NON-ARCHIMEDEAN DIFFERENTIATION

by

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Introduction

This note is a collection of the most important results of a lengthy paper that will appear as a Report, Mathematisch Instituut, Katholieke Universiteit, Nijmegen. There the interested reader may find all the proofs that are missing here, and also a bibliography. Because of the rather complicated formulas and computations that are involved it seems wise to restrict oneself first to \( K \)-valued functions of one single variable. However, a lot of the results can without any problem be carried over to \( E \)-valued functions of one variable, where \( E \) is a \( K \)-Banach space. A generalization to functions: \( K^n \to K^m \) will be less obvious, although it seems clear how to define \( C^k \)-functions in that case. (For example, in order that \( f : K^2 \to K \) is \( C^1 \) one should require (see 3.1) that the difference quotients

\[
\frac{f(x_1, y) - f(x_2, y)}{x_1 - x_2}, \quad \frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2}
\]

can be extended to continuous functions on \( K^3 \). If we take again difference quotients we get four functions of four variables, required to be continuous in order that \( f \) be in \( C^2 \) (see 6.1). It then follows very easily that \( \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \) for \( f \in C^2 \).)

Throughout this note, \( K \) will always be a complete non-archimedean valued field, and \( X \) a non-empty subset of \( K \), without isolated points. We study differentiability properties of functions \( f : X \to K \). Besides the analytic functions we have other examples of differentiable
functions such as the locally constant functions (they have derivative zero everywhere). The function \( f_n : \mathbb{Z}_p \to \mathbb{Z}_p \), defined on \( \mathbb{Z}_p \) is an example of an injective function with zero derivative and which is in Lip_\( \alpha \) for every \( \alpha > 0 \). The function \( f : \mathbb{Z}_p \to \mathbb{Z}_p \) defined via \( f(x) = x \cdot p^n \) if \( |x-p^n| < p^{-2n} \) and \( f(x) = x \) elsewhere has derivative 1 everywhere, but for all \( n \in \mathbb{N} \) \( f(p^n) = f(p^n \cdot p^n) = p^n \cdot p^n \), hence \( f \) is not locally injective at 0.

The above examples show that we do not have a mean value theorem or a local invertibility theorem. We now state the results going from "bad" to "smooth" functions.

1. **Nowhere differentiable functions**

   Let \( BC(X) \) be the algebra of the bounded continuous functions: \( X \to K \), normed by the sup norm \( || \cdot ||_\infty \). We have, analogous to the classical case:

   **THEOREM 1.1.** The collection of those \( f \in BC(X) \) that are somewhere differentiable is of first category in \( BC(X) \) (in the sense of Baire).

   In contrast to the theory of functions on the real line we have

   **THEOREM 1.2.** Let \( X \) be open in \( K \), and let \( f : X \to K \) be a bounded uniformly continuous function, and let \( \varepsilon > 0 \). Then there exists a nowhere differentiable \( g : X \to K \) such that \( g \) has bounded difference quotients, and such that \( ||f-g||_\infty < \varepsilon \).

2. **Differentiability as such**

   Contrary to the classical case we have a nice criterion for a
function to possess an antiderivative:

THEOREM 2.1. Let \( f : X \to K \). Then \( f \) has an antiderivative if and only if \( f \) is of Baire class one, (i.e., \( f \) is the pointwise limit of a sequence of continuous functions).

Further we have Sard-type theorems. If \( K \) is a local field then \( Y \subseteq K \) is called a nullset if it has measure zero in the sense of the (real) Haar measure on \( K \).

THEOREM 2.2. Let \( K \) be a local field and let \( f : X \to K \) be differentiable. Then we have:

1. If \( Y \subseteq X \) is a nullset then \( f(Y) \) is a nullset ("\( f \) has property (N)"")
2. \( \{ f(x) : f'(x) = 0 \} \) is a nullset.

COROLLARY 2.3. If \( f : X \to K \) is differentiable, \( f' = 0 \) almost everywhere, then \( f(X) \) is a nullset.

3. Continuously differentiable functions

If we want the local invertibility theorem to hold for \( C^1 \)-functions we have to take a definition of a \( C^1 \)-function, stronger then just "\( f \) is differentiable and \( f \) is continuous". For \( f : X \to K \), define

\[
\phi^1_f(x,y) = \frac{f(x) - f(y)}{x-y} \quad (x, y \in X, x \neq y).
\]

DEFINITION 3.1. \( f : X \to K \) is in \( C^1(X) \) if \( \phi^1_f \) can (uniquely) be extended to a continuous function \( \overline{\phi^1_f} \) on \( X \times X \).

(Notice that for a real valued function \( f \) defined on an interval
the continuity of $f'$ already guarantees the existence of a continuous $\tilde{f}_1$.

**THEOREM 3.2.** Let $f \in C^1(X)$ and let $a \in X$.

(a) If $f'(a) \neq 0$ then $f$ is locally invertible at $a$. (In fact, $(f'(a))^{-1}f$ is an isometry locally at $a$).

(b) If $X$ is open in $K$ and if $f' \neq 0$ everywhere on $X$ then $f$ is an open mapping.

Let $BC^1(X) = \{f \in C^1(X) : ||f||^1 := ||f||_1 \vee ||f_1||_\infty\}$. Then $BC^1(X)$ is a Banach space with respect to $|| \cdot ||^1$. We may put a locally convex topology on $C^1(X)$ via the defining seminorms $|| \cdot ||^1_C$ where $C$ runs through the compact subsets of $X$:

$$||f||^1_C = \sup_{x \in C} |f(x)| \vee \sup_{x \in C} |\tilde{f}_1f(x,y)|$$

($f \in C^1(X)$).

Let $N^1(X) = \{f \in C^1(X) : f' = 0\}$ and $BN^1(X) = \{f \in BC^1(X) : f' = 0\}$. Then $N^1(X)$ is closed in $C^1(X)$, $BN^1(X)$ is closed in $BC^1(X)$.

**THEOREM 3.3.** The locally linear functions (in $BC^1(X)$) form a dense subset of $C^1(X)$ (of $BC^1(X)$).

The locally constant functions (in $BC^1(X)$) form a dense subset of $N^1(X)$ (of $BN^1(X)$).

**THEOREM 3.4.** If either $X$ is compact or $K$ has discrete valuation then $BC^1(X)$ has an orthonormal base (in the sense of the norm).

If $X$ is not compact and $K$ has dense valuation then, for any $\alpha > 0$, $BC^1(X)$ has no $\alpha$-orthogonal base.
Let us choose real numbers $1 > r_1 > r_2 > \ldots$ with $\lim_{n \to \infty} r_n = 0$, and, for each $n$, let $R_n$ be a full set of representatives of the equivalence relation (in $X$): $x \sim y$ if $|x-y| < r_n$. We can arrange that $R_1 \subset R_2 \subset \ldots$. For each $x \in X$, $n \in \mathbb{N}$, let $x_n \in X$ be determined by:

$$|x_n - x| < r_n, \quad x_n \in R_n.$$

For a continuous $f : X \to \mathbb{K}$ set

$$(Pf)(x) = \sum_{n=1}^{\infty} f(x_n)(x_{n+1} - x_n) \quad (x \in X).$$

**Theorem 3.5.** The map $P$ defined above is a continuous linear map:

$$C(X) \to C^1(X)$$
and its restriction to $BC(X)$ is an isometry: $BC(X) \to BC^1(X)$. $P$ is an antiderivation map, i.e., $(Pf)' = f$ for each $f \in C(X)$.

**Corollary 3.6.** Every continuous function has a $C^1$-antiderivative.

In fact, by passing through the quotient, differentiation yields a map $p : BC^1(X)/BN^1(X) \to BC(X)$ which is a surjective isometry. Moreover, $BN^1(X)$ has an orthogonal complement $(\text{im } P)$ in $BC^1(X)$.

4. $C^1(X)$ for compact $X$

(Throughout section 4, $X$ is compact). The set $\{ |x-y| : x, y \in X \}$ is bounded and has only 0 as an accumulation point, hence it can be written as $\{ r_1, r_2, \ldots \} \cup \{ 0 \}$, where $r_1 > r_2 > \ldots$ and $\lim_{n \to \infty} r_n = 0$.

Let $r_0 = \infty$. For each $i$, let $R_i$ be a full set of representatives in $X$ of the equivalence relation "$x \sim y$ if $|x-y| < r_i$" such that $R_0 \subset R_1 \subset \ldots$. Then $R_i$ is finite for each $i$ and $R_0$ consists only of one single point $a_0$. Let $R = \bigcup_{i=1}^{\infty} R_i$ and define $\nu : R \to \{ 0, 1, 2, \ldots \}$ as follows. For a $a \in R$ let $\nu(a)$ be the nonnegative integer $m$ for which $a \in \mathbb{R}_m \setminus \mathbb{R}_{m-1}$ ($\mathbb{R}_{-1} = \emptyset$ by definition). For each $a \in \mathbb{R}$ let
Let $B_a = \{x \in X : |x-a| < r_v(a)\}$,
and let $e_a$ be the $K$-valued characteristic function of $B_a$. Further, we define
$$a \prec b \iff b \in B_a \quad (a, b \in R)$$
Then we have

**Lemma 4.1.** $(R, \prec)$ is a partially ordered set with a smallest element $a_0$. For each $a \in R$, the set $\{x \in R : x \not\prec a\}$ is finite and linearly ordered by $\prec$.

Define for $a \in R$, $a \neq a_0$: $a_- = \max \{x \in R : x \not\prec a, x \not\prec a_0\}$. Then

**Theorem 4.2.** The set $\{e_a : a \in R\}$ forms an orthonormal base of $C(X)$.

Let $f \in C(X)$ and $f = \sum \lambda_a e_a$ for some $\lambda_a \in K$. Then
$$\lambda_{a_0} = f(a_0) \quad \text{and for } a \neq a_0: \lambda_a = f(a) - f(a_-).$$
The set $\{e_a : a \in R\} \cup \{P_{e_a} : a \in R\}$ (as in 3.5) forms an orthogonal base of $C^1(X)$. Let $f \in C^1(X)$, $f = \sum \lambda_a e_a + \sum \mu_b P_{e_b}$ ($\lambda_a, \mu_b \in K$) in the $|| \cdot ||_1$-norm.

Then $\lambda_{a_0} = f(a_0)$, $\mu_{a_0} = f'(a_0)$ and for $a \neq a_0$:
$$\lambda_a = f(a) - f(a_-) - (a-a_-)f'(a_-)$$
$$\mu_a = f'(a) - f'(a_-).$$

5. Uniform differentiability

There seem to be two natural notions of "uniform differentiability".

Let $f \in C^1(X)$. $f$ is called **uniformly differentiable** if $\lim_{x \to y} f(x, y) = f'(y)$ uniformly in $y$. $f$ is called **strongly uniformly differentiable** if $\Phi f$ is uniformly continuous.

If $X$ is compact both notions are the same and coincide with "continuous differentiable".
THEOREM 5.1. Let $f : X \to K$ be (strongly) uniformly differentiable. Then $f$ has a unique continuous extension $\tilde{f} : \overline{X} \to K$ (\overline{X} is the closure of $X$ in $K$).

This $\tilde{f}$ is (strongly) uniformly differentiable.

THEOREM 5.2. Let $f : X \to K$ be uniformly differentiable. Then each of the following properties implies strong uniform differentiability of $f$:

(a) $\phi_1 f$ is bounded.
(b) Both $f$ and $f'$ are bounded
(c) $X$ is "nice" and $f$ is bounded.

($X$ is called "nice" if for each $r > 0$ there is $s > 0$ such that for every $x \in X$ there is $y \in X$ such that $s \leq |x-y| \leq r$).

The theorems 3.3, 3.5., 3.6. each have an analogon for uniformly differentiable functions.

6. $C^n$-functions

For $n \in \mathbb{N}$, let $V^n X = \{ (x_1, \ldots, x_n) \in X^n : i \neq j \Rightarrow x_i \neq x_j \}$. For $f : X \to K$ we define the $n$th difference quotient $\phi^n f : V^{n+1} X \to K$ inductively as follows $\phi^0 f = f$ and for $(x_1, \ldots, x_{n+1}) \in V^{n+1} X$:

$$\phi^n f(x_1, \ldots, x_{n+1}) = (x_1-x_2)^{-1}(\phi^{n-1} f(x_1, x_3, \ldots, x_{n+1}) - \phi_{n-1} f(x_2, x_3, \ldots, x_{n+1})).$$

Since $V^n X$ is dense in $X^n$ for each $n$ the following definition makes sense.

DEFINITION 6.1. Let $f : X \to K$, $n \in \mathbb{N} \cup \{0\}$. We say that $f \in C^n(X)$ if

$\phi^n f$ can be extended to a continuous function $\tilde{\phi^n f} : X^{n+1} \to K$.

We say that $f \in B^n(X)$ if $\phi^0 f, \ldots, \phi^n f$ are bounded functions.
For \( f \in \mathcal{B}^n(X) \) set

\[
||f||_n = \max_{0 \leq i \leq n} ||\psi_i f||_\infty.
\]

Let \( \mathcal{B}^n(X) = \mathcal{B}^n(X) \cap C^n(X) \), \( \mathcal{C}^\infty(X) = \bigcap_{n=1}^\infty C^n(X) \),

\( \mathcal{B}^\infty(X) = \bigcap_{n=1}^\infty \mathcal{B}^n(X) \).

**Theorem 6.2.** \( C^1(X) \supset C^2(X) \supset \ldots \)

\( \mathcal{B}^1(X) \supset \mathcal{B}^2(X) \supset \mathcal{B}^2(X) \supset \ldots \)

\( \mathcal{B}^n(X) \) is a Banach space with respect to \( || \cdot ||_n \) and \( \mathcal{B}^n(X) \) is closed in \( \mathcal{B}^n(X) \).

For \( f \in C^n(X) \) \((n \geq 1)\) and \( 0 \leq j \leq n \) we define the \( j \)th Hasse derivative of \( f \) by

\[
D_j f(x) = \overline{\phi}_j f(x, \ldots, x) \quad (x \in X).
\]

**Theorem 6.3.** Let \( f \in C^n(X) \). Then for \( 0 \leq j \leq n \) we have \( D_j f \in C^{n-j}(X) \) and if \( i+j \leq n \)

\[
D_i D_j f = \binom{i+j}{i} D_{i+j} f
\]

\( f \) is \( n \) times differentiable in the ordinary sense and for \( 0 \leq i \leq n \) we have

\[
f^{(i)} = i! D_i f.
\]

\( f : X \to K \) is called a spline function of degree \( \leq n \) if for every \( a \in X \) there is a neighbourhood \( U \) of \( a \) such that \( f|U \cap X \) is a polynomial function of degree \( \leq n \). Spline functions are in \( C^\infty(X) \).

**Theorem 6.4.** Let \( f \in C^n(X) \) and \( \varepsilon > 0 \). Then there is a spline function \( g \) of degree \( \leq n \) such that \( f - g \in \mathcal{B}^n(X), ||f - g||_n < \varepsilon. \) If \( D_i f = D_{i+1} f = \ldots = D_n f = 0 \) for some \( i \in \{1, \ldots, n\} \) then \( g \) can be chosen to be of degree \( \leq i-1 \).
THEOREM 6.5. (Local invertibility). Let $f \in C^n(X)$ and $f'(a) \neq 0$ for some $a \in X$. Then there is a neighbourhood $U$ of $a$ ($a \in U \subset X$) such that $f : U \to f(U)$ is a bijection, and such that the local inverse: $f(U) \to U$ is in $C^n(f(U))$.

THEOREM 6.6. (Taylor formula). Let $f \in C^n(X)$. Then for all $x,y \in X$:

$$f(x) = f(y) + (x-y)D_1f(y) + \ldots + (x-y)^{n-1}D_{n-1}f(y) + (x-y)^n R_nf(x,y).$$

The above result leads to another possible notion of "n-times continuously differentiable":

DEFINITION 6.7. Let $f : X \to K$, $n \in \mathbb{IN}$. We say that $f \in C^n(X)$ if there exist functions $D_1f, \ldots, D_{n-1}f : X \to K$ and a continuous $R_nf : X^2 \to K$ such that for all $x,y \in X$

$$f(x) = f(y) + (x-y)D_1f(y) + \ldots + (x-y)^{n-1}D_{n-1}f(y) + (x-y)^n R_nf(x,y).$$

(It follows that the $D_i f$, $R_n f$ are uniquely determined and continuous.

Further we have $C^1(X) \supset C^2(X) \supset \ldots$. It is easy to show that $C^i(X) = C^i(X)$ for $i = 1,2$. Also $C^n(X) \subset C^n(X)$ for all $n$, by 6.6. But we have

EXAMPLE 6.8. Let $X = \{a_n p^n! : a_n \in \{0,1\}\}$, and let $f : X \to K$ be defined via

$$f(\sum a_n p^n!) = \sum a_n p^{3n!}.$$ Then $f \in C^n(X)$ for each $n$, and $D_i f = 0$ for $i = 1,2,4,5,\ldots$ and $D_3 f = 1$. On the other hand, $f \not\in C^3(X)$.

Let $C > 0$ and $\{x_1,\ldots,x_n\}$ a set of $n$ distinct points in $X$. We call $\{x_1,\ldots,x_n\}$ a $C$-polygon if for all $i,j,k,l \in \{1,\ldots,n\}$, $k \neq l$: 
\[ \frac{|x_i - x_j|}{|x_k - x_l|} \leq C. \]

**DEFINITION 6.9.** Let \( n \in \mathbb{N} \). We say that \( X \) has **locally property** \( B_n \) if for each \( a \in X \) there is \( \delta > 0 \) and \( C > 0 \) such that for all \( x_1, x_2 \in X, x_1 \neq x_2 \), \( |x_1 - a| < \delta, |x_2 - a| < \delta \) there exist \( x_3, \ldots, x_n \in X \) such that \( \{x_1, x_2, \ldots, x_n\} \) is a \( C \)-polygon. (By definition, every \( X \) has locally property \( B_1 \) and \( B_2 \).)

We say that \( X \) has **globally property** \( B_n \) if there exists \( C > 0 \) such that for all \( x_1, x_2 \in X, x_1 \neq x_2 \), there exist \( x_3, \ldots, x_n \in X \) such that \( \{x_1, x_2, \ldots, x_n\} \) is a \( C \)-polygon. (Every \( X \) has globally property \( B_1 \) and \( B_2 \).)

For example, a ball in \( K \) has globally property \( B_n \) for each \( n \). Every open (non-empty) subset of \( K \) has locally property \( B_n \), for each \( n \).

Let us call \( BC^n(X) = \{ f \in C^n(X) : ||f||_n^\omega < \omega \} \), where, by definition,

\[ ||f||_n^\omega = \max(||f||_\omega, ||D_1 f||_\omega, \ldots, ||D_{n-1} f||_\omega, ||R_n f||_\omega) \]

(see 6.7). It is very easy to show that \( BC^n(X) \) is a Banach space with respect to \( || \cdot ||_n^\omega \). The main theorem:

**THEOREM 6.10.** If \( X \) has locally property \( B_n \), then \( C^n(X) = C^n(X) \).

Let \( X \) have globally property \( B_n \) \((n \geq 2)\) in the sense that every two-point set can be extended to a \( C \)-polygon. Then \( BC^n(X) = BC^n(X) \) and

\[ ||f||_n^\omega \leq ||f||_n \leq C^2(n-2) ||f||_n^\omega. \]
(In general we have for $f \in BC^N(X) : \|f\|_n = \max_{0 \leq i \leq n} \|D_i f\|_{n-i}^{\nu}$.)

As in 3.5. we want to find an antiderivation map: $C^{n-1}(X) \rightarrow C^n(X)$. We cannot use the map $P$ of 3.5. since one can prove: if $f \in C^1(X)$ then $Pf \in C^2(X)$ if and only if $f' = 0$. Further, if the characteristic of $K$ equals $p \neq 0$ it is easy to see that not every $C^{p-1}$-function has a $C^p$-antiderivative.

**THEOREM 6.11.** Let the characteristic of $K$ be zero and let $r_1 > r_2 \ldots$
as in 3.5 but such that $r_m < \rho r_{m+1}$ for all $m$, some $\rho > 0$.

For $f \in C^{n-1}(X)$, set

$$ P_n f(x) = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \frac{1}{i!} (x_{k+1} - x_k)^i \frac{1}{i!} f(x_k) \quad (x \in X) $$

Then $P_n f \in C^n(X)$ and $(P_n f)' = f$. If $f \in BC^{n-1}(X)$, then $Pf \in BC^n(X)$ and

$$ \|P_n f\|_n \leq c_n \|f\|_{n-1} $$

where

$$ c_n = \max_{1 \leq i \leq n} \frac{1}{i!} \rho^n. $$

It follows that the map $Q_n = n! P_n P_{n-1} \ldots P_1$ sends $C(X)$ into $C^n(X)$, $D_n Q_n$ is the identity on $C(X)$. A computation yields

$$ Q_n = \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} M^{n-i} S_i, $$

where $M$ is the multiplication with $x$ (($Mf)(x) = xf(x)$ for $f \in C(X)$) and where

$$ S_i f(x) = \sum_{k=1}^{\infty} f(x_k) \frac{x_k^i}{(x_{k+1} - x_k)^i} \quad (f \in C(X), x \in X) $$
Note: Theorem 3.4 is also true if we replace $BC^1(X)$ by $BC^n(X)$ ($n \in \mathbb{N}$).

7. $C^\infty$-functions

Spline functions, analytic functions are in $C^\infty(X)$.

Let $f \in C^\infty(X)$ and let $f(a) = 0$ for some $a \in X$. Then $f(x) = (x-a)g(x)$ ($x \in X$) where $g \in C^\infty(X)$.

A derivation on $C^\infty(X)$ is a linear map $\phi : C^\infty(X) \rightarrow C^\infty(X)$ such that

$$\phi(fg) = \phi(f)g + f\phi(g) \quad (f, g \in C^\infty(X)).$$

Any derivation $\phi$ has the form $\frac{d}{dx}g$, where $g \in C^\infty(X)$.

THEOREM 7.1. Let $Y \subset \mathbb{K}$ be a closed subset of $\mathbb{K}$. Then there is $f \in C^\infty(\mathbb{K})$ (with $f' = 0$ everywhere) such that $Y$ is the set of zeros of $f$.

THEOREM 7.2. Let $\lambda_0, \lambda_1, \ldots$ be any sequence in $\mathbb{K}$. Then there exists an $f \in C^\infty(X)$ such that $D_if(0) = \lambda_i$ for all $i$.

Open problem: Let the characteristic of $\mathbb{K}$ be zero. Does every $f \in C^\infty(X)$ have a $C^\infty$-antiderivative?