NON-ARCHIMEDEAN DIFFERENTIATION

by

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Introduction

This note is a collection of the most important results of a lengthy paper that will appear as a Report, Mathematisch Instituut, Katholieke Universiteit, Nijmegen. There the interested reader may find all the proofs that are missing here, and also a bibliography. Because of the rather complicated formulas and computations that are involved it seems wise to restrict oneself first to K-valued functions of one single variable. However, a lot of the results can without any problem be carried over to E-valued functions of one variable, where E is a K-Banach space. A generalization to functions: $K^n \to K^m$ will be less obvious, although it seems clear how to define C^k-functions in that case. (For example, in order that $f : K^2 \to K$ is C^1 one should require (see 3.1) that the difference quotients

$$(x_1,x_2,y) \longmapsto \frac{f(x_1,y)-f(x_2,y)}{x_1-x_2}, \quad (x,y_1,y_2) \longmapsto \frac{f(x,y_1)-f(x,y_2)}{y_1-y_2}$$

can be extended to continuous functions on K^3. If we take again difference quotients we get four functions of four variables, required to be continuous in order that f be in C^2 (see 6.1). It then follows very easily that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ for $f \in C^2$.)

Throughout this note, K will always be a complete non-archimedean valued field, and X a non-empty subset of K, without isolated points. We study differentiability properties of functions $f : X \to K$. Besides the analytic functions we have other examples of differentiable functions such as the locally constant functions (they have derivative zero everywhere). The function $\Sigma a_n p^n \to \Sigma a_n p^{n!}$, defined on $\mathbb{Z}_p$ is an example of an injective function with zero derivative and which is in Lip_α for every $\alpha > 0$. The function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ defined via $f(x) = x - p^{2n}$ if $|x-p^n| < p^{-2n}$ and $f(x) = x$ elsewhere has derivative 1 everywhere, but for all $n \in \mathbb{N}$ $f(p^n) = f(p^n - p^{2n}) = p^n - p^{2n}$, hence f is not locally injective at 0.

The above examples show that we do not have a mean value theorem or a local invertibility theorem. We now state the results going from "bad" to "smooth" functions.

1. Nowhere differentiable functions

Let $BC(X)$ be the algebra of the bounded continuous functions: $X \to K$, normed by the sup norm $\|\ \|_\infty$. We have, analogous to the classical case:

THEOREM 1.1. *The collection of those $f \in BC(X)$ that are somewhere differentiable is of first category in $BC(X)$ (in the sense of Baire).*

In contrast to the theory of functions on the real line we have

THEOREM 1.2. *Let X be open in K, and let $f : X \to K$ be a bounded uniformly continuous function, and let $\varepsilon > 0$. Then there exists a nowhere differentiable $g : X \to K$ such that g has bounded difference quotients, and such that $\|f-g\|_\infty < \varepsilon$.*

2. Differentiability as such

Contrary to the classical case we have a nice criterion for a function to possess an antiderivative:

THEOREM 2.1. *Let f : X → K. Then f has an antiderivative if and only if f is of Baire class one. (i.e., f is the pointwise limit of a sequence of continuous functions).*

Further we have Sard-type theorems. If K is a local field then $Y \subset K$ is called a nullset if it has measure zero in the sense of the (real) Haar measure on K.

THEOREM 2.2. *Let K be a local field and let f : X → K be differentiable. Then we have:*

(1) If $Y \subset X$ is a nullset then f(Y) is a nullset ("f has property (N)")

(2) $\{f(x) : f'(x) = 0\}$ is a nullset.

COROLLARY 2.3. *If f : X → K is differentiable, f' = 0 almost everywhere, then f(X) is a nullset.*

3. Continuously differentiable functions

If we want the local invertibility theorem to hold for C^1-functions we have to take a definition of a C^1-function, stronger then just "f is differentiable and f is continuous". For f : X → K, define

$$\Phi_1 f(x,y) = \frac{f(x)-f(y)}{x-y} \qquad (x,y \in X,\ x \neq y).$$

DEFINITION 3.1. *f : X → K is in $C^1(X)$ if $\Phi_1 f$ can (uniquely) be extended to a continuous function $\overline{\Phi}_1 f$ on X × X.*

(Notice that for a real valued function f defined on an interval the continuity of f' already guarantees the existence of a continuous $\overline{\Phi}_1 f$).

THEOREM 3.2. *Let* $f \in C^1(X)$ *and let* $a \in X$.

(a) *If* $f'(a) \neq 0$ *then* f *is locally invertible at* a. *(In fact,* $(f'(a))^{-1}f$ *is an isometry locally at* a).

(b) *If* X *is open in* K *and if* $f' \neq 0$ *everywhere on* X *then* f *is an open mapping*.

Let $BC^1(X) = \{f \in C^1(X) : ||f||_1 := ||f||_\infty \vee ||\Phi_1 f||_\infty\}$. Then $BC^1(X)$ is a Banach space with respect to $||\ ||_1$. We may put a locally convex topology on $C^1(X)$ via the defining seminorms $||\ ||_{1,C}$ where C runs through the compact subsets of X:

$$||f||_{1,C} = \sup_{x\in C} |f(x)| \vee \sup_{\substack{x\in C\\ y\in C}} |\overline{\Phi}_1 f(x,y)| \qquad (f \in C^1(X)).$$

Let $N^1(X) = \{f \in C^1(X) : f' = 0\}$ and $BN^1(X) = \{f \in BC^1(X) : f' = 0\}$. Then $N^1(X)$ is closed in $C^1(X)$, $BN^1(X)$ is closed in $BC^1(X)$.

THEOREM 3.3. *The locally linear functions (in* $BC^1(X)$*) form a dense subset of* $C^1(X)$ *(of* $BC^1(X)$*).*

The locally constant functions (in $BC^1(X)$*) form a dense subset of* $N^1(X)$ *(of* $BN^1(X)$*).*

THEOREM 3.4. *If either* X *is compact or* K *has discrete valuation then* $BC^1(X)$ *has an orthonormal base (in the sense of the norm).*

If X *is not compact and* K *has dense valuation then, for any* $\alpha > 0$, $BC^1(X)$ *has no* α*-orthogonal base*.

Let us choose real numbers $1 > r_1 > r_2 > \ldots$ with $\lim r_n = 0$, and, for each n, let R_n be a full set of representatives of the equivalence relation (in X): $x \sim y$ if $|x-y| < r_n$. We can arrange that $R_1 \subset R_2 \subset \ldots$ For each $x \in X$, $n \in \mathbb{N}$, let $x_n \in X$ be determined by: $|x_n - x| < r_n$, $x_n \in R_n$. For a continuous $f : X \to K$ set

$$(Pf)(x) = \sum_{n=1}^{\infty} f(x_n)(x_{n+1} - x_n) \qquad (x \in X)$$

THEOREM 3.5. The map P defined above is a continuous linear map: $C(X) \to C^1(X)$ and its restriction to $BC(X)$ is an isometry: $BC(X) \to BC^1(X)$. P is an antiderivation map i.e., $(Pf)' = f$ for each $f \in C(X)$.

COROLLARY 3.6. Every continuous function has a C^1-antiderivative. In fact, by passing through the quotient, differentiation yields a map $\rho : BC^1(X)/BN^1(X) \to BC(X)$ which is a surjective isometry. Moreover, $BN^1(X)$ has an orthogonal complement (im P) in $BC^1(X)$.

4. $C^1(X)$ for compact X

(Throughout section 4, X is compact). The set $\{|x-y| : x,y \in X\}$ is bounded and has only 0 as an accumulation point, hence it can be written as $\{r_1, r_2, \ldots\} \cup \{0\}$, where $r_1 > r_2 > \ldots$ and $\lim r_n = 0$. Let $r_0 = \infty$. For each i, let R_i be a full set of representatives in X of the equivalence relation "$x \sim y$ if $|x-y| < r_i$" such that $R_0 \subset R_1 \subset \ldots$. Then R_i is finite for each i and R_0 consists only of one single point a_0. Let $R = \bigcup_i R_i$ and define $\nu : R \to \{0,1,2,\ldots\}$ as follows. For $a \in R$ let $\nu(a)$ be the nonnegative integer m for which $a \in R_m \setminus R_{m-1}$ ($R_{-1} = \emptyset$ by definition). For each $a \in R$ let

$$B_a = \{x \in X : |x-a| < r_{\nu(a)}\},$$

and let e_a be the K-valued characteristic function of B_a. Further, we define

$$a \lhd b \quad \text{iff} \quad b \in B_a \qquad (a,b \in R)$$

Then we have

LEMMA 4.1. $(R,\lhd)$ *is a partially ordered set with a smallest element* a_0. *For each* $a \in R$, *the set* $\{x \in R : x \lhd a\}$ *is finite and linearly ordered by* $\lhd$.

Define for $a \in R$, $a \neq a_0$: $a_- = \max \{x \in R : x \neq a,\ x \lhd a\}$. Then

THEOREM 4.2. *The set* $\{e_a : a \in R\}$ *forms an orthonormal base of* $C(X)$. *Let* $f \in C(X)$ *and* $f = \Sigma \lambda_a e_a$ *for some* $\lambda_a \in K$. *Then* $\lambda_{a_0} = f(a_0)$ *and for* $a \neq a_0$: $\lambda_a = f(a)-f(a_-)$.
The set $\{e_a : a \in R\} \cup \{Pe_a : a \in R\}$ (P as in 3.5) *forms an orthogonal base of* $C^1(X)$. *Let* $f \in C^1(X)$, $f = \Sigma \lambda_n e_a + \Sigma \mu_b Pe_b$ $(\lambda_a, \mu_b \in K)$ *in the* $\| \ \|_1$*-norm*. *Then* $\lambda_{a_0} = f(a_0)$, $\mu_{a_0} = f'(a_0)$ *and for* $a \neq a_0$:

$$\lambda_a = f(a)-f(a_-)-(a-a_-)f'(a_-)$$

$$\mu_a = f'(a)-f'(a_-).$$

5. Uniform differentiability

There seem to be two natural notions of "uniform differentiability". Let $f \in C^1(X)$. f is called *uniformly differentiable* if $\lim_{x\to y} \Phi_1 f(x,y) = f'(y)$ uniformly in y. f is called *strongly uniformly differentiable* if $\overline{\Phi}_1 f$ is uniformly continuous.

If X is compact both notions are the same and coincide with "continuousl differentiable".

THEOREM 5.1. *Let* $f : X \to K$ *be (strongly) uniformly differentiable. Then* f *has a unique continuous extension* $\bar{f} : \bar{X} \to K$ *(*$\bar{X}$ *is the closure of* X *in* K*).*

This $\bar{f}$ *is (strongly) uniformly differentiable.*

THEOREM 5.2. *Let* $f : X \to K$ *be uniformly differentiable. Then each of the following properties implies strong uniform differentiability of* f:

(a) $\Phi_1 f$ *is bounded.*

(b) *Both* f *and* f' *are bounded*

(c) X *is "nice" and* f *is bounded.*

(X is called "nice" if for each $r > 0$ there is $s > 0$ such that for every $x \in X$ there is $y \in X$ such that $s \leq |x-y| \leq r$).

The theorems 3.3, 3.5., 3.6. each have an analogon for uniformly differentiable functions.

6. C^n-functions

For $n \in \mathbb{N}$, let $\nabla^n X = \{(x_1,\ldots,x_n) \in X^n : i \neq j \Rightarrow x_i \neq j\}$. For $f : X \to K$ we define *the* n^{th} *difference quotient* $\Phi_n f : \nabla^{n+1} X \to K$ inductively as follows $\Phi_0 f = f$ and for $(x_1,\ldots,x_{n+1}) \in \nabla^{n+1} X$:

$$\Phi_n f(x_1,\ldots,x_{n+1}) = (x_1-x_2)^{-1}(\Phi_{n-1} f(x_1,x_3,\ldots,x_{n+1}) - \Phi_{n-1} f(x_2,x_3,\ldots,x_{n+1})).$$

Since $\nabla^n X$ is dense in X^n for each n the following definition makes sense.

DEFINITION 6.1. *Let* $f : X \to K$, $n \in \mathbb{N} \cup \{0\}$. We say that $f \in C^n(X)$ *if* $\Phi_n f$ *can be extended to a continuous function* $\bar{\Phi}_n f : X^{n+1} \to K$.

We say that $f \in B\Delta^n(X)$ *if* $\Phi_0 f,\ldots,\Phi_n f$ *are bounded functions.*

For $f \in B\Delta^n(X)$ *set*

$$||f||_n = \max_{0 \leq i \leq n} ||\Phi_i f||_\infty .$$

Let $BC^n(X) = B\Delta^n(X) \cap C^n(X)$, $C^\infty(X) = \bigcap_{n=1}^{\infty} C^n(X)$,

$BC^\infty(X) = \bigcap_{n=1}^{\infty} BC^n(X)$.

THEOREM 6.2. $C^1(X) \supset C^2(X) \supset \ldots$

$B\Delta^1(X) \supset BC^1(X) \supset B\Delta^2(X) \supset BC^2(X) \supset \ldots$

$B\Delta^n(X)$ *is a Banach space with respect to* $|| \; ||_n$ and

$BC^n(X)$ *is closed in* $B\Delta^n(X)$.

For $f \in C^n(X)$ $(n \geq 1)$ and $0 \leq j \leq n$ we define the j^{th} *Hasse derivative* of f by

$$D_j f(x) = \overline{\Phi}_j f(x,x,\ldots,x) \qquad (x \in X).$$

THEOREM 6.3. *Let* $f \in C^n(X)$. *Then for* $0 \leq j \leq n$ *we have* $D_i f \in C^{n-i}(X)$ *and if* $i+j \leq n$

$$D_i D_j f = \binom{i+j}{i} D_{i+j} f$$

f *is* n *times differentiable in the ordinary sense and for* $0 \leq i \leq n$ *we have*

$$f^{(i)} = i! \, D_i f.$$

$f : X \to K$ is called a *spline function* of degree $\leq n$ if for every $a \in X$ there is a neighbourhood U of a such that $f|U \cap X$ is a polynomial function of degree $\leq n$. Spline functions are in $C^\infty(X)$.

THEOREM 6.4. *Let* $f \in C^n(X)$ *and* $\varepsilon > 0$. *Then there is a spline function* g *of degree* $\leq n$ *such that* $f-g \in BC^n(X)$, $||f-g||_n < \varepsilon$. *If* $D_i f = D_{i+1} f = \ldots = D_n f = 0$ *for some* $i \in \{1,\ldots,n\}$ *then* g *can be chosen to be of degree* $\leq i-1$.

THEOREM 6.5. (*Local invertibility*). *Let* $f \in C^n(X)$ *and* $f'(a) \neq 0$ *for some* $a \in X$. *Then there is a neighbourhood* U *of* a ($a \in U \subset X$) *such that* $f : U \to f(U)$ *is a bijection, and such that the local inverse*: $f(U) \to U$ *is in* $C^n(f(U))$.

THEOREM 6.6. (*Taylor formula*). *Let* $f \in C^n(X)$. *Then for all* $x,y \in X$:

$$f(x) = f(y)+(x-y)D_1f(y)+\ldots+(x-y)^{n-1}D_{n-1}f(y)+(x-y)^n\overline{\Phi}_nf(x,y,\ldots,y).$$

The above result leads to another possible notion of "n-times continuously differentiable":

DEFINITION 6.7. *Let* $f : X \to K$, $n \in \mathbb{N}$. *We say that* $f \in \mathcal{C}^n(X)$ *if there exist functions* $\mathcal{D}_1f,\ldots,\mathcal{D}_{n-1}f : X \to K$ *and a continuous* $\mathcal{R}_nf : X^2 \to K$ *such that for all* $x,y \in X$

$$f(x) = f(y)+(x-y)\mathcal{D}_1f(y)+\ldots+(x-y)^{n-1}\mathcal{D}_{n-1}f(y)+(x-y)^n\mathcal{R}_nf(x,y).$$

(It follows that the $\mathcal{D}_if$, $\mathcal{R}_nf$ are uniquely determined and continuous. Further we have $\mathcal{C}^1(X) \supset \mathcal{C}^2(X) \supset \ldots$). It is easy to show that $\mathcal{C}^i(X) = C^i(X)$ for $i = 1,2$. Also $C^n(X) \subset \mathcal{C}^n(X)$ for all n, by 6.6. But we have

EXAMPLE 6.8. *Let* $X = \{\Sigma\, a_np^{n!} \in \mathbb{Z}_p : a_n \in \{0,1\}\}$, *and let* $f : X \to K$ *be defined via*

$$f(\Sigma\, a_np^{n!}) = \Sigma\, a_np^{3n!}.$$

Then $f \in \mathcal{C}^n(X)$ *for each* n, *and* $\mathcal{D}_if = 0$ *for* $i = 1,2,4,5,\ldots$ and $\mathcal{D}_3f = 1$. *On the other hand*, $f \notin C^3(X)$.

Let $C > 0$ and $\{x_1,\ldots,x_n\}$ a set of n distinct points in X. We call $\{x_1,\ldots,x_n\}$ a *C-polygon* if for all $i,j,k,l \in \{1,\ldots,n\}$, $k \neq l$:

$$\frac{|x_i - x_j|}{|x_k - x_l|} \leq C.$$

DEFINITION 6.9. Let $n \in \mathbb{N}$. We say that X has locally property B_n if for each $a \in X$ there is $\delta > 0$ and $C > 0$ such that for all $x_1, x_2 \in X$, $x_1 \neq x_2$, $|x_1 - a| < \delta$, $|x_2 - a| < \delta$ there exist $x_3, \ldots, x_n \in X$ such that $\{x_1, x_2, \ldots, x_n\}$ is a C-polygon. (By definition, every X has locally property B_1 and B_2).

We say that X has globally property B_n if there exists $C > 0$ such that for all $x_1, x_2 \in X$, $x_1 \neq x_2$, there exist $x_3, \ldots, x_n \in X$ such that $\{x_1, x_2, \ldots, x_n\}$ is a C-polygon. (Every X has globally property B_1 and B_2).

For example, a ball in K has globally property B_n, for each n. Every open (non-empty) subset of K has locally property B_n, for each n.

Let us call $BC^n(X) = \{f \in C^n(X) : \|f\|_n^{\sim} < \infty\}$, where, by definition,

$$\|f\|_n^{\sim} = \max(\|f\|_\infty, \|\mathcal{D}_1 f\|_\infty, \ldots, \|\mathcal{D}_{n-1} f\|_\infty, \|\mathcal{R}_n f\|_\infty)$$

(see 6.7). It is very easy to show that $BC^n(X)$ is a Banach space with respect to $\| \ \|_n^{\sim}$. The main theorem:

THEOREM 6.10. If X has locally property B_n, then $C^n(X) = C^n(X)$.

Let X have globally property B_n $(n \geq 2)$ in the sense that every two-point set can be extended to a C-polygon. Then $BC^n(X) = BC^n(X)$ and

$$\| \ \|_n^{\sim} \leq \| \ \|_n \leq C^{2(n-2)} \| \ \|_n^{\sim}.$$

(In general we have for $f \in BC^n(X)$: $||f||_n = \max_{0 \le i \le n} ||D_i f||^{\sim}_{n-i}$).

As in 3.5. we want to find an antiderivation map: $C^{n-1}(X) \to C^n(X)$. We cannot use the map P of 3.5. since one can prove: if $f \in C^1(X)$ then $Pf \in C^2(X)$ if and only if $f' = 0$. Further, if the characteristic of K equals $p \neq 0$ it is easy to see that not every C^{p-1}-function has a C^p-antiderivative.

THEOREM 6.11. *Let the characteristic of* K *be zero and let* $r_1 > r_2 \ldots$ *as in* 3.5 *but such that* $r_m \le \rho r_{m+1}$ *for all* m, *some* $\rho > 0$. *For* $f \in C^{n-1}(X)$, *set*

$$P_n f(x) = \sum_{k=1}^{\infty} \sum_{i=1}^{n} \frac{1}{i}(x_{k+1}-x_k)^i D_{i-1} f(x_k) \qquad (x \in X)$$

Then $P_n f \in C^n(X)$ *and* $(P_n f)' = f$. *If* $f \in BC^{n-1}(X)$, *then* $Pf \in BC^n(X)$ *and*

$$||P_n f||_n \le c_n ||f||_{n-1}$$

where

$$c_n = \max_{1 \le i \le n} \frac{1}{|i|} \cdot \rho^n .$$

It follows that the map $Q_n = n! P_n P_{n-1} \ldots P_1$ sends $C(X)$ into $C^n(X)$, $D_n Q_n$ is the identity on $C(X)$. A computation yields

$$Q_n = \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} M^{n-i} S_i ,$$

where M is the multiplication with x ($(Mf)(x) = xf(x)$ for $f \in C(X)$) and where

$$S_i f(x) = \sum_{k=1}^{\infty} f(x_k)(x_{k+1}^i - x_k^i) \qquad (f \in C(X),\ x \in X)$$

Note: Theorem 3.4 is also true if we replace $BC^1(X)$ by $BC^n(X)$ ($n \in \mathbb{N}$).

7. C^∞-functions

Spline functions, analytic functions are in $C^\infty(X)$.

Let $f \in C^\infty(X)$ and let $f(a) = 0$ for some $a \in X$. Then $f(x) = (x-a)g(x)$ ($x \in X$) where $g \in C^\infty(X)$.

A *derivation* on $C^\infty(X)$ is a linear map $\phi : C^\infty(X) \to C^\infty(X)$ such that

$$\phi(fg) = \phi(f)g + f\phi(g) \qquad (f,g \in C^\infty(X)).$$

Any derivation ϕ has the form $g\frac{d}{dx}$, where $g \in C^\infty(X)$.

THEOREM 7.1. *Let $Y \subset K$ be a closed subset of K. Then there is $f \in C^\infty(K)$ (with $f' = 0$ everywhere) such that Y is the set of zeros of f.*

THEOREM 7.2. *Let $\lambda_0, \lambda_1, \ldots$ be any sequence in K. Then there exists an $f \in C^\infty(K)$ such that $D_i f(0) = \lambda_i$ for all i.*

Open problem: Let the characteristic of K be zero. Does every $f \in C^\infty(X)$ have a C^∞-antiderivative?