NON-ARCHIMEDEAN DIFFERENTIATION

by

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Introduction

This note is a collection of the most important results of a lengthy paper that will appear as a Report, Mathematisch Instituut, Katholieke Universiteit, Nijmegen. There the interested reader may find all the proofs that are missing here, and also a bibliography. Because of the rather complicated formulas and computations that are involved it seems wise to restrict oneself first to $K$-valued functions of one single variable. However, a lot of the results can without any problem be carried over to $E$-valued functions of one variable, where $E$ is a $K$-Banach space. A generalization to functions: $K^n + K^m$ will be less obvious, although it seems clear how to define $C^k$-functions in that case. (For example, in order that $f : K^2 \to K$ is $C^1$ one should require (see 3.1) that the difference quotients

\[
\frac{f(x_1, y) - f(x_2, y)}{x_1 - x_2}, \quad (x, y) \mapsto \frac{f(x_1, y_2) - f(x, y_2)}{y_1 - y_2}
\]

can be extended to continuous functions on $K^3$. If we take again difference quotients we get four functions of four variables, required to be continuous in order that $f$ be in $C^2$ (see 6.1). It then follows very easily that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ for $f \in C^2$.)

Throughout this note, $K$ will always be a complete non-archimedean valued field, and $X$ a non-empty subset of $K$, without isolated points. We study differentiability properties of functions $f : X \to K$. Besides the analytic functions we have other examples of differentiable
functions such as the locally constant functions (they have derivative zero everywhere). The function $\mathbb{Z}_n p^n \to \mathbb{Z}_n p^{n!}$, defined on $\mathbb{Z}_p$ is an example of an injective function with zero derivative and which is in $\text{Lip}_a$ for every $a > 0$. The function $f : \mathbb{Z}_p + \mathbb{Q}_p$ defined via $f(x) = x - p^{2n}$ if $|x - p^n| < p^{-2n}$ and $f(x) = x$ elsewhere has derivative 1 everywhere, but for all $n \in \mathbb{IN}$ $f(p^n) = f(p^n - p^{2n}) = p^n - p^{2n}$, hence $f$ is not locally injective at 0.

The above examples show that we do not have a mean value theorem or a local invertibility theorem. We now state the results going from "bad" to "smooth" functions.

1. Nowhere differentiable functions

Let $BC(X)$ be the algebra of the bounded continuous functions: $X \to K$, normed by the sup norm $||\cdot||_\infty$. We have, analogous to the classical case:

**THEOREM 1.1.** The collection of those $f \in BC(X)$ that are somewhere differentiable is of first category in $BC(X)$ (in the sense of Baire).

In contrast to the theory of functions on the real line we have

**THEOREM 1.2.** Let $X$ be open in $K$, and let $f : X \to K$ be a bounded uniformly continuous function, and let $\varepsilon > 0$. Then there exists a nowhere differentiable $g : X \to K$ such that $g$ has bounded difference quotients, and such that $||f - g||_\infty < \varepsilon$.

2. Differentiability as such

Contrary to the classical case we have a nice criterion for a
THEOREM 2.1. Let $f : X \rightarrow K$. Then $f$ has an antiderivative if and only if $f$ is of Baire class one. (i.e., $f$ is the pointwise limit of a sequence of continuous functions).

Further we have Sard-type theorems. If $K$ is a local field then $Y \subset K$ is called a nullset if it has measure zero in the sense of the (real) Haar measure on $K$.

THEOREM 2.2. Let $K$ be a local field and let $f : X \rightarrow K$ be differentiable. Then we have:

1. If $Y \subset X$ is a nullset then $f(Y)$ is a nullset ("$f$ has property (N)"").
2. $\{f(x) : f'(x) = 0\}$ is a nullset.

COROLLARY 2.3. If $f : X \rightarrow K$ is differentiable, $f' = 0$ almost everywhere, then $f(X)$ is a nullset.

3. Continuously differentiable functions

If we want the local invertibility theorem to hold for $C^1$-functions we have to take a definition of a $C^1$-function, stronger than just "$f$ is differentiable and $f$ is continuous". For $f : X \rightarrow K$, define

$$\phi_1 f(x,y) = \frac{f(x) - f(y)}{x - y} \quad (x, y \in X, x \neq y).$$

DEFINITION 3.1. $f : X \rightarrow K$ is in $C^1(X)$ if $\phi_1 f$ can (uniquely) be extended to a continuous function $\overline{\phi}_1 f$ on $X \times X$.

(Notice that for a real valued function $f$ defined on an interval
the continuity of $f'$ already guarantees the existence of a continuous $\bar{f}_1$.

**THEOREM 3.2.** Let $f \in C^1(X)$ and let $a \in X$.

(a) If $f'(a) \neq 0$ then $f$ is locally invertible at $a$. (In fact, $(f'(a))^{-1}f$ is an isometry locally at $a$).

(b) If $X$ is open in $K$ and if $f' \neq 0$ everywhere on $X$ then $f$ is an open mapping.

Let $BC^1(X) = \{f \in C^1(X) : \|f\|_1 := \|f\|_\infty \vee \|\bar{f}_1\|_\infty\}$. Then $BC^1(X)$ is a Banach space with respect to $\|\cdot\|_1$. We may put a locally convex topology on $C^1(X)$ via the defining seminorms $\|\cdot\|_{1,C}$ where $C$ runs through the compact subsets of $X$:

$$\|f\|_{1,C} = \sup_{x \in C} |f(x)| \vee \sup_{x \in C} |\bar{f}_1f(x,y)| \quad (f \in C^1(X)).$$

Let $N^1(X) = \{f \in C^1(X) : f' = 0\}$ and $BN^1(X) = \{f \in BC^1(X) : f' = 0\}$. Then $N^1(X)$ is closed in $C^1(X)$, $BN^1(X)$ is closed in $BC^1(X)$.

**THEOREM 3.3.** The locally linear functions (in $BC^1(X)$) form a dense subset of $C^1(X)$ (of $BC^1(X)$).

The locally constant functions (in $BC^1(X)$) form a dense subset of $N^1(X)$ (of $BN^1(X)$).

**THEOREM 3.4.** If either $X$ is compact or $K$ has discrete valuation then $BC^1(X)$ has an orthonormal base (in the sense of the norm).

If $X$ is not compact and $K$ has dense valuation then, for any $\alpha > 0$, $BC^1(X)$ has no $\alpha$-orthogonal base.
Let us choose real numbers $1 > r_1 > r_2 > \ldots$ with $\lim r_n = 0,$ and, for each $n$, let $R_n$ be a full set of representatives of the equivalence relation (in $X$): $x \sim y$ if $|x-y| < r_n$. We can arrange that $R_1 \subset R_2 \subset \ldots$ For each $x \in X$, $n \in \mathbb{N}$, let $x_n \in R_n$ be determined by: $|x_n-x| < r_n$, $x_n \in R_n$. For a continuous $f : X \to \mathbb{K}$ set
\[
(Pf)(x) = \sum_{n=1}^{\infty} f(x_n)(x_{n+1}-x_n) \quad (x \in X)
\]

**Theorem 3.5.** The map $P$ defined above is a continuous linear map:
\[
C(X) \to C^1(X) \text{ and its restriction to } BC(X) \text{ is an isometry: } BC(X) \to BC^1(X) \text{. } P \text{ is an antiderivation map i.e., } (Pf)' = f \text{ for each } f \in C(X).
\]

**Corollary 3.6.** Every continuous function has a $C^1$-antiderivative.

In fact, by passing through the quotient, differentiation yields a map $p : BC^1(X)/BN^1(X) \to BC(X)$ which is a surjective isometry. Moreover, $BN^1(X)$ has an orthogonal complement $(\text{im } P)$ in $BC^1(X)$.

4. $C^1(X)$ for compact $X$

(Throughout section 4, $X$ is compact). The set $\{|x-y| : x,y \in X\}$ is bounded and has only 0 as an accumulation point, hence it can be written as \{r_1, r_2, \ldots\} $\cup$ \{0\}, where $r_1 > r_2 > \ldots$ and $\lim r_n = 0$. Let $x_0 = \infty$. For each $i$, let $R_i$ be a full set of representatives in $X$ of the equivalence relation "$x \sim y$ if $|x-y| < r_i$" such that $R_0 \subset R_1 \subset \ldots$. Then $R_i$ is finite for each $i$ and $R_0$ consists only of one single point $a_0$. Let $R = \bigcup_i R_i$ and define $v : R \to \{0,1,2,\ldots\}$ as follows. For a $a \in R$ let $v(a)$ be the nonnegative integer $m$ for which $a \in R_m \setminus R_{m-1}$ ($R_{-1} = \emptyset$ by definition). For each a $a \in R$ let
and let $e_a$ be the $K$-valued characteristic function of $B_a$. Further, we define

$$a \neq b \iff b \in B_a \quad (a, b \in R)$$

Then we have

**Lemma 4.1.** $(R, q)$ is a partially ordered set with a smallest element $a_0$. For each $a \in R$, the set $\{x \in R : x \neq a\}$ is finite and linearly ordered by $\leq$.

Define for $a \in R$, $a \neq a_0$: $a_- = \max \{x \in R : x \neq a, x \neq a\}$. Then

**Theorem 4.2.** The set $\{e_a : a \in R\}$ forms an orthonormal base of $C(X)$.

Let $f \in C(X)$ and $f = \sum \lambda_a e_a$ for some $\lambda_a \in K$. Then $\lambda_{a_0} = f(a_0)$ and for $a \neq a_0$: $\lambda_a = f(a) - f(a_-)$.

The set $\{e_a : a \in R\} \cup \{p_{e_a} : a \in R\}$ (as in 3.5) forms an orthogonal base of $C^1(X)$. Let $f \in C^1(X)$, $f = \sum \lambda_a e_a + \sum \mu_b p_{e_b}$ ($\lambda_a, \mu_b \in K$) in the $|| \cdot ||_1$-norm. Then $\lambda_{a_0} = f(a_0)$, $\mu_{a_0} = f'(a_0)$ and for $a \neq a_0$:

$$\lambda_a = f(a) - f(a_-) - (a-a_-)f'(a_-)$$

$$\mu_a = f'(a) - f'(a_-).$$

5. Uniform differentiability

There seem to be two natural notions of "uniform differentiability".

Let $f \in C^1(X)$. $f$ is called **uniformly differentiable** if $\lim_{x \rightarrow y} f(x, y) = f'(y)$ uniformly in $y$. $f$ is called **strongly uniformly differentiable** if $f$ is uniformly continuous.

If $X$ is compact both notions are the same and coincide with "continuous differentiable".
THEOREM 5.1. Let $f : X \to K$ be (strongly) uniformly differentiable. Then $f$ has a unique continuous extension $\overline{f} : \overline{X} \to K$ ($\overline{X}$ is the closure of $X$ in $K$).

This $\overline{f}$ is (strongly) uniformly differentiable.

THEOREM 5.2. Let $f : X \to K$ be uniformly differentiable. Then each of the following properties implies strong uniform differentiability of $f$:

(a) $\phi_1 f$ is bounded.

(b) Both $f$ and $f'$ are bounded.

(c) $X$ is "nice" and $f$ is bounded.

($X$ is called "nice" if for each $r > 0$ there is $s > 0$ such that for every $x \in X$ there is $y \in X$ such that $s \leq |x-y| \leq r$).

The theorems 3.3, 3.5., 3.6. each have an analogon for uniformly differentiable functions.

6. $C^n$-functions

For $n \in \mathbb{N}$, let $V^n = \{(x_1, \ldots, x_n) \in X^n : i \neq j \Rightarrow x_i \neq x_j\}$. For $f : X \to K$ we define the $n$th difference quotient $\phi_n f : V^{n+1} \to K$ inductively as follows $\phi_0 f = f$ and for $(x_1, \ldots, x_{n+1}) \in V^{n+1}$:

$$\phi_n f(x_1, \ldots, x_{n+1}) = (x_1 - x_2)^{-1}(\phi_{n-1} f(x_1, x_3, \ldots, x_{n+1}) - \phi_{n-1} f(x_2, x_3, \ldots, x_{n+1})).$$

Since $V^n$ is dense in $X^n$ for each $n$ the following definition makes sense.

DEFINITION 6.1. Let $f : X \to K$, $n \in \mathbb{N} \cup \{0\}$. We say that $f \in C^n(X)$ if $\phi_n f$ can be extended to a continuous function $\overline{\phi_n f} : X^{n+1} \to K$.

We say that $f \in B^n(X)$ if $\phi_0 f, \ldots, \phi_n f$ are bounded functions.
For $f \in B^A_n(X)$ set

$$||f||_n = \max_{0 \leq i \leq n} ||\phi_i f||_\infty.$$ 

Let $BC^n(X) = B^A_n(X) \cap C^n(X), C^\infty(X) = \bigcap_{n=1}^{\infty} C^n(X),$ and $BC^\infty(X) = \bigcap_{n=1}^{\infty} BC^n(X).$

**Theorem 6.2.** $C^1(X) \supset C^2(X) \supset \ldots$

$B^A_1(X) \supset BC_1^1(X) \supset B^A_2(X) \supset BC^2_1(X) \supset \ldots$

$B^A_n(X)$ is a Banach space with respect to $||||_n$ and $BC^n(X)$ is closed in $B^A_n(X).$

For $f \in C^n(X)$ ($n \geq 1$) and $0 \leq j \leq n$ we define the $j$th Hasse derivative of $f$ by

$$D_j f(x) = \bar{\phi}_j f(x,x,\ldots,x) \quad (x \in X).$$

**Theorem 6.3.** Let $f \in C^n(X)$. Then for $0 \leq j \leq n$ we have $D_j f \in C^{n-j}(X)$ and if $i+j \leq n$

$$D_i D_j f = \binom{i+j}{i} D_{i+j} f$$

$f$ is $n$ times differentiable in the ordinary sense and for $0 \leq i \leq n$ we have

$$f^{(i)} = i! D_i f.$$ 

$f : X \to \mathbb{K}$ is called a spline function of degree $\leq n$ if for every $a \in X$ there is a neighbourhood $U$ of $a$ such that $f|U \cap X$ is a polynomial function of degree $\leq n$. Spline functions are in $C^\infty(X)$.

**Theorem 6.4.** Let $f \in C^n(X)$ and $\varepsilon > 0$. Then there is a spline function $g$ of degree $\leq n$ such that $f-g \in BC^n(X)$, $||f-g||_n < \varepsilon$. If $D_i f = D_{i+1} f = \ldots = D_n f = 0$ for some $i \in \{1,\ldots,n\}$ then $g$ can be chosen to be of degree $\leq i-1$. 
THEOREM 6.5. (Local invertibility). Let \( f \in C^n(X) \) and \( f'(a) \neq 0 \) for some \( a \in X \). Then there is a neighbourhood \( U \) of \( a \) \( (a \in U \subseteq X) \) such that \( f : U \rightarrow f(U) \) is a bijection, and such that the local inverse: \( f(U) \rightarrow U \) is in \( C^n(f(U)) \).

THEOREM 6.6. (Taylor formula). Let \( f \in C^n(X) \). Then for all \( x, y \in X \):

\[
    f(x) = f(y) + (x-y)D_1f(y) + \ldots + (x-y)^{n-1}D_{n-1}f(y) + (x-y)^nR_n f(x,y),
\]

The above result leads to another possible notion of "n-times continuously differentiable":

DEFINITION 6.7. Let \( f : X \rightarrow K, n \in \mathbb{N} \). We say that \( f \in C^n(X) \) if there exist functions \( D_1f, \ldots, D_{n-1}f : X \rightarrow K \) and a continuous \( R_n f : X^2 \rightarrow K \) such that for all \( x, y \in X \):

\[
    f(x) = f(y) + (x-y)D_1f(y) + \ldots + (x-y)^{n-1}D_{n-1}f(y) + (x-y)^nR_n f(x,y).
\]

(It follows that the \( D_i f, R_n f \) are uniquely determined and continuous.

Further we have \( C^1(X) \supset C^2(X) \supset \ldots \). It is easy to show that \( C^i(X) = C^i(X) \) for \( i = 1,2 \). Also \( C^n(X) \subseteq C^n(X) \) for all \( n \), by 6.6. But we have

EXAMPLE 6.8. Let \( X = \{a_n^np^n! : a_n \in \{0,1\}\} \), and let \( f : X \rightarrow K \) defined via

\[
    f(a_n^np^n!) = \Sigma a_n^3n!.
\]

Then \( f \in C^n(X) \) for each \( n \), and \( D_if = 0 \) for \( i = 1,2,4,5,\ldots \) and \( D_3f = 1 \). On the other hand, \( f \notin C^3(X) \).

Let \( C > 0 \) and \( \{x_1, \ldots, x_n\} \) a set of \( n \) distinct points in \( X \). We call \( \{x_1, \ldots, x_n\} \) a \( C \)-polygon if for all \( i,j,k,l \in \{1, \ldots, n\}, \ k \neq l \):
DEFINITION 6.9. Let \( n \in \mathbb{N} \). We say that \( X \) has locally property \( B_n \) if for each \( a \in X \) there is \( \delta > 0 \) and \( C > 0 \) such that for all \( x_1, x_2 \in X \), \( x_1 \neq x_2 \), \( |x_1 - a| < \delta \), \( |x_2 - a| < \delta \) there exist \( x_3, \ldots, x_n \in X \) such that \( \{x_1, x_2, \ldots, x_n\} \) is a \( C \)-polygon. (By definition, every \( X \) has locally property \( B_1 \) and \( B_2 \)).

We say that \( X \) has globally property \( B_n \) if there exists \( C > 0 \) such that for all \( x_1, x_2 \in X \), \( x_1 \neq x_2 \), there exist \( x_3, \ldots, x_n \in X \) such that \( \{x_1, x_2, \ldots, x_n\} \) is a \( C \)-polygon. (Every \( X \) has globally property \( B_1 \) and \( B_2 \)).

For example, a ball in \( K \) has globally property \( B_n \) for each \( n \). Every open (non-empty) subset of \( K \) has locally property \( B_n \), for each \( n \).

Let us call \( BC^n(X) = \{ f \in C^n(X) : ||f||_n^\omega < \omega \} \), where, by definition,

\[
||f||_n^\omega = \max(||f||_\omega, ||D_1 f||_\omega, \ldots, ||D_{n-1} f||_\omega, ||R_n f||_\omega)
\]

(see 6.7). It is very easy to show that \( BC^n(X) \) is a Banach space with respect to \( || \cdot ||_n^\omega \). The main theorem:

**THEOREM 6.10.** If \( X \) has locally property \( B_n \), then \( C^n(X) = C^n(X) \).

Let \( X \) have globally property \( B_n \) \( (n > 2) \) in the sense that every two-point set can be extended to a \( C \)-polygon. Then \( BC^n(X) = BC^n(X) \) and

\[
||f||_n^\omega \leq ||f||_n \leq C^{2(n-2)} ||f||_n^\omega.
\]
(In general we have for $f \in BC^n(X)$: $\|f\|_n = \max_{0 \leq i \leq n} \|D_i f\|_{n-i}^\nu$.

As in 3.5. we want to find an antiderivation map: $C^{n-1}(X) \to C^n(X)$. We cannot use the map $P$ of 3.5. since one can prove: if $f \in C^1(X)$ then $Pf \in C^2(X)$ if and only if $f' = 0$. Further, if the characteristic of $K$ equals $p \neq 0$ it is easy to see that not every $C^{p-1}$-function has a $C^p$-antiderivative.

**THEOREM 6.11.** Let the characteristic of $K$ be zero and let $r_1 > r_2 \ldots$

as in 3.5 but such that $r_m < \rho r_{m+1}$ for all $m$, some $\rho > 0$. For $f \in C^{n-1}(X)$, set

$$P_n f(x) = \sum_{k=1}^{\infty} \sum_{i=1}^{n} \frac{1}{i!} (x_{k+1} - x_k)^i D_{i-1} f(x_k) \quad (x \in X)$$

Then $P_n f \in C^n(X)$ and $(P_n f)' = f$. If $f \in BC^{n-1}(X)$, then $Pf \in BC^n(X)$ and

$$\|P_n f\|_n \leq c_n \|f\|_{n-1}$$

where

$$c_n = \max_{1 \leq i \leq n} \frac{1}{i! \cdot i^\rho}. $$

It follows that the map $Q_n = n!P_n P_{n-1} \ldots P_1$ sends $C(X)$ into $C^n(X)$, $D_n Q_n$ is the identity on $C(X)$. A computation yields

$$Q_n = \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} M^{n-i} S_i,$$

where $M$ is the multiplication with $x \ (Mf)(x) = xf(x)$ for $f \in C(X)$) and where

$$S_i f(x) = \sum_{k=1}^{\infty} f(x_k) (x_{k+1} - x_k)^i \quad (f \in C(X), \ x \in X)$$
Note: Theorem 3.4 is also true if we replace BC^1(X) by BC^n(X) (n ∈ IN).

7. C^∞-functions

Spline functions, analytic functions are in C^∞(X).

Let f ∈ C^∞(X) and let f(a) = 0 for some a ∈ X. Then f(x) = (x-a)g(x) (x ∈ X) where g ∈ C^∞(X).

A derivation on C^∞(X) is a linear map φ : C^∞(X) → C^∞(X) such that

φ(fg) = φ(f)g + fφ(g) (f,g ∈ C^∞(X)).

Any derivation φ has the form \( \frac{d}{dx}g \), where g ∈ C^∞(X).

THEOREM 7.1. Let Y ⊂ K be a closed subset of K. Then there is f ∈ C^∞(K) (with f' = 0 everywhere) such that Y is the set of zeros of f.

THEOREM 7.2. Let λ_0, λ_1, ... be any sequence in K. Then there exists an f ∈ C^∞(X) such that D_i f(0) = λ_i for all i.

Open problem: Let the characteristic of K be zero. Does every f ∈ C^∞(X) have a C^∞-antiderivative?