The following full text is a publisher's version.

For additional information about this publication click this link.
http://hdl.handle.net/2066/57030

Please be advised that this information was generated on 2019-01-15 and may be subject to change.
NON-ARCHIMEDEAN DIFFERENTIATION

by

W.H. Schikhof

Introduction

This note is a collection of the most important results of a lengthy paper that will appear as a Report, Mathematisch Instituut, Katholieke Universiteit, Nijmegen. There the interested reader may find all the proofs that are missing here, and also a bibliography. Because of the rather complicated formulas and computations that are involved it seems wise to restrict oneself first to K-valued functions of one single variable. However, a lot of the results can without any problem be carried over to E-valued functions of one variable, where E is a K-Banach space. A generalization to functions: $K^n + K^m$ will be less obvious, although it seems clear how to define $C^K$-functions in that case. (For example, in order that $f : K^2 \to K$ is $C^1$ one should require (see 3.1) that the difference quotients

$$
\left( x_1, x_2, y \right) \mapsto \frac{f(x_1, y) - f(x_2, y)}{x_1 - x_2}, \quad \left( x, y_1, y_2 \right) \mapsto \frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2}
$$

can be extended to continuous functions on $K^3$. If we take again difference quotients we get four functions of four variables, required to be continuous in order that $f$ be in $C^2$ (see 6.1). It then follows very easily that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ for $f \in C^2$.)

Throughout this note, $K$ will always be a complete non-archimedean valued field, and $X$ a non-empty subset of $K$, without isolated points. We study differentiability properties of functions $f : X \to K$. Besides the analytic functions we have other examples of differentiable
functions such as the locally constant functions (they have derivative zero everywhere). The function $\mathbb{Z}_n \times \mathbb{P}^n \to \mathbb{Z}_n \times \mathbb{P}^n$, defined on $\mathbb{Z}_p$, is an example of a function with zero derivative and which is in $\text{Lip}_a$ for every $a > 0$. The function $f : \mathbb{Z}_p \times \mathbb{P}_p$ defined via $f(x) = x-p^{2n}$ if $|x-p^n| < p^{-2n}$ and $f(x) = x$ elsewhere has derivative 1 everywhere, but for all $n \in \mathbb{N}$, $f(p^n) = f(p^{n-2n}) = p^{n-2n}$, hence $f$ is not locally injective at 0.

The above examples show that we do not have a mean value theorem or a local invertibility theorem. We now state the results going from "bad" to "smooth" functions.

1. **Nowhere differentiable functions**

   Let $BC(X)$ be the algebra of the bounded continuous functions: $X \to \mathbb{K}$, normed by the sup norm $|| \cdot ||_\infty$. We have, analogous to the classical case:

   **Theorem 1.1.** The collection of those $f \in BC(X)$ that are somewhere differentiable is of first category in $BC(X)$ (in the sense of Baire).

   In contrast to the theory of functions on the real line we have

   **Theorem 1.2.** Let $X$ be open in $\mathbb{K}$, and let $f : X \to \mathbb{K}$ be a bounded uniformly continuous function, and let $\varepsilon > 0$. Then there exists a nowhere differentiable $g : X \to \mathbb{K}$ such that $g$ has bounded difference quotients, and such that $||f-g||_\infty < \varepsilon$.

2. **Differentiability as such**

   Contrary to the classical case we have a nice criterion for a
THEOREM 2.1. Let $f : X \to K$. Then $f$ has an antiderivative if and only if $f$ is of Baire class one. (i.e., $f$ is the pointwise limit of a sequence of continuous functions).

Further we have Sard-type theorems. If $K$ is a local field then $Y \subseteq K$ is called a nullset if it has measure zero in the sense of the (real) Haar measure on $K$.

THEOREM 2.2. Let $K$ be a local field and let $f : X \to K$ be differentiable. Then we have:

1. If $Y \subseteq X$ is a nullset then $f(Y)$ is a nullset ("$f$ has property (N)"")
2. $\{f(x) : f'(x) = 0\}$ is a nullset.

COROLLARY 2.3. If $f : X + K$ is differentiable, $f' = 0$ almost everywhere, then $f(X)$ is a nullset.

3. Continuously differentiable functions

If we want the local invertibility theorem to hold for $C^1$-functions we have to take a definition of a $C^1$-function, stronger then just "$f$ is differentiable and $f$ is continuous". For $f : X + K$, define

$$
\phi_1 f(x,y) = \frac{f(x) - f(y)}{x - y} \quad (x, y \in X, x \neq y).
$$

DEFINITION 3.1. $f : X + K$ is in $C^1(X)$ if $\phi_1 f$ can (uniquely) be extended to a continuous function $\phi_1 f$ on $X \times X$.

(Notice that for a real valued function $f$ defined on an interval
THEOREM 3.2. Let \( f \in C^1(X) \) and let \( a \in X \).

(a) If \( f'(a) \neq 0 \) then \( f \) is locally invertible at \( a \). (In fact, \( (f'(a))^{-1}f \) is an isometry locally at \( a \)).

(b) If \( X \) is open in \( K \) and if \( f' \neq 0 \) everywhere on \( X \) then \( f \) is an open mapping.

Let \( BC^1(X) = \{ f \in C^1(X) : \| f \|_1 := \| f \|_\infty \vee \| f \|_1 \} \). Then \( BC^1(X) \) is a Banach space with respect to \( \| \|_1 \). We may put a locally convex topology on \( C^1(X) \) via the defining seminorms \( \| \|_1, \), where \( C \) runs through the compact subsets of \( X \):

\[
\| f \|_1, = \sup_{x \in C} |f(x)| \vee \sup_{x \in C} |f(x,y)| \quad (f \in C^1(X)).
\]

Let \( N^1(X) = \{ f \in C^1(X) : f' = 0 \} \) and \( BN^1(X) = \{ f \in BC^1(X) : f' = 0 \} \). Then \( N^1(X) \) is closed in \( C^1(X) \), \( BN^1(X) \) is closed in \( BC^1(X) \).

THEOREM 3.3. The locally linear functions (in \( BC^1(X) \)) form a dense subset of \( C^1(X) \) (of \( BC^1(X) \)).

The locally constant functions (in \( BC^1(X) \)) form a dense subset of \( N^1(X) \) (of \( BN^1(X) \)).

THEOREM 3.4. If either \( X \) is compact or \( K \) has discrete valuation then \( BC^1(X) \) has an orthonormal base (in the sense of the norm).

If \( X \) is not compact and \( K \) has dense valuation then, for any \( \alpha > 0 \), \( BC^1(X) \) has no \( \alpha \)-orthogonal base.
Let us choose real numbers $1 > r_1 > r_2 > \ldots$ with $\lim_{n \to \infty} r_n = 0$, and, for each $n$, let $R_n$ be a full set of representatives of the equivalence relation (in $X$): $x \sim y$ if $|x-y| < r_n$. We can arrange that $R_1 \subseteq R_2 \subseteq \ldots$. For each $x \in X$, $n \in \mathbb{N}$, let $x_n \in X$ be determined by:

\[ |x_n - x| < r_n, \quad x_n \in R_n. \]

For a continuous $f : X \to \mathbb{K}$ set

\[ (Pf)(x) = \sum_{n=1}^{\infty} f(x_n)(x_{n+1} - x_n) \quad (x \in X). \]

**Theorem 3.5.** The map $P$ defined above is a continuous linear map:

\[ C(X) \to C^1(X) \] and its restriction to $BC(X)$ is an isometry: $BC(X) \to BC^1(X)$. $P$ is an antiderivation map i.e., $(Pf)' = f$ for each $f \in C(X)$.

**Corollary 3.6.** Every continuous function has a $C^1$-antiderivative.

In fact, by passing through the quotient, differentiation yields a map $p : BC^1(X)/BN^1(X) \to BC(X)$ which is a surjective isometry. Moreover, $BN^1(X)$ has an orthogonal complement $(\text{im } P)$ in $BC^1(X)$.

### 4. $C^1(X)$ for compact $X$

(Throughout section 4, $X$ is compact). The set $\{|x-y| : x, y \in X\}$ is bounded and has only 0 as an accumulation point, hence it can be written as $\{r_1, r_2, \ldots\} \cup \{0\}$, where $r_1 > r_2 > \ldots$ and $\lim_{n \to \infty} r_n = 0$.

Let $r_0 = \infty$. For each $i$, let $R_i$ be a full set of representatives in $X$ of the equivalence relation "$x \sim y$ if $|x-y| < r_i$" such that $R_0 \subseteq R_1 \subseteq \ldots$. Then $R_i$ is finite for each $i$ and $R_0$ consists only of one single point $a_0$. Let $R = \bigcup_{i=1}^{\infty} R_i$ and define $v : R \to \{0, 1, 2, \ldots\}$ as follows. For a $a \in R$ let $v(a)$ be the nonnegative integer $m$ for which $a \in \bigcap_{j=0}^{m} R_{j}$ \bigcap_{j=0}^{m-1} R_{j} = \emptyset$ by definition. For each $a \in R$ let
\[ B_a = \{ x \in X : |x-a| < r_y(a) \}, \]

and let \( e_a \) be the \( K \)-valued characteristic function of \( B_a \). Further, we define

\[ a \triangleleft b \text{ iff } b \in B_a \quad (a,b \in \mathbb{R}) \]

Then we have

**Lemma 4.1.** \((\mathbb{R}, \triangleleft)\) is a partially ordered set with a smallest element \( a_0 \). For each \( a \in \mathbb{R} \), the set \( \{ x \in \mathbb{R} : x \triangleleft a \} \) is finite and linearly ordered by \( \triangleleft \).

Define for \( a \in \mathbb{R}, a \neq a_0 \):

\[ a_- = \max \{ x \in \mathbb{R} : x \neq a, x \triangleleft a \}. \]

**Theorem 4.2.** The set \( \{ e_a : a \in \mathbb{R} \} \) forms an orthonormal base of \( C(X) \).

Let \( f \in C(X) \) and \( f = \sum \lambda_a e_a \) for some \( \lambda_a \in K \). Then

\[ \lambda_{a_0} = f(a_0) \text{ and for } a \neq a_0 : \lambda_a = f(a) - f(a_-). \]

The set \( \{ e_a : a \in \mathbb{R} \} \cup \{ P e_a : a \in \mathbb{R} \} \) (\( P \) as in 3.5) forms an orthogonal base of \( C^1(X) \). Let \( f \in C^1(X), f = \sum \lambda_a e_a + \sum \mu_b P e_b \) \((\lambda_a, \mu_b \in K)\) in the \( \| \cdot \|_1 \)-norm. Then \( \lambda_{a_0} = f(a_0), \mu_{a_0} = f'(a_0) \) and for \( a \neq a_0 : \)

\[ \lambda_a = f(a) - f(a_-) - (a-a_-) f'(a_-) \]
\[ \mu_a = f'(a) - f'(a_-). \]

5. Uniform differentiability

There seem to be two natural notions of "uniform differentiability". Let \( f \in C^1(X) \). \( f \) is called uniformly differentiable if \( \lim_{x \to y} \phi_i f(x,y) = f'(y) \) uniformly in \( y \). \( f \) is called strongly uniformly differentiable if \( \phi_1 f \) is uniformly continuous.

If \( X \) is compact both notions are the same and coincide with "continuous" differentiable".
THEOREM 5.1. Let \( f : X \to K \) be (strongly) uniformly differentiable. Then
\( f \) has a unique continuous extension \( \overline{f} : \overline{X} \to K \) (\( \overline{X} \) is the closure of \( X \) in \( K \)).
This \( \overline{f} \) is (strongly) uniformly differentiable.

THEOREM 5.2. Let \( f : X \to K \) be uniformly differentiable. Then each of
the following properties implies strong uniform differentiability of \( f \):
(a) \( \phi f \) is bounded.
(b) Both \( f \) and \( f' \) are bounded
(c) \( X \) is "nice" and \( f \) is bounded.
\( (X \) is called "nice" if for each \( r > 0 \) there is \( s > 0 \) such that for
every \( x \in X \) there is \( y \in X \) such that \( s \leq |x-y| \leq r \).

The theorems 3.3, 3.5, 3.6. each have an analogon for uniformly differentiable functions.

6. \( C^n \)-functions

For \( n \in \mathbb{N} \), let \( V^n X = \{(x_1,\ldots,x_n) \in X^n : i \neq j \Rightarrow x_i \neq x_j\} \). For
\( f : X \to K \) we define the \( n \)th difference quotient \( \phi f : V^{n+1} X \to K \)
inductively as follows \( \phi_0 f = f \) and for \( (x_1,\ldots,x_{n+1}) \in V^{n+1} X \):
\[
\phi_n f(x_1,\ldots,x_{n+1}) = (x_1-x_2)^{-1}(\phi_{n-1} f(x_1,x_3,\ldots,x_{n+1}) - \phi_{n-1} f(x_2,x_3,\ldots,x_{n+1})).
\]
Since \( V^n X \) is dense in \( X^n \) for each \( n \) the following definition makes sense.

DEFINITION 6.1. Let \( f : X \to K \), \( n \in \mathbb{N} U \{0\} \). We say that \( f \in C^n(X) \) if
\( \phi_n f \) can be extended to a continuous function \( \overline{\phi_n f} : X^{n+1} \to K \),
We say that \( f \in B^n(X) \) if \( \phi_0 f, \ldots, \phi_n f \) are bounded functions.
For $f \in B^\infty_n(X)$ set
$$
\|f\|_n = \max_{0 \leq i \leq n} \|\varphi_i f\|_\infty.
$$
Let $BC_n(X) = B^\infty_n(X) \cap C^n(X)$, $C^\infty_\infty(X) = \bigcap_{n=1}^\infty C^n(X)$,
$BC^\infty_\infty(X) = \bigcap_{n=1}^\infty BC_n(X)$.

**THEOREM 6.2.** $C^1(X) \supset C^2(X) \supset \ldots$

$B^\infty_1(X) \supset BC^1(X) \supset B^\infty_2(X) \supset BC^2(X) \supset \ldots$

$B^\infty_n(X)$ is a Banach space with respect to $\|\|_n$ and
$BC^\infty_\infty(X)$ is closed in $B^\infty_n(X)$.

For $f \in C^n(X)$ ($n \geq 1$) and $0 \leq j \leq n$ we define the $j^{th}$ Hasse derivative
of $f$ by
$$
D_j^i f(x) = \frac{\partial}{\partial x^i} f(x, x, \ldots, x) \quad (x \in X).
$$

**THEOREM 6.3.** Let $f \in C^n(X)$. Then for $0 \leq j \leq n$ we have $D_j^i f \in C^{n-i}(X)$
and if $i+j \leq n$
$$
D_{i+j}^i f = \binom{i+j}{i} D_i^j f
$$

$f$ is $n$ times differentiable in the ordinary sense and
for $0 \leq i \leq n$ we have
$$
f^{(i)} = i! D_i^i f.
$$

$f : X + K$ is called a spline function of degree $\leq n$ if for every
$a \in X$ there is a neighbourhood $U$ of $a$ such that $f|U \cap X$ is a polynomial
function of degree $\leq n$. Spline functions are in $C^\infty(X)$.

**THEOREM 6.4.** Let $f \in C^n(X)$ and $\epsilon > 0$. Then there is a spline function
$g$ of degree $\leq n$ such that $f-g \in BC^n(X)$, $\|f-g\|_n < \epsilon$. If
$D_i f = D_{i+1}^i f = \ldots = D_n f = 0$ for some $i \in \{1, \ldots, n\}$ then
$g$ can be chosen to be of degree $\leq i-1$. 
THEOREM 6.5. (Local invertibility). Let \( f \in C^n(X) \) and \( f'(a) \neq 0 \) for some \( a \in X \). Then there is a neighbourhood \( U \) of \( a \) (\( a \in U \subset X \)) such that \( f : U \to f(U) \) is a bijection, and such that the local inverse: \( f(U) \to U \) is in \( C^n(f(U)) \).

THEOREM 6.6. (Taylor formula). Let \( f \in C^n(X) \). Then for all \( x, y \in X \):

\[
f(x) = f(y) + (x-y)D_1f(y) + \ldots + (x-y)^{n-1}D_{n-1}f(y) + (x-y)^nR_nf(x,y,\ldots,y).
\]

The above result leads to another possible notion of "n-times continuously differentiable":

DEFINITION 6.7. Let \( f : X \to K \), \( n \in \mathbb{N} \). We say that \( f \in C^n(X) \) if there exist functions \( D_{-1}f, \ldots, D_{n-1}f : X \to K \) and a continuous \( R_nf : X^2 \to K \) such that for all \( x, y \in X \):

\[
f(x) = f(y) + (x-y)D_1f(y) + \ldots + (x-y)^{n-1}D_{n-1}f(y) + (x-y)^nR_nf(x,y,\ldots,y).
\]

(It follows that the \( D_i f, R_n f \) are uniquely determined and continuous.

Further we have \( C^1(X) \supset C^2(X) \supset \ldots \). It is easy to show that \( C^i(X) = C^i(X) \)

for \( i = 1, 2 \). Also \( C^n(X) \subset C^n(X) \) for all \( n \), by 6.6. But we have

EXAMPLE 6.8. Let \( X = \{ a_n p^n : a_n \in \{0,1\} \} \), and let \( f : X \to K \) be defined via

\[
f(\Sigma a_n p^n) = \Sigma a_n p^{3n!}.
\]

Then \( f \in C^n(X) \) for each \( n \), and \( D_i f = 0 \) for \( i = 1,2,4,5,\ldots \) and \( D_3 f = 1 \). On the other hand, \( f \notin C^3(X) \).

Let \( C > 0 \) and \( \{x_1,\ldots,x_n\} \) a set of \( n \) distinct points in \( X \). We call \( \{x_1,\ldots,x_n\} \) a \( C \)-polygon if for all \( i,j,k,l \in \{1,\ldots,n\} \), \( k \neq l \):
\[ \frac{|x_i-x_j|}{|x_k-x_l|} \leq C. \]

**DEFINITION 6.9.** Let \( n \in \mathbb{N} \). We say that \( X \) has **locally property** \( B_n \) if for each \( a \in X \) there is \( \delta > 0 \) and \( C > 0 \) such that for all \( x_1, x_2 \in X, x_1 \neq x_2 \), \( |x_1-a| < \delta, |x_2-a| < \delta \) there exist \( x_3, \ldots, x_n \in X \) such that \( \{x_1, x_2, \ldots, x_n\} \) is a \( C \)-polygon. (By definition, every \( X \) has locally property \( B_1 \) and \( B_2 \)).

We say that \( X \) has **globally property** \( B_n \) if there exists \( C > 0 \) such that for all \( x_1, x_2 \in X, x_1 \neq x_2 \), there exist \( x_3, \ldots, x_n \in X \) such that \( \{x_1, x_2, \ldots, x_n\} \) is a \( C \)-polygon. (Every \( X \) has globally property \( B_1 \) and \( B_2 \)).

For example, a ball in \( K \) has globally property \( B_n \), for each \( n \). Every open (non-empty) subset of \( K \) has locally property \( B_n \), for each \( n \).

Let us call \( BC^n(X) = \{ f \in C^n(X) : ||f||^\omega_n < \omega \} \), where, by definition,

\[
||f||^\omega_n = \max(||f||^\omega, ||D_1 f||^\omega, \ldots, ||D_{n-1} f||^\omega, ||R_n f||^\omega)
\]

(see 6.7). It is very easy to show that \( BC^n(X) \) is a Banach space with respect to \( || \cdot ||^\omega_n \). The main theorem:

**THEOREM 6.10.** If \( X \) has locally property \( B_n \), then \( C^n(X) = C^n(X) \).

Let \( X \) have globally property \( B_n \) \((n > 2)\) in the sense that every two-point set can be extended to a \( C \)-polygon. Then \( BC^n(X) = BC^n(X) \) and

\[
|| ||^\omega_n \leq || '\|| \leq C^{2(n-2)} || ||^\omega_n.
\]
(In general we have for \( f \in B_{C}^{n}(X) \) : \( \|f\|_{n} = \max_{0 < i \leq n} \|D_{1}^{i}f\|_{n-i}^{n} \).

As in 3.5. we want to find an antiderivation map: \( C^{n-1}(X) \to C^{n}(X) \). We cannot use the map \( P \) of 3.5. since one can prove: if \( f \in C^{1}(X) \) then \( Pf \in C^{2}(X) \) if and only if \( f' = 0 \). Further, if the characteristic of \( K \) equals \( p \neq 0 \) it is easy to see that not every \( C^{p-1} \) -function has a \( C^{p} \)-antiderivative.

**Theorem 6.11.** Let the characteristic of \( K \) be zero and let \( r_{1} > r_{2} \ldots \) as in 3.5 but such that \( r_{m} < \rho r_{m+1} \) for all \( m \), some \( \rho > 0 \).

For \( f \in C^{n-1}(X) \), set

\[
P_{n}f(x) = \sum_{k=1}^{n} \sum_{i=1}^{n} \frac{1}{i} (x_{k+1} - x_{k})^{i} D_{1}^{i-1}f(x_{k})
\]

Then \( P_{n}f \in C^{n}(X) \) and \( (P_{n}f)' = f \). If \( f \in B_{C}^{n-1}(X) \), then \( Pf \in B_{C}^{n}(X) \) and

\[
\|P_{n}f\|_{n} \leq c_{n} \|f\|_{n-1}
\]

where

\[
c_{n} = \max_{1 < i \leq n} \frac{1}{i^{\rho} \cdot n^{i}}.
\]

It follows that the map \( Q_{n} = n!P_{n}P_{n-1} \ldots P_{1} \) sends \( C(X) \) into \( C^{n}(X) \), \( D_{n}Q_{n} \) is the identity on \( C(X) \). A computation yields

\[
Q_{n} = \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} M^{n-i}S_{i},
\]

where \( M \) is the multiplication with \( x \) ((\( Mf \))(x) = xf(x) for \( f \in C(X) \)) and where

\[
S_{i}f(x) = \sum_{k=1}^{n} f(x_{k}) (x_{k+1}^{i} - x_{k}^{i})
\]

(\( f \in C(X) \), \( x \in X \)).
Note: Theorem 3.4 is also true if we replace $BC^1(X)$ by $BC^n(X)$ ($n \in \mathbb{N}$).

7. $C^\infty$-functions

Spline functions, analytic functions are in $C^\infty(X)$.

Let $f \in C^\infty(X)$ and let $f(a) = 0$ for some $a \in X$. Then $f(x) = (x-a)g(x)$ ($x \in X$) where $g \in C^\infty(X)$.

A derivation on $C^\infty(X)$ is a linear map $\phi : C^\infty(X) \rightarrow C^\infty(X)$ such that

$$\phi(fg) = \phi(f)g + f\phi(g) \quad (f, g \in C^\infty(X)).$$

Any derivation $\phi$ has the form $\frac{d}{dx}g$, where $g \in C^\infty(X)$.

**THEOREM 7.1.** Let $Y \subset K$ be a closed subset of $K$. Then there is $f \in C^\infty(K)$ (with $f' = 0$ everywhere) such that $Y$ is the set of zeros of $f$.

**THEOREM 7.2.** Let $\lambda_0, \lambda_1, \ldots$ be any sequence in $K$. Then there exists an $f \in C^\infty(K)$ such that $D_i f(0) = \lambda_i$ for all $i$.

**Open problem:** Let the characteristic of $K$ be zero. Does every $f \in C^\infty(X)$ have a $C^\infty$-antiderivative?