact subset. For each \( a \in X \) there is a compact \( C' \supseteq C \) such that \( a \in C' \) and \( a \) is not isolated in \( C' \). By restricting all functions involved to \( C' \) and by applying the above reasoning (where \( X \) is replaced by \( C' \)) we arrive at:

\[ \lim_{\lambda} |f-f_\lambda|_{a,C'} = 0 \text{ and } f|C' \text{ is differentiable at } a, \text{ with derivative } g(a) = \lim_{\lambda} f'_\lambda(a). \]

Since for any sequence \( x_1, x_2, \ldots \) (with \( x_n \neq a \) for each \( n \)) with \( \lim x_n = a \) we can arrange that \( C' \supseteq \{x_1, x_2, \ldots\} \) we may conclude that \( f \) is differentiable at \( a \) and that \( f' = g \).

In order to clarify the notion of convergence in the topology of \( BD(X) \) we define a set \( S \) of differentiable functions to be \textit{equidifferentiable} if for each \( a \in X \) and each \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for all \( f \in S \), \( 0 < |x-a| < \delta \) implies \( |(x-a)^{-1}(f(x)-f(a))-f'(a)| < \varepsilon \). We have:

**THEOREM 3.2.** Let \( f_1, f_2, \ldots \) be a sequence in \( BD(X) \). Then the following conditions (a) and (b) are equivalent for an \( f : X \to K \).

(a) \( \lim f_n = f \) in the topology of \( BD(X) \).

(b) \( \lim f_n = f \) uniformly, \( \lim f'_n \) exists pointwise, \( \{f_1, f_2, \ldots\} \) is equidifferentiable.

**Proof:** (a) \( \rightarrow \) (b). We only have to prove equidifferentiability. Let \( \varepsilon > 0 \), \( a \in X \). For \( n \in \mathbb{N}, x \neq a \) we have

\[ |\phi_1 f_n(x,a) - f'_n(a)| \leq |\phi_1 f_n(x,a) - f(x,a)| + |\phi_1 f(x,a) - f'(a)| + |f'(a) - f'_n(a)| \]

\[ \leq |f_n - f|_a + |\phi_1 f(x,a) - f'(a)|. \]
There is $n \in \mathbb{N}$ such that for $n \geq N$, $|\epsilon_n - \epsilon|_a < \epsilon$. There is $\delta > 0$ such that if $0 < |x-a| < \delta$ then $|\phi^1_f(x,a) - f'(a)| < \epsilon$. Hence, for $0 < |x-a| < \delta$ and $n \geq N$ we have

$$|\phi^1_{\epsilon_n}(x,a) - f'(a)| < \epsilon$$

There is $\delta' > 0$ such that for $0 < |x-a| < \delta'$ we have

$$|\phi^1_{\epsilon_m}(x,a) - f'(a)| < \epsilon$$

for all $m \in \{1, 2, \ldots, N-1\}$. Hence, for $0 < |x-a| < \min(\delta, \delta')$ we find for all $n \in \mathbb{N}$:

$$|\phi^1_{\epsilon_n}(x,a) - f'(a)| < \epsilon$$

which proves the equidifferentiability of $\{\phi^1_{\epsilon_1}, \phi^1_{\epsilon_2}, \ldots\}$.

(b) $\Rightarrow$ (a). It suffices to prove that $\phi^1_{\epsilon_1}, \phi^1_{\epsilon_2}, \ldots$ is a Cauchy sequence in $BD(X)$. Let $a \in X$ and $\epsilon > 0$. There is $\delta > 0$ such that for $0 < |x-a| < \delta$ and all $n \in \mathbb{N}$: $|\phi_{\epsilon_n}(x,a) - f'(a)| < \epsilon$. Choose $N$ such that for all $n, m \geq N$:

$$|\epsilon'_n(a) - \epsilon'_m(a)| < \epsilon$$

and $||\phi_{\epsilon_n}(x,a) - \phi_{\epsilon_m}(x,a)||_\infty < \epsilon\delta$. Then, if $0 < |x-a| < \delta$ we have for $n, m \geq N$

$$|\phi^1_{\epsilon_n}(x,a) - \phi^1_{\epsilon_m}(x,a)| < \epsilon$$

for $|x-a| \geq \delta$ and $n, m \geq N$

$$|\phi^1_{\epsilon_n}(x,a) - \phi^1_{\epsilon_m}(x,a)| < \delta^{-1}||\phi_{\epsilon_n}(x,a) - \phi_{\epsilon_m}(x,a)||_\infty < \delta^{-1}\epsilon\delta = \epsilon.$$ Consequently, for all $x \neq a$, $n, m \geq N$:

$$|\phi^1_{\epsilon_n}(x,a) - \phi^1_{\epsilon_m}(x,a)| < \epsilon,$$

$||\phi_{\epsilon_n}(x,a) - \phi_{\epsilon_m}(x,a)||_\infty < \epsilon$ i.e.

$$||\phi_{\epsilon_n}(x,a) - \phi_{\epsilon_m}(x,a)||_a < \epsilon.$$

The interested reader may try and find a theorem, analogous to 3.2, for convergence in $D(X)$.

It is shown in 1.3 that $f \in N(X)$ does not imply $f$ is locally constant. But we do have the following. (See also 3.11.)

**Theorem 3.3.** Let $f \in BN(X)$. Then there is a sequence $f^1, f^2, \ldots$ of locally constant functions with $\lim f^1_n = f$ in the topology of $BD(X)$. 

---

*Note: The text may contain typographical errors or formatting issues that are not visible in the image.*
Proof: We suppose that we have chosen a center of every ball in $K$. Let $\sigma_n : K \to K$ be a map that assigns to each element of $K$ the center of that ball of radius $\frac{1}{n}$ to which it belongs. Then $|\sigma_n(x) - x| \leq \frac{1}{n}$, $|\sigma_n(x) - \sigma_n(y)| \leq |x - y|$ for all $x, y \in K$ and $\sigma_n$ is locally constant for each $n$. Define $f_n := \sigma_n \circ f$. Then $f_n$ is locally constant for each $n$, and $|\sigma_n \circ f - f|_\infty \leq \frac{1}{n}$, hence $\lim_{n} \sigma_n \circ f = f$ uniformly. Also, $0 = \lim_{n} f'_n = f'$. Let $\varepsilon > 0$. There is $\delta > 0$ such that $|\phi_1 f(x, a)| < \varepsilon$ whenever $0 < |x - a| < \delta$. If $0 < |x - a| < \delta$ (and $f(x) \neq f(a)$) we have

$$|\phi_1 \sigma_n \circ f(x, a)| = |(f(x) - f(a))^{-1}(\sigma_n(f(x)) - \sigma_n(f(a)))| \cdot |f(x, a)| \leq |\phi_1 f(x, a)| < \varepsilon.$$  

Hence the $\sigma_n \circ f$ are equidifferentiable. By 3.2, $\lim_{n} \sigma_n \circ f = f$.

A corresponding theorem for $f \in N(X)$ is very easy to prove; if $f \in N(X)$, then there is (1.2) a locally constant $g$ such that $f - g$ is bounded, hence in $BN(X)$. Application of theorem 3.3 to $f - g$ yields: there exist locally constant $h_1, h_2, \ldots$ such that $h_n \in BN(X)$ for each $n$ and with $\lim_{n} h_n = f - g$ in $BD(X)$. So certainly $f = \lim_{n} (h_n + g)$ in $D(X)$.

Next, we turn to the diagrams

$$\begin{array}{ccc}
\text{D}(X) & \xrightarrow{D} & \text{K}^X \\
\pi & \circlearrowright & \rho \\
\text{D}(X)/N(X) & \xrightarrow{\pi} & \text{K}^X/	ext{BD}(X) \\
\end{array}$$

where $D$ is the differentiation map, $\pi$ the canonical quotient map (we assume that we have the quotient topology on $\text{im } \pi$), and where $\rho$ is the injection making the diagram commutative. The natural problems are

(1) Can we characterize $\text{im } D$?

(2) Can we describe the topology on $\text{im } D$, that makes $\rho$ into an isomorphism of topological vector spaces?
(3) Is $i(-X)$ complemented in $D(X)$? This is equivalent to the problem of existence of a continuous linear $P : \text{im } D \to D(X)$ ("antiderivation map") such that $DP$ is the identity. Of course we have similar questions concerning the second diagram.

We will answer all these questions in the sequel.

It is known in classical analysis that any derivative is of Baire class 1. The same is true in the non-archimedean case. (See Corollary 3.6.)

We say that $f : X \to K$ is of Baire class 1 if there exists a sequence $f_1, f_2, \ldots$ of continuous functions $X \to K$ such that $\lim f_n = f$ pointwise.

**Theorem 3.4.** Let $B^1(X)$ be the collection of Baire class 1 functions $X \to K$. Then

(a) $B^1(X)$ is a $K$-algebra of functions (under pointwise operations).

(b) $B^1(X)$ is uniformly closed.

(c) If $f \in B^1(X)$ then there exist locally constant $f_1, f_2, \ldots : X \to K$ such that $f(x) = \sum_{n=1}^{\infty} f_n(x)$ for all $x \in X$. Moreover, if $f$ is bounded we may assume $||f_n||_\infty \leq ||f||_\infty$ for all $n$.

**Proof:** The proofs of (a), (b) run just as in the classical case (where the base field is $\mathbb{R}$). We only prove (c). Let $f = \lim h_n$ pointwise, where $h_n \in C(X)$. By lemma 1.2 there exists $g_n$, locally constant, with $||h_n - g_n||_\infty < \frac{1}{n}$ for each $n$. Then also $f = \lim g_n$ pointwise. Let $f$ be bounded, define $\tilde{g}_n(x) = g_n(x)$ if $|g_n(x)| < ||f||_\infty$ and $\tilde{g}_n(x) = 0$ if $|g_n(x)| > ||f||_\infty$. Then $\lim \tilde{g}_n = f$ pointwise, $\tilde{g}_n$ is locally constant.

Hence we may assume $||g_n||_\infty \leq ||f||_\infty$ for all $n$. 

Define \( f_1 = g_1, f_n = g_n - g_{n-1} \) (\( n > 1 \)). Then \( f = \sum f_n \) pointwise, \( f_n \) is locally constant, \( ||f_n|| < ||f||_\infty \) for all \( n \).

**Lemma 3.5.** Let \( \varepsilon > 0 \). Then there exists a continuous \( \sigma : X \to X \) with

\[
0 < |\sigma(x) - x| < \varepsilon \quad \text{for all } x \in X.
\]

**Proof:** The equivalence relation "\( x \sim y \) if \( |x-y| < \varepsilon \)" decomposes \( X \) into a disjoint union of balls \( B_1 \) (of the form \( \{x \in X : |x-a| < \varepsilon \} \)). Since \( X \) does not have isolated points we can write each \( B_i \) as a disjoint union of \( U_i, V_i \), where \( U_i, V_i \) are clopen in \( X \), \( U_i \neq \emptyset, V_i \neq \emptyset \). Choose \( a_i \in U_i, b_i \in V_i \) for each \( i \). Let \( x \in X \). Then \( x \in B_i \) for exactly one \( i \). Define \( \sigma(x) = b_i \) if \( x \in U_i \), \( \sigma(x) = a_i \) if \( x \in V_i \). Then certainly \( \sigma(x) \neq x \) and, since \( \sigma(B_i) \subset B_i, |\sigma(x) - x| < \varepsilon \).

**Corollary 3.6.** Let \( f \in D(X) \). Then \( f' \) is of Baire class 1.

**Proof:** For each \( n \in \mathbb{N} \), let \( \sigma_n : X \to X \) be a continuous map with

\[
0 < |\sigma(x) - x| < \frac{1}{n} \quad \text{for all } x. \quad (3.5)
\]

Define:

\[
f_n(x) = f_1f_n(t_n(x), x) \quad (x \in X)
\]

Clearly \( f_n \) is continuous, \( \lim_{n \to \infty} f_n(x) = f'(x) \) for all \( x \).

In contrast to the classical theory, it will turn out that the converse of corollary 3.6 holds in the non-archimedean theory. (Theorem 3.10).

**Lemma 3.7.** Let \( f = \sum f_n \) pointwise. Suppose for each \( n \), \( f_n \) has an anti-derivative \( F_n \) such that \( \lim_{n \to \infty} ||F_n||_a = 0 \) for each \( a \). Then \( \sum F_n \) is convergent in \( BD(X) \), and \( (\sum F_n)' = f \).

**Proof:** That \( F := \sum F_n \) exists in \( BD(X) \) is clear from 3.1 and the fact
that \( n \mapsto \sum_{k=1}^{n} F_k \) is a Cauchy sequence in \( BD(X) \). Again, by 3.1, (continuity of differentiation) it follows that \( F' = \sum_{n=1}^{\infty} F'_n = \sum_{n=1}^{\infty} F_n \) pointwise. Hence, \( F' = f \).

In virtue of 3.4(c) and 3.7, we now prove that locally constant functions have antiderivatives with the right condition for their \( \| \cdot \|_a \) norms (lemma 3.9).

We say that \( f : X \to K \) is locally linear if for each \( a \in X \) there exist \( \delta > 0, \alpha, \beta \in K \) such that \( f \), restricted to \( \{ x \in X : |x-a| < \delta \} \), has the form

\[
x \mapsto \alpha x + \beta
\]

A straightforward argument shows that, if \( f : X \to K \) is locally linear then \( X \) is the disjoint union of balls, such that on each ball \( f \) is linear.

**Lemma 3.8.** Let \( B = \{ x \in X : |x-a| < \rho \} \) and let \( \varepsilon > 0 \). Then there exists a locally linear \( F : X \to K \) such that

\[
F' = \varepsilon \chi_B, \quad \|F\|_\infty \leq \varepsilon, \quad \text{supp } F \subseteq B
\]

\[
|F(x)-F(y)| \leq |x-y| \quad \text{whenever } x, y \in B
\]

\[
|F(x)-F(y)| \leq \varepsilon |x-y| \quad \text{otherwise}
\]

**Proof:** Let \( r = \varepsilon \min(1,\rho) \). By 0.1, \( B \) is a disjoint union of balls \( B_i = \{ x \in X : |x-a_i| < r_i \} \), where \( a_i \in X \) and \( r_i \leq r \) for each \( i \).

Define \( F(x) = x-a_i \) if \( x \in B_i \) for some \( i \) and \( F(x) = 0 \) if \( x \notin B \). Then clearly, \( F' = f \), \( F \) is locally linear, \( \text{supp } F \subseteq B \), \( \|F\|_\infty \leq \varepsilon \). Let \( x, y \in X \). If both \( x, y \) are not in \( B \) then \( F(x)-F(y) = 0 \). If \( x \in B_i \) for some \( i \) and \( y \notin B \), then \( |y-x| > \rho \), hence \( |F(x)-F(y)| = |F(x)| = |x-a_i| \leq r_i < \rho < \varepsilon |x-y| \).

If \( x \in B_i, y \in B_j \) and \( i \neq j \), then \( |F(x)-F(y)| = |x-a_i-y+a_j| \leq \max(|x-a_i|,|y-a_j|) \leq \max(r_i,r_j) < |x-y| \). Finally, if both \( x \) and \( y \) are
LEMMA 3.9. Let $f : X \to K$ be a locally constant function and let $\epsilon > 0$. Then there exists a locally linear $F : X \to K$ such that $F' = f$, $||F||_\infty < \epsilon$, and for all $x, y \in X$

$$|F(x) - F(y)| \leq \max(||f(x)||, \epsilon) |x - y|$$

Proof: $X$ is a disjoint union of balls $S_i$ such that for each $i$, $f$ has the value $c_i$ on $S_i$. Let $\epsilon_i = \epsilon(1 + |c_i|)^{-1}$. By lemma 3.8 there exist locally linear $F_i$ with $F'_i = \epsilon S_i$, supp $F_i \subset S_i$, $||F_i|| \leq \epsilon_i$, and $|F_i(x) - F_i(y)| \leq |x - y|$ if $x, y \in S_i$ and $|F_i(x) - F_i(y)| \leq \epsilon_i |x - y|$ otherwise. Define

$F : X \to K$ by $F(x) = c_i F_i(x)$ whenever $x \in S_i$. Then clearly $F$ is locally linear, $F' = f$, $||F||_\infty \leq \epsilon$ ($|F(x)| = |c_i F_i(x)| \leq \epsilon_i c_i \leq \epsilon$ whenever $x \in S_i$).

Now let $x, y \in X$. Then $x \in S_i$, $y \in S_j$ for some $i, j$. If $i = j$ then

$$|F(x) - F(y)| = |c_i| |F_i(x) - F_i(y)| \leq |c_i| |x - y| = |f(x)| |x - y|.$$ If $i \neq j$, then $F_i(y) = F_j(x) = 0$, so $|F(x)| = |c_i F_i(x)| = |c_i| |F_i(x) - F_i(y)| \leq \epsilon_i |c_i| |x - y| \leq \epsilon |x - y|$. Similarly, $|F(y)| \leq \epsilon |x - y|$. Thus

$$|F(x) - F(y)| \leq \epsilon |x - y|.$$ 

Now let $f \in B^1(X)$ and $\epsilon < ||f||_\infty$. By 3.4(c), $f = \Sigma_{n} f_n$ where $f_n$ locally constant (and $||f_n||_\infty \leq ||f||_\infty$ in case $f$ is bounded). By 3.9, for each $n \in \mathbb{N}$ there is an antiderivative $F_n$ of $f_n$ such that $F_n$ is locally linear, $||F_n||_\infty \leq \frac{\epsilon}{n}$, $|F_n(x) - F_n(y)| \leq \max(||f_n(x)||, \frac{\epsilon}{n}) |x - y|$ for all $x, y \in X$.

Thus, for each $a \in X$, $|F_n(x, a)| \leq \max(||f_n(a)||, \frac{\epsilon}{n})$. Now $\lim f_n(a) = 0$. Hence $\lim_{n \to \infty} |F_n(a)| = 0$. By 3.7, $F := \Sigma_{n} F_n$ exists in $BD(X)$ and $F' = f$.

Also, we have

$$|F(x) - F(y)| \leq \max_{n} |F_n(x) - F_n(y)| \leq \max(||f||_\infty, \epsilon) |x - y| = ||f||_\infty |x - y|$$

We have found
THEOREM 3.10. Let \( f \) be a Haar class 1 function \( : X \rightarrow \mathbb{K} \) and let \( \varepsilon > 0 \).

Then \( f \) has an antiderivative \( F \) with the following properties.

(1) \( F \in \text{BD}(X) \) and it is the limit of a sequence of locally linear functions in \( \text{BD}(X) \).

(2) \( \| F \|_{\infty} \leq \varepsilon \).

(3) If \( f \) is bounded, then \( |\Phi F(x,y)| \leq \|f\|_{\infty} \) for all \( x \neq y \).

In particular \( F \) has bounded difference quotients.

As a corollary we get

THEOREM 3.11. The bounded locally linear functions form a sequentially dense subset of \( \text{BD}(X) \).

Proof: Let \( f \in \text{BD}(X) \). Then \( f' \in B^1(X) \) so, by 3.10, \( f' \) has an antiderivative \( g \) such that \( g = \lim g_n \) in \( \text{BD}(X) \) where \( g_n \) is locally linear for each \( n \).

Since \( (f-g)' = 0 \), theorem 3.3 tells us that \( f-g = \lim h_n \) in \( \text{BD}(X) \), where \( h_n \) is locally constant for each \( n \). Thus, \( f = \lim (g_n + h_n) \), and \( g_n + h_n \) is locally linear for each \( n \).

It is easy to see, that the locally linear functions form a sequentially dense subset of \( D(X) \).

Going back to our diagrams, mentioned after 3.3 which, by now, look as follows

\[
\begin{array}{ccc}
D(X) & \xrightarrow{D} & B^1(X) \\
\pi \downarrow & & \rho \downarrow \\
D(X)/N(X) & \rightarrow & B^1(X) \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{BD}(X) & \xrightarrow{D} & B^1(X) \\
\pi \downarrow & & \rho \downarrow \\
\text{BD}(X)/\text{BN}(X) & \rightarrow & B^1(X) \\
\end{array}
\]

we see that \( D \) is surjective and that \( \rho \) is a bijection (in both diagrams).
THEOREM 3.12. Let $B^1(X)$ be endowed with the topology of pointwise convergence. Then the canonical maps $D(X)/N(X) \to B^1(X)$ and $BD(X)/BN(X) \to B^1(X)$ are isomorphisms of locally convex spaces.

For the proof we need a result which is worth stating separately.

LEMMA 3.13. Let $f$ be a Baire class 1 function : $X \to K$ and let $a \in X$.

Then, for each $\varepsilon > 0$ there exists an antiderivative $F$ of $f$ such that

$$||F||_a \leq \max(\varepsilon, |f(a)|)$$

Proof: By 3.10, $f$ has an antiderivative $G$, such that $G(a) = 0$ and $||G||_\infty \leq \varepsilon' = \max(\varepsilon, |f(a)|)$. There is $\delta < 1$ such that $0 < |x-a| < \delta$ implies $|(x-a)^{-1}(G(x)-G(a))| \leq \varepsilon'$.

Define a locally constant function $H : X \to K$ for which $H(x) = 0$ if $|x-a| < \delta$ and $|H(x)-G(x)| < \delta \varepsilon'$ if $|x-a| > \delta$ and set $F = G-H$. Then $F' = G'-H' = G' = f$. If $|x-a| < \delta$ then $H(x) = 0$ so $|G(x)-H(x)| \leq |G(x)| \leq \varepsilon'$.

We see $||F||_\infty = ||G-H||_\infty \leq \varepsilon'$.

To show that $|\phi F(x,a)| \leq \varepsilon'$ for all $x \in X$ we consider two cases.

$0 < |x-a| < \delta$. Then $|\phi F(x,a)| = |\phi G(x,a)| \leq \varepsilon'$.

$|x-a| > \delta$. Then $|\phi F(x,a)| \leq \varepsilon^{-1} \max(|H(x)-G(x)|, |H(a)-G(a)|) = \varepsilon^{-1} |H(x)-G(x)| \leq \varepsilon^{-1} \cdot \varepsilon' = \varepsilon'$.

Proof of theorem 3.12. It is clear that $\rho : BD(X)/BN(X) \to B^1(X)$ is continuous. We are done if we can prove that $D : BD(X) \to B^1(X)$ is an open mapping. Let $a \in X$. Define

$$U = \{f \in BD(X) : ||f||_a < \varepsilon\}$$

It suffices to show that $DU = \{h \in B^1(X) : |h(a)| < \varepsilon\}$.

That $DU$ is contained in the latter set is clear. Let $h \in B^1(X), |h(a)| < \varepsilon$. 


By Lemma 3.13 there is an antiderivative \( f \) of \( h \) with \( \| f \|_a < \varepsilon \). Hence \( f \in U \) en \( Df = h \).

The corresponding statement for \( D(X)/N(X) \rightarrow B^1(X) \) follows directly from what we have proved. For a set of the form

\[
U' = \{ f : \| f \|_{a,C} < \varepsilon \}
\]

we have \( DU' \subset \{ h \in B^1(X) : |h(a)| < \varepsilon \} \) and \( \{ f \in BD(X) : |f|_a < \varepsilon \} \subset U' \). It follows that \( D : D(X) \rightarrow B^1(X) \) is an open mapping.

Observe that, in spite of the fact that \( D(X) \) is complete and \( N(X) \) is closed, the quotient \( D(X)/N(X) \) may fail to be even sequentially complete. This happens, for instance, if \( X \) is compact. [Proof: \( D(X)/N(X) \neq B^1(X) \).]

The same reasoning as in the classical case yields the fact that \( f \in B^1(X) \) implies continuity of \( f \) for all \( x \in X \) belonging to a complement of a set of first category. Since we may apply the Baire category theorem on \( X \) (\( X \) is complete) it follows that \( f \) is continuous in a dense set of points.

On the other hand, if \( S = \{ x_1, x_2, \ldots \} \) is a countable dense subset of \( X \) then \( \xi_S = \lim_{n \to \infty} \xi_{\{ x_1, x_2, \ldots, x_n \}} \) pointwise, hence \( \xi_S \) is the limit of a sequence of functions in \( B^1(X) \). But \( \xi_S \) is nowhere continuous, hence is not in \( B^1(X) \); \( B^1(X) \) is not sequentially complete.

We now turn to the question of the existence of a decent "primitivation operator": \( P : B^1(X) \rightarrow D(X) \). Suppose we had a continuous linear map \( P : B^1(X) \rightarrow BD(X) \) such that \( DP \) is the identity. Then since the embedding \( BD(X) \rightarrow D(X) \) is continuous \( P \) is also continuous as a map: \( B^1(X) \rightarrow D(X) \).

But we have

**Theorem 3.14.** There is no continuous linear map \( P : B^1(X) \rightarrow D(X) \) such that \( DP \) is the identity on \( B^1(X) \).
Proof: Suppose we had such P. Then we will show that there exists a sequence \( f_1, f_2, \ldots \) in \( B^1(X) \), converging to 0, such that for all \( i \)

\[
|\| f_i \|_{a,C} | \geq 1
\]

for some compact \( C \) and \( a \in X \). Choose \( a \in X \). Since \( a \) is not isolated there exist \( x_1, x_2, \ldots \in X \) such that \( |a - x_1| > |a - x_2| > \ldots \) and \( \lim |a - x_n| = 0 \). Define

\[
U_n = \{ x \in X : |x - x_n| < |x_n - a| \} \quad (n \in \mathbb{N})
\]

Then the \( U_n \) are clopen and mutually disjoint. Let \( h_n = \xi_{U_n} \) for each \( n \).

\( P_n \) cannot be constant on \( U_n \), hence there is \( b_n \in U_n \) such that

\[
\phi_n(b_n) \neq \phi_n(a).
\]

Put \( C = \{ b_1, b_2, \ldots, a \} \). Then \( C \) is compact (since \( \lim b_n = a \)) and

\[
\| \phi_n \|_{a,C} \neq 0
\]

for each \( n \). Choose \( \lambda_n \in \mathbb{K} \) such that

\[
|\lambda_n| \geq \| \phi_n \|_{a,C}^{-1}
\]

for all \( n \). Define \( f_n = \lambda_n h_n \).

Then \( \lim \lambda_n h_n = 0 \) pointwise, whereas

\[
|\| f_n \|_{a,C} | = |\lambda_n| \| \phi_n \|_{a,C} \geq 1.
\]

COROLLARY 3.15. There is no continuous linear \( P : B^1(X) \rightarrow BD(X) \) such that \( DP \) is the identity on \( B^1(X) \).

\( N(X) \) is not complemented in \( D(X) \).

\( \beta N(X) \) is not complemented in \( BD(X) \).

Proof: The first statement follows from the remark preceding Theorem 3.14. Suppose there is a linear space \( S \subset D(X) \) such that \( D(X) = S \oplus N(X) \) as topological vector spaces. Then differentiation, restricted to \( S, \phi : S \oplus B^1(X) \) is a continuous bijection. It is also an open mapping: let \( U \subset S \) be open. Then \( U+N(X) \) is open in \( D(X) \) and since \( D \) is an open mapping on \( D(X), D(U+N(X)) \) is open in \( B^1(X) \). But \( \phi(U) = D(U+N(X)) \), so \( \phi \) is an open mapping. But then \( \phi^{-1} : B^1(X) + S \) is a continuous linear primitivation operator, contradictory to 3.14. A similar proof works for \( \beta N(X) \) in \( BD(X) \).
Looking at theorem 3.10 (3) we see that it is worth investigating the bounded functions, with bounded difference quotients on one hand and the bounded Baire class 1 functions on the other hand.

We denote by $\mathcal{B}^1(X)$ the space of all bounded $f : X \to K$ that have bounded difference quotients, normed by

$$
\|f\|_1 = \max(\|f\|_\infty, \|\phi_1 f\|_\infty) \quad (f \in \mathcal{B}^1(X))
$$

and we define $\mathcal{B}D^1(X) = \{f \in \mathcal{B}^1(X) : f \text{ is differentiable}\} = \mathcal{B}^1(X) \cap \mathcal{B}D(X)$. Similarly $\mathcal{B}N^1(X) = \mathcal{B}D^1(X) \cap \mathcal{B}N(X)$. We denote the space of all bounded Baire class 1 functions on $X$ by $\mathcal{B}B^1(X)$ (with the sup norm).

Then we have the following

**THEOREM 3.16.**

(a) $\mathcal{B}^1(X)$ is a Banach space and $\mathcal{B}D^1(X)$ is a closed subspace.

(b) $\mathcal{B}B^1(X)$ is a Banach space.

(c) Differentiation is a surjective map : $\mathcal{B}D^1(X) \to \mathcal{B}B^1(X)$.

More than that: the map $\varphi$ in the commutative diagram

$$
\begin{array}{c}
\mathcal{B}D^1(X) \\
\downarrow D \quad \pi \\
\mathcal{B}B^1(X) / \mathcal{B}N^1(X) \\
\end{array}
\xymatrix{ & \mathcal{B}B^1(X) \ar[dl]_{\pi} \ar[dr]^{\rho} \ar[r] & \mathcal{B}B^1(X) }$$

is a surjective isometry.

**Proof:** (a) A simple observation shows that $\mathcal{B}^1(X)$ is a K-vector space.

The map

$$f \mapsto (f, \phi_1(f)) \quad (f \in \mathcal{B}^1(X))$$

is an isometry of $\mathcal{B}^1(X)$ into $B(X) \times B(X \times X \Delta)$ (here, for any $Y$, $B(Y)$ is the space of all bounded functions : $Y \to K$ with the sup norm). If $f_1, f_2, \ldots$ is a Cauchy sequence in $\mathcal{B}^1(X)$, then $f_n \to f$ uniformly, $\phi_1 f_n \to g$
uniformly for some \( f : X \to K \) and \( g : X \times X \to K \). But for each \( x, y \in X \), \((x \neq y), \lim_{n \to \infty} \phi_i f_n (x, y) = \phi_i f(x, y)\), so \( \phi_i f_n \to \phi_i f \) pointwise. Therefore, \( \phi_i f = g \), hence \( f \in \mathcal{B}^1(X) \) and \( \lim f_n = f \) in the norm \( || \cdot ||_1 \). To show that \( \mathcal{B}^1(X) \) is closed, let \( f_1, f_2, \ldots \in \mathcal{B}^1(X) \) such that \( \lim f_n = f \in \mathcal{B}^1(X) \).

Let \( a \in X \). Then \( a := \lim f_n(a) \) exists and for \( x \neq a \) and \( n \in \mathbb{N} \):

\[
|\phi_i f(x, a) - a| \leq |\phi_i f(x, a) - \phi_i f_n(x, a)| \vee |\phi_i f_n(x, a) - f'_n(a)| \vee |f'_n(a) - a| \\
\leq |f - f_n|_1 \vee |\phi_i f_n(x, a) - f'_n(a)|.
\]

Given \( \varepsilon > 0 \), choose \( n \in \mathbb{N} \) such that \( ||f - f_n||_1 < \varepsilon \). There is \( \delta > 0 \) such that \( |\phi_i f_n(x, a) - f'_n(a)| < \varepsilon \) whenever \( 0 < |x - a| < \delta \). Then also \( |\phi_i f(x, a) - a| < \varepsilon \); \( f \) is differentiable at \( a \) and \( f'(a) = a \).

(b) is clear.

(c) This is just theorem 3.10 ((2),(3)).

As in the previous case we may question the existence of an anti-derivation \( P : \mathcal{B}^1(X) \to \mathcal{B}^1(X) \). We have

\[ \text{THEOREM 3.17. Let } \mathcal{B}^1(X) \text{ have an orthogonal base (this is true e.g. if } K \text{ has discrete valuation). Then there exists a linear isometry } \]

\[ P : \mathcal{B}^1(X) \to \mathcal{B}^1(X) \]

such that \( DP \) is the identity on \( \mathcal{B}^1(X) \).

Proof: Let \((e_i)_{i \in I}\) be an orthogonal base for \( \mathcal{B}^1(X) \). By Theorem 3.10 (2),(3) there exist \( E_i \in \mathcal{B}^1(X) \) such that \( ||E_i||_1 \leq ||e_i||_\infty \), \( E_i = e_i \) for each \( i \in I \).

For \( f \in \mathcal{B}^1(X) \) there is a unique representation \( f = \sum \lambda_i e_i \) (where \( \lambda_i \in K \) for each \( i \) and \( \lim_i \lambda_i e_i = 0 \)). Set

\[ P(\sum \lambda_i e_i) = \sum \lambda_i E_i \]
The definition is meaningful, since \( \| \lambda_i E_i \|_1 \leq \| \lambda_i e_i \|_\infty \), hence \( \lim \lambda_i E_i = 0 \) in \( BD^1(X) \). Hence \( \Sigma \lambda_i E_i \) exists in \( BD^1(X) \). Further, 
\[
\| Pf \|_1 = \| \Sigma \lambda_i E_i \|_1 \leq \sup_i \| \lambda_i E_i \|_1 \leq \sup_i \| \lambda_i e_i \|_\infty = \| f \|_\infty.
\]
Thus, \( P \) is continuous. For any finite sum \( \Sigma \mu_i e_i = g \) we have \( DPg = D(\Sigma \mu_i E_i) = \Sigma \mu_i e_i = g \); \( DPg = g \) for \( g \) in a dense subset of \( BD^1(X) \) so, by continuity of \( DP \), \( DP \) is the identity. Finally, for any \( f \in BD^1(X) \) we have 
\[ \| f \|_\infty = \| DPf \|_\infty \leq \| Pf \|_1. \]
Together with \( \| Pf \|_1 \leq \| f \|_\infty \) (see above) this means that \( P \) is an isometry.

Note. In spite of the positive result of 3.17, the constructed map \( P \) depends on the choice of the base in \( PBD^1(X) \) and is not canonical in the sense that there exists a simple standard formula that gives us \( Pf \) for every \( f \) (such as \( (Pf)(x) = \int f(t)dt \) for continuous real-valued functions \( f \) on an interval). We will see in 5.4 that we can find such a formula for continuous functions \( f \).

For examples and a study of \( D(X) \) for compact \( X \), see chapter 6.

Note. A special case of 3.10 can be found in [3]. See also [7].
4. SARD-TYPE THEOREMS

Let $f : X \to \mathbb{K}$ be differentiable. Then $\lambda \in \mathbb{K}$ is called a critical value of $f$ if there exists an $x \in X$ for which $f(x) = \lambda$, $f'(x) = 0$. The classical theorem of Sard states that (for functions: $\mathbb{R}^n \to \mathbb{R}^m$) the set of the critical values has Lebesgue measure zero. It turns out that for local fields $\mathbb{K}$ we have similar results for functions $: \mathbb{K} \to \mathbb{K}$.

THROUGHOUT CHAPTER 4 $\mathbb{K}$ IS A LOCAL FIELD.

The additive group of $\mathbb{K}$ being locally compact admits a (real-valued) Haar measure. A set $X \subseteq \mathbb{K}$ is called a null set (or a set of measure zero) if its Haar measure is zero. The class of null sets is a $\sigma$-ideal in $\mathcal{P}(\mathbb{K})$ and includes all countable subsets. If $0$ is the valuation ring of $\mathbb{K}$ and $P$ its maximal ideal (generated by $\pi \in \mathbb{K}$), then $0/P = \mathbb{k}$, the residue class field of $\mathbb{K}$. If $q = \#k$, then there is a valuation on $\mathbb{K}$, compatible with the topology, such that $|\pi| = q^{-1}$. We assume throughout this chapter that the valuation on $\mathbb{K}$ has this property. We normalize the Haar measure $m$ by setting $m(0) = 1$. Then, for any $n \in \mathbb{N}$, $m(\{x \in \mathbb{K} : |x| \leq q^{-n}\}) = q^{-n}$, from which it follows that the measure of any ball equals its diameter. Thus, a set $X \subseteq \mathbb{K}$ is a null set if and only if for each $\varepsilon > 0$ $X$ can be covered by countably many balls for which the sum of the diameters is less than $\varepsilon$.

Since the residue class field of $\mathbb{K}$ is finite and the valuation is discrete, the collection of all balls in $\mathbb{K}$ is countable.

**THEOREM 4.1.** Let $f : X \to \mathbb{K}$ be differentiable. Then $m(\{f(x) : f'(x) = 0\}) = 0$.

**Proof:** Let $A = \{x \in X : f'(x) = 0\}$. We have to show that $m(f(A)) = 0$. 


First assume that \( A \subset \{ x \in K : |x| \leq 1 \} \). Let \( \varepsilon > 0 \). For each \( a \in A \) there exists \( \delta_a \) with \( 0 < \delta_a < 1 \) such that \( |x-a| < \delta_a \), \( x \in X \) implies \( |f(x)-f(a)| \leq \varepsilon |x-a| \). The balls \( B_a(\delta_a) = \{ x \in K : |x-a| < \delta_a \} \) cover \( A \). There exists a (countable) disjoint subcovering \( B_{a_1}(\delta_1), B_{a_2}(\delta_2), \ldots \)

Since \( B_{a_i}(\delta_i) \subset \{ x : |x| \leq 1 \} \) for each \( i \), we have \( \sum \delta_i = \frac{1}{\varepsilon} \) m\( (\bigcup B_{a_i}(\delta_i)) = m(\bigcup B_{a_i}(\delta_i)) < 1 \). If \( x \in A \) then \( x \in B_{a_i}(\delta_i) \) for some \( i \), hence

\[
|f(x)-f(a)| < \varepsilon |x-a| \leq \varepsilon \delta_i, \text{ hence } f(x) \in B_{f(a_i)}(\varepsilon \delta_i), \text{ so }
\]

\[
m(f(A)) \subset m(\bigcup B_{f(a_i)}(\varepsilon \delta_i)) < \sum_{i=1}^{\infty} m(B_{f(a_i)}(\varepsilon \delta_i)) = \varepsilon \sum_{i=1}^{\infty} \delta_i = \varepsilon. \text{ Thus,}
\]

m\( (f(A)) \leq \varepsilon \) for each \( \varepsilon > 0 \): m\( (f(A)) = 0 \).

A similar proof works in case \( A \) is bounded.

If \( A \) is unbounded, let \( A_i = A \cap \{ x \in K : |x| \leq i \} \) (\( i \in \mathbb{N} \)).

Then m\( (f(A_i)) = 0 \) for each \( i \), hence m\( (f(A)) = \lim_{i \to \infty} m(f(A_i)) = 0 \).

**Corollary 4.2.** If \( f : X \to K \), \( f' = 0 \) then m\( (f(X)) = 0 \).

**Theorem 4.3.** Let \( f : X \to K \) be differentiable. If \( Y \subset X \), m\( (Y) = 0 \) then m\( (f(Y)) = 0 \).

**Proof:** For \( n \in \mathbb{N} \), define \( Y_n = \{ y \in Y : |f'(y)| \leq \frac{1}{n} \} \). Then \( Y = \bigcup Y_n \) and it suffices to show that m\( (f(Y_n)) = 0 \). Let \( \varepsilon > 0 \). We cover \( Y_n \) with balls \( B_1, B_2, \ldots \) such that \( \sum m(B_i) < \frac{\varepsilon}{n} \). For each \( a \in Y_n \) there is a ball \( B_a \), contained in \( B_i \) for some \( i \), such that \( x \in B_a \cap X, x \neq a \) implies

\[
|(x-a)^{-1}(f(x)-f(a)) - f'(a)| < 1, \text{ hence } |f(x)-f(a)| \leq n|x-a|. \text{ If } B_a \text{ has}
\]

radius \( \delta \) then we conclude: if \( x \in B_a \cap X \) then \( f(x) \in B_{f(a)}(n\delta) \), so

\[
m(f(B_a \cap X)) \leq m(B_{f(a)}(n\delta)) = n\delta = nm(B_a). \text{ We can construct a (countable)}
\]

disjoint subcovering \( B_{a_1}, B_{a_2}, \ldots \) of the covering by the \( B_a \) (\( a \in Y_n \)).

Since \( B_{a_j} \subset B_j \) for some \( j \) we have \( \sum m(B_{a_i}) < \frac{\varepsilon}{n} \). Now \( Y \subset \bigcup(B_a \cap X) \), so

\[
m(f(Y_n)) \subset \bigcup m(f(B_a \cap X)) \leq \sum m(B_{f(a_i)} \cap X)) \leq \sum n.m(B_{a_i}) = n.\varepsilon = \varepsilon. \text{ Thus,}
\]

m\( (f(Y_n)) = 0 \).
COROLLARY 4.4. If \( f : X \to K \) is differentiable, \( f' = 0 \) almost everywhere (a.e.), then \( m(f(X)) = 0 \).

**Proof:** Let \( A = \{ x : f'(x) = 0 \} \). By (4.1), \( m(A) = 0 \). By 4.3, \( m(f(X \setminus A)) = 0 \).

THEOREM 4.5. Let \( f : X \to K \) have bounded difference quotients. If \( Y \subseteq X \),

\[ m(Y) = 0 \quad \text{then} \quad m(f(Y)) = 0. \]

**Proof:** Let \( M \in \mathbb{R} \) be such that \( |f(x)-f(y)| \leq M|x-y| \) for all \( x,y \in X \), and let \( \varepsilon > 0 \). Then there are \( a_1, a_2, \ldots \in Y \) and \( \varepsilon_1, \varepsilon_2, \ldots \in \mathbb{R}^+ \) such that

\[ Y \subseteq \bigcup_{i=1}^{\infty} B_{a_i}(\varepsilon_i), \quad \sum_{i=1}^{\infty} \varepsilon_i < \frac{\varepsilon}{M}. \]

Let \( z \in f(Y) \), then \( z = f(x) \) for some \( x \in B_{a_i}(\varepsilon_i) \cap Y \), \( |z-f(a_i)| = |f(x)-f(a_i)| \leq M|x-a_i| \leq M\varepsilon_i \). Thus \( z \in B_{f(a_i)}(M\varepsilon_i) \).

Apparently, \( f(Y) \) can be covered by \( B_{f(a_i)}(M\varepsilon_i) \) \( (i = 1,2,\ldots) \), and \( \sum M\varepsilon_i < \varepsilon \).

For \( C^1 \)-functions we have a converse of 4.4.

THEOREM 4.6. Let \( f : X \to K \) be in \( C^1 \). Then the following are equivalent.

(a) \( f' = 0 \) a.e.

(b) \( m(f(X)) = 0 \).

**Proof:** We only have to prove (b) \( \Rightarrow \) (a). Let \( a \in X \) and let \( f'(a) \neq 0 \). Then by 1.7 there is a \( \delta_a > 0 \) such that \( x,y \in B_a(\delta_a) \subseteq X \) implies

\[ |f(x)-f(y)| = |f'(a)| \cdot |x-y|. \]

The inverse of \( f \), restricted to the nullset \( f(B_a(\delta_a) \cap X) \) is a similarity and, by 4.5, \( m(B_a(\delta_a) \cap X) = 0 \). Now \( X \) is covered by the \( B_a(\delta_a) \) but, since the set of all balls in \( X \) is countable, the \( B_a(\delta_a) \) form in fact a countable set \( \{ B_{a_i}(\delta_i) : i = 1,2,\ldots \} \). Thus

\[ X = \bigcup_{i=1}^{\infty} (B_{a_i}(\delta_i) \cap X) \]

is a countable union of nullsets, hence \( X \) itself is a nullset.

COROLLARY 4.7. Let \( X \subseteq X^0 \) (this is the case if, for example, \( X \) is open or \( X \) is the closure of an open set). Then the following are equivalent (for a \( C^1 \)-function \( f : X \to K \)).

(a) \( f' = 0 \) a.e.
(b) $f' = 0$.

(c) $m(f(X)) = 0$.

**Proof:** We only have to prove (a) $\Rightarrow$ (b). If $a \in X^0$ and $f(a) \neq 0$, then by continuity of $f'$, $f' \neq 0$ on a set $\{x \in K : |x-a| < \delta\}$, but this set has positive measure. Thus $f' = 0$ on $X^0$ and, by continuity, also on $X$ since $X^0$ is dense in $X$.

**Note.**

Theorem 4.1 (for functions $k^n + k^m$) appears in [9].
Let $C^1(X)$ be the $K$-algebra of the functions $f : X \to K$ such that the difference quotient $\phi_1^f$ can be extended to a continuous function $\phi_1^f$ (see 1.6). For $f \in C^1(X)$ and a compact $C \subseteq X$, set

$$||f||_{1,C} = \max_{x \in C} |f(x)| \vee \max_{(x,y) \in C^2} |\phi_1^f(x,y)|$$

(the notation $||.||_{1,C}$ is not consistent with the $||.||_{a,C}$ of Chapter 3). Then $||.||_{1,C}$ is a seminorm on $C^1(X)$, and the $||.||_{1,C}$, where $C$ runs through the (non empty) compact subsets of $X$, define a locally convex topology on $C^1(X)$. Unless otherwise stated, we assume $C^1(X)$ to be equipped with this topology. Let $N^1(X) = \{ f \in C^1(X) : f' = 0 \}$.

Let $BC^1(X)$ be the $K$-algebra consisting of the bounded $C^1$-functions on $X$ that have bounded difference quotients (in other words, $BC^1(X) = C^1(X) \cap BA^1(X)$), with the norm $||.||_1$ defined via

$$||f||_1 = ||f||_\infty \vee ||\phi_1^f||_\infty \quad (f \in BC^1(X))$$

(See 3.16). Unless otherwise stated, we assume $BC^1(X)$ to be normed with $||.||_1$. Let $BN^1(X) = \{ f \in BC^1(X) : f' = 0 \}$.

It is clear from the definitions that the norm topology on $BC^1(X)$ is stronger than the topology of $C^1(X)$, restricted to $BC^1(X)$. Also, if $X$ is compact, then $C^1(X) = BC^1(X)$ as topological vector spaces.

We define on $C^0(X) = C(X)$ (the space of all continuous functions : $X \to K$) the topology of uniform convergence on compact subsets; on $BC^0(X) = BC(X)$ (the space of all bounded continuous functions : $X \to K$) the topology of uniform convergence, induced by the norm $f \mapsto ||f||_\infty$. The next theorem is a $C^1$-version of 3.1.
THEOREM 5.1. The spaces $C^1(X)$ and $BC^1(X)$ are complete. $N^1(X)$ is closed in $C^1(X)$ and $BN^1(X)$ is closed in $BC^1(X)$. Differentiation is a continuous map: $C^1(X) \rightarrow C(X)$ and $BC^1(X) \rightarrow BC(X)$.

Proof: For a compact $C \subseteq X$ we have

$$\sup_{x \in C} |f'(x)| \leq \max_{x,y \in C} |\phi_1 f(x,y)| \leq ||f||_{1,C} \quad (f \in C^1(X))$$

and similarly,

$$||f'||_{\infty} \leq ||f||_1 \quad (f \in BC^1(X))$$

Hence, differentiation is a continuous map, from which it follows that $N^1(X)$ and $BN^1(X)$ are closed in $C^1(X)$ and $BN^1(X)$ respectively. Let $f_1, f_2, \ldots$ be a Cauchy sequence in $BC^1(X)$. Then $\lim_{n \to \infty} f_n = f$ in $BN^1(X)$ (3.16 (a)).

Since $\phi_1 f = \lim_{n \to \infty} \phi_1 f_n$ uniformly on $X \times X$, it follows by continuity, that $\lim_{n \to \infty} \phi_1 f_n$ exists uniformly on $X \times X$ and is a continuous extension of $\phi_1 f$.

Hence $f \in BC^1(X)$ i.e. $BC^1(X)$ is complete.

Let $(f_\lambda)$ be a Cauchy net in $C^1(X)$. Then $\lim_{\lambda} f_\lambda = f$ uniformly on compact sets, for some $f : X \rightarrow K$. As previously, we have $\lim_{\lambda} \phi_1 f_\lambda = g$ uniformly on compact subsets of $X \times X$, for some continuous $g$. Also $\phi_1 f = \lim_{\lambda} \phi_1 f_\lambda$ pointwise, hence $\phi_1 f$ has a continuous extension $\phi_1 f = g$. So $f \in C^1(X)$.

That $\lim_{\lambda} f = f$ in the topology of $C^1(X)$ is clear.

Locally linear functions are in $C^1(X)$. We have (see also 3.11)

THEOREM 5.2. Let $f \in C^1(X)$, and let $\varepsilon > 0$. Then there is a locally linear $g : X \rightarrow K$ such that $||f-g||_1 < \varepsilon$.

Proof: For each $a \in X$ there exists $x \in \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ such that if $x,y \in B_a(r)$

$$: = \{ t \in X : |t-a| \leq r, x \neq y, then \phi_1 f(x,y)-f'(a) | < \varepsilon. The balls $B_a(r)$ cover $X$ and by 0.1 we can find a disjoint subcovering, say $B_1 = B_{a_1(r_1)}$, where $i$ runs through some index set $I$. Define $g : X \rightarrow K$ as follows. For
each i set
\[ g(x) = f(a_i) + (x-a_i)f'(a_i) \quad (x \in B_i) \]
Then \( g \) is locally linear. For \( x \in B_i \) we have
\[ |f(x)-g(x)| = \]
\[ = |(x-a_i)(\Phi_i f(x,a_i)-f'(a_i))| \leq r_i e \leq e, \text{ hence } |f-g|_\infty \leq e. \text{ Now let} \]
x, y \in X. If both x, y \in B_i \text{ then}
\[ |(f-g)(x)-(f-g)(y)| = |(x-y)(\Phi_i f(x,y)-f'(a_i))| \leq e|x-y|. \text{ If } x \in B_i, \]
y \in B_j, where \( i \neq j \) then \( |x-a_i| \leq r_i \leq |x-y|; |y-a_j| \leq r_j \leq |x-y|, \text{ hence} \]
\[ |f(x)-g(x)| = |x-a_i| |\Phi_i f(x,a_i)-f'(a_i)| \leq e|x-y|. \text{ Similarly,} \]
\[ |f(y)-g(y)| \leq e|x-y|. \text{ Thus, also } |(f-g)(x)-(f-g)(y)| \leq \]
\[ |f(x)-g(x)| \lor |f(y)-g(y)| \leq e|x-y|. \text{ We obtain } |\Phi_i (f-g)|_\infty \leq e. \text{ Therefore } \]
\[ |f-g|_1 = |f-g|_\infty \lor |\Phi_i (f-g)|_1 \leq e. \]

**COROLLARY 5.2.** Let \( f \in C^1(X) \), \( f' = 0 \) and let \( e > 0 \). Then there is
a locally constant function \( g : X \to K \) with \( |f-g|_1 < e \).

**Proof:** If \( f' = 0 \), the function \( g \), constructed in the proof of 5.2, is
locally constant.

We see, that the locally linear functions in \( BC^1(X) \) form a dense subset
of \( BC^1(X) \) and that the locally linear functions form a sequentially
dense subset of \( C^1(X) \).

As in Chapter 3, we can study the maps involved in the commutative
diagrams

\[
\begin{array}{ccc}
BC^1(X) & \xrightarrow{D} & BC(X) \\
\pi & & \rho \\
BC^1(X)/BN^1(X) & & \\
\end{array}
\quad
\begin{array}{ccc}
C^1(X) & \xrightarrow{D} & C(X) \\
\pi & & \rho \\
C^1(X)/N^1(X) & & \\
\end{array}
\]

(D is the differentiation map, \( \pi \) is the quotient map, and \( \rho \) is the in-
jection making the diagram commutative). We will show first that \( D \) is
surjective by constructing a continuous linear map $P$ such that $DP$ is the identity on $C(X)$.

Let us fix a sequence $r_1, r_2, \ldots$ of real numbers such that

$$1 > r_1 > r_2 \ldots > 0, \lim r_i = 0.$$ 

For each $n$, the equivalence relation "$x \sim y$ iff $|x-y| < r_n$" yields a partition of $X$ into balls. We choose a center in every such ball and we call the set of representatives $R_n$. We can arrange that $R_n \subset R_{n+1}$ for each $n$.

For every $n$, and $x \in X$, let $\sigma_n(x)$ be the element $y$ characterized by $|y-x| < r_n$, $y \in R_n$. We have the following properties of $\sigma_n$.

**Lemma 5.3.** Let $\sigma_n : X \to K$ be defined as above. Then

(i) $\sigma_n$ is locally constant, hence $\sigma_n' = 0$

(ii) $|\sigma_n(x) - x| < r_n$ for all $x \in X$, hence $\sigma_n(x) + x$ uniformly

(iii) $|\phi_n \sigma_n(x, y)| = \begin{cases} 0 & \text{if } 0 < |x-y| < r_n \\ 1 & \text{if } |x-y| \geq r_n \end{cases}$

(iv) $|\phi_n \sigma_n(x, y) - 1| < r_n |x-y|^{-1}$ if $x \neq y$

(v) $\sigma_m \circ \sigma_n = \sigma_t$ where $t = \min(m, n)$.

**Proof:** (i) and (ii) are trivial. If $0 < |x-y| < r_n$ then $\sigma_n(x) = \sigma_n(y)$, hence $\phi_n \sigma_n(x, y) = 0$. If $|x-y| \geq r_n$, then $|\sigma_n(x) - \sigma_n(y)| = \max(|x-y|, |\sigma_n(x) - x|, |\sigma_n(y) - y|) = |x-y|$. This proves (iii). If $x \neq y$ then $|\phi_n \sigma_n(x, y) - 1| = |x-y|^{-1} |\sigma_n(x) - \sigma_n(y)| \leq |x-y|^{-1} \max(|\sigma_n(x) - x|, |\sigma_n(y) - y|) < r_n |x-y|^{-1}$. Thus (iv). Finally if $m \leq n$, we have $\sigma_m(\sigma_n(x)) \in R_m$ and $|x - \sigma_m(\sigma_n(x))| \leq \max(|x - \sigma_n(x)|, |\sigma_n(x) - \sigma_m(\sigma_n(x))|) < \max(r_n, r_m) = r_m$. But also $|x - \sigma_m(x)| < r_m$ and $\sigma_m(x) \in R_m$. Hence $\sigma_m(\sigma_n(x)) = \sigma_m(x)$. If $m \geq n$ then $\sigma_m(\sigma_n(x)) \in R_m$ and $\sigma_n(x) \in R_n \subset R_m$. Now $|\sigma_m(\sigma_n(x)) - \sigma_n(x)| < r_m$. Hence $\sigma_m(\sigma_n(x)) = \sigma_n(x)$. 

In the sequel of this section we write $x_n$ instead of $\sigma_n(x)$.

Let $f : X \to K$ be a continuous function. Define a function $Pf$ by means of the following Riemann-type sum:

$$ (Pf)(x) = \sum_{n=1}^{\infty} f(x_0)(x_{n+1} - x_n) $$

$Pf$ is well-defined since $f$ is bounded on the compact set $\{x, x_1, x_2, \ldots\}$ and $|x_{n+1} - x_n| \leq \max(|x_{n+1} - x_n|, |x_n - x|) < \max(r_{n+1}, r_n) = r_n$, hence

$$ \lim_{n \to \infty} f(x_n)(x_{n+1} - x_n) = 0. $$

We show that $Pf \in C^1$ and that $(Pf)' = f$.

Let $\epsilon > 0$ and $a \in X$. There is $m \in \mathbb{N}$ such that $|x-a| < r_m$, hence $|f(x)-f(a)| < \epsilon$. Now let $x, y \in X$, $x \neq y$ and $|x-a| < r_m$, $|y-a| < r_m$. We are done if we can show

$$ |\phi_1 Pf(x, y) - f(a)| < \epsilon. $$

There is an $s \in \mathbb{N}$, $s > m$, such that $r_{s+1} \leq |x-y| < r_s$. So $x_1 = y_1, x_2 = y_2, \ldots x_s = y_s$ and $x_{s+1} \neq y_{s+1}$.

It follows that

$$ Pf(x) - Pf(y) = \sum_{n=s}^{\infty} f(x_n)(x_{n+1} - x_n) - \sum_{n=s}^{\infty} f(y_n)(y_{n+1} - y_n) $$

$$ x-y = \sum_{n=s}^{\infty} (x_{n+1} - x_n) - \sum_{n=s}^{\infty} (y_{n+1} - y_n) $$

Thus,

$$ Pf(x) - Pf(y) - (x-y)f(a) = \sum_{n=s}^{\infty} (f(x_n) - f(a))(x_{n+1} - x_n) - \sum_{n=s}^{\infty} (f(y_n) - f(a))(y_{n+1} - y_n) $$

$$ = (f(x_s) - f(a))(x_{s+1} - y_{s+1}) + \sum_{n=s}^{\infty} (f(x_n) - f(a))(x_{n+1} - x_n) - \sum_{n=s}^{\infty} (f(y_n) - f(a))(y_{n+1} - y_n) $$

For $n > s$, $|x_n - a| < \max(|x_n - x|, |x-a|) < \max(r_n, r_m) = r_m$, so $|f(x_n) - f(a)| < \epsilon$. Similarly $|f(y_n) - f(a)| < \epsilon$, for $n > s$.

$$ |x_{s+1} - y_{s+1}| \leq \max(|x_{s+1} - x|, |x-y|, |y-y_{s+1}|) \leq \max(r_{s+1}, |x-y|, r_{s+1}) = |x-y| $$

For $n > s$, $|x_{n+1} - x_n| < r_n - r_{s+1} < |x-y|$, similarly $|y_{n+1} - y_n| < |x-y|$ for $n > s$. So we get:

$$ |Pf(x) - Pf(y) - (x-y)f(a)| \leq \epsilon |x-y|, $$

which was to be shown.
THEOREM 5.4. Let \( f \in C(X) \). Then (with \( x_n \) as defined above), let
\[
Pf(x) = \sum_{n=1}^{\infty} f(x_n) (x_{n+1} - x_n).
\]

Then \( Pf \) is a \( C^1 \)-antiderivative of \( f \).

Moreover, \( P : C(X) \to C^1(X) \) is linear and continuous and its restriction \( P : BC(X) \to BC^1(X) \) is an isometry.

Proof: The first part has been shown already. We first prove the last part. Let \( f \in BC(X) \). We have for \( x \in X \):
\[
|f(x_n)| |x_{n+1} - x_n| \leq ||f||_\infty r_n \leq r_1 ||f||_\infty, \text{ hence } ||Pf||_\infty \leq r_1 ||f||_\infty \leq ||f||_\infty.
\]

Now let \( x, y \in X, x \neq y \). We show that
\[
|(x-y)^{-1}(Pf(x) - Pf(y))| \leq ||f||_\infty.
\]

If \( |x-y| > r_1 \) then \( (x-y)^{-1}(Pf(x) - Pf(y)) \leq r_1^{-1} ||Pf||_\infty \leq r_1^{-1} r_1 ||f||_\infty = ||f||_\infty.
\]

If \( r_{s+1} \leq |x-y| < r_s \) for some \( s \), then \( x_1 = y_1, \ldots, x_s = y_s \), hence
\[
Pf(x) - Pf(y) = f(x_s) (x_{s+1} - y_{s+1}) + \sum_{n \geq s} f(x_n) (x_{n+1} - x_n) - \sum_{n \geq s} f(y_n) (y_{n+1} - y_n).
\]

For \( n > s \):
\[
|x_{n+1} - x_n| \leq |x_n - x_{n+1}| \leq |x-y| \text{ and, similarly, } |y_{n+1} - y_n| \leq |x-y|.
\]

Since \( |x_{n+1} - y_{n+1}| \leq |x-y| \), we have:
\[
|Pf(x) - Pf(y)| \leq ||f||_\infty |x-y|.
\]

We may conclude that \( ||Pf||_1 \leq ||f||_\infty \) for all \( f \in BC(X) \). But for any \( f \in BC(X) \):
\[
||f||_\infty = ||DPf||_\infty \leq ||Pf||_1, \text{ so } ||Pf||_1 = ||f||_\infty \text{ for all } f : P \text{ is an isometry.}
\]

Finally, we show that \( P : C(X) \to C^1(X) \) is continuous. Let \( S \subset X \) be compact.

We are done if we can produce a compact set \( T \subset X \) such that
\[
||Pf||_{1, S} \leq \sup \{ |f(x)| : x \in T \}.
\]

By rereading the foregoing proof we see that
\[
\sup_{x \in S} |Pf(x)| \leq \sup \{ |f(x)| : x \in T \}
\]
\[
\sup_{y \in S} |\hat{\phi}_y Pf(x, y)| \leq \sup \{ |f(x)| : x \in T \}
\]
for every \( T \) that contains \( S \) and such that \( x \in T \) implies \( x_n \in T \) for all \( n \).
Now let $S_i = \{x_i : x \in S\}$ (i = 1,2,....). Define
\[ T = S \cup \bigcup_{i=1}^{\infty} S_i. \]

$T$ is compact: Let $(U_i)$ be an open covering of $T$. Then finitely many, say $U_{i_1},...,U_{i_n}$, cover $S$. Then there is $\delta > 0$ such that if $|x-s| < \delta$ for some $s$ then $x \in \bigcup_{i=1}^{n} U_i$.

Hence there is $i_0$ such that $U_{i_1},...,U_{i_n}$ already cover $U S_i$. Now $\bigcup_{i=i_0}^{\infty} S_i$ is a finite set and can be covered by finitely many $U_i$'s. It follows that $T$ is compact. That $x \in T$ implies $x \in T$ for all $n$ follows from 5.3 (v).

Returning to our diagrams we can conclude:

**COROLLARY 5.5.** The map $BC^1(X)/BN^1(X) \overset{\rho}{\to} BC(X)$ is a surjective isometry. Moreover, $BN^1(X)$ is complemented in $BC^1(X)$. In fact we have an orthogonal decomposition
\[ BC^1(X) = BN^1(X) \oplus im P. \]

**Proof:** The spaces $BN^1(X)$ and $im P$ are orthogonal: if $h \in BN^1(X)$ and $f \in BC(X)$, then $||h+Pf||_1 \geq ||D(h+Pf)||_\infty = ||f||_\infty = ||Pf||_1$.

Hence $||h+Pf||_1 = \max(||h||_1,||Pf||_1)$.

$BN^1(X) + im P = BC^1(X)$: Let $f \in BC(X)$. Then $f = (f-Pf') + Pf'$. Now $(f-Pf')' = 0$ and $Pf' \in im P$.

Let $f \in BC^1(X)$. We have to show that $\inf \{||f-h||_1 : h \in BN^1(X)\} = ||Df||_\infty$.

Choose $h = f-Pf'$. Then $||f-h||_1 = ||Pf'||_1 = ||f'||_\infty = ||Df||_\infty$. Hence $\rho$ is an isometry.

**COROLLARY 5.6.** The map $C^1(X)/N^1(X) \overset{\rho}{\to} C(X)$ is an isomorphism of locally convex spaces. We have the decomposition as locally convex spaces
\[ C^1(X) = N^1(X) \oplus im P. \]
Proof: The map $N \otimes \text{im} \, P \mapsto C^1(X)$ sending $(f,g)$ into $f+g$ is obviously continuous. Its inverse is given by $f \mapsto ((I-P)f, Pf)$ and this is also a continuous map.

We can give simple criteria for $BC^1(X)$ to admit an orthogonal base.

**Theorem 5.7.** If either $X$ is compact or $K$ has discrete valuation then $BC^1(X)$ has an orthonormal base.

If $X$ is not compact and $K$ has dense valuation then, for any $\alpha > 0$, $BC^1(X)$ has no $\alpha$-orthogonal base.

**Proof:** If $K$ has discrete valuation then every $K$-Banach space has an orthogonal base (see [6]). If $X$ is compact, then $BC(X)$ and $BC(X \times X)$ have orthogonal bases (see [6]), so has their product $BC(X) \times BC(X \times X)$. The map $f \mapsto (f, \Phi f)$ is a linear isometrical embedding of $BC^1(X)$ into $BC(X) \times BC(X \times X)$. By a theorem of Gruson ([6]), $BC^1(X)$ has an orthogonal base. If either $K$ has discrete valuation or $X$ is compact we have for any $f \in BC^1(X) : ||f||_\infty \in |X|, ||\Phi f||_\infty \in K$, hence $||f||_1 \in |X|$. Thus, by multiplying with suitable scalars, we can transform an orthogonal base into an orthonormal base.

Conversely, suppose $BC^1(X)$ has an $\alpha$-orthogonal base for some $\alpha > 0$. The map $P$ (see 5.4) embeds $BC(X)$ isometrically into $BC^1(X)$. Again, by Gruson's theorem, $BC(X)$ has an $\alpha$-orthogonal base. But it then follows [6] that either $X$ is compact or $K$ has discrete valuation.

**Note:** In case $BC(X)$ has an orthogonal base (that is, if either $X$ is compact or $K$ has discrete valuation) we can construct a map $Q : BC(X) \mapsto BC^1(X)$ such that $DQ$ is the identity in the same way as we did in theorem 3.17 i.e., by choosing suitable antiderivatives for each element of the base. But 5.4 gives us already such a $Q$ for all spaces $BC(X)$.

**Note:** Maps, equal or similar to $P$ of (5.4) can be found in [2],[3],[5],[8].
6. DIFFERENTIABLE AND $C^1$-FUNCTIONS ON COMPACT SETS

Throughout this chapter $X$ will be a compact subset of $K$. As has been remarked before, $D(X) = BD(X)$, $R^1(X) = C^1(X)$. Our aim in this chapter is to construct a more or less standard base of $C^1(X)$ and to apply the results to $X = \{x \in L : |x| \leq 1\}$, where $L$ is a local subfield of $K$, especially the case $X = \mathbb{Q}_p$, $K = \mathbb{Q}_p$.

The set $\{|x-y| : x \in X, y \in Y\}$ is bounded and has only 0 as an accumulation point, hence it can be written as $\{r_1, r_2, \ldots\} \cup \{0\}$, where $r_i \in \mathbb{R}$ for all $i$ and $r_1 > r_2 > \ldots$, $\lim i r_i = 0$. Define $x_0 = \infty$. For each $i$, let $R_i$ be a set of representatives (in $X$) of the equivalence relation: "$x \sim y$ if $|x-y| < r_i$", such that $R_0 \subset R_1 \subset \ldots$. Notice that $R_0$ consists of a single point $a_0$ and that all $R_i$ are finite sets. Let $R = \bigcup_{i=0}^{\infty} R_i$. We define a map $v : R \rightarrow \{0, 1, 2, \ldots\}$ as follows. (For the sake of convenience, let $R_{-1} = \emptyset$). For $a \in R$, let $v(a)$ be the nonnegative integer $m$ for which $a \in R_m \setminus R_{m-1}$. Then $v(a_0) = 0$. For each $a \in R$, let

$$B_a = \{x \in X : |x-a| < r_{v(a)}\}$$

and let $e_a$ be the characteristic function of $B_a$. Our aim is to show, among other things, that the $e_a$ from an orthonormal base of $C(X)$. To this end, we define a partial ordering $\triangleleft$ on $R$ as follows

$$a \triangleleft b \iff b \in B_a \quad \text{(i.e., iff } e_a(b) = 1) \quad (a, b \in R).$$

(Let $x \in X$ and $n \in \{0, 1, 2, \ldots\}$. As in 5.4 we denote by $x_n$ the element of $R_n$ determined by $|x-x_n| < r_n$). We have

**Lemma 6.1.** $(R, \triangleleft)$ is a partially ordered set. $a_0$ is the smallest element of $R$. For all $a, b \in R$, $a \triangleleft b$, $a \neq b$ implies $v(a) < v(b)$.
For each $a \in \mathbb{R}$, the set $\{x \in \mathbb{R} : x < a\}$ is finite and linearly ordered by $\triangleleft$, and it consists of $\{a_0, a_1, a_2, \ldots\}$. Moreover, $a_0 \triangleleft a_1 \triangleleft a_2 \ldots$.

Proof: Clearly $a \triangleleft a$ for each $a \in \mathbb{R}$. If both $a \triangleleft b$, $b \triangleleft a$, $a, b \in \mathbb{R}$ then we may suppose $v(a) = v(b) = n$. Then $a \in R_{v(a)} = R_n$, $b \in R_n$ and (since $b \triangleleft a$) $|b-a| < r_n$. Hence $a = b$. If $a \triangleleft b$, $a, b \in \mathbb{R}$ and $v(a) < v(b)$, then $|b-a| < r_{v(a)} \leq r_{v(b)}$, whence $b \triangleleft a$, hence $b = a$. To prove transitivity of $\triangleleft$, let $a \triangleleft b$, $b \triangleleft c$. Then $v(a) < v(b)$, $|a-b| < r_{v(a)}$, $|b-c| < r_{v(b)}$, hence $|a-c| \leq \max(|a-b|, |b-c|) < \max(r_{v(a)}, r_{v(b)}) = r_{v(a)}$, hence $a \triangleleft c$.

Since $|a-a_0| < \infty$ for each $a \in \mathbb{R}$, we have $a_0 \triangleleft a$ for each $a$: $a_0$ is the smallest element of $\mathbb{R}$. If $a \in \mathbb{R}$ and $n \in \{0,1,2,\ldots\}$ then $|a-a_n| < r_n < r_{v(n)}$ hence $a_n \triangleleft a$. But then also $(a_{n+1}) \triangleleft a_{n+1}$, and since $(a_{n+1}) = a_n$ (see 5.3(v)), we obtain $a_n \triangleleft a_{n+1}$ for all $n$. Thus, $\{a_0, a_1, \ldots\}$ is linearly ordered $(a_0 \triangleleft a_1 \triangleleft a_2 \ldots)$ and is contained in $\{x \in \mathbb{R} : x \triangleleft a\}$. Conversely, if $x \triangleleft a$ for some $x \in \mathbb{R}$, then $v(x) = n$ for some $n$ and $|x-a| < r_n$, $x \in R_n$, hence $x = a_n$. Since $v$ is strictly increasing with respect to $\triangleleft$ we conclude that $\{x \in \mathbb{R}, x \triangleleft a\}$ is finite.

Lemma 6.2. Let $f : X \to K$. Suppose for each $a \in \mathbb{R}$ there exists $\lambda_a \in K$ such that for all $x \in X$

$$f(x) = \sum_{a \in \mathbb{R}} \lambda_a e_a(x)$$

Then the $\lambda_a$ are uniquely determined, and in fact $\lambda_{a_0} = f(a_0)$, $\lambda_a = f(a) - f(a_-)$ for $a \neq a_0$, where $a_- = \max\{x \in \mathbb{R}, x \neq a, x \triangleleft a\}$.

Proof: If $a \neq a_0$ then $e_a(a_0) = 0$, hence $f(a_0) = \lambda_{a_0} e_{a_0}(a_0) = \lambda_{a_0}$. If $b \neq a_0$ then $f(b) = \sum \lambda_a e_a(b) = \sum \lambda_a$. Similarly, $f(b) = \sum \lambda_a$, hence $f(b) - f(b_-) = \lambda_b$.

Let $a \in \mathbb{R}$ and $v(a) = n > 0$. Then $c_{n-1}(a) \in R_{n-1}$, hence $c_{n-1}(a) \neq a$
and, obviously, $\sigma_n(a) = a$. So $a = \sigma_n(a)$. Since $a \in \mathbb{R}^n$, $a \in \mathbb{R}^{n-1} \subset \mathbb{R}^n$, $a \neq a$ we have $|a-a| > r_n$. On the other hand, $|a-a| < r_{n-1}$, so $r_n < |a-a| < r_{n-1}$, which is only possible if $|a-a| = r_n$.

**THEOREM 6.3.** The $e_a$, where $a$ runs through $\mathbb{R}$, form an orthonormal base of $C(\mathbb{X})$. A continuous function is locally constant if and only if it is a finite linear combination of the $e_a$'s.

**Proof:** Let $f \in C(\mathbb{X})$. Then consider the function $\bar{f}$, defined by

$$(*) \quad \bar{f}(x) = \sum \lambda_a e_a(x) \quad (x \in \mathbb{X})$$

where $\lambda_{a_0} = f(a_0)$, $\lambda_a = f(a) - f(a_-)$ for $a \neq a_0$. Since $f$ is (uniformly) continuous $\lim_{a} \lambda_a = 0$, hence the series (*) converges uniformly, so $\bar{f}$ is a continuous function. By lemma 6.2, $f$ and $\bar{f}$ coincide on $\mathbb{R}$, hence $f = \bar{f}$ everywhere. We have $\|f\|_\infty = \|\sum \lambda_a e_a\|_\infty \leq \max \|\lambda_a\| \leq \|f\|_\infty$. Thus, the $e_a$ form an orthonormal base of $C(\mathbb{X})$.

Clearly every finite linear combination of the $e_a$ is locally constant.

Conversely, let $f$ be locally constant, $f \in C(\mathbb{X})$. Then there exists $\delta > 0$ such that $|x-y| < \delta$ implies $f(x) = f(y)$. Thus, if $r_{\delta}(a) < \delta$ then $|a-a_-| < \delta$ hence $f(a) = f(a_-)$. So $f = \sum \lambda_a e_a$, where the sum is taken over those $a \in \mathbb{R}$ for which $r_{\delta}(a) \geq \delta$. But such a $\in \mathbb{R}$ form a finite set.

Our special choice of the base of $C(\mathbb{X})$ is motivated by the fact that it is also of use for differentiable functions. That is, if $f = \sum \lambda_a e_a \in C(\mathbb{X})$ we can compute $\|f\|_c$ (see 3.1) for $c \in \mathbb{X}$ and $\|f\|_1$ in terms of the $\lambda_a$. We first compute $\|e_a\|_c$ for $a \in \mathbb{R}, c \in \mathbb{X}$. There are two possibilities

a) $c \in B_a$. Then $e_a(c) = 1$, whence $\max_{x} |\phi_1 e_a(x,c)| = \max \{(x-c)^{-1} : x \notin B_a\} = \text{dist}(B_a, B^c_a) = r_{\delta}(a)$.

b) $c \notin B_a$. Then $e_a(c) = 0$, whence $\max_{x} |\phi_1 e_a(x,c)| = \max \{|x-c|^{-1} : x \in B_a\} = |a-c|^{-1}$.  


The results of a) and b) together yield
\[ \max_x |\phi_1 e_a(x,c)| = \min(|a-c|^{-1}, v_r^{-1}) \]
(with the convention $0^{-1} = \infty$).

Let us define for $a \in \mathbb{R}, c \in X$
\[ \tau(a,c) = 1 \vee (|a-c|^{-1} \wedge v_r^{-1}) \]
\[ \tau_a = 1 \vee v_r^{-1} \]
Then we have

**THEOREM 6.4.** Let $f = \sum \lambda_a e_a \in C(X)$. Then, for $c \in X$:
\[ ||f||_c = \sup_a |\lambda_a| \tau(a,c) \]
\[ ||f||_1 = \sup_a |\lambda_a| \tau_a \]

**Proof:** The second equality follows from the first by taking the supremum over all $c \in X$. We now prove the first equality. We know that for any $x \in X, x \neq c$ (theorem 6.3) $|f(x)| \leq \max_a |\lambda_a|$. Further:
\[ \sum \lambda_a \phi_1 e_a(x,c) \leq \sup_a |\lambda_a| \phi_1 e_a(x,c) \leq \sup_a |\lambda_a| (|a-c|^{-1} \wedge v_r^{-1}) \]

as we just have seen. It follows that
\[ ||f||_c \leq \sup_a |\lambda_a| \tau(a,c) \]

To prove the opposite inequality, we take $a \in \mathbb{R}$ and show that $|\lambda_a| \tau(a,c) \leq ||f||_c$.

We distinguish two cases (we assume $a \neq a_0$):

a) $c \in B_a$. Then $|a-c| < v_r(a)$, hence $|a-c|^{-1} \wedge v_r^{-1} = v_r^{-1}$. Now,$ \lambda_a | = |f(a)-f(a_\_)| \leq |f(a)-f(c)| \vee |f(a_\_)-f(c)| \leq ||f||_c |a-c| \vee |a_\_a-c|$. Now $|a-c| \leq |a_\_a| \vee |a-c| = v_r(a) \vee |a-c| = v_r(a)$, whence $|\lambda_a| \leq ||f||_c v_r(a)$. Since also (6.3) $|\lambda_a| \leq ||f||_\infty$, we get $|\lambda_a| \tau(a,c) \leq ||f||_c$.

b) $c \notin B_a$. Then $|a-c| \geq v_r(a)$, hence $|a-c|^{-1} \wedge v_r^{-1} = |a-c|^{-1}$. Now
$|a_\_c| \leq |a_\_a| \vee |a-c| = v_r(a) \vee |a-c| = |a-c|$. Hence
\[ |\lambda_a| = |f(a) - f(a_\ast)| \leq |f(a) - f(c)| \vee |f(a_\ast) - f(c)| \leq ||f||_c (|a - c| \vee |a_\ast - c|) \leq ||f||_c |a - c|. \]

We get
\[ |\lambda_a| \tau(a, c) \leq ||f||_c. \]

**COROLLARY 6.5.** Let \( f = \sum \lambda_a e_a \) be in \( C(X) \). Then \( f \in N(X) \) if and only if for every \( c \in X \)
\[ \lim \frac{|\lambda_a| \tau(a, c)}{a} = 0. \]

The set \( \{e_a : a \in \mathbb{R}\} \) forms a base of \( N(X) \) in the following sense. For any \( c \in X \), the \( e_a \) are \( || \cdot ||_c \)-orthogonal and for any \( f \in N(X) \) we have uniquely determined \( \lambda_a \) (\( a \in \mathbb{R}\)) such that
\[ f = \sum \lambda_a e_a \]

in the sense of the topology of \( N(X) \) (i.e.
\[ \lim ||\lambda_a e_a||_c = 0 \] for each \( c \in X \)). Conversely, if \( \lim \lambda_a e_a = 0 \) in \( N(X) \), then \( f := \sum \lambda_a e_a \) is in \( N(X) \).

**Proof:** In the previous pages we have seen that for any \( c \in X \), \( a \in \mathbb{R} \) we have \( ||e_a||_c = \tau(a, c) \). Theorem 6.4 then implies \( || \cdot ||_c \)-orthogonality of the \( e_a \). Clearly any finite linear combination of the \( e_a \) is in \( N(X) \). Let
\[ f = \sum \lambda_a e_a \in C(X) \] and suppose \( \lim |\lambda_a| \tau(a, c) = 0 \) for all \( c \in X \). Let \( \varepsilon > 0 \). Then put \( g = \sum \lambda_a e_a' \), where \( S \) is the finite set \( \{a \in \mathbb{R} : |\lambda_a| \tau(a, c) \geq \varepsilon\} \). Then \( g \in N(X) \) and \( ||f - g||_c \leq \varepsilon \). Thus, \( f \) is in the closure (in the topology of \( D(X) \)) of the set of the locally constant functions. By 3.1 it follows that \( f \in N(X) \). Conversely, let \( f = \sum \lambda_a e_a \in N(X) \). By 3.3, for any \( \varepsilon > 0 \) and \( c \in X \) there exists a locally constant \( g \) for which \( ||g - f||_c < \varepsilon \). By (6.3), \( g = \sum \mu_a e_a \), where \( S \) is a finite set \( ||f - g||_c < \varepsilon \) implies
\[ \sup_{a \in S} |\lambda_a| \tau(a, c) = \sup_{a \in S} ||\lambda_a e_a||_c < \varepsilon \]
so \( \lim |\lambda_a| \tau(a, c) = 0 \).
The other statements of 6.5 now are obvious.

**COROLLARY 6.6.** Let \( f = \sum \lambda_a e_a \) be in \( C(X) \). Then \( f \) has bounded difference quotients if and only if
\[
\sup_a |\lambda_a| \tau_a < \infty.
\]
In fact, for any \( f \in BA^1(X) \), we have
\[
||f||_1 = \sup_a |\lambda_a| \tau_a.
\]

**Proof:** This corollary is an immediate consequence of theorem 6.4.

**COROLLARY 6.7.** Let \( f = \sum \lambda_a e_a \) be in \( C(X) \). Then \( f \in N^1(X) \) if and only if
\[
\lim_a |\lambda_a| \tau_a = 0.
\]
The set \( \{e_a : a \in R\} \) forms an orthogonal base of \( N^1(X) \).

**Proof:** Since \( N^1(X) \) is the \( || \cdot ||_1 \)-closure of the space of the locally constant functions we must have \( \lim_a |\lambda_a| \tau_a = 0 \) for any \( f \in N^1(X) \). Conversely, if \( \lim_a |\lambda_a| \tau_a = 0 \) then there exist locally constant functions \( f_n \) (of the type \( \sum \lambda_a e_a \) where \( S \) is finite) such that \( ||f - f_n||_1 \to 0 \).
Hence \( f \in N^1(X) \). The rest is easy.

**COROLLARY 6.8.** Let \( P : C(X) \to D^1(X) \) be the antiderivation map of 5.4.

Then \( \{e_a : a \in R\} \cup \{Pe_a : a \in R\} \) is an orthogonal base of \( C^1(X) \).

**Proof:** By 6.7 the \( e_a \) are orthogonal. Since the \( e_a \) are also \( || \cdot ||_\infty \)-orthogonal and since \( P \) is an isometry, the \( Pe_a \) form an orthogonal set. The spaces \( N^1(X) \) and \( \text{im } P \) are orthogonal from which it follows that
\[
\{e_a : a \in R\} \cup \{Pe_a : a \in R\} \text{ is an orthogonal set.}
\]

We now want to compute \( Pe_a \). Of course, we change the definition of
p a bit; we define for \( f \in C(X) \)

\[
Pf(x) = \sum_{n=0}^{\infty} f(x_n)(x_{n+1} - x_n) \quad (x \in X)
\]

where \( x_n \in \mathbb{R} \) for each \( n \), \( |x-x_n| < r_n \), and where \( R_n, r_n \) are as in the beginning of this chapter. The results of 5.4 remain valid for our new \( P \), and so does corollary 6.8. Now let \( a, b \in \mathbb{R} \). If \( a \nmid b \), then \( a = b_s \) for some \( s \), and \( (P_{e_a})(b) = \sum_{n=0}^{\infty} e_a(b_n)(b_{n+1} - b_n) = \sum_{n=0}^{\infty} (b_{n+1} - b_n) = b - b_s = b - a \). If not \( a \nmid b \) then \( e_a(b_n) = 1 \) for some \( n \) would imply \( a \nmid b_n \), but also \( b_n \nmid b \), hence \( a \nmid b \), a contradiction. Thus \( P_{e_a}(b) = 0 \). We arrive at

\[
(P_{e_a})(b) = (b-a) e_a(b) \quad (a,b \in \mathbb{R}).
\]

By continuity arguments (\( \mathbb{R} \) is dense in \( X \)) it follows that

\[
(P_{e_a})(x) = (x-a) e_a(x) \quad (x \in X, a \in \mathbb{R}).
\]

Let \( f \in C^1(X) \). Then, by 6.8, there exist \( \lambda_a, \mu_a \) (\( a \in \mathbb{R} \)) such that

\[
(*) \quad f = \sum \lambda_a e_a + \sum \mu_b P_{e_b}
\]

in the sense of the \( \| \cdot \|_1 \)-norm. It is interesting to compute the \( \lambda_a \) and \( \mu_b \) in terms of \( f \).

First, if we differentiate (*), we get:

\[
f' = \sum \mu_b e_b',
\]

so \( \mu_{a_0} = f'(a_0), \mu_b = f'(b) - f'(b) \) if \( b \neq a_0 \).

To compute the \( \lambda_a \), we set

\[
\sum \lambda_a e_a = f - \sum \lambda_b P_{e_b} = g
\]

Now for any \( b, a \in \mathbb{R} \)

\[
(P_{e_b})(a) = (a-b) e_b(a) = \begin{cases} a-b & \text{if } b \nmid a \\ 0 & \text{otherwise.} \end{cases}
\]

In particular, \( (P_{e_b})(a_0) = 0 \), for all \( b \), so we find \( \lambda_{a_0} = f(a_0) \).

If \( a \neq a_0 \), we have

\[
\sum_{b \nmid a} \mu_b(P_{e_b})(a) = \sum_{b \nmid a} (a-b) \mu_b = \sum_{b \nmid a} (a-b) \mu_b
\]

\[
\sum_{b \nmid a} \mu_b(P_{e_b})(a-) = \sum_{b \nmid a} (a-b) \mu_b
\]
Thus \((\sum \mu_b \rho e_b)(a) - (\sum \mu_b \rho e_b)(a) = \sum (a-a_\rho) \mu_b = (a-a_\rho) \sum (f'(b) - f'(b_\rho)) = (a-a_\rho)f'(a_\rho)\). So, \(\lambda_a = g(a) - g(a_\rho) = f(a) - f(a_\rho) - (a-a_\rho)f'(a_\rho)\).

We collect the above results.

**Theorem 6.9.** If \(f = \sum a e a \in C(X)\), then \(Pf(x) = \sum (x-a) e a(x)\) \((x \in X)\).

We have \(Pf(a_0) = 0\) and for \(a \in R, a \neq a_0\)

\[(Pf)(a) - (Pf)(a_0) = (a-a_0)f'(a_0).

If \(f \in C^1(X), f = \sum a e a + \sum \mu_b \rho e_b\) in the \(||\|_1\)-norm, then

\[
\lambda_a = f'(a_0), \mu_a = f'(a_0) \text{ and, for } a \neq a_0, \lambda_a = f(a) - f(a_\rho) - (a-a_\rho)f'(a_\rho).
\]

\[
\mu_a = f'(a) - f'(a_\rho).
\]

**Proof:** Easy.

It is an easy matter now to see what happens if we take \(X = \mathbb{Z}_p\) and \(K = \mathbb{Q}_p\). Then \(\{ |x-y| : x, y \in \mathbb{Z}_p\} \subset \{0, 1, p^{-1}, p^{-2}, \ldots\} \) and we can choose \(R_0 = \{0\}, R_1 = \{0, 1, \ldots, p-1\}, \ldots, R_n = \{0, 1, \ldots, p^n-1\}\). Thus \(R = U R_n = \{0, 1, 2, \ldots\}\). We have \(r_0 = \infty, r_n = p^{-n}\) for \(n > 0\). If \(n \in R, n \neq 0\), there is \(s\) such that

\[n = a_0 + a_1 p + \ldots + a_s p^s\]

\((a_i \in \{0, 1, \ldots, p-1\}, a_s \neq 0)\).

Then, \(\nu(n) = s\), hence \(r_{\nu(n)} = p^{-s}\).

We have \(p^s < n < p^{s+1}\), so \(p^{-s} > \frac{1}{n} > p^{-s-1}\). Thus, if \(|x-n| < \frac{1}{n}\) then \(|x-n| < p^{-s}\). But conversely, if \(|x-n| < p^{-s}\), then \(|x-n| < p^{-s-1} < \frac{1}{n}\). Thus, the base \(e_0, e_1, \ldots\) of \(C(\mathbb{Z}_p)\) can be described as follows: \(e_0 = 1\) and for \(n > 0\)

\[e_n(x) = \begin{cases} 1 & \text{if } |x-n| < \frac{1}{n} \\ 0 & \text{if } |x-n| > \frac{1}{n}. \end{cases}\]

We find:
Let \( f = \sum_{i=0}^{\infty} \lambda_i e_i \in C(\mathbb{Z}_p) \). Then \( f \in \mathcal{N}(\mathbb{Z}_p) \) if and only if for each \( c \in \mathbb{Z}_p \)

\[
\lim_{n \to \infty} |\lambda_n| \left( |n-c|^{-1} \wedge n \right) = 0
\]

\( f \in \mathbb{B}^1(\mathbb{Z}_p) \) if and only if

\[
\sup_n n|\lambda_n| < \infty
\]

\( f \in \mathcal{N}^1(\mathbb{Z}_p) \) if and only if

\[
\lim_{n \to \infty} n|\lambda_n| = 0.
\]

To conclude this chapter, we will discuss the existence of a Haar integral on \( \mathcal{N}^1(\mathbb{X}+\mathbb{K}) \), where \( \mathbb{X} = \{ x \in \mathbb{L} : |x| \leq 1 \} \) and \( \mathbb{L} \) is a local subfield of \( \mathbb{K} \).

The next few lines may serve as an introduction to the problem. Let \( G \) be a compact (abelian, 0-dimensional) group. It is natural to ask the following question: Does there exist a Haar integral i.e. a continuous translation invariant, nonzero linear function: \( C(G) \to \mathbb{K} \)? Or, equivalently, if \( \mathcal{O} \) denotes the ring of open compact subset of \( G \), do we have a Haar measure i.e. a nonzero, translation invariant, (finitely) additive bounded map: \( \mathcal{O} \to \mathbb{K} \)? In [14] it has been shown that the answer is "yes" if and only if \( G \) is \( p \)-finite, where \( p \) is the characteristic of the residue class field \( k \) of \( \mathbb{K} \). (By definition, every \( G \) is 0-finite, for \( p \neq 0 \), \( G \) is \( p \)-finite iff there exists an open subgroup \( H_0 \) such that for all open subgroups \( H \subset H_0 \) the index \( [H_0 : H] \) is not divisible by \( p \)).

For example, if \( G = \mathbb{Z}_p \), \( K = \mathbb{Q}_p \), and if \( m \) were a Haar measure then \( m(p^n \mathbb{Z}_p) = p^{-n} m(\mathbb{Z}_p) \), from which it follows that \( m \) is not bounded, a contradiction. It turns out that a smooth Fourier theory is possible in case the group \( G \) admits a Haar measure. (See [14]). Thus, for example, we have a decent Fourier theory for functions: \( \mathbb{Z}_p \to \mathbb{Q}_p \) (where \( q \neq p \)), but there is nothing of this kind in case \( f : \mathbb{Z}_p \to \mathbb{Q}_p \) (and many people think functions \( f : \mathbb{Z}_p \to \mathbb{Q}_p \) are more interesting than
functions \( f : \mathbb{Z}_p \to \mathfrak{o}_p \). In the sequel we will remove this difficulty partially by showing that, although there is no Haar integral on \( C(\mathbb{Z}_p) \) if \( K = \mathfrak{o}_p \), there does exist a nonzero translation invariant

\[ m : N^1(\mathbb{Z}_p) \to \mathfrak{o}_p, \]

which is continuous with respect to the norm \( \| \|_1 \).

Throughout the rest of this chapter, let \( K \) have characteristic 0, and let its residue class field \( k \) have characteristic \( p \neq 0 \).

Then the closure of the prime field of \( K \) is isomorphic to \( \mathfrak{o}_p \). Let \( L \) be any local subfield of \( K \), and let \( X = \{ x \in L : |x| < 1 \} \). Let \( \Omega(X) \) be the ring of compact open subsets of \( X \).

**Lemma 6.10.** There exists a unique translation invariant additive

\[ m : \Omega(X) \to K \text{ such that } m(X) = 1. \]

**Proof:** We have a real valued Haar measure on \( \Omega(X) \), such that the measure of \( X \) equals 1. Since every open compact subset of \( X \) is a finite union of cosets of \( \{ x \in X : |x| < \rho \} \) for some \( \rho \), it follows that the measure of a set in \( \Omega(X) \) is a rational number, hence can be viewed as an element of \( K \). The uniqueness is easy.

For the time being, let \( T(X) \) denote the linear space of the \( K \)-valued locally constant functions on \( X \). Then, by a standard construction, we may conclude from lemma 6.10 that there exists a \( K \)-linear map

\[ M : T(X) \to K \text{ such that } M(1) = 1 \text{ and } M(f_s) = M(f) \text{ for all } f \in T(X), s \in X. \]

(As usual, \( f_s(x) := f(s+x) \) for \( s, x \in X \).

Sometimes we write \( \int f(x) dm(x) \) instead of \( M(f) \).

We define a convolution \( \ast \) on \( T(X) \) as usual: if \( f, g \in T(X) \) then for each \( x \in X \) the map \( y \mapsto f(x-y)g(y) \) is in \( T(X) \). Let

\[ (f \ast g)(x) = \int f(x-y)g(y) dm(y). \]
It is easy to see that $f \ast g \in T(X)$ and that $\ast$ makes $T(X)$ into a commutative $K$-algebra.

Let us fix $a,b \in R$, $x \in X$, and let us consider the function

$$g : y \mapsto e_a(x-y)e_b(y).$$

Suppose $v(a) \geq v(b)$. Then $g$ is identically zero if $(x-B_a) \cap B_b = \emptyset$, that is, if $|x-a-b| \geq r_{v(b)}$ which means that $e_b(x-a) = 0$. If $e_b(x-a) \neq 0$ then $g$ is the characteristic function of the ball $(x-B_a) \cap B_b = x-B_a$.

Then $fg = m(B_a)$. We find:

$$(e_a \ast e_b)(x) = m(B_a)e_b(x-a).$$

As in chapter 4, we assume that the valuation on $L$ is such that the (real) measure of a ball equals its diameter. Then, if $q$ is the cardinality of the residue class field of $L$, we have for $a \in R$:

$$m(B_a) = m(x \in X : |x-a| < q^{-1}r_{v(a)}) = q^{-1}r_{v(a)} \in K. \text{Thus } |m(B_a)| = q^{-1}r_{v(a)} \in R$$

We find

$$||\phi_1e_a||_\infty = r_{v(a)} = |m(B_a)|q^{-1}$$

It will be convenient to choose a norm on $C^1(X)$, slightly different from $|| \cdot ||_1$, namely

$$||f||_1' = ||f||_\infty \vee q ||\phi_1f||_\infty \quad (f \in C^1(X))$$

(With respect to $|| \cdot ||'$, the $e_a$ still form an orthogonal base of $N^1(X)$).

Clearly, the norm $|| \cdot ||'$ is translation invariant. We have for any $x \in X$, $y \in Y$, $x \neq y$:

$$|(e_a \ast e_b)(x)| = |m(B_a)||e_b(x-a)| \leq q||\phi_1e_a||_\infty||e_b||_\infty \leq ||e_a||_1||e_b||_1'$$

and

$$||\phi_1(e_a \ast e_b)(x,y)|| = |m(B_a)|||\phi_1e_b(x-a,y-a)| \leq q||\phi_1e_a||_\infty||\phi_1e_b||_\infty \leq q^{-1}||e_a||_1'||e_b||_1'.$$

Thus it follows that

$$||e_a \ast e_b||_1' \leq ||e_a||_1'||e_b||_1'$$
for all \( a, b \in \mathbb{R} \). Therefore, for \( f, g \in T(X) \) we have

\[
\|f * g\|_1 \leq \|f\|_1 \|g\|_1.
\]

**LEMMA 6.11.** \( T(X) \) is a normed algebra under convolution and with respect to \( \|\cdot\|_1 \), and \( f \mapsto \int f(x) \, dm(x) \) is a continuous homomorphism with norm 1.

**Proof:** We have proved the first statement already. To prove the second statement, let \( a, b \in \mathbb{R} \), \( v(a) \geq v(b) \). Then

\[
\int (e_a * e_b)(x) \, dm(x) = m(B_a) \int e_b(x-a) \, dm(x) = m(B_a) m(B_b) = \int e_a(x) \, dx . \int e_b(x) \, dx.
\]

By linearity it follows that \( f \) is a homomorphism. The rest is easy.

The bilinear map \( (f, g) \mapsto f * g \) and the integral now can be extended uniquely to a continuous map on \( N^1(X) \times N^1(X) \) and \( N^1(X) \) respectively. We denote these extensions again by \( * \) and \( \int \) respectively. Now if \( f, g \in N^1(X) \) then, for each \( x \in X \) the map \( y \mapsto f(x-y)g(y) \) is also in \( N^1(X) \), hence \( \int f(x-y)g(y) \, dm(y) \) is well-defined. On the other hand, also \( f * g \) is well-defined. It remains to show that

\[
(f * g)(x) = \int f(x-y)g(y) \, dm(y) \quad (x \in X)
\]

To this end it suffices to show that if \( f_n \to f \) in the norm, \( f_n \in T(X) \) for all \( x \), then \( h_n \to h \) in the norm, where

\[
h_n(y) = f_n(x-y)g(y) \quad (y \in X)
\]

\[
h(y) = f(x-y)g(y) \quad (y \in X)
\]

We see: \( \|h-h_n\|_\infty \leq \|f-f_n\|_\infty \|g\|_\infty \), so \( \lim_{n \to \infty} \|h-h_n\|_\infty = 0 \).

Also for \( z, y \in X, z \neq y \):

\[
|\phi_1(h-h_n)(z,y)| = |(z-y)^{-1}||f_n(x-y)g(y)-f_n(x-z)g(y)+f_n(x-z)g(y)-f(x-z)g(z)|
\]

\[
\leq \|g\|_\infty \|\phi_1(f-f_n)\|_\infty + \|f-f_n\|_\infty \|\phi_1 g\|_\infty, \text{ so } \lim_{n \to \infty} \|\phi_1(h-h_n)\|_\infty = 0.
\]
THEOREM 6.12. Let $N^1(X)$ be normed by $|| \cdot ||_1$. Then there exists a unique translation invariant linear map $\mathcal{F} : N^1(X) \to K$, of norm 1, such that $\mathcal{F}1 = 1$. Under convolution $\ast$ defined by

$$(f \ast g)(x) = \int f(x-y) g(y) dm(y) \quad (f, g \in N^1(X))$$

$N^1(X)$ is a Banach algebra.

A character is a continuous homomorphism $\alpha$ of $X$ into the multiplicative group $K^*$ of $K$. Since $X$ is compact it follows that a character maps $X$ into $\{x \in K : |x| = 1\}$. We are particularly interested in those characters that are in $N^1(X)$:

LEMMA 6.13. Let $\alpha$ be a character on $X$. Then the following are equivalent.

1. $\alpha$ is differentiable at 0 and $\alpha'(0) = 0$.
2. $\alpha \in N^1(X)$.
3. $\alpha$ is locally constant.

Proof: (1) $\Rightarrow$ (2): For any $x, y \in X$, $x \neq y$ we have: $|\alpha(x, y)| = |\alpha(x-y)| = |\alpha(x-y, 0)|$. Thus, $\lim_{x-y \to 0} \frac{\alpha(x, y)}{x-y} = 0$ whence $\alpha \in N^1(X)$.

(2) $\Rightarrow$ (3): We prove $\ker \alpha$ is open. Suppose not. Then, for any compact open subgroup $H$ of $X$ there is $s \in H$ with $\alpha(s) \neq 1$. From $\int \alpha(x) \xi_H(x) dm(x) = \alpha(s) \int \alpha(x) \xi_H(x) dm(x)$ it then follows that $\int \xi_H(x) \alpha(x) dm(x) = 0$. But then also for any $a \in X$: $\int \xi_{a+H}(\alpha(x) dm(x) = \alpha(a) \int \alpha(x) \xi_H(x) dm(x) = 0$. Therefore: $\int \alpha = 0$ for all locally constant $f$. By continuity, $\int \alpha = 0$ for all $f \in N^1(X)$. In particular, $1 = \int \alpha^{-1} \alpha = 0$. Contradiction.

(3) $\Rightarrow$ (1): Clear.

For any $\alpha \in N^1(X)$ the map $\phi_\alpha : f \mapsto M(\alpha^{-1})$ is a continuous homomor-
phism: $N^1(X) \to K$. (Proof: $\phi_a$ is linear, $|\phi_a(f)| = |M(fa^{-1})| \leq \|fa^{-1}\|_1$, hence $\phi_a$ is continuous. If $f, g$ are locally constant, then a simple computation shows that $\phi_a(f * g) = \phi_a(f)\phi_a(g)$. Further, we see that $|\phi_a| \leq |a^{-1}|_1'$. But in fact we have $|\phi_a| \leq 1$ (if $|\phi_a(f)| \leq c|f|_1'$ for all $f \in N^1(X)$ then for $n \in \mathbb{N}$ $|\phi_a(f^n)| \leq c|f^n|_1' \leq c|f|_1'^n$, so $|\phi_a(f)| \leq c^{|f|_1'}$).

**Lemma 6.14.** For any character $a \in N^1(X)$ the map $\phi_a: N^1(X) \to K$, defined via

$$\phi_a(f) = \int f(x)a(-x)dm(x) \quad (f \in N^1(X))$$

is a homomorphism of norm 1.

From now on in this chapter we assume that $K$ has sufficiently many roots of unity. That is, if $H$ is an open compact subgroup of $X$ and $\xi \in X/H$, $\xi$ has order $n$, then $x^n = 1$, has $n$ solutions in $K$. (For example an algebraically closed $K$ will do.) It then is an easy matter to show that every locally constant function $f \in N^1(X)$ is a finite linear combination of characters. Corollary 5.2 shows that the linear span of the characters is a $|||_1'$-dense linear subspace of $N^1(X)$. If $\alpha, \beta$ are characters in $N^1(X)$ then $\alpha * \beta = 0$ if $\alpha \neq \beta$ and $\alpha * \alpha = \alpha$. Thus, if $\phi : N^1(X) \to K$ is a nonzero (continuous) homomorphism then there exists a character $a \in N^1(X)$ such that $\phi(a) = 1$, $\phi(\beta) = 0$ if $\beta \neq a, \beta \in N^1(X)$, $\beta$ character. Thus, for a finite linear combination $f$ of characters in $N^1(X)$ we find $\phi(f) = \int f(x)a(-x)dm(x)$. By continuity, the latter formula holds for any $f \in N^1(X)$.

**Theorem 6.15.** Let $K$ have sufficiently many roots of unity, and let $\hat{X}$ be the group of all characters that are in $N^1(X)$. Then we have
(1) The linear span of \( \hat{X} \) is dense in \( N^1(X) \).

(2) There is a bijection between \( \hat{X} \) and the collection of all nonzero homomorphisms: \( N^1(X) \to K \), given by \( \alpha \mapsto \phi_\alpha \) (\( \alpha \in \hat{X} \)), where

\[
\phi_\alpha(f) = \int f(x)\alpha(-x)dm(x) \quad (f \in N^1(X)).
\]

For any \( f \in N^1(X) \) we define its Fourier transform \( f^- : X^- \to K \) by

\[
f^-(\alpha) = \int f(x)\alpha(-x)dx \quad (\alpha \in \hat{X}).
\]

We have \( |f^- (\alpha)| = |\phi_\alpha(f)| \leq ||f||_1 \), so \( f^- \) is a bounded function on \( \hat{X} \).

More than that: if \( f \) is a finite linear combination of characters (i.e. \( f \) is locally constant) then \( f^- \) has finite support. For any \( f \in N^1(X) \) we have \( f = \lim f_n \), where \( f_n \) is locally constant, hence \( f^- = \lim f_n^- \) uniformly: \( f^- \in C_0(X^-) \), where \( C_0(X^-) \) denotes the space of all \( g : X^- \to K \) such that for every \( \varepsilon > 0 \) there exist only finitely many \( \alpha \in X^- \) such that \( |g(\alpha)| > \varepsilon \).

**Theorem 6.16.** Let \( K \) have sufficiently many roots of unity. Then the Fourier transformation \( f \mapsto f^- \), given by

\[
f^-(\alpha) = \int f(x)\alpha(-x)dm(x) \quad (\alpha \in X^-)
\]

is a continuous injection: \( N^1(X) \to C_0(X^-) \).

\([N^1(X)]^-\) is dense in \( C_0(X^-) \) and the inverse of the Fourier transformation \( : \ [N^1(X)]^- \to N^1(X) \) is not continuous.

**Proof:** Suppose \( f^- (\alpha) = 0 \) for all \( \alpha \in X^- \) and for \( f \in N^1(X) \). Then, by 6.15(1) and continuity, we then have \( \int f(x)g(x)dm(x) = 0 \) for all \( g \in N^1(X) \).

Then also \( f \ast g = 0 \) for all \( g \in N^1(X) \). From Lemma 6.17 below it follows that \( f = 0 \). Every function on \( X^- \) having finite support is the Fourier transform of some locally constant function on \( X \), hence \([N^1(X)]^-\) is dense. By 6.18 below there exists a sequence \( \alpha_1, \alpha_2, \ldots \in X^- \) with
\[ ||a_i||_{i'} + \omega. \] Choose \( \lambda_i \in K \) with \( ||\lambda_i||_{i'}^{-1} = ||a_i||_{i'} \), and let \( f_i = \lambda_i a_i \).

Then \( ||f_i^\ast||_{\omega} = ||\lambda_i||_{i'} \to 0 \), whereas \( ||f_i||_{1} = 1 \) for all \( i \).

**Lemma 6.17.** The convolution algebra \( N(X) \) has a bounded approximate identity: In fact, let \( H_n = \{ x \in X : |x| < q^{-n} \} \) and \( e_n = m(H_n)^{-1} \delta_{H_n} (n = 1, 2, \ldots) \). Then \( ||e_n||_{1} = 1 \) for all \( n \) and \( \lim_{n \to \infty} e_n \ast f = f \) for all \( f \in N(X) \).

**Proof:** Clearly, \( ||e_n||_{\omega} = m(H_n)^{-1} \text{dist}(H_n, H_n^C)^{-1} = q^{-n}q^{-n-1} = q^{-1} \), hence \( q||\phi_1 e_n||_{\omega} = 1 \). Clearly \( ||e_n||_{\omega} \leq 1 \) whence \( ||e_n||_{1} = 1 \). For any locally constant function \( f \) we have \( e_n \ast f = f \) for large \( n \). Consider the linear space \( V = \{ f \in N(X) : \lim e_n \ast f = f \} \). Since \( V \) contains the locally constant functions it suffices to show that \( V \) is closed. So, let \( f_n \in V \), \( \lim f_n = f \) in the \( ||||_{1} \)-norm. For any \( n, m \in \mathbb{N} \):

\[
||e_n \ast f - f||_{1} \leq ||e_n \ast f - e_n \ast f_m||_{1} + ||e_n \ast f_m - f_m||_{1} + ||f_m - f||_{1} \\
\leq ||e_n||_{1} ||f||_{1} + ||e_n \ast f_m - f_m||_{1} + ||f_m - f||_{1} \\
\leq ||f_m - f||_{1} + ||e_n \ast f_m - f_m||_{1}.
\]

Let \( \epsilon > 0 \). Then choose \( m \) such that \( ||f_m - f||_{1} < \epsilon \). Since \( f_m \in V \) there is \( N \) such that for \( n \geq N : ||e_n \ast f_m - f_m||_{1} < \epsilon \). So \( ||e_n \ast f - f||_{1} < \epsilon \) for \( n \geq N \).

As in 6.17, let \( H_n = \{ x \in X : |x| < q^{-n} \} \). Let \( \alpha \in X^\ast \), \( \text{Ker} \alpha = H_n \).

Then choose \( x \in H_n \) (if \( n = 1 \), this must be read as \( x \in X \setminus H_1 \)).

Then \( \alpha(x) \neq 1 \), \( px \in H_n \) so \( \alpha(px) = 1 \), hence \( \alpha(x)^p = 1 \). Thus, \( \alpha(x) \) is a primitive \( p \)-th root of unity, and a straightforward argument shows that \( |\alpha(x) - 1|^p = |p| = \frac{1}{q} \). We also have \( |x| = q^{-n+1} \). It follows that

\[ \frac{\alpha(x) - \alpha(0)}{x} = q^pq^{-1}q^{-p} = q^{-1}p \cdot 1. \]

If we choose \( y \notin H_{n-1} \) then \( |y| > q^{-n+2} \), \( |\alpha(y) - 1| < 1 \), whence
\[ |y^{-1}(\alpha(y)-1)| \leq q^{n-2} \leq q^{n-1} \cdot \frac{1}{p-1}. \] It follows that

\[ \max_{x \neq 0} q^{|\alpha(x)-\alpha(0)|} = q^{\frac{n}{p-1}}. \]

But since for any \( x, y \in X, x \neq y \): \(|(x-y)^{-1}(\alpha(x)-\alpha(y))| = |(x-y)^{-1}(\alpha(x-y)-1)|\),

we arrive at

**LEMMA 6.8.** Let \( \alpha \in X^\ast \), Ker \( \alpha = \{ x \in X : |x| \leq q^{-n}\} \) for \( n \in \mathbb{N} \).

Then

\[ ||\alpha||'_1 = q^{n-(p-1)} \cdot \]

**Note.** \( C^1 \)-functions on compact sets, bases in \( C^1 \), are studied for example in [1], [4], [11]. In case \( X \) is a nice subset of \( K \) one has defined other bases of \( C(X) \) (e.g. for \( X = \mathbb{Z}_p \) the functions \( \binom{x}{n} \)) and has obtained results on a connection between regularity of \( f \) and the behaviour of the coefficients. See [1], [5]. Also, the invariant integral on \( N^1(X) \) has been treated in [10], [12], [13].
One can think of two natural notions of "uniform differentiability".

Let \( f : X \to K \).

We say that \( f \in UC^1(X) \) (\( f \) is uniformly differentiable) if \( f \) is differentiable and
\[
\lim_{(x,y) \to (a,a)} \phi^1_f(x,y) = f'(a)
\]
uniformly in \( a \in X \). Equivalently, \( f \in C^1(X) \) and for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for all \( x,y,z,t \) such that the diameter of \( \{x,y,z,t\} < \delta \), we have
\[
| \overline{\phi^1_f(x,y)} - \overline{\phi^1_f(z,t)} | < \varepsilon.
\]

We say that \( f \in SUC^1(X) \) (\( f \) is strongly uniformly differentiable) if \( \phi^1_f \) can be extended to a uniformly continuous function \( \overline{\phi^1_f} \) on \( X \times X \).

Our aim is to show that these two notions are "almost" the same (see 7.4).

**Lemma 7.1.**
1. \( SUC^1(X) \subseteq UC^1(X) \subseteq C^1(X) \).
2. If \( f \in UC^1(X) \) then \( f' \) is uniformly continuous.
3. If \( X \) is compact then \( SUC^1(X) = UC^1(X) = C^1(X) \).

**Proof:** Easy.

It is easy to find an \( f \in UC^1(\mathbb{R}) \setminus SUC^1(\mathbb{R}) \). (Let \( f(x) = 0 \) if \( |x| \leq 1 \) and \( f(x) = p^{-3n} \) if \( |x| = p^n \) (\( n = 1,2,\ldots \)). Then \( \phi^1_f(x,y) = 0 \) if
\[
0 < |x-y| \leq 1, \text{ hence } f \notin UC^1(\mathbb{R}).
\]
If \( |x-y| = \delta, |x| = p^n, |y| = p^n \) then
\[
|\phi^1_f(x,0) - \phi^1_f(y,0)| = |x^{-1}f(x) - y^{-1}f(y)| = p^{3n}|x^{-1} - y^{-1}| = p^n|x-y| = \delta.p^n.
\]
Thus, for any \( \delta > 0 \), \( \sup_{|x-y| = \delta} |\phi^1_f(x,0) - \phi^1_f(y,0)| \) is not bounded:
\( f \notin SUC^1(\mathbb{R}) \).

**Lemma 7.2.** Let \( f : X \to K \).

(i) If \( f \) has bounded difference quotients then
\( f \in \text{UC}^1(X) \) implies \( f \in \text{SUC}^1(X) \).

(ii) If \( f \in \text{UC}^1(X) \) and \( f, f' \) are bounded, then \( f \in \text{BA}^1(X) \).

**Proof:** (i) Let \( M \in \mathbb{R} \) be such that \( |f(x) - f(y)| \leq M|x-y| \) for all \( x, y \in X \).

Let \( \varepsilon > 0 \). There is \( \delta' \) such that diameter \( \{x,y,z,t\} < \delta' \) implies
\[
|\overline{f_1}(x,y) - \overline{f_1}(z,t)| < \varepsilon. 
\]
Let \( \delta := \delta'(M+1)e \wedge 1 \), and let \( |x-y| < \delta \). For any \( z \in X \) we have two cases:

(a) \( |x-z| < \delta' \). Then also \( |y-z| \leq \max(|y-x|,|x-z|) < \max(\delta,\delta') = \delta' \) hence:
\[
|\overline{f_1}(x,z) - \overline{f_1}(y,z)| < \varepsilon.
\]
(b) \( |x-z| \geq \delta' \). Then
\[
|\overline{f_1}(x,z) - \overline{f_1}(y,z)| = |x-z|^{-1}|x-y| |\overline{f_1}(x,y) - \overline{f_1}(z,y)| \leq (\delta')^{-1}(M+1)e \leq \varepsilon. 
\]
So we have shown that for each \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for all \( x, y, z \in X \) for which \( 0 < |x-y| < \delta \)
\[
|\overline{f_1}(x,z) - \overline{f_1}(y,z)| < \varepsilon.
\]

Now let \( x, y, z, t \in X \) such that \( |x-y| < \delta, |z-t| < \delta \). Then
\[
|\overline{f_1}(x,z) - \overline{f_1}(y,t)| \leq |\overline{f_1}(x,z) - \overline{f_1}(y,z)| + |\overline{f_1}(y,z) - \overline{f_1}(y,t)|,
\]
both last terms being smaller than \( \varepsilon \) by the foregoing. Hence \( f \in \text{SUC}^1 \).

(ii) There exists \( \delta > 0 \) such that \( |\overline{f_1}(x,y) - f'(y)| \leq 1 \), whenever \( |x-y| < \delta \).

Hence
\[
|\overline{f_1}(x,y)| \leq ||f'||_\infty + 1 \text{ whenever } |x-y| < \delta. 
\]
If \( |x-y| \geq \delta \) then
\[
|\overline{f_1}(x,y)| \leq \delta^{-1} \sup_{x,y} |f(x) - f(y)| \leq \delta^{-1}||f||_\infty. 
\]
It follows that \( \overline{f_1}f \) is bounded.

We can strengthen 7.2 (ii) if we put an extra condition on \( X \). Let us say that \( X \) has **property (\*)** if there exists a function \( \rho : (0,\infty) \to (0,\infty) \) such that any ball in \( X \) with radius \( r > 0 \) has a diameter \( \geq \rho(r) \). (Or, equivalently, if for each \( r > 0 \) there is \( s > 0 \) such that for every \( x \in X \) there is \( y \in X \) such that \( s \leq |x-y| \leq r \). For example, if \( X \) is the union of a collection of balls in \( K \), all having the same radius, then \( X \) has property (\*).

We have:
LEMMA 7.3. Let $X$ have property $(\ast)$. Then if $f \in \mathcal{C}^1(X)$ and $f$ is bounded, then $f'$ is bounded.

Proof: There is a disjoint covering $(B_i)_{i \in I}$ of $X$, where each $B_i$ is a ball in $X$, having radius $r$ such that $x, y, z \in B_i$ implies $|\phi_1 f(x, y) - f'(z)| < 1$. Suppose $f'$ is unbounded. Then there is a sequence $x_1, x_2, \ldots$ in $X$ with $|f'(x_n)| \geq n$. Since $X$ has property $(\ast)$ there exists $s > 0$ and $y_1, y_2, \ldots \in X$ such that $s \leq |x_n - y_n| \leq r$ for each $n$. We have $|\phi_1 f(x_n, y_n) - f'(x_n)| < 1$, hence $|\phi_1 f(x_n, y_n)| = |f'(x_n)| \geq n$, so

$$s_n \leq |x_n - y_n|^n \leq |f(x_n) - f(y_n)|$$

for all $n$: $\lim_{n \to \infty} |f(x_n) - f(y_n)| = \infty$.

so $f$ is unbounded. Contradiction.

It is easy to construct an example of an $X$ (not having property $(\ast)$) and a bounded $f : X \to K$, uniformly differentiable, such that $f'$ is unbounded.

Let $\rho_1, \rho_2, \ldots \in K$ with $1 > |\rho_1| > |\rho_2| > \ldots$ and $\lim_{n \to \infty} \rho_n = 0$. For every $i \in \mathbb{N}$, let $a_i \in K$ such that $|a_i - a_j| \geq 1$ whenever $i \neq j$. Define $B_i = \{x \in K : |x - a_i| \leq |\rho_i|^2\}$, and let $X = \bigcup_{i=1}^{\infty} B_i$. Then $X$ is open, hence without isolated points. Define $f : X \to K$ as follows. If $x \in X$, then $x \in B_i$ for exactly one $i \in \mathbb{N}$. Then $f(x) = \rho_i^{-1}(x - a_i)$. Then $f$ is bounded (if $x \in B_i$ then $|f(x)| \leq |\rho_i^{-1}| |\rho_i|^2 = |\rho_i| < 1$); $f$ is uniformly differentiable (if $x, y \in X$ and $|x - y| \leq 1$ then both $x, y$ are in $B_i$ for some $i$ and $\phi_1 f(x, y) - f'(x) = \rho_i^{-1}(x - a_i) - 0$; $f'$ is unbounded ($|f'(a_i)| = |\rho_i^{-1}| \to \infty$).

We have

THEOREM 7.4. Let $f \in \mathcal{C}^1(X)$. Each of the following properties implies $f \in \mathcal{SUC}^1(X)$.

(a) $\phi_1 f$ is bounded

(b) Both $f$ and $f'$ are bounded

(c) $X$ has property $(\ast)$, $f$ is bounded.
Theorem 7.5. Let $f : X \to K$ be in UC (in SUC). Then $f$ has a unique extension $\overline{f} : \overline{X} \to K$ such that $\overline{f}$ is in UC (in SUC).

Proof: Suppose first that both $f, f'$ are bounded. Then 7.2 (ii) shows that $f$ is uniformly continuous. Of course, $f$ is uniformly continuous. Then both $f, f'$ can be extended to uniformly continuous functions $\overline{f}, \overline{f}'$ respectively, defined on $\overline{X}$. By a continuity argument we see that for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\varepsilon < |x-y| < \delta$:

$$|(x-y)^{-1}(\overline{f}(x) - \overline{f}(y)) - \overline{f}'(y)| < \varepsilon.$$  

Thus, $\overline{f} \in UC, \overline{f}' = \overline{f}'$. Since $f, f'$ are bounded we have (7.4) $f \in SUC$. Also $\overline{f}$ has bounded difference quotients, so $\overline{f} \in SUC$.

To prove the general case, let $f \in UC$. Then there exists $r > 0$ such that $|x-y| < r, |x-z| < r, x \neq y$ implies $|(x-y)^{-1}(f(x)-f(y))-f'(z)| < 1$.

We can cover $X$ with disjoint balls $(B_i)_{i \in I}$ each having radius $r$. Let $z \in B_i$. For each $x, y \in B_i$ we have $|(x-y)^{-1}(f(x)-f(y))| \leq \max(|f'(z)|, 1)$, hence $f$ has bounded difference quotients on each $B_i$. Thus, $f|B_i$ can be extended to an SUC-function $f_i : \overline{B_i} \to K$. Since $d(B_i, B_j) \geq r$ if $i \neq j$ we have: $X = \bigcup_{i \in I} B_i$. Now define $\overline{f} : \overline{X} \to K$ in the obvious way: if $x \in B_i$, then $\overline{f}(x) = f_i(x)$. Obviously $\overline{f}$ is in UC.

If $f \in SUC$ then $\Phi_1 f$ is uniformly continuous on $X \times X$, and $\Phi_1 \overline{f}$ is continuous on $\overline{X} \times \overline{X}$. Since $X \times X$ is dense in $\overline{X} \times \overline{X}$ and since $\Phi_1 \overline{f}$ is an extension of $\Phi_1 f$ it follows, by a simple continuity argument, that $\overline{f} \in SUC$. 

Proof: 7.2 and 7.3.
Next, we look at the space $BUC^1(X)$, consisting of all $f \in UC^1(X)$ for which

$$||f||_1 := ||f||_{\infty} \vee ||f'||_{\infty} < \infty.$$ 

By 7.4 it follows that an $f \in BUC^1(X)$ is strongly uniformly continuous.

Clearly, $BUC^1(X)$ is a normed linear space.

Let $BUN^1(X) = \{f \in BUC^1(X) : f' = 0\}$.

**Theorem 7.6.** $BUC^1(X)$ is a Banach space with respect to $|| \cdot ||_1$. Differentiation is a continuous linear map: $BUC^1(X) \rightarrow BUC(X)$ with norm 1.

**Proof:** Similar to 5.1.

**Theorem 7.7.** Let $f \mapsto \overline{f}$ be the extension map $UC^1(X) \rightarrow UC^1(\overline{X})$. Then it induces an isomorphism of Banach spaces:

$$BUC^1(X) \cong BUC^1(\overline{X}).$$

**Proof:** Obvious.

Let $f \in BUC(X)$ and let $P$ be the antiderivation map as in 5.4. We expect $Pf$ to be in $BUC^1(X)$. To prove this we only have to show that $Pf$ is uniformly differentiable, and this can be done by using the proof of the first part of theorem 5.4 (only a few modifications are to be made). We leave the details to the reader.

**Theorem 7.8.** Let $f \in BUC(X)$. Then (with $P$ as in 5.4) $Pf \in BUC^1(X)$ and $Pf$ is an antiderivative of $f$. The map $P : BUC(X) \rightarrow BUC^1(X)$ is a linear isometry. In the commutative diagram

$$\begin{array}{c}
BUC^1(X) \xrightarrow{D} BUC(X) \\
\pi \downarrow \quad \downarrow \rho \\
BUC^1(X)/BUN^1(X)
\end{array}$$
(where \( D \) is differentiation and \( \pi \) is the quotient map) we have that \( D \) is surjective and \( p \) is an isomorphism of Banach spaces. \( \text{BUN}^1(X) \) is complemented in \( \text{BUC}^1(X) \). In fact, we have an orthogonal decomposition
\[
\text{BUC}^1(X) = \text{BUN}^1(X) \oplus \text{im} \; p,
\]
given by \( f \leftrightarrow (f-\text{PDF}, \text{PDF}) \).

**Proof:** Similar to 5.4 and 5.5.

Also theorem 5.2 has its "uniform" counter part:

**THEOREM 7.9.** Let \( f \in \text{UC}^1(X) \) and let \( \varepsilon > 0 \). Then there exists a locally linear function \( g \in \text{UC}^1(X) \) such that \( ||f-g||_1 < \varepsilon \).

**Proof:** \( X \) can be covered by disjoint balls \( B_i \) in \( X \) such that all \( B_i \) have the same radius \( r \) and such that for all \( x,y,z \in B_i \) we have
\[|\partial f(x,y)-f'(z)| < \varepsilon.\]
Choose \( a_i \in B_i \) for each \( i \) and define \( g : X \to K \) as follows. For each \( i \) set
\[g(x) = f(a_i)-(x-a_i)f'(a_i) \quad (x \in B_i).\]
Then \( g \) is locally linear and we can use the proof of 5.2 to show that \( ||g-f||_1 < \varepsilon \). It remains to show that \( g \in \text{UC}^1 \). But this is clear since for \( x,y \in X \), \( |x-y| < r \) we have \( x,y \in B_i \) for some \( i \) and
\[\partial f(x,y) = f'(a_i) = f'(x).\]

**COROLLARY 7.10.** The locally linear functions in \( \text{BUC}^1(X) \) form a dense subset of \( \text{BUC}^1(X) \). The locally constant functions in \( \text{BUC}^1(X) \) form a dense subset of \( \text{BUN}^1(X) \).

**Proof:** Let \( f \in \text{BUC}^1(X) \) and \( \varepsilon > 0 \). By 7.9 there is \( g \in \text{UC}^1(X) \) for which \( ||f-g||_1 < \varepsilon \), \( g \) locally linear. It follows that \( ||g||_1 < \varepsilon \) hence \( g \in \text{BUC}^1(X) \). If \( f' = 0 \) then the function \( g \) chosen in the proof of 7.9 is locally constant.
As an example we will discuss here the analytic functions, only in as much they are of interest for our theory.

Let \( f : X \to K \). \( f \) is called **analytic** if there exists a \( a \in X \) and a sequence \( \lambda_0, \lambda_1, \ldots \in K \) such that

\[
(*) \quad f(x) = \sum_{n=0}^{\infty} \lambda_n (x-a)^n \quad (x \in X)
\]

We expect of course that if \( f \) is analytic and if \( b \in X \) then

\[
f(x) = \sum_{n=0}^{\infty} \mu_n (x-b)^n
\]

for some sequence \( \mu_0, \mu_1, \ldots \) in \( K \). This can be proved as follows. From (*) it follows that \( \lim_{n \to \infty} |\lambda_n| |x-a|^n = 0 \). For each \( x \in X \):

\[
|\lambda_n (x-b)^k (b-a)^{n-k}| \leq |\lambda_n| \max(|x-b|,|b-a|)^n \to 0,
\]

from which it follows that \( \sum_{n,k} \lambda_n (x-b)^k (b-a)^{n-k} \) converges for all \( x \in X \).

Hence

\[
f(x) = \sum_{n=0}^{\infty} \lambda_n (x-a)^n = \sum_{n=0}^{\infty} \lambda_n (x-b+b-a)^n = \sum_{n,k} \lambda_n (x-b)^k (b-a)^{n-k} = \sum_{k} \mu_k (x-b)^k
\]

where \( \mu_k = \sum_{n} \lambda_n (b-a)^{n-k} \).

It follows that without loss of generality we may assume that \( 0 \in X \) and that an analytic function \( f \) on \( X \) has the form \( x \mapsto \sum_{n} \lambda_n x^n \).

Define \( V = \{ x \in K : \sum_{n} \lambda_n x^n \text{ is convergent} \} \). Then \( V \supset X \) and

\[
V = \{ x \in K : \lim_{n \to \infty} \lambda_n x^n = 0 \}. \quad \text{Define } \rho = (\lim_{n \to \infty} \sqrt[n]{|\lambda_n|})^{-1} \quad (\text{N.B. } 0^{-1} = \infty).
\]

As in the classical case, \( \sum_{n} \lambda_n x^n \) converges for \( |x| < \rho \), diverges for \( |x| > \rho \).

In our non-archimedean situation we have, in addition, that on

\( \{ x \in K : |x| = \rho \} \) either \( \sum_{n} \lambda_n x^n \) is convergent or \( \sum_{n} \lambda_n x^n \) is divergent at all points of the set. It turns out that our \( f : X \to K \) can be extended to an analytic function \( f : V \to K \), where \( V \supset X \) and where \( V \) has the form \( \{ x \in K : |x| \leq \rho \} \) or \( \{ x \in K : |x| < \rho \} \) for certain \( \rho \in (0,\infty] \).

We have:
THEOREM 7.11. Let $p \in \mathbb{K}$, $p \neq 0$ and let $X = \{x \in \mathbb{K} : |x| < p\}$. Then an analytic function $f : X \to \mathbb{K}$ is in $BUC^1(X)$.

Proof: Let $f(x) = \sum \lambda_n x^n$ ($x \in X$). Since $\lim \lambda_n \rho^n = 0$, we have $|f(x)| = \max |\lambda_n| |x|^n \leq \max |\lambda_n| \rho^n < \infty$ for all $x \in X$. Hence $f$ is bounded. Further, for any $h \in X$

$$|\frac{\phi_1 f(x+h, x)}{n} - \sum \lambda_n x^{n-1}| \leq |h| \max \lambda_n \rho^{n-2},$$

so $f$ is uniformly differentiable. Either 7.2 and 7.3 or a direct proof shows that $\phi_1 f$ is bounded. It follows that $f \in BUC^1(X)$. 

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8. n TIMES CONTINUOUSLY DIFFERENTIABLE FUNCTIONS

In Chapter 1 we have seen that in order to define the space $C^1(X)$ it is more convenient to take \{f : X \to K : \phi_1f\text{ is continuously extendable}\} rather than \{f : X \to K : f'\text{ is continuous}\}. We will define here $C^n$-functions using the same basic idea. First we define difference quotients of higher orders:

For $n \in \{1,2,\ldots\}$, let

$$\nabla^n X = \{(x_1,x_2,\ldots,x_n) \in X^n : x_i \neq x_j \text{ whenever } i \neq j\}.$$  

Then $\nabla^1 X = X$, $\nabla^2 X = X \times X^2$, etc. Since $X$ does not have isolated points, $\nabla^n X$ is dense in $X^n$.

Let $f : X \to K$. We define $\phi_n f : \nabla^{n+1} X \to K$ inductively as follows.

$$\phi_0 f = f,$$

and for any $(x_1,x_2,\ldots,x_{n+1}) \in \nabla^{n+1} X$,

$$\phi_n f(x_1,\ldots,x_{n+1}) = (x_1-x_2)^{-1}(\phi_{n-1} f(x_1,x_3,\ldots,x_{n+1}) - \phi_{n-1} f(x_2,x_3,\ldots,x_{n+1})).$$

We may call $\phi_n f$ the $n^{th}$ difference quotient of $f$. (It is clear that for $n = 1$ this definition of $\phi_1 f$ coincides with the one in Chapter 1).

DEFINITION 8.1. Let $f : X \to K$, and let $n \in \mathbb{N} U \{0\}$. We say that $f \in C^n(X)$ if $\phi_n f$ can (uniquely) be extended to a continuous function $\overline{\phi_n} f$ on $X^n$. We say that $f \in BA^n(X)$ if $\phi_0 f, \phi_1 f, \ldots, \phi_n f$ are bounded functions. For $f \in BA^n(X)$ we put

$$||f||_n = \max_{0 \leq i \leq n} ||\phi_i f||_\infty.$$  

Further, let $BC^n(X) = BA^n(X) \cap C^n(X)$.

Finally, let $C^\infty(X) = \cap_{n=1}^\infty C^n(X)$.
Before starting an investigation of $C^n$-functions we first develop some computational machinery concerning $\Phi_n$.

**Lemma 8.2. (Rules for $\Phi_n$.)** Let $f, g : X \to K$ and let $\lambda, \mu \in K$, $n \in \mathbb{N}$.

I. For $(x, y, z, x_1, \ldots, x_{n-1}) \in \mathbb{V}^{n+2}X$ we have:
\[
(x-y) \Phi_n f(x, y, x_1, \ldots, x_{n-1}) + (y-z) \Phi_n f(y, z, x_1, \ldots, x_{n-1}) = (x-z) \Phi_n f(x, z, x_1, \ldots, x_{n-1})
\]

II. $\Phi_n f$ is a symmetric function of $n+1$ variables.

III. For $(x_1, \ldots, x_{n+1}, a_1, \ldots, a_{n+1}) \in \mathbb{V}^{2n+2}X$ we have
\[
\Phi_n f(x_1, \ldots, x_{n+1}) - \Phi_n f(a_1, \ldots, a_{n+1}) = \sum_{i=1}^{n+1} (x_i - a_i) \Phi_{n+1} f(a_1, \ldots, a_i, \ldots, x_{n+1})
\]

IV. $\Phi_n (\lambda f + \mu g) = \lambda \Phi_n f + \mu \Phi_n g$.

V. For $(x_1, \ldots, x_{n+1}) \in \mathbb{V}^{n+1}X$ we have
\[
\Phi_n (fg)(x_1, \ldots, x_{n+1}) = \sum_{k=0}^{n} \sum_{l=1}^{n+1} \Phi_k f(x_1, \ldots, x_k, x_{k+1}, \ldots, x_{n+1}) \Phi_{n-k} g(x_{k+1}, \ldots, x_{n+1})
\]

VI. For $(x_1, \ldots, x_{n+1}) \in \mathbb{V}^{n+1}X$ we have
\[
\Phi_n f(x_1, \ldots, x_{n+1}) = \prod_{i=1}^{n+1} (x_i - x_j)^{-1} f(x_i)
\]

VII. If $f(x) \neq 0$ for all $x \in X$, then with $g = \frac{1}{f}$, $(x_1, \ldots, x_{n+1}) \in \mathbb{V}^{n+1}X$
\[
\Phi_n g(x_1, \ldots, x_{n+1}) = -f(x_1)^{-1} \sum_{k=0}^{n} \Phi_k f(x_1, \ldots, x_{k+1}) \Phi_{n-k} g(x_{k+1}, \ldots, x_{n+1})
\]

VIII. If $f$ is locally constant then $\Phi_n f = 0$ in a neighborhood of the diagonal.

IX. $\Phi_{n+1} f = 0 \iff f$ is a polynomial function of degree $\leq n$.

X. If $0 \notin X$ and $f(x) = \frac{1}{x}$ for all $x \in X$, then for $(x_1, \ldots, x_{n+1}) \in \mathbb{V}^{n+1}X$
\[
\Phi_n f(x_1, \ldots, x_{n+1}) = (-1)^n \prod_{i=1}^{n+1} x_i^{-1}
\]
Proof: Most of the rules I-X can be proved by induction. We will make some remarks and leave the details to the reader. I is a direct consequence of the definition of $\phi_n f$. The proof of II runs by induction.

The hypothesis that $\phi_{n-1} f$ be symmetric implies the invariance of $\phi_n f(x_1, x_2, \ldots, x_{n+1})$ under permutations of $x_3, \ldots, x_{n+1}$ and of $x_1, x_2$.

Thus it suffices to show that $\phi_n f(x_1, x_2, x_3, \ldots) = \phi_n (x_3, x_2, x_1, \ldots)$ and this is a consequence of I. To prove III, write

$$\phi_n f(x_1, \ldots, x_{n+1}) - \phi_n (a_1, \ldots, a_{n+1}) = \phi_n f(x_1, \ldots, x_{n+1}) - \phi_n f(a_1, x_2, \ldots, x_{n+1}) + \phi_n f(a_1, x_2, \ldots, x_{n+1}) - \phi_n f(a_1, a_2, x_3, \ldots) + \ldots$$ etc. and use the definition of $\phi_{n+1} f$. V, VI go by induction and VII is an application of V and the fact that $\phi_n (fg)(x_1, \ldots, x_{n+1}) = 0$ for $n \geq 1$. To prove VIII, let $a \in X$.

There exists $\delta > 0$ such that $|z - a| < \delta$ implies $f(z) = f(a)$. So if $(x_1, \ldots, x_{n+1}) \in \mathbb{V}^{n+1} X$, $|x_i - a| < \delta$ for all $i$, then $f(x_i) = f(a)$ for all $i$, so, by VI, $\phi_n f(x_1, \ldots, x_{n+1}) = 0$. $X$ is straightforward. We finally prove IX.

Let $X^n$ denote the function $x \mapsto x^n$. By using the product rule V we arrive easily at $\phi_n X^n = 1$, so if $f$ is a polynomial function of degree $\leq n$, then $\phi_{n+1} f = 0$. Conversely if $\phi_{n+1} f = 0$, then $\phi_n f$ is a constant $c$, so $\phi_n (f - cX^n) = 0$. By the induction hypothesis, $f - cX^n$ is a polynomial function of degree $\leq n-1$.

**Theorem 8.3.** Let $n \in \mathbb{N}$. Then $B^n(X)$ is a Banach space with respect to the norm $||f|| = \sup_{0 \leq k \leq n} \max_{(x_1, \ldots, x_{k+1}) \in \mathbb{V}^{k+1} X, \|x_i - x_j\| \leq 1 \text{ all } i, j}$.

**Proof:** The map $f \mapsto (f, \phi_1 f, \phi_2 f, \ldots, \phi_n f)$ is a linear isometry of $B^n(X)$ into $\prod_{k=1}^{n+1} B(\mathbb{V}^k X)$. (Here, for a space $X$, $B(Y)$ denotes the Banach space of...
all bounded functions $Y \to K$, under the sup norm). An argument, similar to that of 3.16, shows that the image of this isometry is closed in the product space.

We prove the second statement by induction on $n$. If $|x-y| \geq 1$ and $f \in B\Delta_1(X)$, then $|\phi_1 f(x,y)| = |(x-y)^{-1}(f(x)-f(y))| \leq |f(x)-f(y)| \leq ||f||_\infty$.

Suppose the statement is true for $n-1$. Then let $f \in B\Delta^n(X)$,

$$(x_1,x_2,\ldots,x_{n+1}) \in \mathcal{V}^{n+1}X \text{ and, say, } |x_1-x_2| \geq 1. \text{ Then } |\phi_n f(x_1,x_2,\ldots,x_{n+1})| =$$

$$= |(x_1-x_2)^{-1}| \cdot |\phi_{n-1} f(x_1,x_3,\ldots,x_{n+1})-\phi_{n-1} f(x_2,x_3,\ldots,x_{n+1})| \leq |\phi_{n-1} f||_\infty$$

$$\leq ||f||_{n-1}, \text{ which proves the theorem.}$$

If $f \in C^n(X)$ for some $n \geq 1$, then by rule III we have for

$$(x_1,x_2,\ldots,x_n,a_1,\ldots,a_n) \in \mathcal{V}^{2n}X$$

$$\phi_{n-1} f(x_1,x_2,\ldots,x_n) = \phi_{n-1} f(a_1,\ldots,a_n) + \sum_{i=1}^n (x_i-a_i) \phi f(a_1,\ldots,a_i,x_{i+1},\ldots,x_n)$$

so that $\phi_{n-1} f$ can be extended to a continuous function $\overline{\phi}_{n-1} f$ on $X^{n-1}$.

Thus we see that

$$(* \quad C(X) \supset C^1(X) \supset C^2(X) \supset \ldots$$

By a simple continuity argument we may conclude that the rules I, II, IV, V, VII, VIII, IX, X remain valid for functions $f,g \in C^n(X)$, where $\phi_k$ is replaced by $\overline{\phi}_k$ and $\mathcal{V}^kX$ by $X^k$ for all occurring $k$. It follows that $C^n(X)$ is a linear space (IV), closed under products (V) and that locally constant functions and polynomial functions are in $C^\infty(X)$ (VIII and IX). If $f \in C^n(X)$ for some $n$ and $f(x) \neq 0$ for all $x \in X$ then $\frac{1}{f} \in C^n(X)$ (VII).

**DEFINITION 8.4. Let $f \in C^n(X)$ for some $n \geq 1$. We define for $0 \leq j \leq n$ the $j$th Hesse derivative of $f$ by**

$$D^j f(x) = \phi_j f(x,x,\ldots,x) \quad (x \in X)$$

(For properties of $D^j f$ e.g., the connection with the ordinary $j$th derivative, see 8.14).
We first turn to $\text{BC}^n(X)$. Since, for fixed $f$, $n \mapsto ||f||_n$ is an increasing sequence we derive from (*)

(**) $\text{BC}(X) \supseteq \text{BC}^1(X) \supseteq \text{BC}^2(X) \supseteq \ldots$

**Theorem 8.5.** Let $n \in \mathbb{N}$. With respect to $||\cdot||_n$, $\text{BC}^n(X)$ is a Banach algebra under pointwise operations.

**Proof:** The completeness follows from the fact that $\text{BC}^n(X)$ is closed in $\text{BA}^n(X)$ and 8.3. (If $f_n \rightarrow f$ in norm and $f_m \in C^n(X)$ for all $m$, then $\overline{f}_n \rightarrow \overline{f}$ uniformly, hence $\lim_{m \rightarrow \infty} \overline{f}_m$ exists uniformly and is a continuous extension of $\overline{f}_n : f \in C^n(X)$). Further, for any $j$ with $0 \leq j \leq n$ we have by $\forall$ for $(x_1, \ldots, x_{j+1}) \in X^{j+1}$

$$||\overline{f}(fg)(x_1, \ldots, x_{j+1})|| \leq \max_{0 \leq k \leq j} ||\overline{f}_k||_\infty ||\overline{f}_{j-k}||_\infty$$

whence

$$||\overline{f}(fg)||_\infty \leq ||f||_j ||g||_j,$$

$(0 \leq j \leq n)$

from which it follows that

$$||fg||_n \leq ||f||_n ||g||_n.$$

There is a connection between $\text{BA}^n(X)$ and $\text{BC}^n(X)$. If $f \in \text{BA}^{n+1}(X)$, then, by rule III, we have for $(a_1, a_2, \ldots, a_{n+1}, x_1, \ldots, x_{n+1}) \in X^{2n+2}$

$$||\overline{f}(x_1, \ldots, x_{n+1}) - \overline{f}(a_1, \ldots, a_{n+1})|| \leq ||f||_{n+1} \max_i ||x_i - a_i||.$$

hence $\overline{f}$ is uniformly continuous and it has therefore a continuous extension $\overline{f}$ to $X^{n+1}$. So we have

$$\text{BA}^1(X) \supseteq \text{BC}^1(X) \supseteq \text{BA}^2(X) \supseteq \text{BC}^2(X) \supseteq \ldots$$

and obviously

$$\text{BC}^\infty(X) := \cap \text{BC}^n(X) = \cap \text{BA}^n(X).$$

If $X$ is compact then $C^\infty(X) = \text{BC}^\infty(X)$, $C^n(X) = \text{BC}^n(X)$ for all $n$. 
**Lemma 8.6.** Let $f \in BC^n(X)$. Then

$$
\|f\|_n = \max_{0 \leq i \leq n-1} \|D_i f\|_\infty \vee \|\phi_i f\|_\infty.
$$

**Proof:** Let $0 \leq k < n$. Then for any $(x_1, \ldots, x_{k+1}) \in X^k$ for which (8.3)

$$
|x_i - x_j| \leq 1 \text{ for all } i, j; \quad \left| \frac{\phi_k f(x_1, \ldots, x_{k+1})}{x_i} \right| \leq \left| \phi_k f(x_1, \ldots, x_{k+1}) - \phi_k f(x_1, \ldots, x_1) \right| \vee \left| \phi_k f(x_1, \ldots, x_{k+1}) \right| \leq \left( \sum_{j=2}^{k+1} \left| x_j - x_1 \right| \phi_k f(x_1, \ldots, x_1, x_j, \ldots, x_{k+1}) \right) \vee \left| D_k f \right|_\infty \leq \sup \left\{ \left| \phi_{k+1} f(x_1, \ldots, x_{k+2}) \right| : \left| x_i - x_j \right| \leq 1 \text{ all } i, j \right\} \vee \left| D_k f \right|_\infty.
$$

Now use induction.

It is possible to put a natural locally convex topology on $C^n(X)$, similar to what we defined for $C^1(X)$ in Chapter 5.

For $f \in C^n(X)$ and a compact $C \subset X$ let

$$
\|f\|_{n,C} = \max \{ \|\phi_i f(u)\| : u \in C^{i+1} \} \quad (0 \leq i \leq n)
$$

and

$$
\|f\|_{n,C} = \max \{ \|\phi_i f\|_C : 0 \leq i \leq n \}.
$$

The seminorms $f \mapsto \|f\|_{n,C}$, where $C$ runs through the (non-empty) compact subsets of $X$ define a locally convex topology on $C^n(X)$. Unless otherwise stated we assume $C^n(X)$ to be equipped with this topology. In case $X$ itself is compact then $BC^n(X) = C^n(X)$ and the above topology coincides with the norm topology. It is easy to prove that $C^n(X)$ is complete. We leave the proof to the reader.

**Theorem 8.7.** (Taylor Formula for $C^n$-functions). Let $f \in C^n(X)$. Then

for all $x, y \in X$

$$
f(x) = f(y) + (x-y)D_1 f(y) + \ldots + (x-y)^{n-1}D_{n-1} f(y) + (x-y)^n \phi_n f(x, y, y, \ldots, y).
$$

**Proof:** For $n = 1$ it is clear. Suppose the theorem is true for $n-1$.

Then let $f \in C^n(X)$. Then also $f \in C^{n-1}(X)$ and
\[ f(x) = \sum_{i=0}^{n-1} (x-y)^i \Phi_i f(y) + (x-y)^n \frac{\partial}{\partial y} f(x, y, \ldots, y) \] for all \( x, y \in X \).

Now \( \Phi_n f(x, y, \ldots, y) - \frac{\partial}{\partial y} f(x, y, \ldots, y) = (x-y)^n \Phi_n f(x, y, \ldots, y) \), which proves the theorem.

One may ask whether there exists a converse of theorem 8.7 in the following sense. Suppose we have \( f : X \to K \) such that there exist continuous functions \( \Phi_1 f, \ldots, \Phi_{n-1} f : X \to K \) and a continuous \( R_n : X \times X \to K \) such that for all \( x, y \in X \)

\[ f(x) = f(y) + (x-y)\Phi_1 f(y) + \cdots + (x-y)^{n-1} \Phi_{n-1} f(y) + (x-y)^n R_n(x, y). \]

Then does it follow that \( f \in C^n(X) \)? Lemma 1.6 shows that the answer is "yes" in case \( n = 1 \). Also for \( n = 2 \) the answer is "yes".

**Lemma 8.8.** Let \( f : X \to K \) and suppose there exists a continuous \( \Phi_1 f : X \to K \) and a continuous \( R_2 : X \times X \to K \) such that for all \( x, y \in X \)

\[ f(x) = f(y) + (x-y)\Phi_1 f(y) + (x-y)^2 R_2(x, y). \]

Then \( f \in C^2(X) \), \( \Phi_1 f = \Phi_2 f \), \( R_2(x, y) = R_2(y, x) \) for all \( x, y \in X \).

**Proof:** First observe that \( f \in C^1(X) \), hence \( \Phi_1 f = \Phi_1 f \). Let \( a \in X \) and let \( \varepsilon > 0 \). Then there exists \( \delta > 0 \) such that if \( |x-a| < \delta, |y-a| < \delta \) then \( |R(x, y) - R(a, a)| < \varepsilon \). Now let \( |x-a| < \delta, |y-a| < \delta, |z-a| < \delta \), \( (x, y, z) \in V^3 X \). We prove: \( |\Phi_2 f(x, y, z) - R(a, a)| < \varepsilon \). Without loss of generality we may assume \( \max \{|x-y|, |x-z|\} \leq |y-z| \). \( \Phi_1 f(x, y) = D_1 f(x) + (y-x) R_2(y, x) \), \( \Phi_1 f(x, z) = D_1 f(x) + (z-x) R_2(z, x) \), hence

\[ \Phi_2 f(x, y, z) = (y-z)^{-1}(\Phi_1 f(x, y) - \Phi_1 f(x, z)) = \frac{y-x}{y-z} R_2(y, x) + \frac{x-z}{y-z} R_2(z, x), \]

so

\[ |\Phi_2 f(x, y, z) - R(a, a)| = |\frac{y-x}{y-z} (R_2(y, x) - R(a, a)) + \frac{x-z}{y-z} (R_2(z, x) - R(a, a))| \leq \varepsilon. \]

It follows that \( \Phi_2 f \) is continuously extendable to \( \Phi_2 f : X^3 \to K \), hence \( f \in C^2(X) \). The rest is easy.
For $n = 3$, however, our problem has a negative solution in the following sense

**EXAMPLE 8.8**. There exists a closed $X \subset \mathbb{Z}_p$ and an $f : X \to \mathbb{Q}_p$ such that $f \in C^2(X)$, $f' = 0$, but for which

$$\lim_{(x,y) \to (a,a)} \frac{f(x)-f(y)}{(x-y)^3} = 1$$

for all $a \in X$. [Thus there is a continuous function $R_3 : X^2 + K$ such that for all $x, y \in X$ we have $R_3(x,x)=1$ and

$$f(x) = f(y) + (x-y)^3 R_3(x,y)$$.]

(If $f$ were in $C^3(X)$ then (8.14): $f'''(x)=3! R_3(x,x) = 6$ which is impossible since $f'=0$).

**Proof**: Let

$$X = \{ \sum_{n=0}^{\infty} a_n p^n! : a_n \in \{0,1\} \}$$. Then $X$ is closed and without isolated points. Define $f : X \to \mathbb{Q}_p$ as follows

$$f(\sum_{n=0}^{\infty} a_n p^n!) = \sum_{n=0}^{\infty} a_n p^3n! \quad (\sum_{n=0}^{\infty} a_n p^n! \in X)$$.

Then $|f(x)-f(y)| = |x-y|^3$ for all $x,y \in X$, so $f \in C^2(X)$, $f' = 0$.

Let $x,y \in X$ such that $|x-y| = p^{-k}$ for some $k \in \mathbb{N}$. We will show that

$$|\frac{f(x)-f(y)}{(x-y)^3} - 1| \leq p^{-k.k!}$$

In fact, let $x = \sum_{n=0}^{\infty} a_n p^n!$, $y = \sum_{n=0}^{\infty} b_n p^n!$, then $f(x)-f(y) = \sum_{n=k}^{\infty} (a_n-b_n)p^{3n!} = (a_k-b_k)p^{3k!} + \sum_{n>k} (a_n-b_n)p^{3n!} = s_k u_k + v_k$ where $|s_k| = p^{-3k!}$, $|u_k| \leq p^{-3(k+1)!}$.

Now $(x-y)^3 = (\sum_{n=0}^{\infty} (a_n-b_n)p^n!)^3 = (a_k-b_k)^3 p^{3k!} + v_k'$ where $|v_k'| \leq p^{-k!(k+3)}$.

Since $a_k, b_k \in \{0,1\}$ we have $a_k-b_k \in \{-1,0,+1\}$ so $(a_k-b_k)^3 = a_k-b_k$.

So we find

$$\frac{f(x)-f(y)}{(x-y)^3} = \frac{s_k u_k}{s_k + v_k}$$

hence,

$$|\frac{f(x)-f(y)}{(x-y)^3} - 1| = \frac{|s_k u_k|}{s_k + v_k} \leq p^{-k.k!}$$
We write for $s > 3$:

$$f(x) = f(y) + (x-y)^3 + (x-y)^s \frac{\partial}{\partial y} (x,y) (x,y \in X, x \neq y)$$

and we show that the $R_s$ defined this way can be extended continuously to $R_s : X^2 \to K$ and that $R_s(x,x) = 0$ for all $x$. From above we know:

$$\text{if } |x-y| = p^k \text{ then } \frac{|f(x) - f(y) - (x-y)^3|}{(x-y)^s} \leq p^{-k} k!.$$ 

Hence

$$\lim_{x-y \to 0} \frac{f(x) - f(y) - (x-y)^3}{(x-y)^s} = 0. \text{ Thus, our function } f \text{ is in } C^3(X) \text{ for all } n \in \mathbb{N}, D_1 f = 0 \text{ for } i = 1,2,4,5,\ldots \text{ and } D_3 f = 1, \text{ whereas } f \text{ fails to be in } C^3(X). \text{ (See 10.1 for the definition of } C^n(X).)$$

It is not accidental that the above set $X$ turns out to be more or less pathological. In Chapter 10 (Theorem 10.7) we will show that we do have a "converse" of 8.7 for a wide class of sets $X$ (including the open sets).

To conclude this first chapter on $C^n$-functions we try to find proofs for simple-looking statements such as

- $f \in C^n$ implies $f' \in C^{n-1}$

- $f \in C^n$ implies $f$ is $n$ times differentiable and $f^{(n)} = \frac{1}{n!} D_n f$. (We should write $n! f^{(n)} = D_n f$ to have any chance since the characteristic of $K$ may be $0$.)

For functions : $IR \to IR$ these statements are trivial. In the non-archimedean case however we have to do a little work.

**Lemma 8.9.** Let $f \in C^{n-1}(X)$ for some $n \in \mathbb{N}$. Then $\phi_n f$ can be extended to
a continuous function \( \tilde{\phi}_n f \) on \( \mathbb{R}^{n+1} \setminus \Delta \). (Here, \( \Delta = \{(x,x,\ldots,x) \in \mathbb{R}^{n+1} : x \in \mathbb{R}\} \).}

**Proof:** For each \( i, j \) with \( 1 \leq i, j \leq n+1 \) and \( i \neq j \) define

\[
U_{ij} = \{(x_1,\ldots,x_{n+1}) \in \mathbb{R}^{n+1} : x_i \neq x_j\}.
\]

Then each \( U_{ij} \) is open in \( \mathbb{R}^{n+1} \) and \( \bigcup_{i,j} U_{ij} = \mathbb{R}^{n+1} \setminus \Delta \). Define \( h_{ij} : U_{ij} \to \mathbb{R} \) via

\[
h_{ij}(x_1,\ldots,x_{n+1}) = (x_i-x_j)^{-1}[\tilde{\phi}_n f(x_1,\ldots,x_{j-1},x_j+1,\ldots,x_{n+1}) - \tilde{\phi}_n f(x_1,\ldots,x_{i-1},x_i+1,\ldots,x_{n+1})].
\]

Then \( h_{ij} \) is a continuous extension of \( \tilde{\phi}_n f \). Define \( \tilde{\phi}_n f(x_1,\ldots,x_{n+1}) = h_{ij}(x_1,\ldots,x_{n+1}) \) whenever \( (x_1,\ldots,x_{n+1}) \in U_{ij} \). It is an easy matter to show that \( \tilde{\phi}_n f \) is well-defined. Since for each \( i,j \) \((i \neq j, 1 \leq i,j \leq n+1)\), \( \tilde{\phi}_n f|U_{ij} = h_{ij} \) is continuous and \( U_{ij} \) is open, we have that \( \tilde{\phi}_n f \) is continuous.

**COROLLARY 8.10.** Rule III of 8.2 has the following extension. If \( f \in C^n(\mathbb{R}) \) then for all \( (x_1,\ldots,x_{n+1}) \in \mathbb{R}^{n+1}, (a_1,\ldots,a_{n+1}) \in \mathbb{R}^{n+1} \)

\[
\tilde{\phi}_n f(x_1,\ldots,x_{n+1}) - \tilde{\phi}_n f(a_1,\ldots,a_{n+1}) = \sum_{i} (x_i-a_i)^{n+1} f(a_1,\ldots,a_i x_i,\ldots,x_{n+1}).
\]

**Proof:** The formula is true for \((x_1,x_2,\ldots,x_{n+1},a_1,\ldots,a_{n+1}) \in \mathbb{R}^{2n+2} \). Both the left and right hand side of the formula defines a continuous function on \((x_1,x_2,\ldots,x_{n+1},a_1,a_2,\ldots,a_{n+1}) \in \mathbb{R}^{2n+2} : x_i \neq a_i \) for all \( i \). Because of

\[
\lim_{x_i \to a_i} \tilde{\phi}_n f(a_1,\ldots,a_i x_i,\ldots,x_{n+1}) = \lim_{x_i \to a_i} (\tilde{\phi}_n f(a_1,\ldots,a_{i-1},x_i,\ldots,x_{n+1}) - \tilde{\phi}_n f(a_1,\ldots,a_i x_i,\ldots,x_{n+1})) = 0
\]

it follows that both the left and right hand expression is continuous on \( \mathbb{R}^{2n+2} \).
LEMMA 8.11. Let $f : X \to K$ and $n \in \mathbb{N} \cup \{0\}$. Then $f \in C^n(X)$ if and only if for each $a \in X$

$$\lim_{(x_1', \ldots, x_{n+1}') \to (a, a, \ldots, a)} (\phi_n f|_n^{n+1} X)(x_1', \ldots, x_{n+1}')$$

exists.

Proof: We only need to prove the "if" part of the lemma. For $n = 0$ it is trivial. Suppose $n > 1$ and let $f$ be such that the limit mentioned above exists.

There exists $\delta > 0$ such that $(x_1, \ldots, x_{n+1}) \in \nu^{n+1} X$, $|x_i - a_i| < \delta$ for all $i$ implies $|\phi_n f(x_1, \ldots, x_{n+1})| \leq 1$. For any $(p_1, \ldots, p_n, q_1, \ldots, q_n) \in \nu^{2n} X$ for which $|p_i - a_i| < \delta$, $|q_i - a_i| < \delta$ for all $i$, we have by rule III

$$|\phi_{n-1} f(p_1, \ldots, p_n) - \phi_{n-1} f(q_1, \ldots, q_n)| \leq |\Sigma (p_i - q_i) \phi_n f(p_1', \ldots, p_n', q_1', \ldots, q_{n+1}')| \leq \max |p_i - q_i|.$$  

Hence $\phi_{n-1} f$ is uniformly continuous, and hence $f \in C^{n-1}(X)$. By 8.9 we can extend $\phi_n f$ continuously to a continuous $\tilde{\phi}_n f : X^{n+1} \Delta$. Define $\tilde{\phi}_n f$ as follows

$$\tilde{\phi}_n f(a, a, \ldots) = \lim_{(x_1', \ldots, x_{n+1}') \to (a_1, \ldots, a_{n+1})} (\phi_n f|_n^{n+1} X)(x_1', \ldots, x_{n+1}')$$

(a $\in X$)

$$\tilde{\phi}_n f = \phi_n f \quad \text{on} \quad X^{n+1} \Delta$$

Then $\tilde{\phi}_n f : X^{n+1} \to K$ and it is an easy matter to show that $\tilde{\phi}_n f$ is continuous.

We say that an $f : X \to K$ is a spline function of degree $< n$ if for every $a \in X$ there is a neighborhood $U$ of $a$ such that $f|U$ is a polynomial function of degree $< n$. Since 8.11 tells us that "$f \in C^n(X)$" is a local property we have
COROLLARY 8.12. Let \( f : X \to K \). If \( f \) is locally \( C^n \) \((C^\infty)\) then \( f \in C^n(X) \) \((C^\infty(X))\). In particular, spline functions are in \( C^n(X) \).

For an \( f \in C^{n-1}(X) \) (where \( n \geq 1 \)) we denote by \( \tilde{f}_n \) the continuous extension of \( f \) to \( \mathbb{V}^{n+1}X \setminus \Delta \) (see 8.9). For such \( f, D_1 f, \ldots, D_{n-1} f \) make sense and we have

**Lemma 8.13.** Let \( f \in C^{n-1}(X) \) and let \((x_1, \ldots, x_{n+1}) \in \mathbb{V}^{n+1}X\). For \( 1 \leq i \leq n \) define \( S_{i,n} = \{(x_{j_1}, x_{j_2}, \ldots, x_{j_{n+1}}) \in X^{n+1} : j_1 \leq j_2 \leq \ldots \leq j_{n+1} \) and \( \{1, 2, \ldots, i+1\} = \{j_1, j_2, \ldots, j_{n+1}\} \}. Then we have for \( 1 \leq i \leq n \)

\[
\phi_i(D_{n-1} f)(x_1, \ldots, x_{i+1}) = \sum_{u \in S_{i,n}} \tilde{f}_n(u).
\]

Before starting with the proof we observe that \( \#S_{i,n} = \binom{n}{i} \). Lemma 8.13 says for example

\[
\phi_1(D_{n-1} f)(x, y) = \tilde{f}_n(x, y, y, \ldots, y) + \tilde{f}_n(x, x, y, \ldots, y) + \ldots + \tilde{f}_n(x, x, \ldots, x, y).
\]

Let \( 2 \leq i \leq n \).

**Proof:** We first prove the lemma for \( i = 1 \). Let \((x, y) \in \mathbb{V}^{2}X\). Then

\[
\phi_1(D_{n-1} f)(x, y) = (x-y)^{-1}(D_{n-1} f(x) - D_{n-1} f(y)) =
\]

\[
= (x-y)^{-1}(\tilde{f}_n f(x, x, \ldots, x) - \tilde{f}_n f(y, y, \ldots, y)) = (8.10) =
\]

\[
= (x-y)^{-1}(\tilde{f}_n f(x, y, y, \ldots, y) + \tilde{f}_n f(x, x, y, \ldots, y) + \ldots + \tilde{f}_n f(x, x, \ldots, x, y)).
\]

Next we prove 8.13 by induction on \( n \). For \( n = 1 \) we have \( i = 1 \), so we are done. Now suppose we have shown 8.13 for \( 1, \ldots, n-1 \). Let \( f \in C^{n-1}(X) \), and let \( 2 \leq i \leq n \). We have
\[
\phi_i(D_{n-i}f)(x_1,\ldots,x_{i+1}) = (x_1-x_2)^{-1} (\phi_{i-1}(D_{n-i}f)(x_1,x_3,\ldots,x_{i+1}) - \\
\phi_{i-1}(D_{n-i}f)(x_2,x_3,\ldots,x_{i+1})) = (induction\ hypothesis) = \\
= (x_1-x_2)^{-1} (\sum_{u \in A} \phi_{n-1}f(u) - \sum_{u \in B} \phi_{n-1}f(u)),
\]

where

\[A = \{(x_1,\ldots,x_n) \in \mathbb{x}^n : j_1 \leq j_2 \leq \cdots \leq j_n \text{ and } \{1,3,4,\ldots,i+1\} = \{j_1,j_2,\ldots,j_n\}\}
\]

\[B = \{(x_1,\ldots,x_n) \in \mathbb{x}^n : j_1 \leq j_2 \leq \cdots \leq j_n \text{ and } \{2,3,4,\ldots,i+1\} = \{j_1,j_2,\ldots,j_n\}\}.
\]

The replacement of \(x_1\) by \(x_2\) induces a bijection of \(A\) onto \(B\). Let \(u = (x_1,x_2,\ldots,x_k,u_{k+1},\ldots)\) be an element of \(A\) and \(u' = (x_2,x_3,\ldots,x_k,u_{k+1},\ldots)\) be its corresponding element of \(B\). (Suppose that \(x_1\) occurs \(k\) times in \(u\)). Then by \(8.10\)

\[\bar{\phi}_{n-1}f(u) - \phi_{n-1}f(u') = (x_1-x_2)[\phi_nf(x_1,x_2,\ldots,x_2,\ldots) + \\
+ \phi_nf(x_1,x_1,x_2,\ldots,x_2,\ldots) + \cdots + \phi_nf(x_1,x_1,\ldots,x_1,x_2,\ldots)] =
\]

\[= (x_1-x_2) \sum_{t \in A_u} \phi_nf(t),
\]

where \(A_u\) is the set consisting of all \((x_1,x_2,\ldots,x_{j_{n+1}}) \in \mathbb{x}^{n+1}\) such that \(j_1 \leq j_2 \leq \cdots \leq j_{n+1}\) and \(\{j_1,j_2,\ldots,j_{n+1}\} = \{1,\ldots,i+1\}\), and such that the sum of the number of times \(x_1\) and \(x_2\) occur is \(k+1\), and \(\phi_nf(t) = (u_{k+1},\ldots,u_n)\). Thus

\[\phi_i(D_{n-i}f)(x_1,\ldots,x_{i+1}) = \sum_{u \in A} \sum_{t \in A_u} \phi_nf(t) = \sum_{t \in S_{0,i,n}} \phi_nf(t).
\]

Lemma 8.13 has several useful corollaries. First our expected

**Theorem 8.14.** Let \(f \in C^n(X)\). Then for \(0 < i < n\) we have \(D_i f \in C^{n-1}(X)\) and if \(i+j < n\)

\[D_i D_j f = \binom{i+j}{i} D_{i+j} f
\]

\(f\) is \(n\) times differentiable and for \(0 < i < n\) we have

\[f^{(i)} = i! D_i f.
\]
Proof: We just proved the formula

\[ (*) \quad \hat{\phi}_i (D_{n-i} f)(x_1, \ldots, x_{i+1}) = \sum_{u \in S_{i,n}} \hat{\phi}_{n} f(u) = \sum_{u \in S_{i,n}} \hat{\phi}_{n} f(u). \]

Since \( f \in C^n(X) \), the right hand expression defines a continuous function on \( x^{i+1} \), so \( \hat{\phi}_i (D_{n-i} f) \) can be extended to a continuous function on \( x^{i+1} \).

This means that \( D_{n-i} f \in C^i(X) \). If in (*) we take limits for \( (x_1, \ldots, x_{i+1}) + (a, \ldots, a) \) for each \( a \in X \) we arrive at

\[ D_{1} D_{n-i} f = \hat{S}_{i,n} D_{n} f = \binom{n}{i} D_{i} f. \]

(If \( i+j \leq n \) then \( f \in C^{i+j}(X) \). Similarly, we get \( D_{1} D_{j} f = \binom{i+j}{i} D_{i} f \).

That \( D_{1} f = f' \) is clear. If \( n > 1 \) then \( D_{1} f \) is differentiable and \( f'' = D_{1} D_{1} f = \binom{2}{1} D_{2} f \). With induction we obtain \( f^i = i! D_{i} f \) for \( 0 \leq i \leq n \).

**COROLLARY 8.15.** If \( f \in C^n(X) \) then \( f' \in C^{n-1}(X) \) \( (n \in \mathbb{N}) \).

The next corollary we need in Chapter 10, but it seems the right place to state it here. For \( f \in C^{n-1}(X) \) define

\[
\begin{align*}
\rho_1 f(x,y) &= \hat{\phi}_n f(x,y,y,\ldots,y) \\
\rho_2 f(x,y) &= \hat{\phi}_n f(x,x,y,\ldots,y) \\
& \vdots \\
\rho_{n} f(x,y) &= \hat{\phi}_n f(x,x,\ldots,y)
\end{align*}
\]

\( (x,y) \in \nabla^2 X \)

**LEMMA 8.16.** Let \( f \in C^{n-1}(X) \) and let \( \rho_1 f, \ldots, \rho_n f \) be as above. Then \( D_{1} f \in C^{n-2}(X), \ldots, D_{n-1} f \in C(X) \) and we have for \( 1 \leq i \leq n-1 \)

\[
\hat{\phi}_i (D_{n-i} f)(x,x,\ldots,x,y) = \sum_{s=1}^{n} \binom{s-1}{i-1} \rho_s f(x,y).
\]

Thus lemma 8.16 says the following

\[
\begin{align*}
\hat{\phi}_1 (D_{n-1} f)(x,y) &= \rho_1 f(x,y) + \rho_2 f(x,y) + \cdots + \rho_n f(x,y) \\
\hat{\phi}_2 (D_{n-2} f)(x,x,y) &= \binom{1}{1} \rho_2 f(x,y) + \binom{2}{1} \rho_3 f(x,y) + \cdots + \binom{n-1}{1} \rho_n f(x,y) \\
\hat{\phi}_{n-1} (D_{1} f)(x,\ldots,x,y) &= \binom{n-2}{n-2} \rho_{n-1} f(x,y) + \binom{n-1}{n-2} \rho_n f(x,y)
\end{align*}
\]

\( (x \neq y) \)
Proof of lemma 8.16. Clearly \( D_1 f \in C^{n-2}(X) \) etc. Hence \( \phi_i(D_{n-i}f) \) makes sense for \( 1 \leq i \leq n \) and it is a matter of a little counting to show that (using 8.13)

\[
\sum_{i} (D_{n-i}f)(x,x,\ldots,x,y) = \lim_{i \to 0} \phi_i(D_{n-i}f)(x_1,\ldots,x_{i+1}) = \sum_{s=1}^{n} (i-1)p_{s-1}f(x,y).
\]

With the help of 8.13 we can also conclude that \( D_i : BC^n(X) \to BC^{n-i}(X) \) is a continuous map. Indeed, if \( f \in BC^n(X) \) we have by 8.13

\[
||\phi_{n-i}(D_1 f)||_\infty \leq ||\phi_0||_\infty \leq ||f||_n.
\]

\[
||D_k f||_\infty \leq ||\phi_k f||_\infty \leq ||f||_n \quad (0 \leq k \leq n)
\]

and consequently, by 8.6,

\[
||D_i f||_{n-i} = \max_{0 \leq k < n-i} ||D_k D_i f||_\infty \vee ||\phi_{n-i}(D_1 f)||_\infty \leq \max_{0 \leq k < n-i} \left( ||(k+i)D_{k+i} f||_\infty \vee ||f||_n \right) \leq ||f||_n
\]

We leave it to the reader to show that also \( D_i : C^n(X) \to C^{n-i}(X) \) is continuous. We have

THEOREM 8.17. Let \( 0 < i < n \). Then the map

\( D_i : BC^n(X) \to BC^{n-i}(X) \)

is continuous, \( ||D_i|| \leq 1 \).

As in Chapter 4 we may ask whether \( D_i : C^n(X) \to C^{n-i}(X) \) is surjective, in particular \( D_1 \). In general the answer is "no": if the characteristic of \( K \) is \( p \neq 0 \) then surjectivity of \( D_1 : C^n(X) \to C^{n-1}(X) \) would imply surjectivity of \( D^p_1 : C^p(X) \to C(X) \), but, as we have seen in 8.14 \( 0 = p! D^p_1 = D^p_1 \). We will investigate the problem in Chapter 11.
The next lemma will turn out to be helpful for the problem of finding a converse of theorem 8.7. It reduces a problem in \( n+1 \) variables into a problem in only two variables (See 10.7).

**LEMMA 8.18.** Let \( f \in C^{n-1}(X) \) and let \( \rho_1 f, \ldots, \rho_n f \) be defined as in 8.16. Let \( (x_1, \ldots, x_{n+1}) \in X^{n+1}. \) Let \( S \) be the set of all triples \((i, j, k)\) for which \( i, j \in \{1, \ldots, n+1\}, k \in \{1, \ldots, n\}, x_i \neq x_j.\)

Then there exists a map \( s \mapsto \lambda_s \) of \( S \) into \( K \) such that

\[
\Phi_n f(x_1, \ldots, x_{n+1}) = \sum_{(i, j, k) \in S} \lambda_{i, j, k} \rho_k f(x_i, x_j)
\]

and for which

\[
\sum_{s \in S} \lambda_s = 1, \quad |\lambda_s| < 1 \text{ for all } s. \quad (\text{Of course, the } \lambda_s \text{ depend on } (x_1, \ldots, x_{n+1}).
\]

Before giving a proof we will first state some corollaries in order to awake some interest for this technical lemma.

**COROLLARY 8.19.** Let \( f \in C^{n-1}(X). \) Then \( f \in C^n(X) \) if and only if there exists a function \( \phi : X \to K \) such that for all \( a \in X \) and for all \( k \in \{1, \ldots, n\} \)

\[
\lim_{(x, y) \to (a, a)} \rho_k f(x, y) = \phi(a).
\]

**Proof:** The "only if" - part is trivial. To show "if", let \( a \in X \) and \( \varepsilon > 0. \) Then there exists \( \delta > 0 \) such that for all \( x, y \in X, x \neq y, \)

\[
0 < |x-a| < \delta, \quad 0 < |y-a| < \delta \quad \text{we have} \quad |\rho_k f(x, y) - \phi(a)| < \varepsilon \quad (k = 1, \ldots, n).
\]

Let \( (x_1, \ldots, x_{n+1}) \in X^{n+1} \) such that \( |x_i - a| < \delta \) for all \( i. \) Then, by lemma 8.18,

\[
|\Phi_n f(x_1, \ldots, x_{n+1}) - \phi(a)| = |\sum_{i, j, k} \rho_k f(x_i, x_j) - \phi(a)| \leq \max_{i, j, k} |\rho_k f(x_i, x_j) - \phi(a)| < \varepsilon. \quad \text{Now apply 8.11.} \]
COROLLARY 8.20. Let $f \in \mathcal{B}^n(X)$ ($n \geq 1$). Then

$$||\phi_n f||_\infty = \max_{1 \leq k \leq n} ||\rho_k f||_\infty.$$  

Proof: We observed earlier that $f \in \mathcal{B}^n(X)$ implies $f \in \mathcal{B}^{n-1}(X)$.

By lemma 8.18 we then have $(x_1, \ldots, x_{n+1}) \in \gamma^{n+1}X)$

$$|\phi_n f(x_1, \ldots, x_{n+1})| \leq \max_k ||\rho_k f||_\infty,$$

so $||\phi_n f||_\infty \leq \max_k ||\rho_k f||_\infty$. The opposite inequality is trivial.

Corollary 8.20 enables us to show that the spline functions of degree $\leq n$ form a dense subset of $\mathcal{B}^n(X)$, $C^n(X)$ respectively (compare 5.2).

For $f \in \mathcal{B}^n(X)$ we have by 8.7

$$f(x) = f(y) + (x-y)D_1 f(y) + \ldots + (x-y)^{n-1}D_{n-1} f(y) + (x-y)^n \rho_n f(x,y) \quad (x,y \in X).$$

Define

$$||f||_n = \max_{0 \leq i \leq n-1} ||D_i f||_\infty \vee ||\rho_i f||_\infty \quad (f \in \mathcal{B}^n(X)).$$

Clearly $||f||_n \leq ||f||_n$. But we have

THEOREM 8.21. Let $f \in \mathcal{B}^n(X)$. Then

$$||f||_n = \max_{0 \leq i \leq n} ||D_i f||_n - i.$$  

Proof: For $0 \leq i \leq n$ we have $||D_i f||_n - i \leq ||D_i f||_n - i \leq ||f||_n - i \leq ||f||_n$. To prove the opposite inequality we proceed by induction on $n$. For $n = 1$ it follows from the definition that $||f||_1 = ||f||_1$ for $f \in \mathcal{B}^1(X)$. Now suppose 8.21 is correct for $k < n$. Let $f \in \mathcal{B}^n(X)$. For $0 \leq i \leq n-1$ we see that $||D_i f||_\infty \leq ||D_i f||_n - i$, so by 8.6 what remains to prove is that $||\phi_n f||_\infty \leq \max_{0 \leq i \leq n} ||D_i f||_n - i$. We obviously have for $i \leq i \leq n-1$

$$||\phi_i (D_{n-i} f)||_\infty \leq ||D_{n-i} f||_i,$$

and, by the induction hypothesis,
From 8.16 it follows that we can write successively $\rho_{n-1} f, \rho_{n-2} f, \ldots, \rho_1 f$ as linear combinations of $\rho_n f$ and the $\phi_i (D_{n-i} f)$, where the coefficients are in the unit disc of $K$. By symmetry, $||\rho_1 f||_\infty = ||\rho_n f||_\infty$.

$$||\phi_n f||_\infty = \max_{1 \leq k \leq n} ||\rho_k f||_\infty \leq \max_{1 \leq i \leq n-1} ||\phi_i (D_{n-i} f)||_\infty \vee ||\rho_1 f||_\infty \leq \max_{0 \leq i \leq n} ||D_if||_{n-i} \vee ||\rho_1 f||_\infty \leq \max_{0 \leq i \leq n} ||D_if||_{n-i}.$$ 

THEOREM 8.22. Let $f \in C^n(X)$ and $\varepsilon > 0$. Then there is a spline function $g$ of degree $\leq n$ (i.e. $g$ is locally a polynomial function of degree $\leq n$) such that $f-g \in BC^n(X)$ and $||f-g||_n < \varepsilon$.

If $D^i f = D^i+1 f = \ldots = D^n f = 0$ for some $i \in \{1, \ldots, n\}$ then $g$ can be chosen to be of degree $\leq i-1$.

COROLLARY 8.23. The spline functions of degree $\leq n$ in $BC^n(X)$ ($C^n(X)$) form a (sequentially) dense subset of $BC^n(X)$ ($C^n(X)$).

If $\chi(K) = 0$ the locally constant functions form a dense subset of $\{f \in BC^n(X) : f' = 0\}$.

Proof of 8.22: For each $a \in X$ there is a ball $B_a(r) = \{x \in X : |x-a| < r\}$ (where $r \in \{1, \frac{1}{3}, \ldots\}$) such that for all $x_1, \ldots, x_{n+1} \in B_a(r)$

$$|\phi_n f(x_1, \ldots, x_{n+1}) - D^n f(a)| < \varepsilon$$

$$|\phi_{n-1} f(x_1, \ldots, x_n) - D_{n-1} f(a)| < \varepsilon$$

$$|\phi_1 f(x_1 x_2) - D_1 f(a)| < \varepsilon$$

$$|D f(x_1) - D f(a)| < \varepsilon$$

(By 8.14, $D f \in C^{n-i}(X)$). The $B_a(r)$ form a covering of $X$ and there is a disjoint subcovering $B_i = B_{a_i}(r_i)$, where $i$ runs through some index set $I$. 

*Notes:*
We define the function \( g : X \to K \) as follows. For each \( i \in I \), let
\[
g(x) = f(a_i) + (x-a_i)D f(a_i) + \ldots + (x-a_i)^n D^n f(a_i) \quad (x \in B_i).
\]

Then clearly \( g \) is locally a polynomial function of degree \( \leq n \), and if
\[D_i f = \ldots = D_n f = 0\]
for some \( i \) then the degree of \( g \) is \( \leq i-1 \). Since
\[f \in C^n(X)\]
we have by the Taylor formula (8.7) for \( x \in B_i \)
\[
f(x) = f(a_i) + (x-a_i)D f(a_i) + \ldots + (x-a_i)^n D^n f(a_i) + (x-a_i)^n D^n f(x,a_i,\ldots,a_i)
\]
Hence, for \( x \in B_i \):
\[
|f(x)-g(x)| = (x-a_i)^n |\frac{\partial^n}{\partial x^n} f(x,a_i,\ldots,a_i) - D^n f(a_i)| \leq r_i^n \varepsilon < \varepsilon.
\]
So: \[||f-g||_\infty < \varepsilon.
\]

We have also the Taylor formula for \( D_i f \) for \( x \in B_i \):
\[
D_i f(x) = D_i f(a_i) + (x-a_i)D_{i+1} f(a_i) + \ldots + (x-a_i)^n D_n f(a_i) +
\]
\[
+ (x-a_i)^{n-1} \frac{\partial}{\partial x_n} D_{i+1} f(x,a_i,\ldots,a_i) \ldots = (8.14) = D_i f(a_i) + 2(x-a_i)D_2 f(a_i) + \ldots
\]
\[
+ (n-1)(x-a_i)^{n-2} D_{n-1} f(a_i) + (x-a_i)^{n-1} \frac{\partial}{\partial x_n} D_{n-1} f(x,a_i,\ldots,a_i) =
\]
\[
D_i g(x) + (x-a_i)^{n-1} \frac{\partial}{\partial x_n} D_{n-1} f(x,a_i,\ldots,a_i) - n D_n f(a_i).
\]
Hence \[|D_i f(x) - D_i g(x)| = |(x-a_i)^{n-1} \frac{\partial}{\partial x_n} D_{n-1} f(x,a_i,\ldots,a_i) - n D_n f(a_i)| \leq r_i^{n-1} \varepsilon < \varepsilon.
\]
It follows that \[||D_i f - D_i g||_\infty < \varepsilon.\] Going on this way we arrive at
\[
|D_j g(x) - D_j f(x)| \leq r_i^{n-j} \varepsilon \quad (x \in B_i) \quad (0 \leq j \leq n-1)
\]

Our first aim is to show that \[||f-g||_n < \varepsilon\] (see the remarks following
8.20). To prove this we only have to show \[||\rho_1 f - \rho_1 g||_\infty < \varepsilon.\] First let
\( x,y \in B_i \). Then \( \rho_1 g(x,y) = D_n f(a_i) \), \( \rho_1 f(x,y) = \frac{\partial}{\partial x_n} f(x,y,y,\ldots,y) \). By (*)
we find \[|\rho_1 g(x,y) - \rho_1 f(x,y)| < \varepsilon\]. Now suppose \( x \in B_i, y \in B_j \) where \( i \neq j \)
and put \( r = r_i \lor r_j \). Then since
\[
\rho_1 f(x,y) = (x-y)^{-n}(f(x) - \Sigma_{s=0}^{n-1} (x-y)^s D_s f(y))
\]
\[
\rho_1 g(x,y) = (x-y)^{-n}(g(x) - \Sigma_{s=0}^{n-1} (x-y)^s D_s g(y))
\]
we have
\[
|\rho_1 f(x,y) - \rho_1 g(x,y)| \leq |x-y|^{-n}(|f(x)-g(x)| + |f(y)-g(y)|) \vee \max_{1 \leq s \leq n} |x-y|^s |D_s f(y) - D_s g(y)|.
\]
Now $|x-y| \geq r$, $|f(x) - g(x)| \leq r^n \varepsilon \leq r^n \varepsilon$, $|f(y) - g(y)| \leq r^n \varepsilon \leq r^n \varepsilon$. For $s \geq 1$:

$$|D_s f(y) - D_s g(y)| \leq r^{n-s} \varepsilon \leq r^{n-s} \varepsilon.$$ Hence we have

$$|\rho_1 f(x, y) - \rho_1 g(x, y)| < \varepsilon.$$

So far, we have proved: if $f \in C^n(X)$, $\varepsilon > 0$, and $B_i = B_i(x_i)$ form a disjoint covering of $X$ such that we have (**) on each $B_i$, then $|f-g| \leq \varepsilon$,

where

$$g(x) = \sum_{j=0}^{n} (x-a_j)^n f(a_j).$$

Now let $j \in \{0, \ldots, n\}$. Then $D_j f \in C^{n-j}(X)$ and for $x_1, \ldots, x_{n-j+1} \in B_i$ we have

\[
\left\{ \begin{array}{c}
|D_{n-j} f(x_1, \ldots, x_{n-j+1}) - D_{n-j} D_j f(a_1)| < \varepsilon \\
\vdots \\
|D_{n-j} f(x_1) - D_{n-j} D_j f(a_1)| < \varepsilon
\end{array} \right.
\]

(Obviously, by lemma 8.13, $\Phi_n D_j f(x_1, \ldots, x_{n-j+1}) = \sum_u \phi_n f(u)$, where in the last sum are \( \binom{n}{j} \) summands and where for each $u = (u_1, \ldots, u_{n+1})$ we have $u_s \in B_i$ for all $s \in \{1, \ldots, n+1\}$. But $D_{n-j} D_j f(a_1) = \binom{n}{j} D_n f(a_1)$.

By (**), we have $|\Phi_n f(u) - D_n f(a_1)| < \varepsilon$. Hence (**) is a consequence of (*) (via 8.13)).

Applying the first result of the above proof for $D_j f$ instead of $f$ we arrive at

$$||D_j f - \tilde{g}||_{n-j} < \varepsilon,$$

where $\tilde{g}$ is defined as

$$\tilde{g}(x) = D_j f(a_1) + (x-a_1) D_1 D_j f(a_1) \ldots (x-a_1)^{n-j} D_{n-j} f(a_1) (x \in B_i).$$

But $\tilde{g}$ is nothing else but $D_j g$ as an easy computation shows. Hence we have found $||D_j f - D_j g||_{n-j} < \varepsilon$ for all $j \in \{0, \ldots, n\}$. Theorem 8.21 then tells us that $||f-g||_n < \varepsilon$. 


COROLLARY 8.24. $C(X) \subseteq C^n(X)$, for every $n$

$BC(X) \subseteq BC^n(X)$, for every $n$.

Proof: Spline functions are in $C(X)$.

We still owe the reader a

Proof of lemma 8.18. Let $P$ be the set of all $(x_1, \ldots, x_{n+1}) \in X^{n+1}$ such that
for which 8.18 is correct. Let $R_i = \{(x_1, \ldots, x_{n+1}) \in X^{n+1} \setminus \Delta : \#(x_1, \ldots, x_{n+1}) = i\}$. Then $X^{n+1} \setminus \Delta = \bigcup_{i=2}^{n+1} R_i$ and the $R_i$ are disjoint. We will show by induction that $R_i \subseteq P$ ($i = 2, \ldots, n+1$). If $u \in R_2$ then $\tilde{\phi}_n f(u)$ equals $\rho_{i}(s,t)$ for some $i$, some $s,t \in X$, so obviously $R_2 \subseteq P$.

Now suppose $R_{i-1} \subseteq P$. Choose $z_1, z_2 \in X$, $z_1 \neq z_2$ and consider the set
$A = \Lambda(z_1, z_2)$ of all elements $(x_1, \ldots, x_{n+1}) \in R_{i-1}$ such that
$z_1, z_2 \in (x_1, \ldots, x_{n+1})$ and such that $|z_1 - z_2| = \max_{i,j} |x_i - x_j|$. Since every element of $R_{i-1}$ lies in $\Lambda(z_1, z_2)$ for some $z_1, z_2$ we are done if we can prove: $\Lambda \subseteq P$. For $u \in \Lambda$, let $\sigma_j(u)$ be the number of times $z_j$ occurs in $u$ ($j = 1, 2$), and let $\sigma(u) = \sigma_1(u) + \sigma_2(u)$. Let $A_k = \{u \in \Lambda : \sigma(u) = k\}$.

Then $\Lambda = \bigcup_{k\geq 2} A_k$. We now prove by induction $A_k \subseteq P$.

If $u \in A_2$, then we may assume $u = (z_1, z_2, z_3, \ldots)$ where $z_1 \neq z_i$ ($i \neq 1$) and $z_2 \neq z_i$ ($i \neq 2$). Then by 8.2, I

$$(z_1 - z_2)^n f(z_1, z_2, \ldots) = (z_1 - z_3)^n f(z_1, z_3, z_3, \ldots) + (z_2 - z_3)^n f(z_2, z_2, z_3, \ldots).$$

Hence

$$\tilde{\phi}_n f(u) = \frac{z_1 - z_3}{z_1 - z_2} \tilde{\phi}_n(v) + \frac{z_2 - z_3}{z_1 - z_2} \tilde{\phi}_n(w) = \lambda \tilde{\phi}_n(v) + \mu \tilde{\phi}_n(w)$$

where $v, w \in R_{i-1}$, $\lambda + \mu = 1$, $|\lambda| \leq 1$, $|\mu| \leq 1$. By our assumption $R_{i-1} \subseteq P$ we see $u \in P$. Thus, $A_2 \subseteq P$. Suppose $A_{k-1} \subseteq P$, and let $u \in A_k$. Then we may assume

$$u = (z_1', \ldots, z_1', z_2', \ldots, z_2', z_3', \ldots)$$

where $z_1$ occurs $l$ times, $z_2$ occurs $m$ times and where $l+m = k$. We write
for simplicity:

\[ u = (z_1^1, z_2^m, z_3, \ldots). \]

Again we have:

\[
\phi_n f(u) = \frac{z_1^1 - z_3^3}{z_1^1 - z_2^2} \phi_n f(z_1^1, z_2^m, z_3, \ldots) + \frac{z_3^3 - z_2^2}{z_1^1 - z_2^2} \phi_n f(z_1^1, z_2^m, z_3, \ldots)
\]

\[ = \lambda \phi_n f(v) + \mu \phi_n f(w), \]

where \( \lambda + \mu = 1, |\lambda| \leq 1, |\mu| \leq 1 \) and where either \( v \in \mathbb{R}_{i-1} \) (in case \( m = 1 \)) or \( v \in \Lambda_{k-1} \). In any case, \( v \in \mathbb{P} \). Similarly, \( w \in \mathbb{P} \). Thus \( u \in \mathbb{P} \), i.e., \( \Lambda_k \subset \mathbb{P} \).

**Note.** The definition of \( C^n(X) \) and some theorems can be found in [1].
9. LOCAL INVERTIBILITY OF $C^{n}$-FUNCTIONS

Let $n \geq 1$ and let $f \in C^{n}(X)$ and suppose $f'(a) \neq 0$ for some $a \in X$. Then we have shown already (1.7) that there is a neighborhood $U$ of $a$ such that $f$ is injective on $U$. $f^{-1} : f(U) \to U$ is continuous so $f^{-1}$ is a $C^{1}$-function (by 1.8). Our problem here is to show that actually $f^{-1} \in C^{n}$. Classically this is very easy: one just differentiates $f \circ g(x) = x$ and one gets $g'(x) = (f'(g(x)))^{-1}$. By the induction hypothesis $g \in C^{n-1}$ and since also $f' \in C^{n-1}$ it follows that $g' = \frac{1}{f' \circ g}$ is in $C^{n-1}$. Then $g \in C^{n}$. But it is just this very last conclusion that cannot be drawn in the non-archimedean case.

**Lemma 9.1.** Let $f : X \to K$ be injective and let $g : f(X) \to X$ be its inverse. For $n \in \mathbb{N}$ ($n \geq 2$), define $S_{n}$ to be the set of the following functions, all defined on $f^{n+1}(X)$:

- $(x_{1}, \ldots, x_{n+1}) \mapsto \phi_{1}g(x_{i_{1}}, x_{i_{2}})$ ($i_{1} < i_{2}$)
- $(x_{1}, \ldots, x_{n+1}) \mapsto \phi_{2}g(x_{i_{1}}, x_{i_{2}}, x_{i_{3}})$ ($i_{1} < i_{2} < i_{3}$)

- ...

- $(x_{1}, \ldots, x_{n+1}) \mapsto \phi_{n-1}g(x_{i_{1}}, \ldots, x_{i_{n}})$ ($i_{1} < i_{2} \ldots < i_{n}$)

and

- $(x_{1}, \ldots, x_{n+1}) \mapsto \phi_{1}f(g(x_{i_{1}}), g(x_{i_{2}}))$ ($i_{1} < i_{2}$)

- ...

- $(x_{1}, \ldots, x_{n+1}) \mapsto \phi_{n-1}f(g(x_{1}), \ldots, g(x_{i_{n}}))$ ($i_{1} < i_{2} \ldots < i_{n}$)

- $(x_{1}, \ldots, x_{n+1}) \mapsto \phi_{n}f(g(x_{1}), \ldots, g(x_{n+1}))$.

Let $R_{n}$ be the ring, generated by the elements of $S_{n}$.

Then $\phi_{n}g \in R_{n}$.

**Proof:** We first show that $\phi_{2}g \in R_{2}$. We have the identity

$$
\phi_{2}g(x_{1}, x_{2}, x_{3}) = -\phi_{2}f(g(x_{1}), g(x_{2}), g(x_{3}))\phi_{1}g(x_{1}, x_{2})\phi_{1}g(x_{2}, x_{3})
$$
for all \((x_1,x_2,x_3) \in \mathcal{V}^3x\). Hence \(\phi_2g\) is a product of functions in \(S_2\) i.e., \(\phi_2g \in R_2\). A simple inspection tells us that if \(h \in S_{n-1}\) then the function

\[
(x_1, \ldots, x_{n+1}) \mapsto h(x_1, \ldots, x_n) \quad ((x_1, \ldots, x_{n+1}) \in \mathcal{V}^{n+1}x)
\]

is in \(S_n\). It is also true if we replace \(S_{n-1}, S_n\) by \(R_{n-1}, R_n\) respectively.

Now suppose \(9.1\) is correct for \(n-1\). Then for \((x_1, \ldots, x_{n+1}) \in \mathcal{V}^{n+1}x\):

\[
\phi_n g(x_1, \ldots, x_{n+1}) = (x_1-x_2)^{-1}(\phi_{n-1} g(x_1, x_3, \ldots, x_{n+1}) - \phi_{n-1} g(x_2, x_3, \ldots, x_{n+1}))
\]

The induction hypothesis states that \(\phi_{n-1} g \in R_{n-1}\). Thus, we are done if we can prove: if \(h \in R_{n-1}\) then \(Ah \in R_n\), where

\[
Ah(x_1, \ldots, x_{n+1}) = (x_1-x_2)^{-1}(h(x_1, x_3, \ldots, x_{n+1}) - h(x_2, x_3, \ldots, x_{n+1}))
\]

To show this, consider the set \(B = \{h \in R_{n-1} : Ah \in R_n\}\). We must prove: \(B = R_{n-1}\). We will do this by showing that \(B\) is a ring containing \(S_{n-1}\).

\(A\) is a linear map, so it is clear that \(B\) is an additive group. Let \(h,t \in B\). Then we have for \((x_1, \ldots, x_{n+1}) \in \mathcal{V}^{n+1}x\):

\[
\Delta(ht)(x_1, \ldots, x_{n+1}) = t(x_1, x_3, \ldots, x_{n+1})Ah(x_1, x_2, \ldots, x_{n+1}) +
+ h(x_2, x_3, \ldots, x_{n+1})At(x_1, x_2, \ldots, x_{n+1})
\]

By a previous remark \((x_1, \ldots, x_{n+1}) \mapsto t(x_1, x_3, \ldots, x_{n+1})\) and \((x_1, \ldots, x_{n+1}) \mapsto h(x_2, x_3, \ldots, x_{n+1})\) are in \(B_n\). Since \(h,t \in B\) we have \(\Delta h, \Delta t \in R_n\). Thus \(\Delta(ht) \in R_n\) i.e. \(ht \in B\). Therefore, \(B\) is a ring. We now must prove \(S_{n-1} \subseteq B\). Let \(h \in S_{n-1}\) be of the first type i.e.

\[
h(x_1, \ldots, x_n) = \phi_j g(x_{i_1}, \ldots, x_{i_{j+1}}) \quad (1 \leq j \leq n-2)
\]

Then if \(1 \not\in \{i_1, \ldots, i_{j+1}\}\) then \(h(x_1, x_3, \ldots, x_{n+1}) = h(x_2, x_3, \ldots, x_{n+1})\), so \(0 = Ah \in R_n\). If \(1 \in \{i_1, \ldots, i_{j+1}\}\) then \(i_1 = 1\) and for \((x,y,x_2, \ldots, x_n) \in \mathcal{V}^{n+1}x\) we have...
h(x,x_2,...,x_n) = \phi_j g(x,x_2,...,x_{j+1})

h(y,x_2,...,x_n) = \phi_j g(y,x_2,...,x_{j+1})

hence

\Delta h(x,y,x_2,...,x_n) = \phi_{j+1} g(x,y,x_2,...,x_{j+1})

Thus \Delta h \in R_n implying h \in B.

If h is of the second type i.e.,

h(x_1,...,x_n) = \phi_j f(g(x_1),...,g(x_j)) (1 \leq j \leq n-1)

then again \Delta h = 0 if 1 \notin \{i_1,...,i_{j+1}\}.

If 1 = i_j then for (x,y,x_2,...,x_n) \in \mathbb{V}^{n+1} X we have

h(x,x_2,...,x_n) = \phi_j f(g(x),g(x_2),...,g(x_{j+1}))

h(y,x_2,...,x_n) = \phi_j f(g(y),g(x_2),...,g(x_{j+1}))

hence

\Delta h(x,y,x_2,...,x_n) = \phi_{j+1} f(g(x),g(y),g(x_2),...,g(x_{j+1})) \phi_1 g(x,y).

It follows that \Delta h, being a product of two \mathbb{S}_n-\text{functions}, is in \mathbb{R}_n

which implies h \in B.

**THEOREM 9.2.** (LOCAL INVERTIBILITY OF C^n-\text{FUNCTIONS}). Let f \in C^n(X) and

let f'(a) \neq 0 for some a \in X, Then there is a neighborhood U of a (a \in U \subset X) such that f : U \rightarrow f(U) in a bijection.

The inverse g : f(U) \rightarrow U is in C^n(f(U)).

**Proof:** For n = 1, see Lemma 1.8. Suppose the theorem is true for n-1.

Let f \in C^n(X), f'(a) \neq 0. By the induction hypothesis we have a neighborhood U of a such that

\[ g : f(U) \rightarrow U \]

is in C^{n-1}. We apply Lemma 9.1 to f and g, defined on U, f(U) respectively. Hence \phi_n g \in R_n. R_n consists of functions that are continuously extendable to \mathbb{V}^{n+1} f(U). Hence \phi_n g has a continuous extension to f(U)^{n+1}:

\[ g \in C^n(f(U)). \]
COROLLARY 9.3. Let $f \in C(X)$. If a local inverse of $f$ is continuous then it is in $C(X)$. We will use tricks, similar to those of 9.1, to show that the composition of two $C^n$-functions is again $C^n$. 

LEMMA 9.4. Let $g : X \to K$ be continuous, let $Y \supset g(X)$ be a set without isolated points, and let $f : Y \to K$ be in $C^n(Y)$. Let $S'_n$ be the set of the following functions, all defined on $V^{n+1}X$

$$(x_1,\ldots,x_{n+1}) \mapsto \phi_1 g(x_{i_1},x_{i_2}) \quad (i_1 < i_2)$$

$$(x_1,\ldots,x_{n+1}) \mapsto \phi_2 g(x_{i_1},x_{i_2},x_{i_3}) \quad (i_1 < i_2 < i_3)$$

and

$$\ldots$$

$$(x_1,\ldots,x_{n+1}) \mapsto \phi_n g(x_1,\ldots,x_{n+1})$$

and

$$(x_1,\ldots,x_{n+1}) \mapsto \phi_1 f(g(x_{i_1}),g(x_{i_2})) \quad (i_1 < i_2)$$

and

$$\ldots$$

$$(x_1,\ldots,x_{n+1}) \mapsto \phi_n f(g(x_1),\ldots,g(x_{n+1})).$$

Let $R'_n$ be the ring, generated by the elements of $S'_n$. Then $\phi_n (f \circ g) \in R'_n$. 

Proof: First observe that the set $S'_n$ is well-defined, since $g$ is continuous and $f \in C^n(X)$. For $n = 1$ we have

$$\phi_1 (f \circ g)(x_1,x_2) = \phi_1 f(g(x_1),g(x_2)) \phi_1 g(x_1,x_2) \quad ((x_1,x_2) \in V^2 X)$$

so that $\phi_1 (f \circ g) \in R'_1$. For the induction step, we may proceed just as in the proof of 9.1: It suffices to show that if $h \in R'_{n-1}$ then $Ah \in R'_n$, and the methods to do this are the same as in 9.1 and will be omitted.

THEOREM 9.5. Let $g \in C^n(X), f \in C^n(Y)$ where $g(X) \subset Y$. Then $f \circ g \in C^n(X)$. 


Proof: Observe that all the functions in $S'_n$ (hence in $R'_n$) are continuously extendable to $X^{n+1}$. By 9.4, \( \phi_n(f \circ g) \in R'_n \) hence continuously extendable: \( f \circ g \in C^n(X) \).

COROLLARY 9.6. Composition of $C^\infty$-functions are $C^\infty$-functions
10. THE SPACE $C^n(X)$. IS $C^n(X) = C^n(X)$?

In order to answer the question under which circumstances for $f : X \to K$ having a Taylor formula up to order $n$ implies $f \in C^n(X)$ (see the remarks following 8.7) we introduce the spaces $C^n(X)$.

**DEFINITION 10.1.** Let $f : X \to K$ and let $n \in \mathbb{N}$. We say that $f \in C^n(X)$

if there exist functions $D_1f, \ldots, D_{n-1}f : X \to K$ and a continuous function $R_n f : X \times X \to K$ such that for all $x, y \in X$

\[
(*) \quad f(x) = f(y) + \sum_{i=1}^{n-1} (x-y)^i D_i f(y) + (x-y)^n R_n f(x, y).
\]

We see that, given $f \in C^n(X)$ the functions $D_1f, \ldots, R_n f$ are uniquely determined. (This follows by a simple induction argument). Less trivial is the following

**LEMMA 10.2.** Let $f \in C^n(X)$. Then $D_1f, \ldots, D_{n-1}f$ are continuous.

**Proof:** From (*) we obtain (take lim) that $f$ is continuous. Thus the map

\[
(x, y) \mapsto (x-y)D_1 f(y) + \ldots + (x-y)^{n-1} D_{n-1} f(y)
\]

is continuous. It is a polynomial in $x$ of degree $\leq n-1$ with coefficients that are functions of $y$. The coefficient of $x^{n-1}$ is $D_{n-1} f(y)$. That $D_{n-1} f$ is continuous and, by the same token also $D_{n-2} f, \ldots, D_1 f$ follows from:

Let $a_0, \ldots, a_{n-1} : X \to K$. If $(x, y) \mapsto a_0(y) + a_1(y)x + \ldots + a_{n-1}(y)x^{n-1}$

is continuous then the $a_i$ are continuous.

(Proof: choose $x_1, \ldots, x_n$ such that $x_i \neq x_j$ whenever $i \neq j$ and $x_i \neq 0$

for all $i$. Then

\[
\begin{pmatrix}
1 & x_1 & \ldots & x_1^{n-1} \\
\vdots & & & \\
1 & x_n & \ldots & x_n^{n-1}
\end{pmatrix}
\begin{pmatrix}
a_0(y) \\
a_1(y) \\
a_{n-1}(y)
\end{pmatrix}
=
\begin{pmatrix}
\phi_1(y) \\
\vdots \\
\phi_{n-1}(y)
\end{pmatrix}
\]
where $\phi_i$ are continuous functions. Since the matrix is invertible, the $a_i$ are continuous).

For $f \in C^n(X)$, put (compare 8.21)

$$||f||_n = \max(||f||_{\infty}, ||D_1 f||_{\infty}, \ldots, ||D_{n-1} f||_{\infty}, ||R_n f||_{\infty}).$$

(If $f \in BC^n(X)$ this definition coincides with the one given previously), and let $BC^n(X)$ be the space of all $f$ for which $||f||_n < \infty$. Then $BC^n(X)$ is a Banach space with respect to $||\cdot||_n$.

**Lemma 10.3.** $C^1(X) = C^1(X)$, $C^2(X) = C^2(X)$; $BC^1(X) = BC^1(X)$, $BC^2(X) = BC^2(X)$ and $||f||_1 = ||f||_1$; $||f||_2 = ||f||_2$. In general we have

- $C^1(X) \supset C^2(X) \supset \ldots$
- $BC^1(X) \supset BC^2(X) \supset \ldots$
- $C^n(X) \subset BC^n(X)$ for all $n$
- $BC^n(X) \subset BC^n(X)$ for all $n$, and $||f||_n \leq ||f||_n$ for all $f \in BC^n(X)$.

**Proof:** The first three equalities are clear from the definitions or 8.8. In the proof of 8.8 it is shown that for $f \in C^2(X): \phi_2 f(x,y,z)$ is a convex combination of $R_n f(y,x)$ and $R_n f(z,x)$, so $||\phi_2 f||_{\infty} \leq ||f||_2$. It follows that $||f||_2 = ||f||_2$.

Let $f \in C^{n+1}(X)$. Then (with $R_n f(x,y) = D_{n+1} f(y) + (x-y) R_{n+1} f(y)$) we see that for all $x,y \in X: f(x) = f(y) + (x-y) D_1 f(y) + \ldots + (x-y)^{n-1} D_{n-1} f(y) + (x-y)^n R_n f(x,y)$, hence $f \in C^n(X)$. Since $||\cdot||_{n+1} \geq ||\cdot||_n$, we have also $BC^{n+1}(X) \subset BC^n(X)$. The last two statements follow from 8.7 and 8.21 respectively.

For $(x_1, \ldots, x_n) \in \mathbb{V}^n X$ we define $C(x_1, \ldots, x_n) = \sup \{ \frac{|x_i - x_j|}{|x_k - x_l|} : 1 \leq i,j,k,l \leq n, k \neq l \}$.  


LEMMA 10.4. Let $H_n$ be the set of all symmetric functions $f: \mathbb{X}^n \to \mathbb{K}$ satisfying

$$(x-y)f(x,y,x_1,\ldots,x_{n-1})+(y-z)f(y,z,x_2,\ldots,x_{n-1}) =$$

$$(x-z)f(x,z,x_1,\ldots,x_{n-1})$$

(whenever meaningful). For $f \in H_n$, let $\overline{f}: \mathbb{V}^2 \mathbb{X} \to \mathbb{K}$ be defined via

$$\overline{f}(x,y) = f(x,x,\ldots,x,y) \quad ((x,y) \in \mathbb{V}^2 \mathbb{X}).$$

Let $n \geq 3$, $f \in H_n$, $(x_1,\ldots,x_n) \in \mathbb{V}^n \mathbb{X}$, $1 \leq k \leq n$. Then there exist $\lambda_{ij} \in \mathbb{K}$ $(0 \leq i,j \leq n, i \neq j)$ such that

$$f(x_1^k x_2^{n+1-k}) = \sum_{i,j} \lambda_{ij} \overline{f}(x_1,x_j)$$

$$\sum_{i,j} \lambda_{ij} = 1, \quad |\lambda_{ij}| \leq C^2(n-2)_{(x_1,\ldots,x_n)} \quad \text{for all } i,j.$$  

(Here, as in 8.18, by $(x_1^k x_2^{n+1-k})$ we mean the element

$(z_1,z_2,\ldots,z_{n+1}) \in \mathbb{X}^{n+1}$ for which $z_s = x_1$ in case

$1 \leq s \leq k$ and $z_s = x_2$ in case $k+1 \leq s \leq n+1$).

Proof: By induction on $n$. First let $n = 3$. The expressions $f(x_1^k x_2 x_3)$ and $f(x_1 x_2 x_3)$ are already of the form $\overline{f}(\ldots)$.

An easy computation shows

$$f(x_1 x_2 x_3) = \frac{x_1 x_3 - x_2 x_1}{x_1 x_2 x_3} f(x_1 x_2) + \frac{x_1 x_3 - x_3 x_2}{x_1 x_2 x_3} f(x_1 x_3) + \frac{x_3 x_2 - x_1 x_3}{x_1 x_2 x_3} f(x_2 x_3).$$

Indeed, the sum of the coefficients equals 1 and the absolute value of each coefficient is less than $C^2_{(x_1, x_2, x_3)} = C^2_{(x_1, x_2, x_3)}$.

Suppose 10.4 is shown for $n-1$. We prove 10.4 for $n$. In case $k = 1$ or $n$ there is nothing to prove. Let $1 < k < n$. Then

$$(x_1 - x_2)f(x_1^k x_2^{n+1-k}) = (x_1 - x_n)f(x_n^k x_1^{n+1-k}) + (x_n - x_1)f(x_n^{k-1} x_1 x_2^{n+1-k}).$$

Now $(s_1,\ldots,s_n) \mapsto f(x_n, s_1,\ldots,s_n)$ is in $H_{n-1}$. By the induction hypothesis,
we have
\[ f(x_n, x_{n-k+1}, x_2) = \sum_{i,j=1}^{n-1} \lambda_{ij} f(x_n, x_i, \ldots, x_i, x_j) \]
where \( \sum_{i,j=1}^{n-1} \lambda_{ij} = 1 \) and \( |\lambda_{ij}| \leq C^{2(n-3)} \). A similar result holds for
\[ f(x_n, x_{k-1}, x_{n+1-k}). \]
Thus we find
\[ f(x_1, x_2, \ldots, x_{n-k+1}) = \sum_{i,j=1}^{n-1} \mu_{ij} f(x_n, x_i, \ldots, x_i, x_j) \]
where \( \sum_{i,j=1}^{n-1} \mu_{ij} = 1 \) and each \( |\mu_{ij}| \leq C^{2(n-3)+1} \).
But for each \( i, j \) we have
\[ f(x_n, x_i, \ldots, x_i, x_j) = (x_n - x_i)^{-1}\left[ (x_n - x_i) f(x_n, x_i, \ldots, x_i) + \right. \]
\[ + (1 - x_n - x_i) f(x_n, x_i, \ldots, x_i) \].
So we get
\[ f(x_1, x_2, \ldots, x_{n-k+1}) = \sum_{i,j=1}^{n-1} \tau_{ij} f(x_n, x_i, \ldots, x_i), \]
where \( \sum_{i,j=1}^{n-1} \tau_{ij} = 1 \) and each \( |\tau_{ij}| \leq C^{2(n-3)+2} \).

**DEFINITION 10.5.** Let \( n \in \mathbb{N}, n \geq 3 \). We say that \( X \) has property \( B_n \) if
for each \( a \in X \) there exists \( \delta > 0 \) and a constant \( C > 0 \) such that for all \( x_1, x_2 \in X \) with \( |x_1 - a| < \delta \), \( |x_2 - a| < \delta \), \( x_1 \neq x_2 \) there exist \( x_3, \ldots, x_n \) such that \( \{x_1, x_2, \ldots, x_n\} \) is a C-polygon i.e., \( C(x_1, \ldots, x_n) \leq C \).

By convention, every \( X \) has the properties \( B_1 \) and \( B_2 \).

We say that \( X \) has property \( B_n \) uniformly if there exists \( C > 0 \) such that for all \( x_1, x_2 \in X \), \( x_1 \neq x_2 \) there exists \( x_3, \ldots, x_n \) such that \( \{x_1, x_2, \ldots, x_n\} \) is a C-polygon.

As an example, let \( X = \{x \in X : |x| \leq 1\} \). We will show that \( X \) has property \( B_n \) uniformly for every \( n \). It suffices to show that for \( x_1, x_2 \in X \)
with \(|x_1 - x_2| = 1\) we can extend \(\{x_1, x_2\}\) to a C-polygon. If the residue class field of \(K\) is infinite, then for every \(n\) we can choose \(x_3, \ldots, x_n\) such that \(|x_i - x_j| = 1\) whenever \(i \neq j, 1 \leq i, j \leq n\). So in this case \(X\) has property \(B_n\) uniformly for every \(n\) and we may take \(C = 1\). If the valuation of \(K\) is dense then for each \(\epsilon > 0\) and \(n \in \mathbb{N}\) there exists \(x_3, \ldots, x_n \in K\) such that \(1 - \epsilon < |x_3| < |x_4| \ldots < |x_n| < 1\).

It follows that \(\min\{|x_i - x_j|, 1 \leq i, j \leq n, i \neq j\} = |x_3| > 1 - \epsilon\). Hence \(\{x_1, x_2, \ldots, x_n\}\) is a C-polygon where \(C = (1 - \epsilon)^{-1}\). So also in this case \(X\) has property \(B_n\) for each \(n\) and we may choose for \(C\) any number \(> 1\).

If the valuation of \(K\) is discrete and the residue class field is finite then \(K\) is a local field. Let the residue class field have \(q\) elements, and let \(\pi\) be the smallest value that is greater than 1. Write \(n\) in base \(q\): 

\[
n = a_0 + a_1 q + \ldots + a_s q^s \quad (a_i \in \{0, 1, \ldots, q-1\}, a_s \neq 0)
\]

The equivalence relation \(x \sim y\) if \(|x - y| \leq \pi^{-s-1}\) divides \(X\) into \(q^{s+1}\) classes, having distances \(\geq \pi^{-s}\) to one another. Hence we can choose \(x_3, \ldots, x_n\) such that \(\inf\{|x_i - x_j|, i \neq j\} \geq \pi^{-s}\). It follows that for all \(n\) \(X\) has property \(B_n\) uniformly, and, given \(n\), we can choose for \(C\) the number \(\pi^s\), where \(s = \lfloor \frac{\log n}{\log q} \rfloor\).

Since every ball \(\{x \in K : |x-a| \leq r\} (r \in |K|)\) can be obtained by \(\{x \in K : |x| \leq 1\}\) via a similarity we find

**Lemma 10.6.** If \(X\) is a ball in \(K\) then \(X\) has property \(B_n\) uniformly for every \(n\). If \(X\) is an open subset of \(K\) then \(X\) has property \(B_n\) for every \(n\).

**Proof:** The second statement follows from the first and from the fact that having property \(B_n\) is a local property. The first statement has
already been proved for balls of the type \( \{ x \in K : |x-a| \leq r \} \) where \( r \in |K| \). If \( X = \{ x \in K : |x-a| < r \} \) and \( x_1, x_2 \in X, x_1 \neq x_2 \), then \( \{ x \in K : |x-x_1| \leq |x_1-x_2| \} \subset X \) and we can apply the foregoing.

Now we can prove the main theorem of this chapter:

**Theorem 10.7.** Let \( X \) have property \( B_n \). Then \( C^n(X) = C(X) \).

**Proof:** By induction on \( n \). The theorem is true for \( n = 1, 2 \) (10.3). Suppose \( C^{n-1}(X) = C(X) \). We only need to prove \( C^n(X) \subset C(X) \). So let \( f \in C^n(X) \). Then by 10.3 we have also \( f \in C^{n-1}(X) \), hence, by the induction hypothesis, \( f \in C^{n-1}(X) \). Then \( \rho_1 f, \ldots, \rho_n f \) are well-defined (see the definition preceding 8.16). Let \( a \in X \). We will show that

\[
\lim_{i \to \infty} \rho_i f(x,y) \text{ exists and does not depend on } i. \quad (\text{Then, by 8.19 we are done}).
\]

Since \( f \in C^n(X) \) and \( f \in C^{n-1}(X) \):

\[
f(x) = f(y) + \sum_{i=1}^{n-1} (x-y)^i \frac{\partial_i f(y)}{i!} + (x-y)^{n-1} R_n f(x,y)
\]

By the uniqueness of the \( D_i \) we have \( D_i f = D_i f (1 \leq i \leq n-2) \) and

\[
(x-y)^n \frac{\partial_{n-1} f(y)}{n!} + (x-y)^{n-1} R_n f(x,y) = (x-y)^n \frac{\partial_{n-1} f(x,y)}{n!}.
\]

Take \( \lim \) at both sides. We find \( D_{n-1} f(y) = D_{n-1} f(y) \) and hence

\[
R_n f(x,y) = (x-y)^{-1} (\frac{\partial_{n-1} f(x,y)}{n!} - \frac{\partial_{n-1} f(y)}{n!}) = \frac{\partial_{n-1} f(x,y)}{n!} = \rho_1 f(x,y).
\]

Since \( f \in C^n(X) \), \( R_n f \) is continuous, so \( a := \lim_{(x,y) \to (a,a)} R_n f(x,y) \) exists.

Since \( X \) has property \( B_n \) there is \( r \) and \( C \geq 1 \) such that for each \( x_1, x_2 \) with \( |x_1-a| < r, |x_2-a| < r \) there exist \( x_3, \ldots, x_n \in X \) with
Let $\epsilon > 0$. Then there is $\delta < \epsilon$ such that $|x-a| < \delta$, $|y-a| < \delta$, $x \neq y$ implies $|p_1 f(x,y) - a| < \epsilon C^{-2(n-2)}$. Now we claim: for all $i = 1, \ldots, n$, $|x-a| < \delta C^{-1}$, $|y-a| < \delta C^{-1}$, $x \neq y$ implies $|p_i f(x,y) - a| < \epsilon$.

To see this, choose $|x_1-a| < \delta C^{-1}$, $|x_2-a| < \delta C^{-1}$, $x_1 \neq x_2$. There are $x_3, \ldots, x_n \in X$ such that $|x_m-x_j| \leq C|x_k-x_1|$ for all $m,j,k,l$, $k \neq l$.

For any $k \in \{3, \ldots, n\}$ we have $|x_k-a| \leq |x_k-x_1| \lor |x_1-a| \leq C|x_1-x_2| \lor |x_1-a| < C\delta C^{-1} \lor \delta C^{-1} = \delta$. It follows that for all $k, l$, $k \neq l$

$$|p_1 f(x_k,x_1) - a| < \epsilon C^{-2(n-2)}.$$

By Lemma 10.4 we may write (for all $i$ with $1 \leq i \leq n$)

$$p_i f(x_1,x_2) = \sum_{k,l} \lambda_{kl} p_1(x_k,x_1),$$

where $\sum_{k,l} \lambda_{kl} = 1$, $|\lambda_{kl}| \leq C^{2(n-2)}$.

Hence

$$|p_i f(x_1,x_2) - a| = |\sum_{k,l} \lambda_{kl} (p_1(x_k,x_1) - a)| \leq C^{2(n-2)} \epsilon C^{-2(n-2)} = \epsilon.$$

Without extra assumptions on $X$ we can only prove the following.

(For $f \in C^n(X)$, let $D^i_n f(x) = R^i_n f(x,x)$ $(x \in X)$).

**Theorem 10.8.** Let $n \in \mathbb{N}$ and $f \in C^n(X)$. If $D^i_1 f \in C^{n-1}(X)$ $(1 \leq i \leq n-1)$ and if $D^i_1 D^j_1 f = \binom{i+j}{i} D^{i+j}_1 f$ (for all $i,j \in \mathbb{N}$, $i+j \leq n$), then $f \in C^n(X)$.

This theorem looks "better" if we state it in a special case:

**Corollary 10.9.** Let the characteristic of $K$ be 0. Suppose $f \in C^n(X)$,

$$i! D^i_1 f = f(i) \quad (0 \leq i \leq n), \quad f' \in C^{n-1}(X).$$

Then $f \in C^n(X)$.

In the sequel we will use 10.9 in a different form: Let $K$ have characteristic 0, and let $f \in C^{n-1}(X)$. Suppose $F$ is an antiderivative of $f$ such that there exists a continuous function $S : X \times X \to K$, zero on the
diagonal such that for all \(x,y \in X\)

\[
P(x) = P(y) + (x-y)f(y) + \ldots + (x-y)^n \frac{f^{(n-1)}(y)}{n!} + (x-y)^n S(x,y).
\]

Then \(P \in C^n(X)\).

We first prove 10.9 to be a corollary of 10.8. By induction, \(f^{(i)} \in C^{n-1}(X)\) hence \(D_i f = \frac{1}{i!} f^{(i)} \in C^{n-1}(X) (1 \leq i \leq n-1)\). If \(i, j \in \mathbb{N}\) and \(i+j \leq n\), then \(D_j f \in C^{n-1}\), hence \(D_i (D_j f) = \frac{1}{i!} (D_j f)^{(i)} = \frac{1}{i!} \frac{1}{j!} f^{(i+j)}\), and

\[
D_{i+j} f = \frac{1}{(i+j)!} f^{(i+j)}, \quad \text{so} \quad D_i D_j f = \binom{i+j}{i} D_{i+j} f.
\]

Proof of theorem 10.8. By induction on \(n\). For \(n = 1\) it is trivial. Suppose 10.8 is correct for \(n-1\) and let \(f \in C^n(X)\), \(D_i f \in C^{n-1}(X) (1 \leq i \leq n-1)\),

\[
D_i D_j f = \binom{i+j}{i} D_{i+j} f (i+j \leq n). \quad \text{Then the induction hypothesis tells us that} \quad f \in C^{n-1}(X). \quad \text{Thus,} \quad D_i f = D_i f (1 \leq i \leq n-1) \quad \text{and we have for all} \quad x,y:\n\]

\[
f(x) = f(y) + (x-y)D_1 f(y) + \ldots + (x-y)^n D_{n-1} f(y) + (x-y)^n f(x,y,y,\ldots,y).
\]

Let \(a \in X\).

Since \(f \in C^n(X)\) we know that \(\alpha := \lim_{x \to a} \phi_n f(x,y,\ldots,y) = \lim_{x \to a} \rho_i f(x,y) = \lim_{y \to a} \rho_i f(x,y)\) exists,

\[
\begin{align*}
\lim_{x \to a} \rho_i f(x,y) &= \lim_{y \to a} \rho_i f(x,y), \\
\lim_{x \to a} \rho_i f(x,y) &= \lim_{y \to a} \rho_i f(x,y).
\end{align*}
\]

Since \(D_{n-1} f \in C^{n-1}\) \((1 \leq i \leq n-1)\) it follows that

\[
(\rho_i) D_n f(a) = D_{n-1} f(a) = \lim_{x \to a} \phi_n (D_{n-1} f)(x,x,\ldots,x,y)
\]

exists for \(1 \leq i \leq n-1\). From Lemma 8.16 we obtain inductively that

also \(\alpha_i := \lim_{x \to a} \rho_i f(x,y)\) exists for \(1 \leq i \leq n\), and we have the equations:

\[
(\rho_i) (\phi_n f(a)) = \sum_{s=1}^{n} \binom{n}{s-1} \alpha_s (1 \leq i \leq n),
\]

where \(\alpha_n = D_n f(a)\). By taking \(i = n, n-1, \ldots\) successively we arrive at

\[\alpha_s = D_n f(a)\] for each \(s\). Now apply 8.19. It follows that \(f \in C^n(X)\).
COROLLARY 10.10. Let \( f : X \to K, n \in \mathbb{N} \). Then the following are equivalent:

(a) \( f \in C^n(X), \ D_1 f = D_2 f = \ldots = D_n f = 0 \).

(b) \( \lim_{x \to a} (x-y)^{-n} (f(x) - f(y)) = 0 \) for each \( a \in X \).

(c) \( f \in C^n(X), \ D_1 f = D_2 f = \ldots = D_n f = 0 \).

If, in addition, \( X \) has property \( B_n \), and the characteristic of \( K \) is zero, then (a), (b), (c) are equivalent to

(d) \( \lim_{x \to a} \frac{f(x) - f(y)}{(x-y)^n} \) exists for each \( a \in X \).

(e) \( f \in C^n(X), \ f' = 0 \).

Proof: (a) \( \Rightarrow \) (c), (c) \( \Rightarrow \) (b) are clear from the definitions. To show (b) \( \Rightarrow \) (a), observe that \( f(x) = f(y) + (x-y)^n R(x,y) \) where \( R \) is a continuous function on \( X \times X \) and \( R(x,x) = 0 \) for all \( x \). Hence \( f \) satisfies the conditions of 10.8 : \( f \in C^n(X) \).

Suppose (d). Then there is a continuous \( R : X \times X \to K \) such that

\[
f(x) = f(y) + (x-y)^n R(x,y) \quad (x,y \in X).
\]

It follows that \( f \in C^n(X) = C^n(X) \) and that \( D_1 f = \ldots = D_{n-1} f = 0 \). But

\[
D_n f(x) = \frac{D_n f(x)}{n!} = 0.
\]

COROLLARY 10.11. If \( f \in C^\infty(X), \ f' = 0 \) and the characteristic of \( K \) is zero, then

\[
(*) \quad \lim_{x \to a} \frac{f(x) - f(y)}{(x-y)^n} = 0 \quad (a \in X)
\]

for every \( n \in \mathbb{N} \). Conversely, if \( f \) is a function for which \( (*) \) holds for every \( n \) then \( f \in C^\infty \) and \( f' = 0 \).
Note. The reader may recall example 8.8bis and investigate its role with respect to the statements in 10.10 and 10.11.

THEOREM 10.12. Let $n > 2$ and let $X$ have property $B_n$ uniformly in the sense that every set of two elements can be extended to a $C$-polygon. Then $BC^n(X) = BC^n(X)$ and $\| \|_n \leq c^{2(n-2)} \| \|_n^\sim$.

Proof: Obviously we have a norm-decreasing embedding $BC^n(X) \hookrightarrow BC^n(X)$.

Let $f \in BC^n(X)$. Then, by 10.7, $f \in C^n(X)$, so we only have to prove $\| f \|_n \leq c^{2(n-2)} \| f \|_n^\sim$. Obviously, $\| \rho f \|_\infty \leq \| f \|_n^\sim$. By lemma 10.4 we have ($1 \leq i \leq n$)

$$\rho_i f(x_1, x_2) = \sum_{k,l} \rho_1(x_k, x_l) \lambda_{kl}$$

where $\sum \lambda_{kl} = 1$, $|\lambda_{kl}| \leq c^{2(n-2)}$ for all $k, l$. Thus: $\max_{1 \leq i \leq n} \| \rho_i f \|_\infty \leq c^{2(n-2)} \| f \|_n^\sim$.

Now apply 8.20 and 8.6.
11. ANTIDERIVATIVES OF $C^n$-FUNCTIONS

In this Chapter we will discuss the question whether an $f \in C^n(X)$ possesses an antiderivative $F \in C^{n+1}(X)$ and, in case the answer is "yes", we also will ask whether we can construct a continuous linear antiderivation map : $C^n(X) \rightarrow C^{n+1}(X)$. (For $n = 0$ we did so in Chapter 5.) (For real-valued functions defined on an interval the whole problem is trivial.)

Our first idea is to use the map $P$ of Chapter 5 (see 5.4). But it turns out that this $P$ does not work for $n > 1$. It is worth investigating in detail what goes wrong:

**Theorem 11.1.** For $f \in C^1(X)$, let (with $x_n$ as in 5.4)

$$ (Pf)(x) = \sum_{n=1}^{\infty} f(x_n)(x - x_n). $$

Then $Pf \in C^2(X)$ implies $f' = 0$.

**Proof:** If the characteristic of $K$ happens to be 2 then $f' = D_1D_2Pf = 2!D_2Pf = 0$, so suppose the characteristic of $K \neq 2$. From the definition of $Pf$ it follows that (since $(x_n)^k = x_n^k$ for all $x \in X$; $k,n \in \mathbb{N}$)

$$ (Pf)(x_k) = \sum_{n<k} f(x_n)(x_{n+1} - x_n), \quad (k \in \mathbb{N}, k > 1) $$

$$ (Pf)(x_1) = 0. $$

Hence

$$ (*) \quad Pf(x_{k+1}) - Pf(x_k) = f(x_k)(x_{k+1} - x_k) \quad (k \in \mathbb{N}). $$

Since $Pf \in C^2(X)$ there exists a continuous $H : X^2 + K$ such that

$$ (**) \quad Pf(x_{k+1}) - Pf(x_k) = (x_{k+1} - x_k)f(x_k) + (x_{k+1} - x_k)h(x_{k+1}, x_k) \quad (k \in \mathbb{N}) $$

and such that $H(x,x) = D_2 Pf(x) = \frac{1}{2}D_1D_2 f(x) = \frac{1}{2}f'(x)$, for all $x \in X$. 

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From (*), (**) we infer

\[ H(x_{k+1}, x_k) = 0 \quad (x \in X, k \in \mathbb{N}, x_{k+1} \neq x_k) \]

Now let \( a \in X \). We show that \( H(a, a) = 0 \). Let \( \varepsilon > 0 \). Then there is a ball \( B \subseteq \{ x : |x-a| < \varepsilon \} \) that does not contain \( a \). There is \( k \in \mathbb{N} \) such that \( R_k \cap B \neq 0 \) and \( R_{k+1} \cap B \neq R_k \cap B \) (with \( R_k \) as in the definition preceding 5.3). So choose \( x \in R_{k+1} \setminus R_k, x \in B \). Then \( x = x_{k+1}, x_k \neq x_{k+1} \) (since \( x \notin R_k, x_k \in R_k \), \( |x-a| < \varepsilon \). It follows that \( (a, a) \) can be approximated by elements of the type \((x_{k+1}, x_k)\) where \( x \in X, k \in \mathbb{N}, x_{k+1} \neq x_k \). By continuity of \( H \), \( H(a, a) = 0 \).

Formula (*) may give us a clue to finding the antiderivation map : \( C^n(X) \to C^{n+1}(X) \). Since \( Pf \in C^1(X) \) we have

\[ Pf(x_{n+1}) - Pf(x_n) = (x_{n+1} - x_n)^T \bar{\Phi}_1 Pf(x_{n+1}, x_n). \]

Hence \( \bar{\Phi}_1 Pf(x_{n+1}, x_n) = f(x_n) \) for all \( x \in X, n \in \mathbb{N} \). Conversely, if \( F \) is a \( C^1 \)-antiderivative of \( f \) such that \( F(x_1) = 0 \) and

\[ \bar{\Phi}_1 F(x_{n+1}, x_n) = f(x_n) \quad (x \in X, n \in \mathbb{N}) \]

then \( F = Pf \).

We may use a similar idea when trying to construct a \( C^{n+1} \)-antiderivative \( F \) of \( f \in C^n(X) \). Suppose \( F \) is such an antiderivative. Then for any \( x \in X, k \in \mathbb{N} \)

\[ F(x_{k+1}) - F(x_k) = \sum_{i=1}^{n+1} (x_{k+1} - x_k)^T D_i F(x_k) + (x_{k+1} - x_k)^T n+1 \bar{\Phi}_n+1 F(x_{k+1}, x_k, \ldots, x_k) \]

\[ - D_{n+1} F(x_k) \]

hence (suppose the characteristic of \( K \) is 0, and require that

\[ \bar{\Phi}_{n+1} F(x_{k+1}, x_k, \ldots, x_k) = D_{n+1} F(x_k) \]

\[ F(x_{k+1}) - F(x_k) = \sum_{i=1}^{n+1} \frac{1}{i} (x_{k+1} - x_k)^T D_{i-1} f(x_k). \]
If we also require that $F(x^*) = 0$ then we obtain for any $x \in X$

$$F(x) = \sum_{k=1}^{\infty} \frac{1}{k!} (x_{k+1} - x_k)^i D_{i-1} f(x_k).$$

Thus, we define $P_n f(x) = \sum_{k=1}^{\infty} \frac{1}{k!} (x_{k+1} - x_k)^i D_{i-1} f(x_k)$ ($f \in C^{n-1}(X), x \in X$).

First we see that $P_n f(x)$ is well defined for every $x \in X$ since the series involved is convergent. We define a map $R : \mathbb{V}^2 X + K$ via

$$R_n f(x) = \sum_{i=1}^{n} \frac{(x-y)^i}{i!} f(i-1)(y) + (x-y)^n R(x,y) \quad ((x,y) \in \mathbb{V}^2 X)$$

Suppose we can show that $R$ can be extended to a continuous $\overline{R} : X^2 + K$, where $\overline{R}(x,x) = 0$ for all $x \in X$. Then clearly $(P_n f)' = f$ and by Corollary 10.9 we then have $P_n f \in C^n(X)$. Thus, we have to show for any $x \in X$

$$\lim_{x \to a} R(x,y) = 0.$$  

We write for each $i \in \{1, \ldots, n\}$

$$D_{i-1} f(x_k) = \sum_{j=0}^{n-i} \frac{1}{j!} (x_k - y)^j D_j D_{i-1} f(y) + (x_k - y)^{n-1} A_{i-1}(x_k, y),$$

where $A_{i-1}$ is a continuous function, zero on the diagonal, viz.

$$A_{i-1}(x,y) = \sum_{i=1}^{n-i} \frac{1}{j!} (x_k - y)^j D_j D_{i-1} f(y) - D_{n-j} D_{i-1} f(y)$$

and we substitute this in the defining formula for $P_n f$. We get

$$P_n f(x) = \sum_{k=1}^{\infty} \sum_{i=1}^{n} \sum_{j=0}^{n-i} \frac{1}{i!} (x_k - x_{k-1})^i (x_k - y)^j (y^{i+j-1}) D_{i+j-1} f(y) + \sum_{k=1}^{\infty} \frac{1}{i!} (x_k - x_{k-1})^i (x_k - y)^{n-1} A_{i-1}(x_k, y)$$

$$= A + B.$$  

Now

$$\sum_{i=1}^{n} \frac{1}{i!} (x_k - x_{k-1})^i (x_k - y)^j \sum_{v=1}^{n} \frac{v}{v} D_{v-1} f(y) =$$

$$= \sum_{i=1}^{n} \sum_{v=1}^{n} (x_k - y)^v \sum_{i=1}^{n} \frac{1}{v} D_{v-1} f(y).$$
Thus \( A = \sum_{v=1}^{n} \sum_{k=1}^{\infty} (x_{k+1} - y)^{v} - (x_{k} - y)^{v} \frac{1}{v} D_{v-1} f(y) = \)
\[ = \sum_{v=1}^{n} (x - y)^{v} \frac{1}{v} D_{v-1} f(y) - \sum_{v=1}^{n} (x_{1} - y)^{v} \frac{1}{v} D_{v-1} f(y). \]

Similarly we can also write \( \sum_{n} f(y) = A_{1} + B_{1} \) and we get
\[ A_{1} = \sum_{v=1}^{n} (y - y)^{v} \frac{1}{v} D_{v-1} f(y) - \sum_{v=1}^{n} (y_{1} - y)^{v} \frac{1}{v} D_{v-1} f(y). \]

So, if we suppose \( x_{1} = y_{1} \) then:
\[ A - A_{1} = \sum_{v=1}^{n} (x - y)^{v} \frac{1}{v} D_{v-1} f(y). \]

It follows that \((x - y)^{n} R(x, y) = B - B_{1}\). Let \( \epsilon > 0 \). Then there is \( s \in \mathbb{N} \)
such that for \( i \in \{1, \ldots, n\} \)
\[ |A_{i-1}(x, y)| < \epsilon \]
whenever \( |x-a| < r_{s}, |y-a| < r_{s} \).

We choose such \( x, y, x \neq y \). Then \( x_{1} = y_{1}, \ldots, x_{m} = y_{m}, x_{m+1} \neq y_{m+1} \) for
some \( m > s \). Then \( r_{m+1} \leq |x-y| < r_{m} \).

We have
\[ B - B_{1} = \sum_{k>m}^{n} \frac{1}{i!} (x_{k+1} - x_{k})^{i} (x_{k} - y)^{n-i} A_{i-1}(x_{k}, y) - \frac{1}{i!} (y_{k+1} - y_{k})^{i} (y_{k} - y)^{n-i} A_{i-1}(y_{k}, y). \]

For \( k > m \) we have:
\[ |x_{k+1} - x_{k}| \leq |x_{m+1} - x_{m}| \leq r_{m}, |x_{k} - y| \leq |x_{k} - x| \lor |x - y| < \]
\[ \leq |x_{m} - x| \lor |x - y| \leq r_{m}. \]
Similarly, \( |y_{k+1} - y_{k}| \leq r_{m}, |y_{k} - y| \leq r_{m} \).

We find
\[ \left| B - B_{1} \right| \leq \max_{1 \leq i \leq n} \frac{1}{i!} r_{m}^{i} \varepsilon, \] so
\[ |R(x, y)| = \frac{|B - B_{1}|}{(x - y)^{n}} \leq \max_{1 \leq i \leq n} \frac{1}{i!} \cdot \left( \frac{r_{m}}{r_{m+1}} \right)^{n} \varepsilon. \]

If \( \frac{r_{m}}{r_{m+1}} \) is bounded then \( \lim_{(x, y) \to (a, a)} R(x, y) = 0 \). Thus we have
\[ \text{THEOREM 11.2. Let the characteristic of } K \text{ be zero. Let } r_{1}, r_{2}, \ldots \text{ be as} \]
\[ \text{in 5.3 such that there is } \rho > 1 \text{ for which } r_{m} \leq \rho r_{m+1} \text{ for} \]
\[ \text{all } m \in \mathbb{N}. \text{ (For each } \rho > 1 \text{ we can construct such } r_{1}, r_{2}, \ldots).} \]
With $x_n$ as in 5.4, for $f \in C^{n-1}(X)$, set

$$P_n f(x) = \sum_{k=1}^{\infty} \sum_{i=1}^{n} \frac{1}{i} (x_{k+1} - x_k) D_{i-1} f(x) \quad (x \in X)$$

Then $P_n f \in C^n(X)$ and $(P_n f)' = f$.

Of course $P_n$ is a linear map $C^{n-1}(X) \to C^n(X)$. We expect $P_n$ to map $BC^{n-1}(X)$ into $BC^n(X)$ and so we estimate the norm of $P_n$. Now by 8.21, for $f \in C^{n-1}(X)$

$$\|P_n f\|_n = \max_{0 \leq i \leq n} \|D_i P_n f\|_{n-i}.$$  

If $i > 0$ then $D_i P_n f = i! D_i^1 P f = i! D_i^{-1} f = i D_{i-1} f$, hence by 8.17

$$\|D_i P_n f\|_{n-i} = \|i D_{i-1} f\|_{n-i} \leq \|D_{i-1} f\|_{n-i} \leq \|f\|_{n-1}.$$  

If $i = 0$, then $\|P_n f\|_n = \max_{0 \leq i < n} \|D_i P_n f\|_\infty \vee \|R_{P_n} f\|_\infty$.

Again we have $\|D_i P_n f\|_\infty \leq \|f\|_{n-1}$ for $0 \leq i \leq n$, so we only need to estimate $\|P_n f\|_\infty$ and $\|R_{P_n} f\|_\infty$.

Since we have chosen $1 > r_1 > r_2$ etc. we get $|x_{k+1} - x_k| \leq 1$ for all $x \in X$, $k \in \mathbb{N}$, so

$$|P_n f(x)| \leq \max_{1 \leq i \leq n} \frac{1}{i} \|D_{i-1} f\|_\infty \leq \max_{1 \leq i \leq n} \frac{1}{i} \|f\|_{n-1}.$$  

For $x, y \in X$, $x \neq y$ we have

$$|R_{P_n} f(x, y)| \leq |R_{P_n} f(x, y) - D_{P_n} f(y)| + |D_{P_n} f(y)|$$

and we have seen already that $|D_{P_n} f(y)| \leq \|f\|_{n-1}$. The other term, $R_{P_n} f(x, y) - D_{P_n} f(y)$ is what we called $R(x, y)$ in the proof of 11.2. Thus, borrowing the notations of that proof we get

$$(*) \quad R(x, y) = (x-y)^{-n} \sum_{k=1}^{\infty} \sum_{i=1}^{n} \frac{1}{i} (x_{k+1} - x_k)\frac{1}{i} (x_k - y) \h_{i-1} (x_k, y) - \frac{1}{i} (y_{k+1} - y_k)\frac{1}{i} (y_k - y) \h_{i-1} (y_k, y)),$$

where $|\Lambda_{i-1}(x_k, y)| \leq \|f\|_{n-1}$, $|\Lambda_{i-1}(y_k, y)| \leq \|f\|_{n-1}$ (see the proof of 11.2).
If $r_{m+1} \leq |x-y| < r_m$ for some $m \in \mathbb{N}$ we can see from the proof of 11.2 that

$$|R(x,y)| \leq \max_{1 \leq i \leq n} \frac{1}{|i|} \cdot \left(\frac{r_m}{r_{m+1}}\right)^n \cdot |f|_{n-1} \leq \max_{1 \leq i \leq n} \frac{1}{|i|} \cdot \rho^n \cdot |f|_{n-1}.$$  

In case $|x-y| \geq r_1$, then $|\left(x_{k+1} - x_k\right)^i (x_k - y)^{n-i}| \leq r_1^n$ ($k \in \mathbb{N}$, $1 \leq i \leq n$) hence

$$|R(x,y)| \leq \max_{1 \leq i \leq n} \frac{1}{|i|} \cdot |f|_{n-1}.$$  

**Theorem 11.3.** Let the characteristic of $K$ be zero and let $P_n : C^{n-1}(X) \to C^n(X)$ as in 11.2. Then

$$||P_n f||_n \leq C_n ||f||_{n-1} \quad (f \in BC^{n-1}(X))$$  

where

$$C_n = \max_{1 \leq i \leq n} \frac{1}{|i|} \cdot \rho^n$$

Thus, for each $\varepsilon > 0$ there exists a continuous linear map

$$P_{n,\varepsilon} : BC^{n-1}(X) \to BC^n(X)$$

such that $DP_{n,\varepsilon}$ is the identity on $BC^{n-1}(X)$ and

$$||P_{n,\varepsilon}|| \leq 1 + \varepsilon.$$  

Now let us suppose that the characteristic of $K$ equals $p \neq 0$. Then we have the following

**Theorem 11.4.** Let the characteristic of $K$ be $p \neq 0$. Then if $n < p$ and $\varepsilon > 0$ there is a linear map $P_n : C^{n-1}(X) \to C^n(X)$ such that $(P_n f)' = f$ for all $f \in C^{n-1}(X)$ and such that the restriction of $P_n$ to $BC^{n-1}(X)$ maps into $BC^n(X)$ and has norm less than $1 + \varepsilon$. If $n \geq p$ such a map does not exist: differentiation: $C^n(X) \to C^{n-1}(X)$ is not surjective.

**Proof:** If $n < p$ we can just formally take over the definitions and
reasonings of 11.2 and 11.3. If \( n \geq p \), consider the function \( f : x \mapsto x^{p-1} \) and suppose there is \( F \in \mathcal{C}^n(X) \) such that \( F' = f \). Since \( F \in \mathcal{C}^p(X) \) we have
\[
1 = D_{p-1}D_{p-1}D_1F = pD_{p-1}D_1F = 0,
\]
a contradiction.

**Note.** In case the characteristic of \( K \) is zero, define for \( n \in \mathbb{N} \)
\[
Q_n = n! P_n P_{n-1} \cdots P_1
\]
then \( Q_n : \mathcal{C}^0(X) \to \mathcal{C}^n(X) \) and \( D_nQ_n \) is the identity on \( \mathcal{C}^0(X) \). A tedious but elementary computation shows that
\[
(*) \quad Q_n = \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} M_i S_i
\]
where
\[
(Mf)(x) = xf(x) \quad (x \in X, f : X \to K)
\]
and, for each \( i \),
\[
(S_i f)(x) = \sum_{k=1}^{\infty} f(x_k) (x_{k+1}^i - x_k^i) \quad (x \in X, f \in \mathcal{C}(X)).
\]
[Note that (*) also makes sense in case the characteristic of \( K \) is non-zero. We are too lazy to show that in this case \( Q_n \) (now defined by formula (*)) is a right inverse of \( D_n \).]
12. THE SPACE $C^\infty(X)$

First we introduce three spaces of $C^\infty$-functions and their topologies.

(1) On $C^\infty(X)$ (see definition 8.1) we define seminorms $||\phi_i||_n, C$ $(n \in \mathbb{N}, C \subseteq X$ compact) via

$$||\phi_i||_n, C = \max_{0 \leq i \leq n} ||\phi_i f||_C \quad (f \in C^\infty(X))$$

(see the definition following 8.6). The seminorms $||\phi_i||_n, C$ where $n$ runs through $\mathbb{N}$ and $C$ runs through the class of non-empty compact subsets of $X$ define a locally convex topology on $C^\infty(X)$. From now on we always assume $C^\infty(X)$ to be equipped with this topology.

(2) On $BC^\infty(X)$ (see definition 8.1) we define the norms $||\phi_i||_n$ $(n \in \mathbb{N})$ via

$$||\phi_i||_n = \max_{0 \leq i \leq n} ||\phi_i f||_{\infty} \quad (f \in BC^\infty(X))$$

The norms $||\phi_i||_n$, where $n$ runs through $\mathbb{N}$ define a locally convex topology on $BC^\infty(X)$. From now on we assume $BC^\infty(X)$ to be equipped with this topology.

(3) Finally, let us call $B C^\infty(X) = \{f \in BC^\infty(X) : \sup_n ||f||_n < \infty\}$. Then $B C^\infty(X)$ is a linear space, normed via

$$||f||^\infty = \sup_n ||\phi_i||_n \quad (f \in BC^\infty(X))$$

From now on we assume $B C^\infty(X)$ to be normed with $||\phi_i||$. 

**Theorem 12.1.** Let $A$ be any of the spaces $C^\infty(X)$, $BC^\infty(X)$, $B C^\infty(X)$.

Then $A$ is complete. Differentiation is a continuous linear map $A \to A$. Let $N = \{f \in A : f' = 0\}$. Then $N$ is closed.

**Proof:** Let $(f_\lambda)$ be a Cauchy net in $C^\infty(X)$. Then, for each $n \in \mathbb{N}$, $(f_\lambda)$
is a Cauchy net in $C^n(X)$. Since $C^n(X)$ is complete, there is $f_n \in C^n(X)$ such that $f^\lambda_n + f_n$ in the topology of $C^n(X)$. Then also $f^\lambda_n + f_n$ uniformly on compact sets, hence $f_m = f_n$ for all $n,m \in \mathbb{N}$. Hence $f \in C^\infty(X)$ and $f^\lambda_n + f$ in the topology of $C^n(X)$ for each $n \in \mathbb{N}$. Thus $f^\lambda_n + f$ in $C^\infty(X)$: $C^\infty(X)$ is complete.

Let $(f^\lambda)$ be a Cauchy net in $BC^\infty(X)$. Then, for each $n \in \mathbb{N}$, $(f^\lambda_n)$ is a Cauchy net in $BC^n(X)$, so $f^\lambda_n + f$ in the topology of each $BC^n(X)$ for some $f \in BC^\infty(X)$. Then, $BC^\infty(X)$ is complete.

Let $f_1, f_2, \ldots$ be a Cauchy sequence in $BC^\infty(X)$. Then there is $f \in C^\infty(X)$ such that $f_k + f$ in each $BC^n(X)$. Let $\varepsilon > 0$. There is $N$ such that $k, l \geq N$ implies $||f_k - f_l||^\infty < \varepsilon$. For any $n \in \mathbb{N}$ and $k, l \in \mathbb{N}$:

$$||f_k - f||_n \leq ||f_k - f_1||_n + ||f_1 - f_k||_n.$$ 

Hence, by taking $\lim_{l \to \infty}$ at both sides we get

$$||f_k - f||_n \leq \varepsilon.$$ 

Thus, $||f_k - f||^\infty < \varepsilon$: $\lim f_k = f$ in the sense of $BC^\infty(X)$.

The rest of the proof is left to the reader.

**Lemma 12.2.** Let $f_1, f_2, \ldots$ be a sequence in $C^\infty(X)$ such that $f_m \in BC^m(X)$ for each $m$ and such that $\lim_{m \to \infty} ||f_m||_m = 0$. Then $\sum f_m \in C^\infty(X)$.

**Proof:** Clearly, for $m \geq n$ we have $||f_m||_n \leq ||f_m||_m$, so $f_m \in BC^n(X)$ for $m \geq n$ and $\lim_{m \to \infty} ||f_m||_n = 0$. This means that $\sum f_m$ converges in $BC^n(X)$, so $\sum f_m$ is an element of $C^n(X)$. But we can do this for every $n \in \mathbb{N}$, so that $\sum f_m \in C^\infty(X)$.

Theorem 8.22, applied to an $f \in C^\infty(X)$, tells us that for every $n \in \{0,1,2,\ldots\}$ we can find a spline function $f_m$ of degree $\leq n$ such that $||f - f^m||_n < \frac{1}{n}$. For $m > n$ we have $||f - f^m||_n \leq ||f - f^m||_m < \frac{1}{m}$, so
the sequence \( f_n, f_{n+1}, \ldots \) converges to \( f \) in the \( || \cdot ||_{n,x} \)-norm. It follows that the sequence \( f_1, f_2, \ldots \) converges to \( f \) in each seminorm \( || \cdot ||_{n,C} \) (\( C \subset X \) compact). We can repeat the above reasoning for any \( n \in \mathbb{N} \), so \( \lim f_n = f \) in the sense of \( C^\infty (X) \). We have

**Theorem 12.3.** The spline functions form a sequentially dense subset of \( C^\infty (X) \). If \( f \in C^\infty (X) \), \( D^j f = 0 \) for \( j > i \), then there is a sequence of spline functions \( f_1, f_2, \ldots \) such that \( \lim f_k = f \) in the sense of \( C^\infty (X) \) and such that the degree of each \( f_k \) is \( \leq i-1 \).

To prove a theorem, similar to 12.3 for \( BC^\infty (X) \), we have to go back to the proof of 8.22. Suppose we have \( f \in BC^\infty (X) \). Then for each \( n, \Phi_n f \) is uniformly continuous, since \( \Phi_{n+1} f \) is bounded. So, the balls \( B_a(r) \) in the proof of 8.22 can be chosen to be all of the same radius \( r \leq 1 \).

The approximating spline function \( g \) then has the following property:

- If \( |x-y| \leq r \) then \( g(x) = g(y) + \sum_{i=0}^{n} (x-y)D^i g(y) \), and if \( |x-y| > r \) then \( |g(x)-g(y) - \sum_{i=0}^{n} (x-y)D^i g(y)| \leq ||f||_n \). Thus, for any \( k \geq n \) we find for \( k \neq i \)

\[
\Phi_k g(x,y,y,\ldots,y) = (x-y)^k (g(x)-g(y)-\sum_{i=0}^{n} (x-y)D^i g(y)) \quad (x \neq y)
\]

that \( |\Phi_k g(x,y,y,\ldots,y)| \leq r^k ||f||_n \). It follows that \( ||g||_k \leq \infty \).

Similarly, \( ||D^i g||_k \leq \infty \) for all \( i \in \{0,1,\ldots,n\} \), \( k \geq n \). Hence \( ||g||_k \leq \infty \) for each \( k \in \mathbb{N} \). We have proved: If \( f \in BC^\infty (X) \), \( n \in \mathbb{N} \) then there exists a spline function \( g_n \) of degree \( \leq n \), such that \( g_n \in BC^\infty (X) \) and \( ||f-g_n||_n \leq \frac{1}{n} \). It is now easy to see that \( g_1, g_2, \ldots \) converges to \( f \) in the sense of \( BC^\infty (X) \), for every \( n \). Thus:

**Theorem 12.4.** The spline functions in \( BC^\infty (X) \) form a sequentially dense
subset of $BC^\infty(X)$. If $f \in BC^\infty(X)$, $D_j f = 0$ for $j \geq i$, then there is a sequence of spline functions $f_1, f_2, \ldots$ such that $f_k \in BC^\infty(X)$, degree of $f_k < i-1$ for all $k$ and $\lim f_k = f$ in $BC^\infty(X)$.

**COROLLARY 12.5.** The locally constant functions in $C^\infty(X)$ (BC $^\infty(X)$) form a dense subset of \{f $\in C^\infty(X)$ : $D_1 f = D_2 f = \ldots = 0$\} (f $\in BC^\infty(X)$ : $D_1 f = D_2 f = \ldots = 0$). (In case the characteristic of $K$ is 0 then $D_1 f = D_2 f = \ldots = 0$ is equivalent to $f' = 0$).

One may wonder if we have a similar theorem for $BC^\infty(X)$. That the answer is "no" follows from the following

**THEOREM 12.6.** Let $X = \{x \in K : |x| < 1\}$. Then $BC^\infty(X)$ is the space $H(X)$ of all analytic functions $f : X \to K$:

$$f = \sum a_n x^n$$

for which $\sup |a_n| < \infty$ and $||f||^\infty = \sup |a_n|$.

**Proof:** Let $f \in BC^\infty(X)$. Define a function $g$ via

$$g(x) = \sum_{i=0}^{\infty} x^i D_i f(0) \quad (|x| < 1)$$

Clearly, $g \in H(X)$. Since for each $k, n \in \mathbb{N}$, $\phi_n(x^k)$ is a polynomial (in $n+1$ variables) with coefficients in $K$ we have $||\phi_n(x^k)||_\infty \leq 1$. We find for $n \in \mathbb{N}$

$$||\phi_n(g)||_\infty = ||\sum_{i=0}^{\infty} \phi_n(x^i) D_i f(0) ||_\infty \leq \sup_{i=0} |D_i f(0)| \leq ||f||^\infty.$$

It follows that $||g||_n$ is uniformly bounded hence $g \in BC^\infty(X)$.

Now $f(x) = \sum_{i=0}^s x^i D_i f(0) + x^{s+1} R_{s+1}(x, 0)$ where $|R_{s+1}(x, 0)| \leq ||f||^\infty$. We
see \( f(x) = \sum_{k=0}^{\infty} a_k x^k \), \( f = g \). Conversely, if \( g \in \mathcal{H}(X) \), \( g = \sum a_k x^k \) then \( |\phi_n(g)|_{\infty} = |\|a_k \phi_n(x^k)\|_{\infty} \leq \sup_n |a_n| \). Hence \( g \in \mathcal{B}^\infty(X) \) and \( |g|_{\infty} \leq \sup |a_n| \). The opposite inequality is clear.

We see that in case \( X = \{x \in \mathbb{R} : |x| < 1\} \) spline functions are polynomials and one can see easily that they do not form a dense subset of \( \mathcal{H}(X) \).

**COROLLARY 12.7.** **Analytic functions are** \( \mathcal{C}^\infty \).

**Proof:** Left to the reader.

Let \( 0 \in X \) and let \( f \in \mathcal{C}^{n+1}(X) \) such that \( f(0) = 0 \) for some \( n \subset \{0,1,2,\ldots\} \). Let us define the function: \( \frac{f}{x} \) via:

\[
\frac{f}{x}(x) = \begin{cases} 
\frac{f(x)}{x} & \text{if } x \neq 0 \\
 f'(0) & \text{if } x = 0 
\end{cases}
\]

Then clearly \( \frac{f}{x} \) is continuous. For \( (x_1,\ldots,x_n) \in \mathbb{R}^n \) we have

\[
\phi_{n+1}(x_1,x_2,\ldots,x_n,0) = \phi\left(\frac{f}{x}\right)(x_1,\ldots,x_n)
\]

(Proof: for \( n = 0 \) it is just the definition of \( \frac{f}{x} \). The induction step is very easy).

It follows that \( \phi\left(\frac{f}{x}\right) \) can be extended to a continuous function on \( \mathbb{R}^n \).

We have

**LEMMA 12.8.** Let \( f \in \mathcal{C}^\infty(X) \), \( f(a) = 0 \) for some \( a \in X \). Then

\( f = (x-a)g \) where \( g \in \mathcal{C}^\infty(X) \).

**Proof:** Simple.

**THEOREM 12.9.** Let \( f \in \mathcal{C}^\infty(X) \) and let \( f(a) = D_1 f(a) = \ldots = D_{n-1} f(a) = 0 \) for some \( a \in X \), \( n \in \mathbb{N} \). Then \( f = (x-a)^n g \), where \( g \in \mathcal{C}^\infty(X) \).

**Proof:** For \( n = 1 \) the statement is just 12.8. Suppose \( f = (x-a)^{n-1} h \)
where $h \in C^\infty(X)$, $f(a) = D_1 f(a) = \ldots = D_{n-1} f(a) = 0$. Now

$$D_{n-1} f = D_{n-1} (x-a)^{n-1} h = \sum_{i=0}^{n-1} D_i (x-a)^{n-1} \frac{D_{n-i-1} h}{D_{n-i-1} h} = h(a).$$

By lemma 12.8, $h = (x-a)g$ for some $g \in C^\infty(X)$.

A simple consequence is that we can describe all derivations on $C^\infty(X)$. Let $a \in X$ and let $\phi : C^\infty(X) \to K$ be a linear map satisfying

$$\phi(fg) = \phi(f)g + \phi(g)f, \quad (f, g \in C^\infty(X)).$$

Then $\phi(1) = \phi(1.1) = \phi(1) + \phi(1)$, so $\phi(1) = 0$. Let $\phi(x-a) = a$.

For $f \in C^\infty(X)$ we have $f = f(a) + (x-a)g$ for some $g \in C^\infty$, $f'(a) = g(a)$.

It follows that $\phi(f) = \phi(x-a)g = \phi(x-a)g(a) = af'(a)$. Hence, $\phi$ has the form

$$f \mapsto c f'(a)$$

for some constant $c \in K$.

A derivation on $C^\infty(X)$ is a linear map $\phi : C^\infty(X) \to C^\infty(X)$ such that

$$\phi(fg) = \phi(f)g + \phi(g)f, \quad (f, g \in C^\infty(X)).$$

**Theorem 12.10.** Let $\phi : C^\infty(X) \to C^\infty(X)$ be a derivation. Then there is a $g \in C^\infty(X)$ such that

$$\phi = g \frac{d}{dx}.$$

**Proof:** By the foregoing remark for each $a \in X$ there is an element $g(a)$ of $K$ such that $f \mapsto \phi(f)(a)$ has the form

$$f \mapsto g(a)f'(a).$$

Hence $\phi = g \frac{d}{dx}$ for some $g : X \to K$. Since $\phi(x) = g$, $g \in C^\infty(X)$.

Let $U \subset K$ be a clopen set such that $\inf(|x-y| : x \in U, y \notin U) = 0$. Then $\xi_U \in C^\infty(K)$ and for each $n \in \mathbb{N}$ we have $||\xi_U||_n = \max_{0 \leq i \leq n} |D_i \xi_U|^{(n)} = \sup_{x \neq y} |(x-y)^{-n}| |\xi_U(x) - \xi_U(y)| \leq n^r$. Thus,
for each $\varepsilon > 0$ we can find a $c \in K$ ($c \neq 0$) such that $|c \varepsilon y|_n < \varepsilon$.

Now let $Y \subseteq K$ be a closed subset. Let

$$U_0 = \{x \in K : d(x,Y) \geq 1\}$$
$$U_n = \{x \in K : \frac{1}{n+1} \leq d(x,Y) < \frac{1}{n}\} \quad (n \in \mathbb{N})$$

Then for each $n \in \{0,1,2,\ldots\}$ we have $d(U_n, U_n^c) > 0$. Define $f_n : K \to K$ such that $f_n = 0$ on $U_n^c$, $f_n$ is constant, nonzero, on $U_n$, $|f_n|_n < \frac{1}{n}$. By Lemma 12.2 $f := \sum f_n \in C^\omega(K)$. It follows that $f(x) = 0 \iff x \in Y$ and since $f' = 0$ on $Y^c$ and on $Y^0$, we must have $f' = 0$ everywhere. We have found:

**THEOREM 12.11.** Let $Y \subseteq K$ be a closed subset of $K$. Then there exists $f \in C^\omega(K)$ (with $f' = 0$) such that $Y$ is the set of zeros of $f$.

Finally, we will prove the non-archimedean version of Borel's Theorem:

**THEOREM 12.12.** Let $\lambda_0, \lambda_1, \ldots$ be any sequence in $K$. Then there exists a $C^\omega$-function $f : K \to K$ such that

$$D_i f(0) = \lambda_i \quad (i = 0, 1, 2, \ldots).$$

For the proof we need two simple lemmas on polynomials:

**LEMMA 12.13.** Let $P : x \mapsto \lambda_0 x^0 + \lambda_1 x^1 + \ldots + \lambda_m x^m$ be a polynomial function defined on $K$. For $n \leq m$ let $R_n$ be defined via

$$P(x) = P(y) + (x-y)D_1 P(y) + \ldots + (x-y)^{n-1} D_{n-1} P(y) + (x-y)^n R_n P(x,y) \quad (x,y \in K)$$

(see 8.7). Then we have for all $x, y \in K$, $|x| < 1$, $|y| < 1$

$$|R_n P(x,y) - \lambda_n| \leq \max(|x|, |y|) \cdot \max_{0 \leq i \leq m} |\lambda_i|.$$
Proof: Since $P$ has degree $\leq m$, we have $P_{m+1} = 0$, hence

$$P(x) = P(y) + (x-y)D_1 P(y) + \ldots + (x-y)^m D_m P(y) \quad (x,y \in K)$$

So, $R_n P(x,y) = D_n P(y) + (x-y)D_{n+1} P(y) + \ldots + (x-y)^{m-n} D_m P(y)$.

We have

$$D_n P(y) = D_n P(0) + y D_1 D_n P(0) + \ldots$$

and for $k > n$ $D_k P(y) = D_k P(0) + y D_1 D_k P(0) + \ldots$

It is easy to see that $|D_i D_j P(0)| \leq \max(|\lambda_0|, \ldots, |\lambda_m|)$ for all $i,j$, so we have for $|x| \leq 1$, $|y| \leq 1$:

$$|R_n P(x,y) - D_n P(y)| \leq |x-y| \max(|\lambda_0|, \ldots, |\lambda_m|)$$

and

$$|D_n P(y) - D_n P(0)| \leq |y| \max(|\lambda_0|, |\lambda_1|, \ldots, |\lambda_m|),$$

which proves 12.13.

**Lemma 12.14.** Let $Q : x \mapsto \lambda_{m+1,1} x^{m+1} + \ldots + \lambda_{s,1} x^s$. Then for all $x \in K$,

$$|x| \leq 1 \text{ and } i \leq m$$

$$|D_i Q(x)| \leq |x|^{m+1-i} \max(|\lambda_{m+1,1}|, \ldots, |\lambda_{s,1}|).$$

**Proof:** Obvious, since $D_i Q(x) = \lambda_{m+1,1} x^{m+1-i} + \lambda_{m+2,1} x^{m+2-i} + \ldots + \lambda_{s,1} x^s$.

**Proof of 12.12.** Choose $r_1, r_2, \ldots \in \mathbb{R}$ such that $1 > r_1 > r_2 > \ldots$ and

$$r_n \leq \frac{n}{\max_{0 \leq i \leq n} \left| \frac{\lambda_i}{\lambda_i^1} \right| + 1} \quad (n \geq 1)$$

For each $n \geq 1$ let $P_n(x) = \lambda_0 x + \lambda_1 x^2 + \ldots + \lambda_n x^n \quad (x \in K)$. Define a function $f : K \to K$ as follows

$$f(x) = \begin{cases} 0 & \text{if } |x| > r_1 \\ P_n(x) & \text{if } r_{n+1} < |x| \leq r_n \quad (n \geq 1) \\ \lambda_0 & \text{if } x = 0 \end{cases}$$

It is clear that $f$ is a spline function on $K \setminus \{0\}$. We show by induction
that \( f \in C^n(K) \) and \( D_i f(0) = \lambda_i \) \((0 \leq i \leq n)\). If \( r_{k+1} < |x| \leq r_k \) for some \( k \), then \( |f(x) - \lambda_0| = |P_k(x) - \lambda_0| = |\lambda_1 x + \ldots + \lambda_k x^k| \leq |x| \max |\lambda_i| < \frac{1}{k} \).

Thus, since \( \lim r_k = 0 \), \( f \) is continuous at 0. Suppose the statement is true for \( n-1 \). Define \( R_n \) by

\[
(*) \quad f(x) = f(y) + (x-y)D_1f(y) + \ldots + (x-y)^{n-1}D_{n-1}f(y) + (x-y)^nR_n f(x,y) \quad (x \neq y)
\]

By 10.7 \((K \text{ has property } B_n \text{ for all } n)\) in order to prove \( f \in C^n(K) \) it suffices to show that \( R_n \) can be extended to a continuous function on \( K^2 \). Since \( f \in C^\infty \) on \( K^* \) and since \( R_n f \) is continuous on \( K^2 \setminus \Delta \) (induction hypothesis) we have that \( R_n f \) can continuously be extended to \( K^2 \setminus \{0\} \). So we are done if we can show

\[
\lim_{x \to 0, y \to 0} R_n f(x,y) = \lambda_n.
\]

We prove:

If \( |x| \leq r_k, \quad |y| \leq r_k, \quad x \neq y, k \geq n \) then \( |R_n f(x,y) - \lambda_n| \leq \frac{1}{k} \).

Consider the following cases

A.1 \( r_{m+1} < |x| \leq r_m \) and \( r_{m+1} < |y| \leq r_m \) for some \( m \geq k \)

A.2 \( x = 0 \) and \( r_{m+1} < |y| \leq r_m \) for some \( m \geq k \)

A.3 \( r_{m+1} < |x| \leq r_m \) and \( y = 0 \) for some \( m \geq k \)

B.1 \( r_{m+1} < |x| \leq r_m \) and \( r_{s+1} < |y| \leq r_s \) for some \( s, m \) where \( s > m \geq k \)

B.2 \( r_{m+1} < |x| \leq r_m \) and \( r_{s+1} < |y| \leq r_s \) for some \( s, m \) where \( m > s \geq k \).

In the case of type \( A \) we have \( f(x) = P_m(x) \) and \( f(y) = P_m(y) \). Comparing the formula

\[
P_m(x) = P_m(y) + (x-y)D_1P_m(y) + \ldots + (x-y)^{n-1}D_{n-1}P_m(y) + (x-y)^nR_nP_m(x,y)
\]

with formula (*) we obtain \( D_i P_m(y) = D_i f(y) \) \((1 \leq i \leq n-1)\). (In case A.1 and A.2 this is clear since \( P_m(y) = f(y) \) for all \( y \) with
\[ r_m + 1 < |y| \leq r_m \] in case A.1 our induction hypothesis says that
\[ D_i f(0) = \lambda_i \quad (0 \leq i \leq n-1), \] and clearly \[ D_i P(0) = \lambda_i. \] Thus, in the

\[ \text{type A cases we have } R_n P_m(x,y) = R_n f(x,y) \quad (x \neq y). \] Thus, using 12.3,

we get
\[ |R_n f(x,y) - \lambda_n| = |R_n P_m(x,y) - \lambda_n| \leq r_m \max(|\lambda_0|, \ldots, |\lambda_m|) \leq \frac{1}{m} \leq \frac{1}{k}. \]

In case B.1 we have \( f(x) = P_m(x) \) and \( f(y) = P_s(y) + Q(y) \),

where \( Q(y) = \lambda_{m+1} x^{m+1} + \ldots + \lambda_s x^s \). Further we have \( |x-y| = r_m \). Thus,

\[ |R_n f(x,y) - \lambda_n| = \left| \frac{P_m(x) - \sum_{i=0}^{n} (x-y)^i D_i P_m(y)}{(x-y)^n} \right| / \lambda_n \left| \frac{\sum_{i=0}^{n} (x-y)^i D_i Q(y)}{(x-y)^n} \right| \leq \frac{|P_m(x) - P_s(y)|}{|x-y|^{n-1}}. \]

By Lemma 12.13 we have
\[ |R_n f(x,y) - \lambda_n| \leq \max(|x|, |y|) \max(|\lambda_0|, \ldots, |\lambda_m|) \leq r_m \max(|\lambda_0|, \ldots, |\lambda_m|) \leq \frac{1}{m} \leq \frac{1}{k}. \]

And by Lemma 12.14 we have
\[ \frac{|D_i Q(y)|}{|x-y|^{n-1}} \leq r_m \max(|\lambda_0|, \ldots, |\lambda_s|) \leq \frac{1}{s} \leq \frac{1}{k}. \] This proves \( |R_n f(x,y) - \lambda_n| < \frac{1}{k} \) in case

B.1.

Finally, if we are in case B.2 then \( f(x) = P_m(x) = P_s(x) + Q(x) \), where

\( Q(x) = \lambda_{m+1} x^{m+1} + \ldots + \lambda_s x^s. \) More or less as in B.1 we see

\[ |R_n f(x,y) - \lambda_n| = \left| \frac{P_s(x) - \sum_{i=0}^{n-1} (x-y)^i D_i P_s(y)}{(x-y)^n} \right| / \lambda_n \left| \frac{Q(x)}{(x-y)^n} \right|. \]

Now
\[ \frac{|Q(x)|}{|x-y|^{n-1}} \leq \frac{|x|^{m+1} \max(|\lambda_0|, \ldots, |\lambda_s|)}{r_s} \leq r_s \max(|\lambda_0|, \ldots, |\lambda_s|) \leq \frac{1}{s} \leq \frac{1}{k}. \]

The rest is obvious.
BIBLIOGRAPHY


