Introduction

This note is a collection of the most important results of a lengthy paper that will appear as a Report, Mathematisch Instituut, Katholieke Universiteit, Nijmegen. There the interested reader may find all the proofs that are missing here, and also a bibliography. Because of the rather complicated formulas and computations that are involved it seems wise to restrict oneself first to $K$-valued functions of one single variable. However, a lot of the results can without any problem be carried over to $E$-valued functions of one variable, where $E$ is a $K$-Banach space. A generalization to functions: $K^n + K^m$ will be less obvious, although it seems clear how to define $C^k$-functions in that case. (For example, in order that $f : K^2 + K$ is $C^1$ one should require (see 3.1) that the difference quotients

$$
\frac{f(x_1,y) - f(x_2,y)}{x_1 - x_2}, \quad (x_2, y) \to \frac{f(x_1,y) - f(x_2,y)}{x_1 - x_2}
$$

$$
\frac{f(x_1,y) - f(x,y)}{y_1 - y_2}, \quad (x_1, y_1) \to \frac{f(x_1,y) - f(x,y)}{y_1 - y_2}
$$

can be extended to continuous functions on $K^3$. If we take again difference quotients we get four functions of four variables, required to be continuous in order that $f$ be in $C^2$ (see 6.1). It then follows very easily that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ for $f \in C^2$.)

Throughout this note, $K$ will always be a complete non-archimedean valued field, and $X$ a non-empty subset of $K$, without isolated points. We study differentiability properties of functions $f : X \to K$. Besides the analytic functions we have other examples of differentiable
functions such as the locally constant functions (they have derivative zero everywhere). The function \( I_a \): \( \mathbb{R} \to \mathbb{R} \), defined on \( \mathbb{R} \), is an example of an injective function with zero derivative and which is in \( \text{Lip}_a \) for every \( a > 0 \). The function \( f : \mathbb{R} \to \mathbb{R} \) defined via 
\[
f(x) = x - p^n \quad \text{if} \quad |x - p^n| < p^{-2n} \quad \text{and} \quad f(x) = x \quad \text{elsewhere}
\] 
has derivative 1 everywhere, but for all \( n \in \mathbb{N} \) \( f(p^n) = f(p^n - p^{n+1}) = p^{-1} \), hence \( f \) is not locally injective at 0.

The above examples show that we do not have a mean value theorem or a local invertibility theorem. We now state the results going from "bad" to "smooth" functions.

1. **Nowhere differentiable functions**

Let \( BC(X) \) be the algebra of the bounded continuous functions: \( X \to K \), normed by the sup norm \( \| \cdot \|_\infty \). We have, analogous to the classical case:

**THEOREM 1.1.** The collection of those \( f \in BC(X) \) that are somewhere differentiable is of first category in \( BC(X) \) (in the sense of Baire).

In contrast to the theory of functions on the real line we have

**THEOREM 1.2.** Let \( X \) be open in \( K \), and let \( f : X \to K \) be a bounded uniformly continuous function, and let \( \varepsilon > 0 \). Then there exists a nowhere differentiable \( g : X \to K \) such that \( g \) has bounded difference quotients, and such that \( \| f-g \|_\infty < \varepsilon \).

2. **Differentiability as such**

Contrary to the classical case we have a nice criterion for a
function to possess an antiderivative:

**THEOREM 2.1.** Let \( f : X \rightarrow K \). Then \( f \) has an antiderivative if and only if \( f \) is of Baire class one, (i.e., \( f \) is the pointwise limit of a sequence of continuous functions).

Further we have Sard-type theorems. If \( K \) is a local field then \( Y \subset K \) is called a nullset if it has measure zero in the sense of the (real) Haar measure on \( K \).

**THEOREM 2.2.** Let \( K \) be a local field and let \( f : X \rightarrow K \) be differentiable. Then we have:

1. If \( Y \subset X \) is a nullset then \( f(Y) \) is a nullset ("\( f \) has property (N)"")
2. \( \{ f(x) : f'(x) = 0 \} \) is a nullset.

**COROLLARY 2.3.** If \( f : X \rightarrow K \) is differentiable, \( f' = 0 \) almost everywhere, then \( f(X) \) is a nullset.

3. Continuously differentiable functions

If we want the local invertibility theorem to hold for \( C^1 \)-functions we have to take a definition of a \( C^1 \)-function, stronger then just "\( f \) is differentiable and \( f \) is continuous". For \( f : X \rightarrow K \), define

\[
\phi_1 f(x,y) = \frac{f(x) - f(y)}{x - y} \quad (x,y \in X, x \neq y).
\]

**DEFINITION 3.1.** \( f : X \rightarrow K \) is in \( C^1(X) \) if \( \phi_1 f \) can (uniquely) be extended to a continuous function \( \phi_1 f \) on \( X \times X \).

(Notice that for a real valued function \( f \) defined on an interval
the continuity of $f'$ already guarantees the existence of a continuous

\[ \Phi_1(f). \]

**THEOREM 3.2.** Let $f \in C^1(X)$ and let $a \in X$.

(a) If $f'(a) \neq 0$ then $f$ is locally invertible at $a$. (In fact, $(f'(a))^{-1}f$ is an isometry locally at $a$).

(b) If $X$ is open in $K$ and if $f' \neq 0$ everywhere on $X$ then $f$ is an open mapping.

Let $BC^1(X) = \{ f \in C^1(X) : \| f \|_1 := \| f \|_\infty \wedge \| \Phi_1 f \|_\infty \}$. Then $BC^1(X)$ is a Banach space with respect to $\| \|_1$. We may put a locally convex topology on $C^1(X)$ via the defining seminorms $\| \|_{1,C}$ where $C$ runs through the compact subsets of $X$:

\[ \| f \|_{1,C} = \sup_{x \in C} |f(x)| \vee \sup_{x \in C} |\Phi_1 f(x,y)| \quad (f \in C^1(X)). \]

Let $N^1(X) = \{ f \in C^1(X) : f' = 0 \}$ and $BN^1(X) = \{ f \in BC^1(X) : f' = 0 \}$. Then $N^1(X)$ is closed in $C^1(X)$, $BN^1(X)$ is closed in $BC^1(X)$.

**THEOREM 3.3.** The locally linear functions (in $BC^1(X)$) form a dense subset of $C^1(X)$ (of $BC^1(X)$).

The locally constant functions (in $BC^1(X)$) form a dense subset of $N^1(X)$ (of $BN^1(X)$).

**THEOREM 3.4.** If either $X$ is compact or $K$ has discrete valuation then $BC^1(X)$ has an orthonormal base (in the sense of the norm).

If $X$ is not compact and $K$ has dense valuation then, for any $\alpha > 0$, $BC^1(X)$ has no $\alpha$-orthogonal base.
Let us choose real numbers \( 1 > r_1 > r_2 > \ldots \) with \( \lim_{n \to \infty} r_n = 0 \), and, for each \( n \), let \( R_n \) be a full set of representatives of the equivalence relation (in \( X \)): \( x \sim y \) if \( |x - y| < r_n \). We can arrange that \( R_1 \subset R_2 \subset \ldots \). For each \( x \in X \), \( n \in \mathbb{N} \), let \( x \in X \) be determined by:

\[
|x_n - x| < r_n, \quad x \in R_n.
\]

For a continuous \( f : X \to \mathbb{K} \) set

\[
(Pf)(x) = \sum_{n=1}^{\infty} f(x_n)(x_{n+1} - x_n) \quad (x \in X)
\]

**Theorem 3.5.** The map \( P \) defined above is a continuous linear map:

\[
C(X) \to C^1(X) \quad \text{and its restriction to } BC(X) \text{ is an isometry: } BC(X) \to BC^1(X). \quad P \text{ is an antiderivation map i.e., } (Pf)' = f \text{ for each } f \in C(X).
\]

**Corollary 3.6.** Every continuous function has a \( C^1 \)-antiderivative.

In fact, by passing through the quotient, differentiation yields a map \( p : BC^1(X)/BN^1(X) \to BC(X) \) which is a surjective isometry. Moreover, \( BN^1(X) \) has an orthogonal complement \( \text{im } P \) in \( BC^1(X) \).

### 4. \( C^1(X) \) for compact \( X \)

(Throughout section 4, \( X \) is compact). The set \( \{ |x - y| : x, y \in X \} \) is bounded and has only 0 as an accumulation point, hence it can be written as \( \{ r_1, r_2, \ldots \} \cup \{ 0 \} \), where \( r_1 > r_2 > \ldots \) and \( \lim_{n \to \infty} r_n = 0 \).

Let \( r_0 = \infty \). For each \( i \), let \( R_i \) be a full set of representatives in \( X \) of the equivalence relation "\( x \sim y \) if \( |x - y| < r_i \)" such that \( R_0 \subset R_1 \subset \ldots \). Then \( R_i \) is finite for each \( i \) and \( R_0 \) consists only of one single point \( a_0 \). Let \( R = \bigcup_i R_i \) and define \( v : R \to \{ 0, 1, 2, \ldots \} \) as follows. For a \( a \in R \) let \( v(a) \) be the nonnegative integer \( m \) for which

\[ a \in R_m \setminus R_{m-1} \quad (R_{-1} = \emptyset \text{ by definition}). \]
and let \( e_a \) be the \( K \)-valued characteristic function of \( B_a \). Further, we define

\[
a \not\preceq b \text{ iff } b \in B_{a} \quad (a,b \in \mathbb{R})
\]

Then we have

**Lemma 4.1.** \((\mathbb{R}, \preceq)\) is a partially ordered set with a smallest element \( a_0 \). For each \( a \in \mathbb{R} \), the set \( \{x \in \mathbb{R} : x \not\preceq a\} \) is finite and linearly ordered by \( \preceq \).

Define for \( a \in \mathbb{R} \), \( a \neq a_0 \):

\[ a_- = \max \{x \in \mathbb{R} : x \not\preceq a, x \not\preceq a_0\} \]

Then

**Theorem 4.2.** The set \( \{e_a : a \in \mathbb{R}\} \) forms an orthonormal base of \( C(X) \).

Let \( f \in C(X) \) and \( f = \sum \lambda_a e_a \) for some \( \lambda_a \in K \). Then

\[ \lambda_{a_0} = f(a_0) \text{ and for } a \neq a_0 : \lambda_a = f(a) - f(a_-). \]

The set \( \{e_a : a \in \mathbb{R}\} \cup \{P_{a} : a \in \mathbb{R}\} \) (\( P \) as in 3.5) forms an orthogonal base of \( C^1(X) \). Let \( f \in C^1(X) \), \( f = \sum \lambda_a e_a + \sum \mu_b P_b \) \((\lambda_a, \mu_b \in K)\) in the \( \| \cdot \|_1 \)-norm. Then \( \lambda_{a_0} = f(a_0) \), \( \mu_{a_0} = f'(a_0) \)

and for \( a \neq a_0 \):

\[ \lambda_a = f(a) - f(a_-) - (a-a_)f'(a_-) \]
\[ \mu_a = f'(a) - f'(a_-). \]

5. Uniform differentiability

There seem to be two natural notions of "uniform differentiability".

Let \( f \in C^1(X) \). \( f \) is called **uniformly differentiable** if \( \lim_{x\to y} f(x,y) = f'(y) \)

uniformly in \( y \). \( f \) is called **strongly uniformly differentiable** if \( f \) is uniformly continuous.

If \( X \) is compact both notions are the same and coincide with "continuous differentiable".
THEOREM 5.1. Let \( f : X \to K \) be (strongly) uniformly differentiable. Then

\( f \) has a unique continuous extension \( \overline{f} : \overline{X} \to K \) (\( \overline{X} \) is the closure of \( X \) in \( K \)).

This \( \overline{f} \) is (strongly) uniformly differentiable.

THEOREM 5.2. Let \( f : X \to K \) be uniformly differentiable. Then each of

the following properties implies strong uniform differentiability of \( f \):

(a) \( \phi_1 f \) is bounded.
(b) Both \( f \) and \( f' \) are bounded.
(c) \( X \) is "nice" and \( f \) is bounded.

(\( X \) is called "nice" if for each \( r > 0 \) there is \( s > 0 \) such that for every \( x \in X \) there is \( y \in X \) such that \( s \leq |x-y| \leq r \)).

The theorems 3.3, 3.5., 3.6. each have an analogon for uniformly differentiable functions.

6. \( C^n \)-functions

For \( n \in \mathbb{N} \), let \( \mathcal{V}^n X = \{(x_1, \ldots, x_n) \in X^n : i \neq j \Rightarrow x_i \neq x_j\} \). For \( f : X \to K \) we define the \( n \)th difference quotient \( \phi^n f : \mathcal{V}^{n+1} X \to K \)

inductively as follows \( \phi_0 f = f \) and for \( (x_1, \ldots, x_{n+1}) \in \mathcal{V}^{n+1} X \):

\[
\phi_n f(x_1, \ldots, x_{n+1}) = (x_1 - x_2)^{-1} (\phi_{n-1} f(x_1, x_3, \ldots, x_{n+1}) - \phi_{n-1} f(x_2, x_3, \ldots, x_{n+1})).
\]

Since \( \mathcal{V}^n X \) is dense in \( X^n \) for each \( n \) the following definition makes sense.

DEFINITION 6.1. Let \( f : X \to K, n \in \mathbb{N} \cup \{0\} \). We say that \( f \in C^n(X) \) if

\( \phi^n f \) can be extended to a continuous function \( \overline{\phi^n f} : X^{n+1} \to K \).

We say that \( f \in B^n(X) \) if \( \phi f, \ldots, \phi^n f \) are bounded functions.
For $f \in B^0_n(X)$ set

$$||f||_n = \max_{0 \leq i \leq n} ||\phi_i f||_\infty.$$ 

Let $B^0_n(X) = B^0_n(X) \cap C^n(X)$, $C^\infty(X) = \bigcap_{n=1}^\infty C^n(X)$,

$BC^\infty(X) = \bigcap_{n=1}^\infty BC^n(X)$. 

**THEOREM 6.2.** $C^1(X) \supset C^2(X) \supset \ldots$

$B^1_0(X) \supset BC^1(X) \supset B^2_0(X) \supset BC^2(X) \supset \ldots$

$B^0_n(X)$ is a Banach space with respect to $|| ||_n$ and $BC^n(X)$ is closed in $B^0_n(X)$. 

For $f \in C^n(X)$ ($n \geq 1$) and $0 \leq j \leq n$ we define the $j^{th}$ Hasse derivative of $f$ by

$$D_j f(x) = \phi_j f(x, x, \ldots, x) \quad (x \in X).$$

**THEOREM 6.3.** Let $f \in C^n(X)$. Then for $0 \leq j \leq n$ we have $D_j f \in C^{n-i}(X)$ and if $i+j \leq n$

$$D_i D_j f = \binom{i+j}{i} D_{i+j} f$$

$f$ is $n$ times differentiable in the ordinary sense and for $0 \leq i \leq n$ we have

$$f^{(i)} = i! D_i f.$$ 

$f : X \to K$ is called a spline function of degree $\leq n$ if for every $a \in X$ there is a neighbourhood $U$ of $a$ such that $f|U \cap X$ is a polynomial function of degree $\leq n$. Spline functions are in $C^\infty(X)$. 

**THEOREM 6.4.** Let $f \in C^n(X)$ and $\varepsilon > 0$. Then there is a spline function $g$ of degree $\leq n$ such that $f-g \in BC^n(X)$, $||f-g||_n < \varepsilon$. If $D_i f = D_{i+1} f = \ldots = D_n f = 0$ for some $i \in \{1, \ldots, n\}$ then $g$ can be chosen to be of degree $\leq i-1$. 
THEOREM 6.5. (Local invertibility). Let $f \in C^n(X)$ and $f'(a) \neq 0$ for some $a \in X$. Then there is a neighbourhood $U$ of $a$ $(a \in U \subset X)$ such that $f : U \to f(U)$ is a bijection, and such that the local inverse: $f(U) \to U$ is in $C^n(f(U))$.

THEOREM 6.6. (Taylor formula). Let $f \in C^n(X)$. Then for all $x, y \in X$:

$$f(x) = f(y) + (x-y)D_1f(y) + \ldots + (x-y)^{n-1}D_{n-1}f(y) + (x-y)^n R_nf(x, y, \ldots, y).$$

The above result leads to another possible notion of "n-times continuously differentiable":

DEFINITION 6.7. Let $f : X \to \mathbb{R}$, $n \in \mathbb{N}$. We say that $f \in C^n(X)$ if there exist functions $D_1f, \ldots, D_{n-1}f : X \to \mathbb{R}$ and a continuous $R_nf : X^2 \to \mathbb{R}$ such that for all $x, y \in X$

$$f(x) = f(y) + (x-y)D_1f(y) + \ldots + (x-y)^{n-1}D_{n-1}f(y) + (x-y)^n R_nf(x, y).$$

(It follows that the $D_i f, R_nf$ are uniquely determined and continuous.

Further we have $C^1(X) \supset C^2(X) \supset \ldots$). It is easy to show that $C^i(X) = C^i(X)$ for $i = 1, 2$. Also $C^n(X) \subset C^n(X)$ for all $n$, by 6.6. But we have

EXAMPLE 6.8. Let $X = \{\Sigma a_n p^n! : a_n \in \{0,1\}\}$, and let $f : X \to \mathbb{R}$ be defined via

$$f(\Sigma a_n p^n!) = \Sigma a_n p^{3n!}.$$ 

Then $f \in C^n(X)$ for each $n$, and $D_i f = 0$ for $i = 1, 2, 4, 5, \ldots$ and $D_3 f = 1$. On the other hand, $f \notin C^3(X)$.

Let $C > 0$ and $\{x_1, \ldots, x_n\}$ a set of $n$ distinct points in $X$. We call $\{x_1, \ldots, x_n\}$ a $C$-polygon if for all $i, j, k, l \in \{1, \ldots, n\}$, $k \neq l$: 
DEFINITION 6.9. Let \( n \in \mathbb{N} \). We say that \( X \) has **locally property** \( B_n \) if for each \( a \in X \) there is \( \delta > 0 \) and \( C > 0 \) such that for all \( x_1, x_2 \in X \), \( x_1 \neq x_2 \), \( |x_1 - a| < \delta \), \( |x_2 - a| < \delta \) there exist \( x_3, \ldots, x_n \in X \) such that \( \{x_1, x_2, \ldots, x_n\} \) is a \( C \)-polygon. (By definition, every \( X \) has **locally property** \( B_1 \) and \( B_2 \)).

We say that \( X \) has **globally property** \( B_n \) if there exists \( C > 0 \) such that for all \( x_1, x_2 \in X \), \( x_1 \neq x_2 \), there exist \( x_3, \ldots, x_n \in X \) such that \( \{x_1, x_2, \ldots, x_n\} \) is a \( C \)-polygon.

(Every \( X \) has **globally property** \( B_1 \) and \( B_2 \)).

For example, a ball in \( K \) has globally property \( B_n \), for each \( n \). Every open (non-empty) subset of \( K \) has locally property \( B_n \), for each \( n \).

Let us call \( BC^n(X) = \{ f \in \mathcal{C}^n(X) : \| f \|^n < \infty \} \), where, by definition,

\[
\| f \|^n = \max(\| f \|, \| D_1 f \|, \ldots, \| D_n f \|)
\]

(see 6.7). It is very easy to show that \( BC^n(X) \) is a Banach space with respect to \( \| f \|^n \). The main theorem:

**THEOREM 6.10.** If \( X \) has locally property \( B_n \), then \( C^n(X) = \mathcal{C}^n(X) \).

Let \( X \) have globally property \( B_n \) (\( n \geq 2 \)) in the sense that every two-point set can be extended to a \( C \)-polygon. Then \( BC^n(X) = BC^n(X) \) and

\[
\| f \|^n \leq \| f \|^n \\
| f | \leq C^2(n-2) \| f \|
\]
(In general we have for $f \in B^{n}(X)$ : $||f||_{n} = \max_{0 \leq i \leq n} ||D_{i}f||_{n-i}$.)

As in 3.5. we want to find an antiderivation map: $C^{n-1}(X) \to C^{n}(X)$. We cannot use the map $P$ of 3.5. since one can prove: if $f \in C^{1}(X)$ then $Pf \in C^{2}(X)$ if and only if $f' = 0$. Further, if the characteristic of $K$ equals $p \neq 0$ it is easy to see that not every $C^{p-1}$-function has a $C^{p}$-antiderivative.

**THEOREM 6.11.** Let the characteristic of $K$ be zero and let $r_{1} > r_{2} \ldots$ as in 3.5 but such that $r_{m} < \rho r_{m+1}$ for all $m$, some $\rho > 0$.

For $f \in C^{n-1}(X)$, set

$$P_{n}f(x) = \sum_{k=1}^{\infty} \sum_{i=1}^{n} \frac{1}{i} (x_{k+1} - x_{k})^{i} D_{i-1}f(x_{k})$$

$x \in X$.

Then $P_{n}f \in C^{n}(X)$ and $(P_{n}f)' = f$. If $f \in B^{n-1}(X)$, then $Pf \in B^{n}(X)$ and

$$||P_{n}f||_{n} \leq c_{n}||f||_{n-1}$$

where

$$c_{n} = \max_{1 \leq i \leq n} \frac{1}{i^{i} \cdot n^{i}}.$$

It follows that the map $Q_{n} = n! P_{n} P_{n-1} \ldots P_{1}$ sends $C(X)$ into $C^{n}(X)$, $D_{n}Q_{n}$ is the identity on $C(X)$. A computation yields

$$Q_{n} = \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} M^{n-i} S_{i},$$

where $M$ is the multiplication with $x$ ($(Mf)(x) = xf(x)$ for $f \in C(X)$) and where

$$S_{i}f(x) = \sum_{k=1}^{\infty} f(x_{k}) (x_{k+1}^{i} - x_{k}^{i})$$

$(f \in C(X), x \in X)$. 

Note: Theorem 3.4 is also true if we replace $BC^1(X)$ by $BC^n(X)$ ($n \in \mathbb{N}$).

7. $C^\infty$-functions

Spline functions, analytic functions are in $C^\infty(X)$.

Let $f \in C^\infty(X)$ and let $f(a) = 0$ for some $a \in X$. Then $f(x) = (x-a)g(x)$ ($x \in X$) where $g \in C^\infty(X)$.

A derivation on $C^\infty(X)$ is a linear map $\phi : C^\infty(X) \to C^\infty(X)$ such that

$$\phi(fg) = \phi(f)g + f\phi(g) \quad (f, g \in C^\infty(X)).$$

Any derivation $\phi$ has the form $g \frac{d}{dx}$, where $g \in C^\infty(X)$.

THEOREM 7.1. Let $Y \subset K$ be a closed subset of $K$. Then there is $f \in C^\infty(K)$ (with $f' = 0$ everywhere) such that $Y$ is the set of zeros of $f$.

THEOREM 7.2. Let $\lambda_0, \lambda_1, \ldots$ be any sequence in $K$. Then there exists an $f \in C^\infty(K)$ such that $D^i f(0) = \lambda_i$ for all $i$.

Open problem: Let the characteristic of $K$ be zero. Does every $f \in C^\infty(X)$ have a $C^\infty$-antiderivative?