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NON-ARCHIMEDEAN DIFFERENTIATION

by

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Introduction

This note is a collection of the most important results of a lengthy paper that will appear as a Report, Mathematisch Instituut, Katholieke Universiteit, Nijmegen. There the interested reader may find all the proofs that are missing here, and also a bibliography. Because of the rather complicated formulas and computations that are involved it seems wise to restrict oneself first to K-valued functions of one single variable. However, a lot of the results can without any problem be carried over to E-valued functions of one variable, where E is a K-Banach space. A generalization to functions $K^n + K^m$ will be less obvious, although it seems clear how to define $C^k$-functions in that case. (For example, in order that $f : K^2 \to K$ is $C^1$ one should require (see 3.1) that the difference quotients

$$
\frac{f(x_1, y) - f(x_2, y)}{x_1 - x_2}, \quad \frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2}
$$

can be extended to continuous functions on $K^3$. If we take again difference quotients we get four functions of four variables, required to be continuous in order that $f$ be in $C^2$ (see 6.1). It then follows very easily that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ for $f \in C^2$.)

Throughout this note, $K$ will always be a complete non-archimedean valued field, and $X$ a non-empty subset of $K$, without isolated points. We study differentiability properties of functions $f : X \to K$. Besides the analytic functions we have other examples of differentiable
functions such as the locally constant functions (they have derivative zero everywhere). The function \( f_n : \mathbb{R} \to \mathbb{R} \), defined on \( \mathbb{R} \), is an example of an injective function with zero derivative and which is in \( \text{Lip}_a \) for every \( a > 0 \). The function \( f : \mathbb{R} \to \mathbb{R} \) defined via
\[
f(x) = x - p^n \text{ if } |x - p^n| < p^{-2n} \quad \text{and} \quad f(x) = x \text{ elsewhere}
\]
has derivative 1 everywhere, but for all \( n \in \mathbb{N} \) \( f(p^n) = f(p^n - p^n) = p^n - p^{2n} \), hence \( f \) is not locally injective at 0.

The above examples show that we do not have a mean value theorem or a local invertibility theorem. We now state the results going from "bad" to "smooth" functions.

1. Nowhere differentiable functions

Let \( BC(X) \) be the algebra of the bounded continuous functions: \( X \to \mathbb{R} \), normed by the sup norm \( || \cdot ||_\infty \). We have, analogous to the classical case:

**Theorem 1.1.** The collection of those \( f \in BC(X) \) that are somewhere differentiable is of first category in \( BC(X) \) (in the sense of Baire).

In contrast to the theory of functions on the real line we have

**Theorem 1.2.** Let \( X \) be open in \( \mathbb{R} \), and let \( f : X \to \mathbb{R} \) be a bounded uniformly continuous function, and let \( \varepsilon > 0 \). Then there exists a nowhere differentiable \( g : X \to \mathbb{R} \) such that \( g \) has bounded difference quotients, and such that \( ||f-g||_\infty < \varepsilon \).

2. Differentiability as such

Contrary to the classical case we have a nice criterion for a
THEOREM 2.1. Let $f : X \to K$. Then $f$ has an antiderivative if and only if $f$ is of Baire class one. (i.e., $f$ is the pointwise limit of a sequence of continuous functions).

Further we have Sard-type theorems. If $K$ is a local field then $Y \subseteq K$ is called a nullset if it has measure zero in the sense of the (real) Haar measure on $K$.

THEOREM 2.2. Let $K$ be a local field and let $f : X \to K$ be differentiable. Then we have:

1. If $Y \subseteq X$ is a nullset then $f(Y)$ is a nullset ("$f$ has property (N)"")

2. $\{f(x) : f'(x) = 0\}$ is a nullset.

COROLLARY 2.3. If $f : X \to K$ is differentiable, $f' = 0$ almost everywhere, then $f(X)$ is a nullset.

3. Continuously differentiable functions

If we want the local invertibility theorem to hold for $C^1$-functions we have to take a definition of a $C^1$-function, stronger than just "$f$ is differentiable and $f$ is continuous". For $f : X \to K$, define

$$\phi^1 f(x,y) = \frac{f(x)-f(y)}{x-y} \quad (x,y \in X, \ x \neq y).$$

DEFINITION 3.1. $f : X \to K$ is in $C^1(X)$ if $\phi^1 f$ can (uniquely) be extended to a continuous function $\overline{\phi^1 f}$ on $X \times X$.

(Notice that for a real valued function $f$ defined on an interval
the continuity of \( f' \) already guarantees the existence of a continuous \( \Phi_1 \).

**THEOREM 3.2.** Let \( f \in C^1(X) \) and let \( a \in X \).

(a) If \( f'(a) \neq 0 \) then \( f \) is locally invertible at \( a \). (In fact, \( (f'(a))^{-1}f \) is an isometry locally at \( a \).)

(b) If \( X \) is open in \( K \) and if \( f' \neq 0 \) everywhere on \( X \) then \( f \) is an open mapping.

Let \( BC^1(X) = \{ f \in C^1(X) : ||f||_1 := ||f||_\infty \vee ||f_1||_\infty \} \). Then \( BC^1(X) \) is a Banach space with respect to \( || \cdot \||_1 \). We may put a locally convex topology on \( C^1(X) \) via the defining seminorms \( || \cdot ||_{1,C} \) where \( C \) runs through the compact subsets of \( X \):

\[
||f||_{1,C} = \sup_{x \in C} |f(x)| \vee \sup_{x \in C} |\Phi_1 f(x,y)| \quad (f \in C^1(X)).
\]

Let \( N^1(X) = \{ f \in C^1(X) : f' = 0 \} \) and \( BN^1(X) = \{ f \in BC^1(X) : f' = 0 \} \). Then \( N^1(X) \) is closed in \( C^1(X) \), \( BN^1(X) \) is closed in \( BC^1(X) \).

**THEOREM 3.3.** The locally linear functions (in \( BC^1(X) \)) form a dense subset of \( c^1(X) \) (of \( BC^1(X) \)).

The locally constant functions (in \( BC^1(X) \)) form a dense subset of \( N^1(X) \) (of \( BN^1(X) \)).

**THEOREM 3.4.** If either \( X \) is compact or \( K \) has discrete valuation then \( BC^1(X) \) has an orthonormal base (in the sense of the norm).

If \( X \) is not compact and \( K \) has dense valuation then, for any \( \alpha > 0 \), \( BC^1(X) \) has no \( \alpha \)-orthogonal base.
Let us choose real numbers $1 > r_1 > r_2 > \ldots$ with $\lim_{n \to \infty} r_n = 0$, and, for each $n$, let $R_n$ be a full set of representatives of the equivalence relation (in $X$): $x \sim y$ if $|x-y| < r_n$. We can arrange that $R_1 \subseteq R_2 \subseteq \ldots$. For each $x \in X$, $n \in \mathbb{N}$, let $x_n \in X$ be determined by:

$$|x_n - x| < r_n, \quad x_n \in R_n.$$  

For a continuous $f : X \to \mathbb{K}$ set

$$\left( P f \right)(x) = \sum_{n=1}^{\infty} f(x_n)(x_{n+1} - x_n) \quad (x \in X)$$

**Theorem 3.5.** The map $P$ defined above is a continuous linear map:

$C(X) \to C^1(X)$ and its restriction to $BC(X)$ is an isometry: $BC(X) \to BC^1(X)$. $P$ is an antiderivation map, i.e., $(Pf)' = f$ for each $f \in C(X)$.

**Corollary 3.6.** Every continuous function has a $C^1$-antiderivative.

In fact, by passing through the quotient, differentiation yields a map $p : BC^1(X)/BN^1(X) \to BC(X)$ which is a surjective isometry. Moreover, $BN^1(X)$ has an orthogonal complement $(\text{im } P)$ in $BC^1(X)$.

### 4. $C^1(X)$ for compact $X$

(Throughout section 4, $X$ is compact). The set $\{|x-y| : x,y \in X\}$ is bounded and has only 0 as an accumulation point, hence it can be written as $\{r_1,r_2,\ldots\} \cup \{0\}$, where $r_1 > r_2 > \ldots$ and $\lim_{n \to \infty} r_n = 0$.

Let $r_0 = \infty$. For each $i$, let $R_i$ be a full set of representatives in $X$ of the equivalence relation "$x \sim y$ if $|x-y| < r_i$" such that $R_0 \subseteq R_1 \subseteq \ldots$. Then $R_i$ is finite for each $i$ and $R_0$ consists only of one single point $a_0$. Let $R = \bigcup_{i=1}^{\infty} R_i$ and define $v : R \to \{0,1,2,\ldots\}$ as follows. For $a \in R$ let $v(a)$ be the nonnegative integer $m$ for which $a \in R_m \setminus R_{m-1}$ ($R_{-1} = \emptyset$ by definition). For each $a \in R$ let
\[ B_a = \{ x \in X : |x-a| < r_y(a) \}, \]

and let \( e_a \) be the \( K \)-valued characteristic function of \( B_a \). Further, we define

\[ a \triangleq b \text{ iff } b \in B_a \quad (a,b \in R) \]

Then we have

**LEMMA 4.1.** \((R,\triangleq)\) is a partially ordered set with a smallest element \( a_0 \). For each \( a \in R \), the set \( \{ x \in R : x \triangleq a \} \) is finite and linearly ordered by \( \triangleq \).

Define for \( a \in R, a \neq a_0 \):

\[ a_- = \max \{ x \in R : x \not\triangleq a, x \triangleq a \}. \]

**THEOREM 4.2.** The set \( \{ e_a : a \in R \} \) forms an orthonormal base of \( C(X) \).

Let \( f \in C(X) \) and \( f = \sum \lambda_a e_a \) for some \( \lambda_a \in K \). Then

\[ \lambda_a = f(a_0) \text{ and for } a \neq a_0 : \lambda_a = f(a) - f(a_-). \]

The set \( \{ e_a : a \in R \} \cup \{ p_e_a : a \in R \} \) (\( P \) as in 3.5) forms an orthogonal base of \( C^1(X) \). Let \( f \in C^1(X), f = \sum \lambda_a e_a + \sum \mu_b p_b \) \((\lambda_a, \mu_b \in K)\) in the \( \| \ | \_1\)-norm. Then \( \lambda_a = f(a_0), \mu_a = f'(a_0) \) and for \( a \neq a_0 \):

\[ \lambda_a = f(a) - f(a_-) - (a-a_-)f'(a_-) \]
\[ \mu_a = f'(a) - f'(a_-). \]

5. **Uniform differentiability**

There seem to be two natural notions of "uniform differentiability". Let \( f \in C^1(X) \). \( f \) is called uniformly differentiable if \( \lim_{x \to y} \phi_y f(x,y) = f'(y) \) uniformly in \( y \). \( f \) is called strongly uniformly differentiable if \( \phi_y f \) is uniformly continuous.

If \( X \) is compact both notions are the same and coincide with "continuous differentiable".
THEOREM 5.1. Let $f : X \to K$ be (strongly) uniformly differentiable. Then $f$ has a unique continuous extension $\bar{f} : \overline{X} \to K$ ($\overline{X}$ is the closure of $X$ in $K$).

This $\bar{f}$ is (strongly) uniformly differentiable.

THEOREM 5.2. Let $f : X \to K$ be uniformly differentiable. Then each of the following properties implies strong uniform differentiability of $f$:

(a) $\phi_1 f$ is bounded.

(b) Both $f$ and $f'$ are bounded.

(c) $X$ is "nice" and $f$ is bounded.

($X$ is called "nice" if for each $r > 0$ there is $s > 0$ such that for every $x \in X$ there is $y \in X$ such that $s \leq |x-y| \leq r$).

The theorems 3.3, 3.5., 3.6. each have an analogon for uniformly differentiable functions.

6. $\mathbb{C}^n$-functions

For $n \in \mathbb{N}$, let $\mathcal{V}^n X = \{(x_1, \ldots, x_n) \in X^n : i \neq j \Rightarrow x_i \neq x_j\}$. For $f : X \to K$ we define the $n\text{th}$ difference quotient $\phi_n f : \mathcal{V}^{n+1} X \to K$ inductively as follows $\phi_0 f = f$ and for $(x_1, \ldots, x_{n+1}) \in \mathcal{V}^{n+1} X$:

$$\phi_n f(x_1, \ldots, x_{n+1}) = (x_1 - x_2)^{-1}(\phi_{n-1} f(x_1, x_3, \ldots, x_{n+1}) - \phi_{n-1} f(x_2, x_3, \ldots, x_{n+1})).$$

Since $\mathcal{V}^n X$ is dense in $X^n$ for each $n$ the following definition makes sense.

DEFINITION 6.1. Let $f : X \to K$, $n \in \mathbb{N} \cup \{0\}$. We say that $f \in \mathbb{C}^n(X)$ if $\phi_n f$ can be extended to a continuous function $\Phi_n f : X^{n+1} \to K$.

We say that $f \in R^n(X)$ if $\phi_0 f, \ldots, \phi_n f$ are bounded functions.
For $f \in B^\infty_0(X)$ set
\[
|f|_n = \max_{0 \leq i \leq n} |f_i f|_\infty.
\]
Let $BC_0^n(X) = B_0^n(X) \cap C^n(X)$, $C_0^\infty(X) = \bigcap_{n=1}^\infty C^n(X)$,
$BC_0^\infty(X) = \bigcap_{n=1}^\infty BC^n(X)$.

**THEOREM 6.2.** $C^1(X) \supset C^2(X) \supset \ldots$

$B_0^1(X) \supset BC^1_0(X) \supset B_0^2(X) \supset BC^2(X) \supset \ldots$

$B_0^n(X)$ is a Banach space with respect to $||| \cdot |||_n$ and
$BC^n(X)$ is closed in $B_0^n(X)$.

For $f \in C^n(X)$ ($n \geq 1$) and $0 \leq j \leq n$ we define the $j^{th}$ Hasse derivative
of $f$ by
\[
D^j f(x) = \frac{\partial^j f}{\partial x_1 \partial x_2 \ldots \partial x_j}(x \in X).
\]

**THEOREM 6.3.** Let $f \in C^n(X)$. Then for $0 \leq j \leq n$ we have $D^j f \in C^{n-j}(X)$
and if $i+j \leq n$
\[
D^i D^j f = \binom{i+j}{i} D^{i+j} f
\]
f is $n$ times differentiable in the ordinary sense and
for $0 \leq i \leq n$ we have
\[
f^{(i)} = i! D^i f.
\]

t : $X \to \mathbb{K}$ is called a spline function of degree $\leq n$ if for every
$a \in X$ there is a neighbourhood $U$ of $a$ such that $f|U \cap X$ is a polynomial
function of degree $\leq n$. Spline functions are in $C^\infty(X)$.

**THEOREM 6.4.** Let $f \in C^n(X)$ and $\varepsilon > 0$. Then there is a spline function
$g$ of degree $\leq n$ such that $f-g \in BC^n(X)$, $||f-g||_n < \varepsilon$. If
$D^i f = D^i+1 f = \ldots = D^n f = 0$ for some $i \in \{1, \ldots, n\}$ then
g can be chosen to be of degree $\leq i-1$. 
THEOREM 6.5. (Local invertibility). Let \( f \in C^n(X) \) and \( f'(a) \neq 0 \) for some \( a \in X \). Then there is a neighbourhood \( U \) of \( a \) (\( a \in U \subset X \)) such that \( f : U \rightarrow f(U) \) is a bijection, and such that the local inverse: \( f(U) \rightarrow U \) is in \( C^n(f(U)) \).

THEOREM 6.6. (Taylor formula). Let \( f \in C^n(X) \). Then for all \( x,y \in X \):

\[
f(x) = f(y) + (x-y)f_1(y) + \ldots + (x-y)^{n-1}f_{n-1}(y) + (x-y)^n\frac{\partial^n f(x,y)}{\partial x^n}.
\]

The above result leads to another possible notion of "\( n \)-times continuously differentiable":

DEFINITION 6.7. Let \( f : X \rightarrow K \), \( n \in \mathbb{N} \). We say that \( f \in C^n(X) \) if there exist functions \( D_1f, \ldots, D_{n-1}f : X \rightarrow K \) and a continuous \( R_n : \mathbb{R}^2 \rightarrow K \) such that for all \( x,y \in X \):

\[
f(x) = f(y) + (x-y)f_1(y) + \ldots + (x-y)^{n-1}f_{n-1}(y) + (x-y)^nR_n f(x,y).
\]

(It follows that the \( D_i f \), \( R_n f \) are uniquely determined and continuous. Further we have \( C^1(X) \supset C^2(X) \supset \ldots \). It is easy to show that \( C^i(X) = C^i(X) \) for \( i = 1,2 \). Also \( C^n(X) \subset C^n(X) \) for all \( n \), by 6.6. But we have

EXAMPLE 6.8. Let \( X = \{ \Sigma a_n p^n! : a_n \in \{0,1\} \} \), and let \( f : X \rightarrow K \) be defined via

\[
f(\Sigma a_n p^n!) = \Sigma a_n p^{3n!}.
\]

Then \( f \in C^n(X) \) for each \( n \), and \( D_i f = 0 \) for \( i = 1,2,4,5,\ldots \) and \( D_3 f = 1 \). On the other hand, \( f \notin C^3(X) \).

Let \( C > 0 \) and \( \{x_1,\ldots,x_n\} \) a set of \( n \) distinct points in \( X \). We call \( \{x_1,\ldots,x_n\} \) a \( C \)-polygon if for all \( i,j,k,l \in \{1,\ldots,n\}, k \neq l \):
\[ \frac{|x_i - x_j|}{|x_k - x_1|} \leq C. \]

**Definition 6.9.** Let \( n \in \mathbb{N} \). We say that \( X \) has locally property \( B_n \) if for each \( a \in X \) there is \( \delta > 0 \) and \( C > 0 \) such that for all \( x_1, x_2 \in X, x_1 \neq x_2, |x_1 - a| < \delta, |x_2 - a| < \delta \) there exist \( x_3, \ldots, x_n \in X \) such that \( \{x_1, x_2, \ldots, x_n\} \) is a \( C \)-polygon. (By definition, every \( X \) has locally property \( B_1 \) and \( B_2 \)).

We say that \( X \) has globally property \( B_n \) if there exists \( C > 0 \) such that for all \( x_1, x_2 \in X, x_1 \neq x_2 \), there exist \( x_3, \ldots, x_n \in X \) such that \( \{x_1, x_2, \ldots, x_n\} \) is a \( C \)-polygon. (Every \( X \) has globally property \( B_1 \) and \( B_2 \)).

For example, a ball in \( K \) has globally property \( B_n \), for each \( n \). Every open (non-empty) subset of \( K \) has locally property \( B_n \), for each \( n \).

Let us call \( BC_n(X) = \{ f \in C^n(X) : ||f||_n^\omega < \infty \} \), where, by definition,

\[ ||f||_n^\omega = \max(||f||_\omega, ||D_1 f||_\omega, \ldots, ||D_{n-1} f||_\omega, ||R_n f||_\omega) \]

(see 6.7). It is very easy to show that \( BC_n(X) \) is a Banach space with respect to \( || \cdot ||_n^\omega \). The main theorem:

**Theorem 6.10.** If \( X \) has locally property \( B_n \), then \( C^n(X) = BC_n(X) \).

Let \( X \) have globally property \( B_n \) \((n > 2)\) in the sense that every two-point set can be extended to a \( C \)-polygon. Then \( BC_n(X) = BC_n(X) \) and

\[ ||f||_n^\omega \leq ||f||_n \leq C^{2(n-2)} ||f||_n^\omega. \]
(In general we have for $f \in BC^n(X)$: $\|f\|_n = \max_{0 \leq i \leq n} \|D_i f\|_{n-i}^\nu$).

As in 3.5, we want to find an antiderivation map: $C^{n-1}(X) \to C^n(X)$. We cannot use the map $P$ of 3.5. since one can prove: if $f \in C^1(X)$ then $Pf \in C^2(X)$ if and only if $f' = 0$. Further, if the characteristic of $K$ equals $p \neq 0$ it is easy to see that not every $C^p-1$-function has a $C^p$-antiderivative.

**Theorem 6.11.** Let the characteristic of $K$ be zero and let $r_1 > r_2 \ldots$ as in 3.5 but such that $r_m < \rho r_{m+1}$ for all $m$, some $\rho > 0$.

For $f \in C^{n-1}(X)$, set

$$P_n f(x) = \sum_{k=1}^{\infty} \sum_{i=1}^{n} \frac{1}{i!} (x_{k+1} - x_k)^i D_{i-1} f(x_k)$$

$(x \in X)$

Then $P_n f \in C^n(X)$ and $(P_n f)' = f$. If $f \in BC^{n-1}(X)$, then

$$\|P_n f\|_n \leq c_n \|f\|_{n-1}$$

where

$$c_n = \max_{1 \leq i \leq n} \frac{1}{i! \cdot \rho^n}.$$

It follows that the map $Q_n = n! P_n P_{n-1} \ldots P_1$ sends $C(X)$ into $C^n(X)$, $D_n Q_n$ is the identity on $C(X)$. A computation yields

$$Q_n = \sum_{i=1}^{n} (-1)^{i+1} \frac{n_i}{i!} S_i,'$$

where $M$ is the multiplication with $x$ ($(Mf)(x) = xf(x)$ for $f \in C(X)$) and where

$$S_i f(x) = \sum_{k=1}^{\infty} f(x_k) (x_k^i - x_k^{i+1})$$

$(f \in C(X), x \in X)$
Note: Theorem 3.4 is also true if we replace $BC^1(X)$ by $BC^n(X)$ ($n \in \mathbb{N}$).

7. $C^\infty$-functions

Spline functions, analytic functions are in $C^\infty(X)$.

Let $f \in C^\infty(X)$ and let $f(a) = 0$ for some $a \in X$. Then $f(x) = (x-a)g(x)$ ($x \in X$) where $g \in C^\infty(X)$.

A derivation on $C^\infty(X)$ is a linear map $\phi : C^\infty(X) \to C^\infty(X)$ such that

$$\phi(fg) = \phi(f)g + f\phi(g) \quad (f, g \in C^\infty(X)).$$

Any derivation $\phi$ has the form $\frac{d}{dx}g$, where $g \in C^\infty(X)$.

THEOREM 7.1. Let $Y \subset K$ be a closed subset of $K$. Then there is $f \in C^\infty(K)$ (with $f' = 0$ everywhere) such that $Y$ is the set of zeros of $f$.

THEOREM 7.2. Let $\lambda_0, \lambda_1, \ldots$ be any sequence in $K$. Then there exists an $f \in C^\infty(X)$ such that $D_1 f(0) = \lambda_i$ for all $i$.

Open problem: Let the characteristic of $K$ be zero. Does every $f \in C^\infty(X)$ have a $C^\infty$-antiderivative?