NON-ARCHIMEDEAN DIFFERENTIATION

by

W.H. Schikhof

Introduction

This note is a collection of the most important results of a lengthy paper that will appear as a Report, Mathematisch Instituut, Katholieke Universiteit, Nijmegen. There the interested reader may find all the proofs that are missing here, and also a bibliography. Because of the rather complicated formulas and computations that are involved it seems wise to restrict oneself first to \( K \)-valued functions of one single variable. However, a lot of the results can without any problem be carried over to \( E \)-valued functions of one variable, where \( E \) is a \( K \)-Banach space. A generalization to functions: \( K^n + K^m \) will be less obvious, although it seems clear how to define \( C^k \)-functions in that case. (For example, in order that \( f : K^2 \to K \) is \( C^1 \) one should require (see 3.1) that the difference quotients

\[
\frac{f(x_1, y_1) - f(x_2, y_1)}{x_1 - x_2}, \quad \frac{f(x_1, y_2) - f(x_1, y_2)}{y_1 - y_2}
\]

can be extended to continuous functions on \( K^3 \). If we take again difference quotients we get four functions of four variables, required to be continuous in order that \( f \) be in \( C^2 \) (see 6.1). It then follows very easily that

\[
\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}
\]

for \( f \in C^2 \).

Throughout this note, \( K \) will always be a complete non-archimedean valued field, and \( X \) a non-empty subset of \( K \), without isolated points. We study differentiability properties of functions \( f : X \to K \). Besides the analytic functions we have other examples of differentiable
functions such as the locally constant functions (they have derivative zero everywhere). The function $\sum_a p^n \to \sum_a p^n$, defined on $\mathbb{R}$, is an example of an injective function with zero derivative and which is in $\text{Lip}_a$ for every $a > 0$. The function $f : \mathbb{R} \to \mathbb{R}$ defined via $f(x) = x - p^{2n}$ if $|x - p^n| < p^{-2n}$ and $f(x) = x$ elsewhere has derivative 1 everywhere, but for all $n \in \mathbb{N}$ $f(p^n) = f(p^n - p^{2n}) = p^n - p^{2n}$, hence $f$ is not locally injective at 0.

The above examples show that we do not have a mean value theorem or a local invertibility theorem. We now state the results going from "bad" to "smooth" functions.

1. **Nowhere differentiable functions**

Let $BC(X)$ be the algebra of the bounded continuous functions $X \to \mathbb{R}$, normed by the sup norm $|| \cdot ||_\infty$. We have, analogous to the classical case:

**THEOREM 1.1.** The collection of those $f \in BC(X)$ that are somewhere differentiable is of first category in $BC(X)$ (in the sense of Baire).

In contrast to the theory of functions on the real line we have

**THEOREM 1.2.** Let $X$ be open in $\mathbb{R}$, and let $f : X \to \mathbb{R}$ be a bounded uniformly continuous function, and let $\varepsilon > 0$. Then there exists a nowhere differentiable $g : X \to \mathbb{R}$ such that $g$ has bounded difference quotients, and such that $||f-g||_\infty < \varepsilon$.

2. **Differentiability as such**

Contrary to the classical case we have a nice criterion for a
function to possess an antiderivative:

**THEOREM 2.1.** Let \( f : X \to K \). Then \( f \) has an antiderivative if and only if \( f \) is of Baire class one. (i.e., \( f \) is the pointwise limit of a sequence of continuous functions).

Further we have Sard-type theorems. If \( K \) is a local field then \( Y \subseteq K \) is called a nullset if it has measure zero in the sense of the (real) Haar measure on \( K \).

**THEOREM 2.2.** Let \( K \) be a local field and let \( f : X \to K \) be differentiable. Then we have:

1. If \( Y \subseteq X \) is a nullset then \( f(Y) \) is a nullset ("\( f \) has property (N)"")
2. \( \{ f(x) : f'(x) = 0 \} \) is a nullset.

**COROLLARY 2.3.** If \( f : X \to K \) is differentiable, \( f' = 0 \) almost everywhere, then \( f(X) \) is a nullset.

### 3. Continuously differentiable functions

If we want the local invertibility theorem to hold for \( \mathcal{C}^1 \)-functions we have to take a definition of a \( \mathcal{C}^1 \)-function, stronger than just "\( f \) is differentiable and \( f \) is continuous". For \( f : X \to K \), define

\[
\phi_1 f(x,y) = \frac{f(x)-f(y)}{x-y} \quad (x,y \in X, x \neq y).
\]

**DEFINITION 3.1.** \( f : X \to K \) is in \( \mathcal{C}^1(X) \) if \( \phi_1 f \) can (uniquely) be extended to a continuous function \( \bar{\phi}_1 f \) on \( X \times X \).

(Notice that for a real valued function \( f \) defined on an interval
the continuity of $f'$ already guarantees the existence of a continuous $\bar{f}'$.

**Theorem 3.2.** Let $f \in C^1(X)$ and let $a \in X$.

(a) If $f'(a) \neq 0$ then $f$ is locally invertible at $a$. (In fact, $(f'(a))^{-1}f$ is an isometry locally at $a$).

(b) If $X$ is open in $K$ and if $f' \neq 0$ everywhere on $X$ then $f$ is an open mapping.

Let $BC^1(X) = \{f \in C^1(X) : \|f\|_1 := \|f\|_\infty \vee \|f_1\|_\infty\}$. Then $BC^1(X)$ is a Banach space with respect to $\|\|$. We may put a locally convex topology on $C^1(X)$ via the defining seminorms $\|\|_{1,C}$ where $C$ runs through the compact subsets of $X$:

$$\|f\|_{1,C} = \sup_{x \in C} |f(x)| \vee \sup_{x \in C} \sup_{y \in C} |\bar{f}(x,y)|$$

(f $\in C^1(X)$).

Let $N^1(X) = \{f \in C^1(X) : f' = 0\}$ and $BN^1(X) = \{f \in BC^1(X) : f' = 0\}$. Then $N^1(X)$ is closed in $C^1(X)$, $BN^1(X)$ is closed in $BC^1(X)$.

**Theorem 3.3.** The locally linear functions (in $BC^1(X)$) form a dense subset of $C^1(X)$ (of $BC^1(X)$).

The locally constant functions (in $BC^1(X)$) form a dense subset of $N^1(X)$ (of $BN^1(X)$).

**Theorem 3.4.** If either $X$ is compact or $K$ has discrete valuation then $BC^1(X)$ has an orthonormal base (in the sense of the norm).

If $X$ is not compact and $K$ has dense valuation then, for any $\alpha > 0$, $BC^1(X)$ has no $\alpha$-orthogonal base.
Let us choose real numbers \(1 > r_1 > r_2 > \ldots\) with \(\lim r_n = 0\), and, for each \(n\), let \(R_n\) be a full set of representatives of the equivalence relation (in \(X\)): \(x \sim y\) if \(|x-y| < r_n\). We can arrange that \(R_1 \subset R_2 \subset \ldots\). For each \(x \in X\), \(n \in \mathbb{N}\), let \(x_n \in X\) be determined by:
\[
|x_n - x| < r_n, \quad x_n \in R_n.
\]
For a continuous \(f : X \to \mathbb{K}\) set
\[
(Pf)(x) = \sum_{n=1}^{\infty} f(x_n) (x_{n+1} - x_n) \quad (x \in X).
\]

**Theorem 3.5.** The map \(P\) defined above is a continuous linear map:
\[
C(X) \to C^1(X) \text{ and its restriction to } BC(X) \text{ is an isometry: } BC(X) \to BC^1(X).
\]
\(P\) is an antiderivation map i.e., \((Pf)' = f\) for each \(f \in C(X)\).

**Corollary 3.6.** Every continuous function has a \(C^1\)-antiderivative.

In fact, by passing through the quotient, differentiation yields a map \(p : BC^1(X)/BN^1(X) \to BC(X)\) which is a surjective isometry. Moreover, \(BN^1(X)\) has an orthogonal complement \((\text{im } P)\) in \(BC^1(X)\).

### 4. \(C^1(X)\) for compact \(X\)

(Throughout section 4, \(X\) is compact). The set \(|x-y| : x, y \in X\) is bounded and has only 0 as an accumulation point, hence it can be written as \(\{r_1, r_2, \ldots\} \cup \{0\}\), where \(r_1 > r_2 > \ldots\) and \(\lim r_n = 0\).

Let \(r_0 = \infty\). For each \(i\), let \(R_i\) be a full set of representatives in \(X\) of the equivalence relation "\(x \sim y\) if \(|x-y| < r_i\)" such that \(R_0 \subset R_1 \subset \ldots\). Then \(R_i\) is finite for each \(i\) and \(R_0\) consists only of one single point \(a_0\). Let \(R = \bigcup_1^\infty R_i\) and define \(v : R \to \{0, 1, 2, \ldots\}\) as follows. For a \(a \in R\) let \(v(a)\) be the nonnegative integer \(m\) for which \(a \in R_m \setminus R_{m-1}\) (\(R_{-1} = \emptyset\) by definition). For each a \(\in R\) let
$B_a = \{x \in X : |x-a| < r_b(a)\}$,

and let $e_a$ be the $K$-valued characteristic function of $B_a$. Further, we define

$$a \preceq b \iff b \in B_a \quad (a,b \in R)$$

Then we have

**Lemma 4.1.** $(R, \preceq)$ is a partially ordered set with a smallest element $a_0$. For each $a \in R$, the set $\{x \in R : x \preceq a\}$ is finite and linearly ordered by $\preceq$.

Define for $a \in R$, $a_\preceq = \max \{x \in R : x \preceq a, x \not\equiv a\}$. Then

**Theorem 4.2.** The set $\{e_a : a \in R\}$ forms an orthonormal base of $C(X)$.

Let $f \in C(X)$ and $f = \sum \lambda_a e_a$ for some $\lambda_a \in K$. Then

$$\lambda_{a_0} = f(a_0) \quad \text{and for } a \neq a_0: \quad \lambda_a = f(a) - f(a\preceq).$$

The set $\{e_a : a \in R\} \cup \{P_x e_a : a \in R\}$ (as in 3.5) forms an orthogonal base of $C^1(X)$. Let $f \in C^1(X)$, $f = \sum \lambda_a e_a + \sum \mu_b P_b$ ($\lambda_a, \mu_b \in K$) in the $\|\cdot\|_1$-norm. Then $\lambda_{a_0} = f(a_0)$, $\mu_{a_0} = f'(a_0)$ and for $a \neq a_0$:

$$\lambda_a = f(a) - f(a\preceq) - (a-a\preceq)f'(a\preceq)$$

$$\mu_a = f'(a) - f'(a\preceq).$$

5. Uniform Differentiability

There seem to be two natural notions of "uniform differentiability".

Let $f \in C^1(X)$. $f$ is called uniformly differentiable if $\lim_{x \to y} f(x,y) = f'(y)$ uniformly in $y$. $f$ is called strongly uniformly differentiable if $f$ is uniformly continuous.

If $X$ is compact both notions are the same and coincide with "continuously differentiable".
THEOREM 5.1. Let $f : X \to K$ be (strongly) uniformly differentiable. Then $f$ has a unique continuous extension $\bar{f} : \overline{X} \to K$ ($\overline{X}$ is the closure of $X$ in $K$).

This $\bar{f}$ is (strongly) uniformly differentiable.

THEOREM 5.2. Let $f : X \to K$ be uniformly differentiable. Then each of the following properties implies strong uniform differentiability of $f$:

(a) $\phi_1 f$ is bounded.

(b) Both $f$ and $f'$ are bounded.

(c) $X$ is "nice" and $f$ is bounded.

($X$ is called "nice" if for each $r > 0$ there is $s > 0$ such that for every $x \in X$ there is $y \in X$ such that $s \leq |x-y| \leq r$).

The theorems 3.3, 3.5, 3.6. each have an analogon for uniformly differentiable functions.

6. $C^n$-functions

For $n \in \mathbb{N}$, let $\mathcal{V}^n X = \{(x_1, \ldots, x_n) \in X^n : i \neq j \implies x_i \neq x_j\}$. For $f : X \to K$ we define the $n$th difference quotient $\phi_n f : \mathcal{V}^{n+1} X \to K$ inductively as follows $\phi_0 f = f$ and for $(x_1, \ldots, x_{n+1}) \in \mathcal{V}^{n+1} X$:

$$\phi_n f(x_1, \ldots, x_{n+1}) = (x_1 - x_2)^{-1}(\phi_{n-1} f(x_1, x_3, \ldots, x_{n+1}) - \phi_{n-1} f(x_2, x_3, \ldots, x_{n+1})).$$

Since $\mathcal{V}^n X$ is dense in $X^n$ for each $n$ the following definition makes sense.

DEFINITION 6.1. Let $f : X \to K$, $n \in \mathbb{N} \cup \{0\}$. We say that $f \in C^n(X)$ if $\phi_n f$ can be extended to a continuous function $\overline{\phi}_n f : X^{n+1} \to K$.

We say that $f \in B^n(X)$ if $\phi_0 f, \ldots, \phi_n f$ are bounded functions.
For \( f \in B^n(X) \) set
\[
\|f\|_n = \max_{0 \leq i < n} \|\phi_i f\|_\infty.
\]

Let \( BC^n(X) = B^n(X) \cap C^n(X) \), \( C^n(X) = \bigcap_{n=1}^{\infty} C^n(X) \),
\( BC^n(X) = \bigcap_{n=1}^{\infty} BC^n(X) \).

**THEOREM 6.2.** \( C^1(X) \supset C^2(X) \supset \ldots \)

\( B^1(X) \supset BC^1(X) \supset B^2(X) \supset BC^2(X) \supset \ldots \)

\( B^n(X) \) is a Banach space with respect to \( \|\|_n \) and
\( BC^n(X) \) is closed in \( B^n(X) \).

For \( f \in C^n(X) \) \((n \geq 1) \) and \( 0 \leq j \leq n \) we define the \( j^{\text{th}} \) Hasse derivative of \( f \) by
\[
D_j f(x) = \phi_j f(x, x, \ldots, x) \quad (x \in X).
\]

**THEOREM 6.3.** Let \( f \in C^n(X) \). Then for \( 0 \leq j \leq n \) we have \( D_j f \in C^{n-j}(X) \)
and if \( i+j \leq n \)
\[
D_i D_j f = \binom{i+j}{i} D_{i+j} f
\]
\( f \) is \( n \) times differentiable in the ordinary sense and
for \( 0 \leq i \leq n \) we have
\[
f^{(i)} = i! D_i f.
\]

\( f : X \rightarrow K \) is called a spline function of degree \( \leq n \) if for every \( a \in X \) there is a neighbourhood \( U \) of \( a \) such that \( f|U \cap X \) is a polynomial function of degree \( \leq n \). Spline functions are in \( C^\infty(X) \).

**THEOREM 6.4.** Let \( f \in C^n(X) \) and \( \varepsilon > 0 \). Then there is a spline function \( g \) of degree \( \leq n \) such that \( f-g \in BC^n(X) \), \( \|f-g\|_n < \varepsilon \). If \( D_i f = D_{i+1} f = \ldots = D_n f = 0 \) for some \( i \in \{1, \ldots, n\} \) then \( g \) can be chosen to be of degree \( \leq i-1 \).
THEOREM 6.5. (Local invertibility). Let $f \in C^n(X)$ and $f'(a) \neq 0$ for some $a \in X$. Then there is a neighbourhood $U$ of $a$ ($a \in U \subset X$) such that $f : U \to f(U)$ is a bijection, and such that the local inverse: $f(U) \to U$ is in $C^n(f(U))$.

THEOREM 6.6. (Taylor formula). Let $f \in C^n(X)$. Then for all $x, y \in X$:

$$f(x) = f(y) + (x-y)D_1f(y) + \ldots + (x-y)^{n-1}D_{n-1}f(y) + (x-y)^n R_n f(x, y, \ldots, y).$$

The above result leads to another possible notion of "n-times continuously differentiable":

DEFINITION 6.7. Let $f : X \to \mathbb{K}$, $n \in \mathbb{N}$. We say that $f \in C^n(X)$ if there exist functions $D_1f, \ldots, D_{n-1}f : X \to \mathbb{K}$ and a continuous $R_n f : X^2 \to \mathbb{K}$ such that for all $x, y \in X$:

$$f(x) = f(y) + (x-y)D_1f(y) + \ldots + (x-y)^{n-1}D_{n-1}f(y) + (x-y)^n R_n f(x, y).$$

(It follows that the $D_i f$, $R_n f$ are uniquely determined and continuous. Further we have $C^1(X) \supset C^2(X) \supset \ldots$). It is easy to show that $C^i(X) = C^i(X)$ for $i = 1, 2$. Also $C^n(X) \subset C^n(X)$ for all $n$, by 6.6. But we have

EXAMPLE 6.8. Let $X = \{ \Sigma a_n^p^n : a_n \in \{0,1\} \}$, and let $f : X \to \mathbb{K}$ be defined via

$$f(\Sigma a_n^p^n) = \Sigma a_n^p 3^n!.$$ 

Then $f \in C^n(X)$ for each $n$, and $D_i f = 0$ for $i = 1, 2, 4, 5, \ldots$ and $D_3 f = 1$. On the other hand, $f \notin C^3(X)$.

Let $C > 0$ and $\{x_1, \ldots, x_n\}$ a set of $n$ distinct points in $X$. We call $\{x_1, \ldots, x_n\}$ a $C$-polygon if for all $i, j, k, l \in \{1, \ldots, n\}, k \neq l$:  

DEFINITION 6.9. Let $n \in \mathbb{N}$. We say that $X$ has locally property $B_n$ if for each $a \in X$ there is $\delta > 0$ and $C > 0$ such that for all $x_1, x_2 \in X$, $x_1 \neq x_2$, $|x_1 - a| < \delta$, $|x_2 - a| < \delta$ there exist $x_3, \ldots, x_n \in X$ such that \{x_1, x_2, \ldots, x_n\} is a C-polygon. (By definition, every $X$ has locally property $B_1$ and $B_2$).

We say that $X$ has globally property $B_n$ if there exists $C > 0$ such that for all $x_1, x_2 \in X$, $x_1 \neq x_2$, there exist $x_3, \ldots, x_n \in X$ such that \{x_1, x_2, \ldots, x_n\} is a C-polygon.

(Every $X$ has globally property $B_1$ and $B_2$).

For example, a ball in $K$ has globally property $B_n$, for each $n$. Every open (non-empty) subset of $K$ has locally property $B_n$, for each $n$.

Let us call $BC^n(X) = \{f \in C^n(X) : ||f||_n^\omega < \infty\}$, where, by definition,

$$||f||_n^\omega = \max(||f||_\omega, ||D_1 f||_\omega, \ldots, ||D_{n-1} f||_\omega, ||R_n f||_\omega)$$

(see 6.7). It is very easy to show that $BC^n(X)$ is a Banach space with respect to $|| \cdot ||_n^\omega$. The main theorem:

THEOREM 6.10. If $X$ has locally property $B_n$, then $C^n(X) = C^n(X)$.

Let $X$ have globally property $B_n$ ($n > 2$) in the sense that every two-point set can be extended to a C-polygon. Then $BC^n(X) = BC^n(X)$ and

$$|| ||^\omega_n \leq || ||_n \leq C^{2(n-2)} || ||_n^\omega.$$
(In general we have for \( f \in BC^n(X) \): \( \|f\|_n = \max_{0 \leq i \leq n} \|D_i f\|_{n-i} \).

As in 3.5. we want to find an antiderivation map: \( C^{n-1}(X) \to C^n(X) \). We cannot use the map \( P \) of 3.5. since one can prove: if \( f \in C^1(X) \) then \( Pf \in C^2(X) \) if and only if \( f' = 0 \). Further, if the characteristic of \( K \) equals \( p \neq 0 \) it is easy to see that not every \( C^p \)-function has a \( C^p \)-antiderivative.

**THEOREM 6.11.** Let the characteristic of \( K \) be zero and let \( r_1 > r_2 \ldots \) as in 3.5 but such that \( r_m < \rho r_{m+1} \) for all \( m \), some \( \rho > 0 \). For \( f \in C^{n-1}(X) \), set

\[
P_n f(x) = \sum_{k=1}^{n} \sum_{i=1}^{k} \frac{1}{i^{k+1}} (x_{k+1} - x_k)^i D_i f(x_k) \quad (x \in X)
\]

Then \( P_n f \in C^n(X) \) and \( (P_n f)' = f \). If \( f \in BC^{n-1}(X) \), then \( Pf \in BC^n(X) \) and

\[
\|P_n f\|_n \leq c_n \|f\|_{n-1}
\]

where

\[
c_n = \max_{1 \leq i \leq n} \frac{1}{i \cdot i^p}.
\]

It follows that the map \( Q_n = n! P_n P_{n-1} \ldots P_1 \) sends \( C(X) \) into \( C^n(X) \), \( D_n Q_n \) is the identity on \( C(X) \). A computation yields

\[
Q_n = \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} n^{-i} S_i^1,
\]

where \( M \) is the multiplication with \( x \) ((\( Mf \))(x) = xf(x) for \( f \in C(X) \)) and where

\[
S_i^1 f(x) = \sum_{k=1}^{\infty} f(x_k) (x_{k+1}^i - x_k^i) \quad (f \in C(X), \ x \in X)
\]
Note: Theorem 3.4 is also true if we replace $BC^1(X)$ by $BC^n(X)$ ($n \in \mathbb{N}$).

7. $C^\infty$-functions

Spline functions, analytic functions are in $C^\infty(X)$.

Let $f \in C^\infty(X)$ and let $f(a) = 0$ for some $a \in X$. Then $f(x) = (x-a)g(x)$ ($x \in X$) where $g \in C^\infty(X)$.

A derivation on $C^\infty(X)$ is a linear map $\phi: C^\infty(X) \to C^\infty(X)$ such that

$$\phi(fg) = \phi(f)g + f\phi(g) \quad (f, g \in C^\infty(X)).$$

Any derivation $\phi$ has the form $\frac{d}{dx}g$, where $g \in C^\infty(X)$.

THEOREM 7.1. Let $Y \subset K$ be a closed subset of $K$. Then there is $f \in C^\infty(K)$ (with $f' = 0$ everywhere) such that $Y$ is the set of zeros of $f$.

THEOREM 7.2. Let $\lambda_0, \lambda_1, \ldots$ be any sequence in $K$. Then there exists an $f \in C^\infty(X)$ such that $D_1 f(0) = \lambda_1$ for all $i$.

Open problem: Let the characteristic of $K$ be zero. Does every $f \in C^\infty(X)$ have a $C^\infty$-antiderivative?