THE SPECIAL AUTOMORPHISM GROUP OF $R[t]/(t^m)[x_1, \cdots, x_n]$ AND COORDINATES OF A SUBRING OF $R[t][x_1, \ldots, x_n]$

ARNO VAN DEN ESSEN, STEFAN MAUBACH AND STÉPHANE VÉNÉREAU

Abstract. Let R be a ring. The Special Automorphism Group $\mathrm{SAut}_R R[x_1, \cdots, x_n]$ is the set of all automorphisms with determinant of the Jacobian equal to 1. It is shown that the canonical map of $\mathrm{SAut}_{R[t]} R[t][x_1, \cdots, x_n]$ to $\mathrm{SAut}_{R_m} R_m[x_1, \cdots, x_n]$ where $R_m := R[t]/(t^m)$ and $\mathbb{Q} \subset R$ is surjective. This result is used to study a particular case of the following question: if A is a subring of a ring B and $f \in A^{[n]}$ is a coordinate over B does it imply that f is a coordinate over A? It is shown that if $A = R[t^m, t^{m+1}, \ldots] \subset R[t] = B$ then the answer to this question is "yes".

Also, a question on the Vénéreau polynomial is settled, which indicates another "coordinate-like property" of this polynomial.

1. Introduction

Some notations: Let R be a commutative ring with one, as all the other rings in this paper. We will denote individual variables by small letters, and lists of variables by capitals: $X := x_1, \ldots, x_n$. The polynomials in these variables with coefficients in R form a ring, and even an algebra over R, that we will denote $R[X]$, $R[x_1, \ldots, x_n]$ or $R^{[n]}$ as well.

A coordinate in $R^{[n]}$ (also called "variable"[1]) is a polynomial $f \in R^{[n]}$ for which one can find $f_2, \ldots, f_n \in R^{[n]}$ such that $R[f, f_2, \ldots, f_n] = R^{[n]}$. One of the central problems in affine algebraic geometry is the question under what conditions a polynomial is a coordinate. One of the conjectures in regard to this problem is the

Abhyankar-Sathaye Conjecture (AS(n)): Let $f \in \mathbb{C}^{[n+1]}$ and assume that $\mathbb{C}^{[n+1]}/(f) \cong \mathbb{C}^{[n]}$. Then f is a coordinate.

Notice that the converse of the conjecture is true. When f satisfies the condition $R^{[n+1]}/(f) \cong R^{[n]}$, then f is called a *hyperplane* (resp. *plane* or *line*, according to dimensions).

A lot of work has been done on attempts to solve this conjecture, and also on the problem of classifying coordinates (see for example [1, 4, 6, 8, 11] and many others). One of these works, [6], studied hyperplanes in $\mathbb{C}[x, y, z, u]$ with a prescribed form. Some of them could be proved to be coordinates but some others could not; two of the simplest examples of such polynomials with an indefinite status are the (now

[1]This can be somewhat confusing. Sometimes, the word "variable" is exclusively used for the coordinate system that one is working with, i.e. if one writes $R[x, y]$ then x and y are variables, whereas $x + y^2$ is a coordinate (and not a variable). But sometimes, especially from a geometric viewpoint, it is natural to choose no coordinate system and view objects globally, and in these cases "variable" is used synonymous with "coordinate".

called) Vénéreau polynomials $y + x^n[xz + y(yu + z^2)]$ where $n = 1$ or $n = 2$. At the same time, independent of this work, Berson (in [1]) studied coordinates in 3 and 4 variables, but from the other side: he tried to classify which polynomials were coordinates. The Vénéreau polynomials appeared in his work as polynomials that he could not show to be a coordinate. So the Vénéreau polynomials are right in the middle: maybe they are coordinates, by a (complicated?) transformation that no one has found yet, or they are not, giving a counterexample to AS(3).

In the present paper, we describe a result which indicates that the Vénéreau polynomial $y + x[xz + y(yu + z^2)]$ resembles a coordinate in another respect (see below). As so very often in mathematics, perhaps even more important than this result are the methods which we use to achieve this (the "main theorem" of this paper comes from the methods). We study the following object:

Definition 1.1. *The Special Automorphism Group of $R[X]$, where R is a ring, is the set*

$$\mathrm{SAut}_R R[X] := \{\varphi \in \mathrm{Aut}_R R[X] \mid \det(\mathrm{Jac}(\varphi)) = 1\}.$$

In section 3 we show our

Main Theorem. *Let R be a ring containing the field of rational numbers $\mathbb{Q}$, m a positive integer and $R_m := R[t]/(t^m)$. Then the map*

$$\mathrm{SAut}_{R[t]} R[X] \longrightarrow \mathrm{SAut}_{R_m} R_m[X]$$

induced by the canonical morphism $R[t] \to R[t]/(t^m) = R_m$ is surjective.

This result is then used in section 4 to study coordinates in $R[t^{\geq m}][X]$, where $R[t^{\geq m}] := R[t^m, t^{m+1}, \ldots] = R[t^m, t^{m+1}, \ldots, t^{2m-1}]$. It is shown that if $f \in R[t^{\geq m}][X]$ is a coordinate in $R[t][X]$, then f is a coordinate in $R[t^{\geq m}][X]$.

It is worth mentioning that there exists an equivalent formulation of the Jacobian Conjecture in terms of the automorphism group of $R_m[X]$ (see [3]). Study of the automorphism group of $R_m[X]$ as in this paper and the paper [7] can help in giving a good foundation for an attack on the Jacobian Conjecture.

2. Consequences of the main theorem

Let $A \subset B$ be rings. Considering a polynomial $f \in A[X] \subset B[X]$ the expression *over A* resp. B will mean "considered as a polynomial in $A[X]$ resp. $B[X]$".

Definition 2.1. *Let $A \subset B$ be rings. We will say that "B does not introduce new coordinates over A" if for any $n \in \mathbb{N}^*$:*
$f \in A[x_1, \ldots, x_n]$ is a coordinate over B $\Longrightarrow$ f is a coordinate over A.

It is shown in [10] that $B = \mathbb{Z}[t]$ introduces new coordinates over $A = \mathbb{Z}[t^2, t^3, 3t]$, so this does occur. Of course, an interesting general question is: under what conditions does B not introduce new coordinates over A? With

$$A = R[t^{\geq m}] = R[t^m, t^{m+1}, \ldots] = R[t^m, t^{m+1}, \ldots, t^{2m-1}] \subset B = R[t]$$

we get in section 4 the following

Corollary 2.2. *Let R be a ring containing $\mathbb{Q}$ and m a positive integer. Then $R[t]$ does not introduce new coordinates over $R[t^{\geq m}]$.*

This is a corollary of the following theorem, which in turn requires the main result of this paper (see section 3). This theorem will also be proved in section 4.

Theorem 2.3. *Let R be a ring containing $\mathbb{Q}$ and m a positive integer. Let $f = f(x,Y)$ be a polynomial in $R[t^{\geq m}][x,Y] = R[t^{\geq m}][x,y_1,\cdots,y_n] = R[t^{\geq m}]^{[n+1]}$ such that*

- $f \equiv x \mod t^m \cdot R[t][x,Y]$;
- *f is a hyperplane over $R[t]$.*

Then f is a hyperplane over $R[t^{\geq m}]$.

The above theorem has consequences for the Vénéreau polynomial, $y + x[xz + y(yu+z^2)]$: at the end of [9] (see also [5]) the following question is asked:

Question 2.4. *Is $A[y,z,u]/(y + x[xz + y(yu+z^2)] - c)$ isomorphic to $A^{[2]}$ where $A = \mathbb{C}[x,c]/(x^2 - c^3)$?*

An answer of "no" would settle this Vénéreau polynomial to not be a coordinate, proving that it is a counterexample to the Abhyankar-Sathaye conjecture. Let us shortly explain why this is the case, and see where this question came from:

Note that, when $g \in \mathbb{C}^{[n]}$, "$f \in \mathbb{C}^{[n]}$ is a g-coordinate" means that there exist $f_3,\ldots,f_n \in \mathbb{C}^{[n]}$ such that $\mathbb{C}[f,g,f_3,\ldots,f_n] = \mathbb{C}^{[n]}$. As shown in [9, 5, 6, 11] (and mentioned in the introduction), the "Vénéreau" polynomial $y+x[xz+y(yu+z^2)] \in \mathbb{C}[x][y,z,u]$ fulfills a bunch of necessary conditions to be a coordinate and even an x-coordinate of $\mathbb{C}[x][y,z,u]$ (e.g. it is an x-plane, meaning that $\mathbb{C}[x][y,z,u]/(f) \cong_{\mathbb{C}[x]} \mathbb{C}[x]^{[2]}$) however it is not yet known if it is a coordinate. In view of lemma 4.3 the problem is to decide if the quotient $\mathbb{C}[x,c][y,z,u]/(y + x[xz + y(yu + z^2)] - c)$ is x,c-isomorphic to $\mathbb{C}[x,c]^{[2]}$ i.e. if $y+x[xz+y(yu+z^2)]-c$ is an x,c-plane. The idea is to replace $\mathbb{C}[x,c]$ by quotients of the form $A = \mathbb{C}[x,c]/(p)$ where $p \in \mathbb{C}[x,c]$ and to check if $y+x[xz+y(yu+z^2)]-c$ is then an A-plane. If this is not the case, then the Vénéreau polynomial is no coordinate. Now if $x_0 \in \mathbb{C}$ and $p = x - x_0$, then the polynomial $y+x_0[x_0z+y(yu+z^2)]-c$ is an A-plane (since $y+x_0[x_0z+y(yu+z^2)]$ is a coordinate of $\mathbb{C}[y,z,u]$). Similarly, in the case that $p = c - c_0$ for any $c_0 \in \mathbb{C}$, the polynomial $y + x[xz + y(yu + z^2)] - c_0$ is an x-plane in the variables y,z,u. In [9] one could find the answer for some other p's. The simplest case that was not settled was the "cusp" $p = x^2 - c^3$.

A consequence of theorem 2.3 is that the answer to the question is "yes":

Corollary 2.5. *$A[y,z,u]/(y+x[xz+y(yu+z^2)]-c) \cong A^{[2]}$ where $A = \mathbb{C}[x,c]/(x^2 - c^3)$.*

Proof. Let us change notation to match notations of theorem 2.3: replace x,c,y,z,u by t^2,t^3,x,y_1,y_2, thus replace $\mathbb{C}[x,c]/(x^2-c^3)$ by $\mathbb{C}[t^2,t^3] = \mathbb{C}[t^{\geq 2}]$ and replace $y+x[xz+y(yu+z^2)]-c$ by $x+t^3[t^3y_1+x(xy_2+y_1^2)]-t^2$. As required in theorem 2.3 (with $R = \mathbb{C},$, $Y = (y_1,y_2)$, $m = 2$ and $f = x + t^3[t^3y_1 + x(xy_2+y_1^2)] - t^2$) one has $f \equiv x \mod t^2 \cdot \mathbb{C}[t][x,Y]$. The condition "$f$ is an hyperplane over $\mathbb{C}[t]$" is not obvious to check but can be retrieved from e.g. [6]. Then we get the desired conclusion that f is a (hyper)plane of $\mathbb{C}[t^2,t^3][x,y_1,y_2]$. □

3. Surjectivity of the Special Automorphism Group

The following notations and assumptions are fixed throughout the rest of the article:

- R is a ring containing $\mathbb{Q}$ i.e. R is a $\mathbb{Q}$-algebra;
- m is a positive integer and $R_m := R[t]/(t^m)$;
- if $f \in R[t][X]$ then $f \mod t^m$ denotes the polynomial in $R_m[X]$ obtained from f by reduction modulo t^m;
- if no confusion (about the value of m) is possible, we will denote this polynomial by $\bar{f}$;
- similarly, if $F = (F_1, \ldots, F_n) \in R[t][X]^n$ we define $\bar{F} := (\bar{F}_1, \ldots, \bar{F}_n)$.

In this section we prove the

Main Theorem. *The map*

$$\begin{array}{ccc} \mathrm{SAut}_{R[t]}R[t][X] & \longrightarrow & \mathrm{SAut}_{R_m}R_m[X] \\ F & \longmapsto & \bar{F} \end{array}$$

is surjective.

Note that without the "S" in SAut the corresponding map is not surjective anymore. The reason is that the map $R[t][X]^\times \to R_m[X]^\times$, where $^\times$ denotes the set of invertible elements, is not surjective. For example there is no $p \in R[t][X]^\times$ such that $\bar{p} = 1 + \bar{t} \in R_m[X]^\times$ and consequently there is no $F \in \mathrm{Aut}_{R[t]}R[t][X]$ such that $\bar{F} = ((1+\bar{t})x_1, x_2, \cdots, x_n) \in \mathrm{Aut}_{R_m}R_m[X]$.
Before we can prove the Main Theorem we need some preparations.

Lemma 3.1. *Let $H = (H_1, \ldots, H_n) \in R[X]^n$ and $G = (G_1, \ldots, G_n) \in R[X]^n$. Put $h := X + \epsilon^{m-1}H$ and $g := X + \epsilon^{m-1}G$, where $\epsilon = \bar{t} \in R_m$.Then*

1. $h \circ g = X + \epsilon^{m-1}(H + G)$
2. $h \in \mathrm{Aut}_{R_m}R_m[X]$ *(with inverse $X - \epsilon^{m-1}H$)*
3. $h \in \mathrm{SAut}_{R_m}R_m[X]$ *iff* $\frac{\partial}{\partial x_1}H_1 + \ldots + \frac{\partial}{\partial x_n}H_n = 0$

Proof. (1) and (2) are immediate. To see (3) just observe that $\det(\mathrm{Jac}(h)) = 1 + \epsilon^{m-1}(\frac{\partial}{\partial x_1}H_1 + \ldots + \frac{\partial}{\partial x_n}H_n)$. □

Lemma 3.2. *Let $(H_1, H_2) \in R[x,y]^2$ and $f = (x,y) + \epsilon^{m-1}(H_1, H_2) \in \mathrm{SAut}_{R_m}R_m[x,y]$. Then there exists $G \in \mathrm{SAut}_{R[t]}R[t][x,y]$ such that $\bar{G} = f$.*

Proof. *(i)* Since $\det(\mathrm{Jac}(f)) = 1$ we get $\frac{\partial}{\partial x}H_1 + \frac{\partial}{\partial y}H_2 = 0$ (by 3.1(3)). So there exists $P \in R[x,y]$ with $H_1 = P_y$ and $H_2 = P_x$ (see [2], 1.3.53). So $f = (x,y) + \epsilon^{m-1}(P_y, -P_x)$. Since P is a sum of monomials, it follows from 3.1(1) that we may assume that $P = rx^iy^j$ for some $r \in R$ and $i, j \geq 0$. It is well-known that each monomial x^iy^j is a $\mathbb{Q}$-linear combination of polynomials of the form L^d, where $d = i + j$ and $L = x + qy$ with $q \in \mathbb{Q}$ (see for example Exercise 1, paragraph 5.2 in [2]). Therefore, again by 3.1(1), we may assume that $P = rL^d$ for some $r \in R$ and $q \in \mathbb{Q}$.
(ii) Finally consider the derivation $D = t^{m-1}((rL^d)_y\partial_x - (rL^d)_x\partial_y)$
$(= t^{m-1}rdL^{d-1}(q\partial_x - \partial_y))$. Then D is a locally nilpotent derivation on $R[t][x,y]$. So $G = \exp(D) = (x,y) + t^{m-1}(P_y, -P_x) \in \mathrm{Aut}_{R[t]}R[t][x,y]$. From the special form of P it follows that $\det(\mathrm{Jac}(G)) = 1$. So $G \in \mathrm{SAut}_{R[t]}R[t][x,y]$. Since $H_1 = P_y$ and $H_2 = -P_x$ it follows that $\bar{G} = f$, as desired. □

Corollary 3.3. *Let $F = X + t^{m-1}H \in \mathrm{End}_{R[t]}R[t][X]$ with $H \in R[X]^n$. If $\bar{F} \in \mathrm{SAut}_{R_m}R_m[X]$, then there exists $F_* \in \mathrm{SAut}_{R[t]}R[t][X]$ such that $\bar{F}_* = \bar{F}$.*

Proof. By induction on n. The case $n=1$ is obvious, so let $n \geq 2$. Put $x := x_1, y = x_2$ and $A := R[x_3,\ldots,x_n]$. (So $A = R$ if $n = 2$.) Choose $K_2 \in R[X] = A[x,y]$ such that $\frac{\partial}{\partial x}H_1 + \frac{\partial}{\partial y}K_2 = 0$. Then by 3.1(3), $(x,y) + \epsilon^{m-1}(H_1,K_2)$ satisfies the hypothesis of 3.2 (with A instead of R). So there exists $G \in \mathrm{SAut}_{A[t]}A[t][x,y]$ with $\bar{G} = (x,y) + \epsilon^{m-1}(H_1,K_2)$. Obviously G defines an element of $\mathrm{SAut}_{R[t]}[x_1,\ldots,x_n]$, which we also denote by G. Then by 3.1(1) we get

$$\overline{G^{-1}\circ F} = \bar{G}^{-1}\circ\bar{F} = X + \epsilon^{m-1}(0,\tilde{H}_2,\ldots,\tilde{H}_n)$$

for some $\tilde{H}_i \in R[X]$. Now the result follows from the induction hypothesis. □

Proof (of the main theorem) By induction on m. The case $m=1$ is obvious, so let $m \geq 2$ and assume that the theorem holds for $m-1$. Let $f \in \mathrm{SAut}_{R_m}R_m[X]$. So $f = \bar{F}$ where $F = F_0 + tF_1 + \ldots + t^{m-1}F_{m-1}$ for some $F_i \in R[X]^n$. Reducing $\bar{F}$ modulo the element $\bar{t}^{m-1}$ ($\in R_m$) it follows that $f_* := F_* \mod t^{m-1}$ belongs to $\mathrm{SAut}_{R_{m-1}}R_{m-1}[X]$, where $F_* = F_0 + tF_1 + \ldots + t^{m-2}F_{m-2}$. (Use that $R_m/(\bar{t}^{m-1}) \cong R[t]/(t^{m-1}) = R_{m-1}$.) By the induction hypothesis there exists $G \in \mathrm{SAut}_{R[t]}R[t][X]$ such that $G = F_* \mod t^{m-1}$. So $G^{-1}\circ F \equiv X + t^{m-1}H \mod t^m$ for some $H \in R[X]^n$. Finally, by 3.3 there exists $\tilde{G} \in \mathrm{SAut}_{R[t]}R[t][X]$ such that $\tilde{G} \equiv X + t^{m-1}H \mod t^m \equiv G^{-1}\circ F \mod t^m$. So $G\circ\tilde{G} \in \mathrm{SAut}_{R[t]}R[t][X]$ has the desired property that $G\circ\tilde{G} \equiv F \mod t^m$.

4. Hyperplanes and coordinates of $R[t^2,t^3][x,Y]$

In this section we will prove the results announced in section 2. Recall that $R[t^{\geq m}]$ denotes $R[t^m,t^{m+1},\cdots] = R[t^m,t^{m+1},\cdots,t^{2m-1}]$.

Remark 4.1. *One has*

$$R[t^{\geq m}] \subset R[t] \twoheadrightarrow R_m$$

and $\forall a \in R[t]$, $a \in R[t^{\geq m}]$ if and only if $\bar{a} \in R$.

The following two lemmas are well-known:

Lemma 4.2. *Let S be any ring. A polynomial $f = f(X) \in S[X]$ is a hyperplane resp. a coordinate if and only if its canonical image in $(S/\nu)[X]$ is, where ν is the nilradical of S.*

Lemma 4.3. *Let S be any ring. A polynomial $f \in S[X] = S^{[n]}$ is a coordinate if and only if $S[X]$ is $S[f]$-isomorphic to $S[f]^{[n-1]}$ (the prefix '$S[f]$' means 'as $S[f]$-algebras'). This condition is equivalent to the following: $f - c$ is a $S[c]$-hyperplane of $S[c][X]$ where c is an additional indeterminate (here again the prefix '$S[c]$' means that the isomorphism required in the definition of 'hyperplane' is a $S[c]$-isomorphism: $S[c][X]/(f-c) \simeq_{S[c]} S[c]^{[n-1]}$).*

In the proof of theorem 2.3 we will need the following easy lemma.

Lemma 4.4. *Let $Z = (Z_1,\cdots,Z_n) \in R[X]^n = R[x_1,\cdots x_n]^n$ be such that*

$$R[X]/(Z_1) = R[\widetilde{Z}_2,\cdots,\widetilde{Z}_n] \simeq R^{[n-1]}$$

where $\widetilde{\ }$ denotes the image by the canonical epimorphism : $R[X] \to R[X]/(Z_1)$.

Then the jacobian determinant of Z with respect to X, $\widetilde{\mathrm{j}_X(Z)}$, i.e. the determinant of the jacobian matrix, $\widetilde{(\frac{\partial Z_i}{\partial x_j})}$, is an invertible element of $R[X]/(Z_1)$.

Proof. By assumption

$$\begin{aligned} R[X] &= R[Z_2, \cdots, Z_n] + (Z_1) = R[Z_2, \cdots, Z_n] + Z_1 \cdot R[X] \\ &= R[Z_2, \cdots, Z_n] + Z_1 \cdot (R[Z_2, \cdots, Z_n] + Z_1 \cdot R[X]) \\ &= R[Z] + (Z_1^2) \end{aligned}$$

hence there exists $P = (P_1, \cdots, P_n) \in R[X]^n$ such that

$$X = P(Z) + Z_1^2 \cdot R[X]^n$$

but then

$$\begin{aligned} \mathrm{Id} = \mathrm{Jac}_X(X) &= \mathrm{Jac}_X(P(Z)) + Z_1 \cdot M \text{ (for some } M \in \mathrm{Mat}_{n\times n}(R[X])) \\ \mathrm{Id} &= \mathrm{Jac}_X(P)(Z) \cdot \mathrm{Jac}_X(Z) + Z_1 \cdot M \end{aligned}$$

and the conclusion follows. □

Now we can give the proof of 2.3. Notice that in this lemma, the Main Theorem is used.

Proof (of theorem 2.3). In view of lemma 4.2 one may assume that $R[t]$ and hence R is reduced. By assumptions there exists $G = (G_1, \dots, G_n) \in R[t][x, Y]^n$ such that

$$R[t][x, Y]/(f) = R[t][\widetilde{G}_1, \dots, \widetilde{G}_n] \simeq R[t]^{[n]}$$

where $\widetilde{\ }$ denotes the image by the canonical epimorphism : $R[t][x, Y] \to R[t][x, Y]/(f)$. By lemma 4.4,

$$\widetilde{\mathrm{j}_{x,Y}(f, G)} \in R[t][x, Y]/(f)^{\times}$$

but since $R[t][x, Y]/(f) \simeq R[t]^{[n]}$ and $R[t]^{[n]\times} = R^{\times}$ (R is reduced!) we have

$$\widetilde{\mathrm{j}_{x,Y}(f, G)} \in R^{\times} .$$

Up to multiplying G_1 by the inverse of this latter scalar (which does not modify $R[t][\widetilde{G}_1, \dots, \widetilde{G}_n]$) one may therefore assume that

$$\widetilde{\mathrm{j}_{x,Y}(f, G)} = 1.$$

Now remark that since $\overline{f} = \overline{x}$ one may identify

$$\overline{R[t][x, Y]/(f)} = R_m[x, Y]/(f) = R_m[x, Y]/(x)$$

with $R_m[Y] = R_m^{[n]}$ by taking $\overline{x}$ to 0. So we have

$$R_m[\overline{G(0, Y)}] = R_m[Y]$$

with

$$\overline{\widetilde{1}} = \overline{\widetilde{\mathrm{j}_{x,Y}(f, G)}} = \overline{\widetilde{\mathrm{j}_{x,Y}(x, G)}} = \overline{\widetilde{\mathrm{j}_Y(G)}} = \overline{\mathrm{j}_Y(G(0, Y))}.$$

Hence $\overline{G(0, Y)} \in \mathrm{SAut}_{R_m} R_m[Y]$ and by the Main Theorem there exists $H \in \mathrm{SAut}_{R[t]} R[t][Y]$ such that $\overline{H} = \overline{G(0, Y)}$. This automorphism extends naturally to

$R[t,x][Y]$ and we get $R[t][\widetilde{G}] = R[t][H^{-1}(\widetilde{G})]$ with $\overline{H^{-1}(\widetilde{G})} = \overline{H}^{-1} \circ \overline{G(0,Y)} = \overline{Y}$. Hence with $G' := H^{-1}(G)$ instead of G one has

$$R[t][x,Y]/(f) = R[t][\widetilde{G'}]$$

with $\overline{G'} = \overline{Y} \mod (\overline{x}) = \overline{Y} \mod (\overline{f})$. Up to adding to G' some multiple of f, which does not affect $R[t][\widetilde{G'}]$ one can hence assume that $\overline{G'} = \overline{Y}$ and, in view of remark 4.1, $G' \in R[t^{\geq m}][x,Y]^n$. Let $(p,Q_1,\cdots,Q_n) = (p,Q) \in R[t][x_1,\cdots,x_n]^{n+1}$ be such that $\widetilde{x} = p(\widetilde{G'})$ and $\widetilde{Y} = Q(\widetilde{G'})$. One has then $0 = \overline{p}(\overline{G'(0,Y)}) = \overline{p}(Y)$ and $Y = \overline{Q}(\overline{G'(0,Y)}) = \overline{Q}(Y)$ therefore $(p,Q) \in R[t^{\geq m}][X]^{n+1}$ (again by 4.1). Hence we have

$$R[t^{\geq m}][x,Y]/(f) = R[t^{\geq m}][\widetilde{G'}] \simeq R[t^{\geq m}]^{[n]}$$

i.e. f is an hyperplane of $R[t^{\geq m}][x,Y]$. $\square$

Now we prove corollary 2.2:

proof (of corollary 2.2). In order to fit the notations of theorem 2.3 we replace Y by x,Y which is harmless. Let $p \in R[t^{\geq m}][x,Y]$ be a variable over $R[t]$. We have to show that p is a variable over $R[t^{\geq m}]$ i.e. a variable in $R[t^{\geq m}][x,Y]$. In view of lemma 4.3 this amounts to show that $p-c$ is a $R[t^{\geq m}][c]$-hyperplane of $R[t^{\geq m}][c][x,Y]$.

By assumptions p is a coordinate of $R[t][x,Y]$ hence $p-c$ is a $R[t,c]$-coordinate of $R[t,c][x,Y]$ and $p-c \mod (t)$ is a $R[c]$-coordinate of $R[c][x,Y]$ i.e. there exists $\alpha \in \mathrm{Aut}_{R[c]} R[c][x,y]$ such that $\alpha(x) = p-c \mod (t)$ i.e $\alpha^{-1}(p-c) = x \mod (t)$. This automorphism has a natural extension to $R[t^{\geq m}][c][x,Y]$. It is now sufficient to prove that $f := \alpha^{-1}(p-c)$ is an $R[t^{\geq m}][c]$-hyperplane of $R[t^{\geq m}][c][x,Y]$. We have $f = x \mod (t)$ i.e. $f - x \in (t)$ but f and x are in $R[t^{\geq m}][c][x,Y]$ hence $f - x \in (t) \cap R[t^{\geq m}][c][x,Y] = (t^m)$ i.e. $\overline{f} = \overline{x}$ and theorem 2.3 (with $R[c]$ instead of R) concludes. $\square$

References

[1] J.Berson, *Polynomial coordinates and their behavior in higher dimensions,* Ph.D thesis, University of Nijmegen, november 2004.

[2] A. van den Essen, *Polynomial Automorphisms and the Jacobian Conjecture,* Progress in Mathematics, Vol 190, Birkhäuser, Basel, (2000)

[3] A. van den Essen,*On Bass' inverse degree approach to the Jacobian Conjecture and exponential automorphisms,* Contemporary Mathematics, Vol. 264, (2000), 207-214

[4] S. Kaliman, *Polynomials with generic* $\mathbb{C}^2$*-fibers are equivalent to a linear polynomial*, Pacific J.Math 203:1, 161-189 (2002)

[5] S. Kaliman, M. Zaidenberg, *Vénéreau polynomials and related fiber bundles*, J. Pure Appl. Algebra 192, No.1-3, 275-286 (2004).

[6] S. Kaliman, M. Zaidenberg, S. Vénéreau, *Simple birational extensions of the polynomial algebra* $\mathbb{C}[3]$, Trans. Amer. Math. Soc, 356, No.2, 509-555 (2004).

[7] S.Maubach, *The automorphism group of* $\mathbb{C}[T]/(T^m)[X_1,\ldots,X_n]$. Com.Algebra 30, no.2. (2000) 619-629

[8] S.Maubach, *The commuting derivations conjecture.* J.Pure Appl. Algebra 179 (2003), No.1-2. 159-168.[13B25]

[9] S. Vénéreau, *Is* $y+x[xz+y(yu+z^2)]$ *an x-variable of* $\mathbb{C}[x][y,z,u]$?, Problem session, Workshop on group action on rational varieties, (2002), 6p.

[10] S. Vénéreau, *New bad lines in* $R[x,y]$ *and optimization of the Epimorphism Theorem*, to appear in J. Algebra, 24p.

[11] S. Vénéreau, *Automorphismes et variables de l'anneau de polynômes* $A[y_1,\ldots,y_n]$, Ph.D. Thesis, Université Grenoble I, Institut Fourier, 2001.

Author's addresses

Arno van den Essen
Department of Mathematics
Radboud University Nijmegen
Toernooiveld, 6525ED Nijmegen
The Netherlands
essen@math.ru.nl

Stefan Maubach
Department of Mathematics
University of Texas at Brownsville
80 Fort Brown, Brownsville, TX 78520
USA
stefan.maubach@utb.edu

Stéphane Vénéreau
Mathematisches Institut
Universität Basel
Rheinsprung 21, CH-4051 Basel
Switzerland
stephane.venereau@unibas.ch