NON-ARCHIMEDEAN DIFFERENTIATION

by

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Introduction

This note is a collection of the most important results of a lengthy paper that will appear as a Report, Mathematisch Instituut, Katholieke Universiteit, Nijmegen. There the interested reader may find all the proofs that are missing here, and also a bibliography. Because of the rather complicated formulas and computations that are involved it seems wise to restrict oneself first to $K$-valued functions of one single variable. However, a lot of the results can without any problem be carried over to $E$-valued functions of one variable, where $E$ is a $K$-Banach space. A generalization to functions: $K^n + K^m$ will be less obvious, although it seems clear how to define $C^k$-functions in that case. (For example, in order that $f : K^2 + K$ is $C^1$ one should require (see 3.1) that the difference quotients

$$
\left( \frac{f(x_1, y) - f(x_2, y)}{x_1 - x_2}, \frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2} \right)
$$

can be extended to continuous functions on $K^3$. If we take again difference quotients we get four functions of four variables, required to be continuous in order that $f$ be in $C^2$ (see 6.1). It then follows very easily that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ for $f \in C^2$.

Throughout this note, $K$ will always be a complete non-archimedean valued field, and $X$ a non-empty subset of $K$, without isolated points. We study differentiability properties of functions $f : X \rightarrow K$. Besides the analytic functions we have other examples of differentiable
functions such as the locally constant functions (they have derivative zero everywhere). The function $\mathbb{Z}_n \cdot p^n \to \mathbb{Z}_n \cdot p^n!$, defined on $\mathbb{Z}_p$ is an example of an injective function with zero derivative and which is in $\text{Lip}_a$ for every $a > 0$. The function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ defined via $f(x) = x-p^{-2n}$ if $|x-p^n| < p^{-2n}$ and $f(x) = x$ elsewhere has derivative 1 everywhere, but for all $n \in \mathbb{N}$ $f(p^n) = f(p^n - p^n) = p^n - p^n$, hence $f$ is not locally injective at 0.

The above examples show that we do not have a mean value theorem or a local invertibility theorem. We now state the results going from "bad" to "smooth" functions.

1. Nowhere differentiable functions

Let $BC(X)$ be the algebra of the bounded continuous functions $X \to K$, normed by the sup norm $|| \cdot ||_\infty$. We have, analogous to the classical case:

**THEOREM 1.1.** The collection of those $f \in BC(X)$ that are somewhere differentiable is of first category in $BC(X)$ (in the sense of Baire).

In contrast to the theory of functions on the real line we have

**THEOREM 1.2.** Let $X$ be open in $K$, and let $f : X \to K$ be a bounded uniformly continuous function, and let $\varepsilon > 0$. Then there exists a nowhere differentiable $g : X \to K$ such that $g$ has bounded difference quotients, and such that $||f - g||_\infty < \varepsilon$.

2. Differentiability as such

Contrary to the classical case we have a nice criterion for a
function to possess an antiderivative:

**THEOREM 2.1.** Let $f : X \rightarrow K$. Then $f$ has an antiderivative if and only if $f$ is of Baire class one. (i.e., $f$ is the pointwise limit of a sequence of continuous functions).

Further we have Sard-type theorems. If $K$ is a local field then $Y \subset K$ is called a nullset if it has measure zero in the sense of the (real) Haar measure on $K$.

**THEOREM 2.2.** Let $K$ be a local field and let $f : X \rightarrow K$ be differentiable. Then we have:

1. If $Y \subset X$ is a nullset then $f(Y)$ is a nullset ("$f$ has property $(N)$")
2. $\{f(x) : f'(x) = 0\}$ is a nullset.

**COROLLARY 2.3.** If $f : X \rightarrow K$ is differentiable, $f' = 0$ almost everywhere, then $f(X)$ is a nullset.

3. Continuously differentiable functions

If we want the local invertibility theorem to hold for $C^1$-functions we have to take a definition of a $C^1$-function, stronger then just "$f$ is differentiable and $f$ is continuous". For $f : X \rightarrow K$, define

$$\Phi_1 f(x,y) = \frac{f(x) - f(y)}{x-y} \quad (x,y \in X, x \neq y).$$

**DEFINITION 3.1.** $f : X \rightarrow K$ is in $C^1(X)$ if $\Phi_1 f$ can (uniquely) be extended to a continuous function $\Phi_1 f$ on $X \times X$.

(Notice that for a real valued function $f$ defined on an interval
the continuity of $f'$ already guarantees the existence of a continuous $\Phi_1 f$.

**THEOREM 3.2.** Let $f \in C^1(X)$ and let $a \in X$.

(a) If $f'(a) \neq 0$ then $f$ is locally invertible at $a$. (In fact, $(f'(a))^{-1} f$ is an isometry locally at $a$).

(b) If $X$ is open in $K$ and if $f' \neq 0$ everywhere on $X$ then $f$ is an open mapping.

Let $BC^1(X) = \{ f \in C^1(X) : \| f \|_1 := \| f \|_\infty \vee \| \Phi_1 f \|_\infty \}$. Then $BC^1(X)$ is a Banach space with respect to $\| \|_1$. We may put a locally convex topology on $C^1(X)$ via the defining seminorms $\| \|_{1,C}$ where $C$ runs through the compact subsets of $X$:

$$\| f \|_{1,C} = \sup_{x \in C} |f(x)| \vee \sup_{x \in C} |\Phi_1 f(x,y)| \quad (f \in C^1(X)).$$

Let $N^1(X) = \{ f \in C^1(X) : f' = 0 \}$ and $BN^1(X) = \{ f \in BC^1(X) : f' = 0 \}$. Then $N^1(X)$ is closed in $C^1(X)$, $BN^1(X)$ is closed in $BC^1(X)$.

**THEOREM 3.3.** The locally linear functions (in $BC^1(X)$) form a dense subset of $C^1(X)$ (of $BC^1(X)$).

The locally constant functions (in $BC^1(X)$) form a dense subset of $N^1(X)$ (of $BN^1(X)$).

**THEOREM 3.4.** If either $X$ is compact or $K$ has discrete valuation then $BC^1(X)$ has an orthonormal base (in the sense of the norm). If $X$ is not compact and $K$ has dense valuation then, for any $\alpha > 0$, $BC^1(X)$ has no $\alpha$-orthogonal base.
Let us choose real numbers $1 > r_1 > r_2 > \ldots$ with $\lim n \to \infty \lim n r_n = 0$, and, for each $n$, let $R_n$ be a full set of representatives of the equivalence relation (in $X$): $x \sim y$ if $|x-y| < r_n$. We can arrange that $R_1 \subset R_2 \subset \ldots$. For each $x \in X$, $n \in \mathbb{N}$, let $x_n \in X$ be determined by:

$$|x_n-x| < r_n', x_n \in R_n.$$ For a continuous $f : X \to \mathbb{K}$ set

$$(Pf)(x) = \sum_{n=1}^{\infty} f(x_n)(x_n-x_n) \quad (x \in X).$$

**THEOREM 3.5.** The map $P$ defined above is a continuous linear map:

$$C(X) \to C^1(X)$$ and its restriction to $BC(X)$ is an isometry: $BC(X) \to BC^1(X)$. $P$ is an antiderivation map i.e., $(Pf)' = f$ for each $f \in C(X)$.

**COROLLARY 3.6.** Every continuous function has a $C^1$-antiderivative.

In fact, by passing through the quotient, differentiation yields a map $p : BC^1(X)/BN^1(X) \to BC(X)$ which is a surjective isometry. Moreover, $BN^1(X)$ has an orthogonal complement (im $P$) in $BC^1(X)$.

4. $C^1(X)$ for compact $X$

(Throughout section 4, $X$ is compact). The set $\{|x-y| : x,y \in X\}$ is bounded and has only 0 as an accumulation point, hence it can be written as $\{r_1,r_2,\ldots\} \cup \{0\}$, where $r_1 > r_2 > \ldots$ and $\lim n r_n = 0$. Let $r_0 = \infty$. For each $i$, let $R_i$ be a full set of representatives in $X$ of the equivalence relation "$x \sim y$ if $|x-y| < r_i$" such that $R_0 \subset R_1 \subset \ldots$. Then $R_i$ is finite for each $i$ and $R_0$ consists only of one single point $a_0$. Let $R = \bigcup_i R_i$ and define $v : R \to \{0,1,2,\ldots\}$ as follows. For $a \in R$ let $v(a)$ be the nonnegative integer $m$ for which $\in R_m \setminus R_{m-1}$ ($R_{-1} = \emptyset$ by definition). For each $a \in R$ let
\[ B_a = \{ x \in X : |x-a| < r_v(a) \}, \]

and let \( e_a \) be the \( K \)-valued characteristic function of \( B_a \). Further, we define

\[ a \triangleleft b \iff b \in B_a \quad (a,b \in R) \]

Then we have

**Lemma 4.1.** \((R,\triangleleft)\) is a partially ordered set with a smallest element \( a_0 \). For each \( a \in R \), the set \( \{ x \in R : x \triangleleft a \} \) is finite and linearly ordered by \( \triangleleft \).

Define for \( a \in R \), \( a \neq a_0 \):

\[ a_- = \max \{ x \in R : x \neq a, x \triangleleft a \} \]

**Theorem 4.2.** The set \( \{ e_a : a \in R \} \) forms an orthonormal base of \( C(X) \).

Let \( f \in C(X) \) and \( f = \sum \lambda \ e_a \) for some \( \lambda_a \in K \). Then

\[ \lambda_{a_0} = f(a_0) \quad \text{and for } a \neq a_0 : \quad \lambda_a = f(a) - f(a_-). \]

The set \( \{ e_a : a \in R \} \cup \{ p_{e_a} : a \in R \} \) (\( p \) as in 3.5) forms an orthogonal base of \( C^1(X) \). Let \( f \in C^1(X) \), \( f = \sum \lambda_{e_a} + \sum \mu_{p_{e_a}} \) \((\lambda_a, \mu_b \in K)\) in the \( || \cdot ||_1 \)-norm. Then \( \lambda_{a_0} = f(a_0) \), \( \mu_{a_0} = f'(a_0) \)

and for \( a \neq a_0 \):

\[ \lambda_a = f(a) - f(a_-) - (a-a_-) f'(a_-) \]

\[ \mu_a = f'(a) - f'(a_-). \]

5. Uniform differentiability

There seem to be two natural notions of "uniform differentiability".

Let \( f \in C^1(X) \). \( f \) is called uniformly differentiable if \( \lim \limits_{x \to y} f(x,y) = f'(y) \)

uniformly in \( y \). \( f \) is called strongly uniformly differentiable if \( f \) is uniformly continuous.

If \( X \) is compact both notions are the same and coincide with "continuous differentiable".

...
THEOREM 5.1. Let $f : X \to K$ be (strongly) uniformly differentiable. Then $f$ has a unique continuous extension $\tilde{f} : \overline{X} \to K$ (\(\overline{X}\) is the closure of $X$ in $K$).

This $\tilde{f}$ is (strongly) uniformly differentiable.

THEOREM 5.2. Let $f : X \to K$ be uniformly differentiable. Then each of the following properties implies strong uniform differentiability of $f$:

(a) $\Phi_1^f$ is bounded.

(b) Both $f$ and $f'$ are bounded

(c) $X$ is "nice" and $f$ is bounded.

($X$ is called "nice" if for each $r > 0$ there is $s > 0$ such that for every $x \in X$ there is $y \in X$ such that $s \leq |x-y| \leq r$).

The theorems 3.3, 3.5., 3.6. each have an analogon for uniformly differentiable functions.

6. $C^n$-functions

For $n \in \mathbb{N}$, let $V^n_X = \{(x_1, \ldots, x_n) \in X^n : i \neq j \Rightarrow x_i \neq x_j\}$. For $f : X \to K$ we define the $n$th difference quotient $\phi_n^f : V^{n+1}_X \to K$ inductively as follows $\phi_0^f = f$ and for $(x_1, \ldots, x_{n+1}) \in V^{n+1}_X$:

$$\phi_n^f(x_1, \ldots, x_{n+1}) = (x_1-x_2)^{-1}(\phi_{n-1}^f(x_1, x_3, \ldots, x_{n+1}) - \phi_{n-1}^f(x_2, x_3, \ldots, x_{n+1})).$$

Since $V^n_X$ is dense in $X^n$ for each $n$ the following definition makes sense.

DEFINITION 6.1. Let $f : X \to K$, $n \in \mathbb{N} \cup \{0\}$. We say that $f \in C^n(X)$ if $\phi_n^f$ can be extended to a continuous function $\Phi_n^f : X^{n+1} \to K$.

We say that $f \in B^n(X)$ if $\phi_0^f, \ldots, \phi_n^f$ are bounded functions.
For \( f \in B^A_n(X) \) set
\[
\| f \|_n = \max_{0 \leq i \leq n} \| \Phi_i f \|_\infty.
\]

Let \( BC^\infty_n(X) = B^\infty_n(X) \cap C^\infty_n(X) \), \( C^\infty(X) = \bigcap_{n=1}^\infty C^n(X) \),
\( BC^\infty(X) = \bigcap_{n=1}^\infty BC^n(X) \).

**THEOREM 6.2.** \( C^1(X) \supset C^2(X) \supset \ldots \)
\( B^A_1(X) \supset BC^1(X) \supset B^A_2(X) \supset BC^2(X) \supset \ldots \)

\( B^A_n(X) \) is a Banach space with respect to \( \| \|_n \) and
\( BC^n(X) \) is closed in \( B^A_n(X) \).

For \( f \in C^n(X) \) \((n \geq 1)\) and \( 0 \leq j \leq n \) we define the \( j^{\text{th}} \) Hasse derivative
of \( f \) by
\[
D_j f(x) = \Phi_j f(x, x, \ldots, x) \quad (x \in X).
\]

**THEOREM 6.3.** Let \( f \in C^n(X) \). Then for \( 0 \leq j \leq n \) we have \( D_j f \in C^{n-j}(X) \)
and if \( i+j \leq n \)
\[
D_i D_j f = \binom{i+j}{i} D_{i+j} f
\]
f is \( n \) times differentiable in the ordinary sense and
for \( 0 \leq i \leq n \) we have
\[
f^{(i)} = i! \, D_i f.
\]

\( f : X \to \mathbb{K} \) is called a spline function of degree \( \leq n \) if for every
\( a \in X \) there is a neighbourhood \( U \) of \( a \) such that \( f|U \cap X \) is a polynomial
function of degree \( \leq n \). Spline functions are in \( C^\infty(X) \).

**THEOREM 6.4.** Let \( f \in C^n(X) \) and \( \epsilon > 0 \). Then there is a spline function
\( g \) of degree \( \leq n \) such that \( f-g \in BC^n(X) \), \( \| f-g \|_n < \epsilon \). If
\[
D_i f = D_{i+1} f = \ldots = D_n f = 0 \quad \text{for some} \quad i \in \{1, \ldots, n\}
\]
then \( g \) can be chosen to be of degree \( \leq i-1 \).
THEOREM 6.5. (Local invertibility). Let \( f \in C^n(X) \) and \( f'(a) \neq 0 \) for some \( a \in X \). Then there is a neighbourhood \( U \) of \( a \) (\( a \in U \subset X \)) such that \( f : U \to f(U) \) is a bijection, and such that the local inverse: \( f(U) \to U \) is in \( C^n(f(U)) \).

THEOREM 6.6. (Taylor formula). Let \( f \in C^n(X) \). Then for all \( x, y \in X \):

\[
 f(x) = f(y) + (x-y)D_1f(y) + \ldots + (x-y)^{n-1}D_{n-1}f(y) + (x-y)^nR_nf(x,y),
\]

The above result leads to another possible notion of "n-times continuously differentiable":

DEFINITION 6.7. Let \( f : X \to K, n \in \mathbb{N} \). We say that \( f \in C^n(X) \) if there exist functions \( D_1f, \ldots, D_{n-1}f : X \to K \) and a continuous \( R_nf : X^2 \to K \) such that for all \( x, y \in X \):

\[
 f(x) = f(y) + (x-y)D_1f(y) + \ldots + (x-y)^{n-1}D_{n-1}f(y) + (x-y)^nR_nf(x,y).
\]

(It follows that the \( D_i f, R_n f \) are uniquely determined and continuous.

Further we have \( C^1(X) \supset C^2(X) \supset \ldots \). It is easy to show that \( C^i(X) = C^i(X) \) for \( i = 1, 2 \). Also \( C^n(X) \subset C^n(X) \) for all \( n \), by 6.6. But we have

EXAMPLE 6.8. Let \( X = \{ \Sigma a_n p^n! : a_n \in \{0,1\} \} \), and let \( f : X \to K \) be defined via

\[
 f(\Sigma a_n p^n!) = \Sigma a_n p^{3n!}.
\]

Then \( f \in C^n(X) \) for each \( n \), and \( D_i f = 0 \) for \( i = 1, 2, 4, 5, \ldots \) and \( D_3 f = 1 \). On the other hand, \( f \not\in C^3(X) \).

Let \( C > 0 \) and \( \{x_1, \ldots, x_n\} \) a set of \( n \) distinct points in \( X \). We call \( \{x_1, \ldots, x_n\} \) a \textit{C-polygon} if for all \( i, j, k, l \in \{1, \ldots, n\}, \) \( k \neq l \).
\[
\frac{|x_i - x_j|}{|x_k - x_1|} \leq C.
\]

**Definition 6.9.** Let \( n \in \mathbb{N} \). We say that \( X \) has locally property \( B_n \) if for each \( a \in X \) there is \( \delta > 0 \) and \( C > 0 \) such that for all \( x_1, x_2 \in X, x_1 \neq x_2, |x_1 - a| < \delta, |x_2 - a| < \delta \) there exist \( x_3, \ldots, x_n \in X \) such that \( \{x_1, x_2, \ldots, x_n\} \) is a \( C \)-polygon. (By definition, every \( X \) has locally property \( B_1 \) and \( B_2 \).)

We say that \( X \) has globally property \( B_n \) if there exists \( C > 0 \) such that for all \( x_1, x_2 \in X, x_1 \neq x_2 \), there exist \( x_3, \ldots, x_n \in X \) such that \( \{x_1, x_2, \ldots, x_n\} \) is a \( C \)-polygon. (Every \( X \) has globally property \( B_1 \) and \( B_2 \).)

For example, a ball in \( K \) has globally property \( B_n \) for each \( n \). Every open (non-empty) subset of \( K \) has locally property \( B_n \), for each \( n \).

Let us call \( BC^n(X) = \{ f \in C^n(X) : \|f\|_n < \infty \} \), where, by definition,

\[
||f||_n^\omega = \max(||f||_\infty, ||D_1 f||_\infty, \ldots, ||D_{n-1} f||_\infty, ||R_n f||_\infty)
\]

(see 6.7). It is very easy to show that \( BC^n(X) \) is a Banach space with respect to \( || \cdot ||_n^\omega \). The main theorem:

**Theorem 6.10.** If \( X \) has locally property \( B_n \), then \( C^n(X) = C^n(X) \).

Let \( X \) have globally property \( B_n \) \((n \geq 2) \) in the sense that every two-point set can be extended to a \( C \)-polygon. Then \( BC^n(X) = BC^n(X) \) and

\[
|| ||_n^\omega \leq || ||_n \leq C^{2(n-2)} || ||_n^\omega.
\]
(In general we have for $f \in B_{C}^{n}(X) : ||f||_{n} = \max_{0 < i < n} ||D_{i}f||_{n-i}$.)

As in 3.5, we want to find an antiderivation map: $C^{n-1}(X) \rightarrow C^{n}(X)$. We cannot use the map $P$ of 3.5. since one can prove: if $f \in C^{1}(X)$ then $Pf \in C^{2}(X)$ if and only if $f' = 0$. Further, if the characteristic of $K$ equals $p \neq 0$ it is easy to see that not every $C^{p-1}$-function has a $C^{p}$-antiderivative.

**THEOREM 6.11.** Let the characteristic of $K$ be zero and let $r_{1} > r_{2} \ldots$

as in 3.5 but such that $r_{m} < \rho r_{m+1}$ for all $m$, some $\rho > 0$. For $f \in C^{n-1}(X)$, set

$$P_{n}f(x) = \sum_{k=1}^{n} \sum_{i=1}^{k} \frac{1}{i} (x_{k+1} - x_{k})^i D_{i-1}f(x_{k}) \quad (x \in X)$$

Then $P_{n}f \in C^{n}(X)$ and $(P_{n}f)' = f$. If $f \in B_{C}^{n-1}(X)$, then $Pf \in B_{C}^{n}(X)$ and

$$||P_{n}f||_{n} \leq c_{n} ||f||_{n-1}$$

where

$$c_{n} = \max_{1 < i < n} \frac{1}{i} \cdot i^{\rho}.$$ 

It follows that the map $Q_{n} = n! P_{n} P_{n-1} \ldots P_{1}$ sends $C(X)$ into $C^{n}(X)$, $D_{n}Q_{n}$ is the identity on $C(X)$. A computation yields

$$Q_{n} = \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} M^{n-i} S_{i},$$

where $M$ is the multiplication with $x$ ($(Mf)(x) = xf(x)$ for $f \in C(X)$) and where

$$S_{i}f(x) = \sum_{k=1}^{n} f(x_{k}) (x_{k+1} - x_{k})^{i} \quad (f \in C(X), x \in X)$$
Note: Theorem 3.4 is also true if we replace \( BC^1(X) \) by \( BC^n(X) \) (\( n \in \mathbb{N} \)).

7. \( C^\infty \)-functions

Spline functions, analytic functions are in \( C^\infty(X) \).

Let \( f \in C^\infty(X) \) and let \( f(a) = 0 \) for some \( a \in X \). Then \( f(x) = (x-a)g(x) \) \((x \in X)\) where \( g \in C^\infty(X) \).

A derivation on \( C^\infty(X) \) is a linear map \( \phi : C^\infty(X) \rightarrow C^\infty(X) \) such that

\[
\phi(fg) = \phi(f)g + f\phi(g) \quad (f, g \in C^\infty(X)).
\]

Any derivation \( \phi \) has the form \( \frac{d}{dx}g(x) \) where \( g \in C^\infty(X) \).

**THEOREM 7.1.** Let \( Y \subseteq K \) be a closed subset of \( K \). Then there is \( f \in C^\infty(K) \) (with \( f' = 0 \) everywhere) such that \( Y \) is the set of zeros of \( f \).

**THEOREM 7.2.** Let \( \lambda_0, \lambda_1, \ldots \) be any sequence in \( K \). Then there exists an \( f \in C^\infty(X) \) such that \( D_i f(0) = \lambda_i \) for all \( i \).

**Open problem:** Let the characteristic of \( K \) be zero. Does every \( f \in C^\infty(K) \) have a \( C^\infty \)-antiderivative?