NON-ARCHIMEDEAN DIFFERENTIATION

by

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Introduction

This note is a collection of the most important results of a lengthy paper that will appear as a Report, Mathematisch Instituut, Katholieke Universiteit, Nijmegen. There the interested reader may find all the proofs that are missing here, and also a bibliography. Because of the rather complicated formulas and computations that are involved it seems wise to restrict oneself first to K-valued functions of one single variable. However, a lot of the results can without any problem be carried over to E-valued functions of one variable, where E is a K-Banach space. A generalization to functions: $K^n \times K^m$ will be less obvious, although it seems clear how to define $C^k$-functions in that case. (For example, in order that $f : K^2 \times K$ is $C^1$ one should require (see 3.1) that the difference quotients

\[
\frac{f(x_1, y) - f(x_2, y)}{x_1 - x_2}, \quad \frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2}
\]

can be extended to continuous functions on $K^3$. If we take again difference quotients we get four functions of four variables, required to be continuous in order that $f$ be in $C^2$ (see 6.1). It then follows very easily that

\[
\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}
\]

for $f \in C^2$.)

Throughout this note, $K$ will always be a complete non-archimedean valued field, and $X$ a non-empty subset of $K$, without isolated points. We study differentiability properties of functions $f : X \rightarrow K$. Besides the analytic functions we have other examples of differentiable
functions such as the locally constant functions (they have derivative zero everywhere). The function \( I_{n}^{a} p^{n} \rightarrow I_{n}^{a} p^{n'} \), defined on \( \mathbb{Z}_{p} \) is an example of an injective function with zero derivative and which is in \( \text{Lip}_{a} \) for every \( a > 0 \). The function \( f : \mathbb{Z}_{p} \rightarrow \mathbb{Q}_{p} \) defined via 
\[ f(x) = x - p^{2n} \text{ if } |x - p^{n}| < p^{-2n} \text{ and } f(x) = x \text{ elsewhere} \]
has derivative 1 everywhere, but for all \( n \in \mathbb{N} \) \( f(p^{n}) = f(p^{n} - p^{2n}) = p^{n} - p^{2n} \), hence \( f \) is not locally injective at 0.

The above examples show that we do not have a mean value theorem or a local invertibility theorem. We now state the results going from "bad" to "smooth" functions.

1. Nowhere differentiable functions

Let \( BC(X) \) be the algebra of the bounded continuous functions on \( X \rightarrow K \), normed by the sup norm \( || ||_{\infty} \). We have, analogous to the classical case:

\[ \text{THEOREM 1.1. The collection of those } f \in BC(X) \text{ that are somewhere differentiable is of first category in } BC(X) \text{ (in the sense of Baire).} \]

In contrast to the theory of functions on the real line we have

\[ \text{THEOREM 1.2. Let } X \text{ be open in } K, \text{ and let } f : X \rightarrow K \text{ be a bounded uniformly continuous function, and let } \varepsilon > 0. \text{ Then there exists a nowhere differentiable } g : X \rightarrow K \text{ such that } ||f-g||_{\infty} < \varepsilon. \]

2. Differentiability as such

Contrary to the classical case we have a nice criterion for a
function to possess an antiderivative:

**THEOREM 2.1.** Let \( f : X \rightarrow K \). Then \( f \) has an antiderivative if and only if \( f \) is of Baire class one, (i.e., \( f \) is the pointwise limit of a sequence of continuous functions).

Further we have Sard-type theorems. If \( K \) is a local field then \( Y \subseteq K \) is called a nullset if it has measure zero in the sense of the (real) Haar measure on \( K \).

**THEOREM 2.2.** Let \( K \) be a local field and let \( f : X \rightarrow K \) be differentiable. Then we have:

1. If \( Y \subseteq X \) is a nullset then \( f(Y) \) is a nullset ("\( f \) has property (N)"")
2. \( \{ f(x) : f'(x) = 0 \} \) is a nullset.

**COROLLARY 2.3.** If \( f : X \rightarrow K \) is differentiable, \( f' = 0 \) almost everywhere, then \( f(X) \) is a nullset.

3. Continuously differentiable functions

    If we want the local invertibility theorem to hold for \( C^1 \)-functions we have to take a definition of a \( C^1 \)-function, stronger then just "\( f \) is differentiable and \( f \) is continuous". For \( f : X \rightarrow K \), define

    \[
    \phi_1 f(x,y) = \frac{f(x) - f(y)}{x-y} \quad (x,y \in X, x \neq y).
    \]

**DEFINITION 3.1.** \( f : X \rightarrow K \) is in \( C^1(X) \) if \( \phi_1 f \) can (uniquely) be extended to a continuous function \( \bar{\phi}_1 f \) on \( X \times X \).

    (Notice that for a real valued function \( f \) defined on an interval
the continuity of $f'$ already guarantees the existence of a continuous $\Phi(f)$.

**THEOREM 3.2.** Let $f \in C^1(X)$ and let $a \in X$.

(a) If $f'(a) \neq 0$ then $f$ is locally invertible at $a$. (In fact, $(f'(a))^{-1}f$ is an isometry locally at $a$).

(b) If $X$ is open in $K$ and if $f' \neq 0$ everywhere on $X$ then $f$ is an open mapping.

Let $BC^1(X) = \{f \in C^1(X) : \|f\|_1 := \|f\|_\infty + \|\Phi_1 f\|_\infty\}$. Then $BC^1(X)$ is a Banach space with respect to $\|\|$. We may put a locally convex topology on $C^1(X)$ via the defining seminorms $\|\|_1,c$ where $C$ runs through the compact subsets of $X$:

$$\|f\|_{1,c} = \sup_{x \in C} |f(x)| + \sup_{x \in C} \sup_{y \in C} |\Phi_1 f(x,y)| \quad (f \in C^1(X)).$$

Let $N^1(X) = \{f \in C^1(X) : f' = 0\}$ and $BN^1(X) = \{f \in BC^1(X) : f' = 0\}$. Then $N^1(X)$ is closed in $C^1(X)$, $BN^1(X)$ is closed in $BC^1(X)$.

**THEOREM 3.3.** The locally linear functions (in $BC^1(X)$) form a dense subset of $C^1(X)$ (of $BC^1(X)$).

The locally constant functions (in $BC^1(X)$) form a dense subset of $N^1(X)$ (of $BN^1(X)$).

**THEOREM 3.4.** If either $X$ is compact or $K$ has discrete valuation then $BC^1(X)$ has an orthonormal base (in the sense of the norm).

If $X$ is not compact and $K$ has dense valuation then, for any $\alpha > 0$, $BC^1(X)$ has no $\alpha$-orthogonal base.
Let us choose real numbers \(1 > r_1 > r_2 > \ldots\) with \(\lim r_n = 0\), and, for each \(n\), let \(R_n\) be a full set of representatives of the equivalence relation (in \(X\)): \(x \sim y\) if \(|x-y| < r_n\). We can arrange that \(R_1 \subset R_2 \subset \ldots\) For each \(x \in X, n \in \mathbb{N}\), let \(x_n \in X\) be determined by:

\[|x_n - x| < r_n, x_n \in R_n.\]

For a continuous \(f : X \to \mathbb{K}\) set

\[
(Pf)(x) = \sum_{n=1}^{\infty} f(x_n)(x_{n+1} - x_n) \quad (x \in X)
\]

THEOREM 3.5. The map \(P\) defined above is a continuous linear map:

\(C(X) \to C^1(X)\) and its restriction to \(BC(X)\) is an isometry: \(BC(X) \to BC^1(X)\). \(P\) is an antiderivation map i.e., \((Pf)' = f\) for each \(f \in C(X)\).

COROLLARY 3.6. Every continuous function has a \(C^1\)-antiderivative.

In fact, by passing through the quotient, differentiation yields a map \(p : BC^1(X)/BN^1(X) \to BC(X)\) which is a surjective isometry. Moreover, \(BN^1(X)\) has an orthogonal complement \((\text{im } P)\) in \(BC^1(X)\).

4. \(C^1(X)\) for compact \(X\)

(Throughout section 4, \(X\) is compact). The set \(|x-y| : x, y \in X\) is bounded and has only 0 as an accumulation point, hence it can be written as \(\{r_1, r_2, \ldots\} \cup \{0\}\), where \(r_1 > r_2 > \ldots\) and \(\lim r_n = 0\).

Let \(r_0 = \infty\). For each \(i\), let \(R_i\) be a full set of representatives in \(X\) of the equivalence relation "\(x \sim y\) if \(|x-y| < r_1\)" such that \(R_0 \subset R_1 \subset \ldots\). Then \(R_i\) is finite for each \(i\) and \(R_0\) consists only of one single point \(a_0\). Let \(R = \bigcup_i R_i\) and define \(v : R \to \{0, 1, 2, \ldots\}\) as follows. For a \(a \in R\) let \(v(a)\) be the nonnegative integer \(m\) for which \(a \in R_m \setminus R_{m-1}\) \((R_{-1} = \emptyset\) by definition). For each a \(a \in R\) let
\[ B_a = \{ x \in X : |x-a| < r_{\nu}(a) \}, \]

and let \( e_a \) be the \( K \)-valued characteristic function of \( B_a \). Further, we define

\[ a \not\preceq b \text{ iff } b \in B_a \quad (a,b \in \mathbb{R}) \]

Then we have

**Lemma 4.1.** \((\mathbb{R}, \rho)\) is a partially ordered set with a smallest element \( a_0 \). For each \( a \in \mathbb{R} \), the set \( \{ x \in \mathbb{R} : x \not\preceq a \} \) is finite and linearly ordered by \( \not\preceq \).

Define for \( a \in \mathbb{R} \), \( a \neq a_0 \):

\[ a_- = \max \{ x \in \mathbb{R} : x \not\preceq a, x \not\not\preceq a \}. \]

**Theorem 4.2.** The set \( \{ e_a : a \in \mathbb{R} \} \) forms an orthonormal base of \( C(X) \).

Let \( f \in C(X) \) and \( f = \sum \lambda a \mathbb{e}_a \) for some \( \lambda a \in K \). Then:

\[ \lambda a_0 = f(a_0) \text{ and for } a \neq a_0: \lambda a = f(a) - f(a_-). \]

The set \( \{ e_a : a \in \mathbb{R} \} \cup \{ P e_a : a \in \mathbb{R} \} \) (\( P \) as in 3.5) forms an orthogonal base of \( C^1(X) \). Let \( f \in C^1(X) \), \( f = \sum \lambda a \mathbb{e}_a + \sum \mu b P e_b \) \((\lambda a, \mu b \in K)\) in the \( \| \|_1 \)-norm. Then:

\[ \lambda a_0 = f(a_0), \mu a_0 = f'(a_0) \]

and for \( a \neq a_0:\)

\[ \lambda a = f(a) - f(a_-) - (a-a_-)f'(a_-) \]
\[ \mu a = f'(a) - f'(a_-). \]

5. Uniform differentiability

There seem to be two natural notions of "uniform differentiability".

Let \( f \in C^1(X) \). \( f \) is called uniformly differentiable if \( \lim_{x \to y} f(x,y) = f'(y) \) uniformly in \( y \). \( f \) is called strongly uniformly differentiable if \( \Phi f \) is uniformly continuous.

If \( X \) is compact both notions are the same and coincide with "continuous differentiable".
THEOREM 5.1. Let $f : X \to K$ be (strongly) uniformly differentiable. Then $f$ has a unique continuous extension $\overline{f} : \overline{X} \to K$ ($\overline{X}$ is the closure of $X$ in $K$).

This $\overline{f}$ is (strongly) uniformly differentiable.

THEOREM 5.2. Let $f : X \to K$ be uniformly differentiable. Then each of the following properties implies strong uniform differentiability of $f$:

(a) $\phi_1 f$ is bounded.

(b) Both $f$ and $f'$ are bounded

(c) $X$ is "nice" and $f$ is bounded.

$X$ is called "nice" if for each $r > 0$ there is $s > 0$ such that for every $x \in X$ there is $y \in X$ such that $s \leq |x-y| \leq r$.

The theorems 3.3, 3.5., 3.6. each have an analogon for uniformly differentiable functions.

6. $C^n$-functions

For $n \in \mathbb{N}$, let $\mathcal{V}_n = \{(x_1, \ldots , x_n) \in X^n : i \neq j \Rightarrow x_i \neq x_j \}$. For $f : X \to K$ we define the $n$th difference quotient $\phi_n f : \mathcal{V}_{n+1} \to K$ inductively as follows $\phi_0 f = f$ and for $(x_1, \ldots , x_{n+1}) \in \mathcal{V}_{n+1}$:

$$\phi_n f(x_1, \ldots , x_{n+1}) = (x_1-x_2)^{-1}(\phi_{n-1} f(x_1, x_3, \ldots , x_{n+1}) - \phi_{n-1} f(x_2, x_3, \ldots , x_{n+1})).$$

Since $\mathcal{V}_n$ is dense in $X^n$ for each $n$ the following definition makes sense.

DEFINITION 6.1. Let $f : X \to K$, $n \in \mathbb{N} \cup \{0\}$. We say that $f \in C^n(X)$ if $\phi_n f$ can be extended to a continuous function $\overline{\phi}_n f : X^{n+1} \to K$.

We say that $f \in \mathcal{B}^n(X)$ if $\phi_0 f, \ldots , \phi_n f$ are bounded functions.
For $f \in B^\infty_n(X)$ set

$$||f||_n = \max_{0 \leq i \leq n} ||\Phi_i f||_\infty.$$  

Let $B^\infty_n(X) = B^\infty_n(X) \cap C^n(X)$, $C^\infty(X) = \bigcap_{n=1}^\infty C^n(X)$, $BC^\infty_n(X) = \bigcap_{n=1}^\infty BC^n(X)$.

**Theorem 6.2.** $C^1(X) \supset C^2(X) \supset \ldots$

$B^\infty_1(X) \supset BC^\infty_1(X) \supset B^\infty_2(X) \supset BC^\infty_2(X) \supset \ldots$

$B^\infty_n(X)$ is a Banach space with respect to $|| ||_n$ and $BC^\infty_n(X)$ is closed in $B^\infty_n(X)$.

For $f \in C^n(X)$ ($n \geq 1$) and $0 \leq j \leq n$ we define the $j^{th}$ Hasse derivative of $f$ by

$$D_j f(x) = \Phi_j f(x, x, \ldots, x) \quad (x \in X).$$  

**Theorem 6.3.** Let $f \in C^n(X)$. Then for $0 \leq j \leq n$ we have $D_j f \in C^{n-j}(X)$ and if $i+j \leq n$

$$D_i D_j f = \binom{i+j}{i} D_{i+j} f$$

$f$ is $n$ times differentiable in the ordinary sense and for $0 \leq i \leq n$ we have

$$f^{(i)} = i! D_i f.$$  

$f : X \to \mathbb{K}$ is called a spline function of degree $\leq n$ if for every $a \in X$ there is a neighbourhood $U$ of $a$ such that $f|_U \cap X$ is a polynomial function of degree $\leq n$. Spline functions are in $C^\infty(X)$.

**Theorem 6.4.** Let $f \in C^n(X)$ and $\varepsilon > 0$. Then there is a spline function $g$ of degree $\leq n$ such that $f-g \in B^n(X)$, $||f-g||_n < \varepsilon$. If $D_i f = D_{i+1} f = \ldots = D_n f = 0$ for some $i \in \{1, \ldots, n\}$ then $g$ can be chosen to be of degree $\leq i-1.$
THEOREM 6.5. (Local invertibility). Let $f \in C^n(X)$ and $f'(a) \neq 0$ for some $a \in X$. Then there is a neighbourhood $U$ of $a$ ($a \in U \subseteq X$) such that $f : U \to f(U)$ is a bijection, and such that the local inverse: $f(U) \to U$ is in $C^n(f(U))$.

THEOREM 6.6. (Taylor formula). Let $f \in C^n(X)$. Then for all $x,y \in X$:

$$f(x) = f(y) + (x-y)D_1f(y) + \ldots + (x-y)^{n-1}D_{n-1}f(y) + (x-y)^nR_nf(x,y)$$

The above result leads to another possible notion of "n-times continuously differentiable":

DEFINITION 6.7. Let $f : X \to \mathbb{K}$, $n \in \mathbb{N}$. We say that $f \in C^n(X)$ if there exist functions $D_1f, \ldots, D_{n-1}f : X \to \mathbb{K}$ and a continuous $R_nf : \mathbb{K}^2 \to \mathbb{K}$ such that for all $x,y \in X$:

$$f(x) = f(y) + (x-y)D_1f(y) + \ldots + (x-y)^{n-1}D_{n-1}f(y) + (x-y)^nR_nf(x,y).$$

(It follows that the $D_if$, $R_nf$ are uniquely determined and continuous.

Further we have $C^1(X) \supset C^2(X) \supset \ldots$. It is easy to show that $C^i(X) = C^i(X)$ for $i = 1,2$. Also $C^n(X) \subset C^n(X)$ for all $n$, by 6.6. But we have

EXAMPLE 6.8. Let $X = \{ \Sigma a_n p_n! : a_n \in \{0,1\} \}$, and let $f : X \to \mathbb{K}$ be defined via

$$f(\Sigma a_n p_n!) = \Sigma a_n p_n^{3n!}.$$ 

Then $f \in C^n(X)$ for each $n$, and $D_if = 0$ for $i = 1,2,4,5,\ldots$ and $D_3f = 1$. On the other hand, $f \notin C^3(X)$.

Let $C > 0$ and $\{x_1, \ldots, x_n\}$ a set of $n$ distinct points in $X$. We call $\{x_1, \ldots, x_n\}$ a C-polygon if for all $i,j,k,l \in \{1, \ldots, n\}$, $k \neq l$: 
\[ \frac{|x_i - x_j|}{|x_k - x_1|} \leq C. \]

**Definition 6.9.** Let \( n \in \mathbb{N} \). We say that \( X \) has locally property \( B_n \) if for each \( a \in X \) there is \( \delta > 0 \) and \( C > 0 \) such that for all \( x_1, x_2 \in X \), \( x_1 \neq x_2 \), \( |x_1 - a| < \delta \), \( |x_2 - a| < \delta \) there exist \( x_3, \ldots, x_n \in X \) such that \( \{x_1, x_2, \ldots, x_n\} \) is a \( C \)-polygon. (By definition, every \( X \) has locally property \( B_1 \) and \( B_2 \)).

We say that \( X \) has globally property \( B_n \) if there exists \( C > 0 \) such that for all \( x_1, x_2 \in X \), \( x_1 \neq x_2 \), there exist \( x_3, \ldots, x_n \in X \) such that \( \{x_1, x_2, \ldots, x_n\} \) is a \( C \)-polygon. (Every \( X \) has globally property \( B_1 \) and \( B_2 \)).

For example, a ball in \( K \) has globally property \( B_n \) for each \( n \). Every open (non-empty) subset of \( K \) has locally property \( B_n \), for each \( n \).

Let us call \( BC^n(X) = \{ f \in C^n(X) : ||f||^\wedge_n < \omega \} \), where, by definition,

\[ ||f||^\wedge_n = \max(||f||^\wedge, ||D^1 f||^\wedge, \ldots, ||D^n f||^\wedge) \]

(see 6.7). It is very easy to show that \( BC^n(X) \) is a Banach space with respect to \( || \cdot ||^\wedge_n \). The main theorem:

**Theorem 6.10.** If \( X \) has locally property \( B_n \), then \( C^n(X) = BC^n(X) \).

Let \( X \) have globally property \( B_n \) \( (n > 2) \) in the sense that every two-point set can be extended to a \( C \)-polygon. Then \( BC^n(X) = BC^n(X) \) and

\[ || \cdot ||^\wedge_n \leq || \cdot ||_n \leq C^2(n-2) || \cdot ||^\wedge_n. \]
(In general we have for \( f \in \text{BC}^n(X) : \|f\|_n = \max_{0 < i \leq n} \|D_i f\|_{n-i}^\nu \).)

As in 3.5. we want to find an antiderivation map: \( \text{C}^{n-1}(X) \to \text{C}^n(X) \).
We cannot use the map \( P \) of 3.5. since one can prove: if \( f \in \text{C}^1(X) \) then \( Pf \in \text{C}^2(X) \) if and only if \( f' = 0 \). Further, if the characteristic of \( K \) equals \( p \neq 0 \) it is easy to see that not every \( \text{C}^{p-1} \)-function has a \( \text{C}^p \)-antiderivative.

**Theorem 6.11.** Let the characteristic of \( K \) be zero and let \( r_1 > r_2 \ldots \)
as in 3.5 but such that \( r_m < \rho r_{m+1} \) for all \( m \), some \( \rho > 0 \).
For \( f \in \text{C}^{n-1}(X) \), set

\[
P_n f(x) = \sum_{k=1}^{\infty} \frac{1}{(k+1)!} (x_{k+1} - x_k) D_{k-1} f(x_k) 
\]

Then \( P_n f \in \text{C}^n(X) \) and \((P_n f)' = f \). If \( f \in \text{BC}^{n-1}(X) \), then \( Pf \in \text{BC}^n(X) \) and

\[
\|P_n f\|_n \leq c_n \|f\|_{n-1}
\]

where

\[
c_n = \max_{1 < i \leq n} \frac{1}{i!} \rho^n.
\]

It follows that the map \( Q_n = n! P_n P_{n-1} \ldots P_1 \) sends \( \text{C}(X) \) into \( \text{C}^n(X) \),
\( \text{D}_n Q_n \) is the identity on \( \text{C}(X) \). A computation yields

\[
Q_n = \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} M^{-i} S_i,
\]

where \( M \) is the multiplication with \( x \) \((Mf)(x) = xf(x) \text{ for } f \in \text{C}(X)\)
and where

\[
S_i f(x) = \sum_{k=1}^{\infty} f(x_k) (x_{k+1} - x_k) (x_k) (x_{k+1} - x_k) (x_k) 
\]

\((f \in \text{C}(X), x \in X)\)
Note: Theorem 3.4 is also true if we replace $BC^1(X)$ by $BC^n(X)$ ($n \in \mathbb{N}$).

7. $C^\infty$-functions

Spline functions, analytic functions are in $C^\infty(X)$.

Let $f \in C^\infty(X)$ and let $f(a) = 0$ for some $a \in X$. Then $f(x) = (x-a)g(x)$ ($x \in X$) where $g \in C^\infty(X)$.

A derivation on $C^\infty(X)$ is a linear map $\phi : C^\infty(X) \to C^\infty(X)$ such that

$$\phi(fg) = \phi(f)g + f\phi(g) \quad (f, g \in C^\infty(X)).$$

Any derivation $\phi$ has the form $\frac{d}{dx}g$, where $g \in C^\infty(X)$.

**THEOREM 7.1.** Let $Y \subset K$ be a closed subset of $K$. Then there is $f \in C^\infty(K)$ (with $f' = 0$ everywhere) such that $Y$ is the set of zeros of $f$.

**THEOREM 7.2.** Let $\lambda_0, \lambda_1, \ldots$ be any sequence in $K$. Then there exists an $f \in C^\infty(X)$ such that $D_i f(0) = \lambda_i$ for all $i$.

Open problem: Let the characteristic of $K$ be zero. Does every $f \in C^\infty(X)$ have a $C^\infty$-antiderivative?