THE SPECIAL AUTOMORPHISM GROUP OF $R[t]/(t^m)[x_1, \ldots, x_n]$
AND COORDINATES OF A SUBRING OF $R[t][x_1, \ldots, x_n]$

ARNO VAN DEN ESSEN, STEFAN MAUBACH AND STÉPHANE VÉNÉRÉAU

Abstract. Let $R$ be a ring. The Special Automorphism Group $\text{SAut}_R R[x_1, \ldots, x_n]$ is the set of all automorphisms with determinant of the Jacobian equal to 1. It is shown that the canonical map of $\text{SAut}_R R[t][x_1, \ldots, x_n]$ to $\text{SAut}_{R_n} R_n[x_1, \ldots, x_n]$ where $R_n := R[t]/(t^m)$ and $Q \subset R$ is surjective. This result is used to study a particular case of the following question: if $A$ is a subring of a ring $B$ and $f \in A^n$ is a coordinate over $B$ does it imply that $f$ is a coordinate over $A$? It is shown that if $A = R[t^m, t^{m+1}, \ldots] \subset R[t] = B$ then the answer to this question is “yes”.

Also, a question on the Vénéréau polynomial is settled, which indicates another “coordinate-like property” of this polynomial.

1. Introduction

Some notations: Let $R$ be a commutative ring with one, as all the other rings in this paper. We will denote individual variables by small letters, and lists of variables by capitals: $X := x_1, \ldots, x_n$. The polynomials in these variables with coefficients in $R$ form a ring, and even an algebra over $R$, that we will denote $R[X]$, $R[x_1, \ldots, x_n]$ or $R[n]$ as well.

A coordinate in $R^n$ (also called “variable”\textsuperscript{1}) is a polynomial $f \in R^n$ for which one can find $f_2, \ldots, f_n \in R^n$ such that $R[f, f_2, \ldots, f_n] = R^n$. One of the central problems in affine algebraic geometry is the question under what conditions a polynomial is a coordinate. One of the conjectures in regard to this problem is the Abhyankar-Sathaye Conjecture (AS(n)):

\textbf{Abhyankar-Sathaye Conjecture (AS(n))}: Let $f \in \mathbb{C}^{n+1}$ and assume that $\mathbb{C}^{(n+1)}/(f) \cong \mathbb{C}^n$. Then $f$ is a coordinate.

Notice that the converse of the conjecture is true. When $f$ satisfies the condition $R^{(n+1)}/(f) \cong R^n$, then $f$ is called a hyperplane (resp. plane or line, according to dimensions).

A lot of work has been done on attempts to solve this conjecture, and also on the problem of classifying coordinates (see for example [1, 4, 6, 8, 11] and many others). One of these works, [6], studied hyperplanes in $\mathbb{C}[x, y, z, u]$ with a prescribed form. Some of them could be proved to be coordinates but some others could not; two of the simplest examples of such polynomials with an indefinite status are the (now

\textsuperscript{1}This can be somewhat confusing. Sometimes, the word “variable” is exclusively used for the coordinate system that one is working with, i.e. if one writes $R[x, y]$ then $x$ and $y$ are variables, whereas $x + y^2$ is a coordinate (and not a variable). But sometimes, especially from a geometric viewpoint, it is natural to choose no coordinate system and view objects globally, and in these cases “variable” is used synonymous with “coordinate”.

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The Special Automorphism Group of $R[X]$, where $R$ is a ring, is the set
$$\text{SAut}_R R[X] := \{ \varphi \in \text{Aut}_R R[X] \mid \det(\text{Jac}(\varphi)) = 1 \}.$$

In section 3 we show our

Main Theorem. Let $R$ be a ring containing the field of rational numbers $\mathbb{Q}$, $m$ a positive integer and $R_m := R[t]/(t^m)$. Then the map
$$\text{SAut}_{R[t]} R[X] \rightarrow \text{SAut}_{R_m} R_m[X]$$
induced by the canonical morphism $R[t] \rightarrow R[t]/(t^m) = R_m$ is surjective.

This result is then used in section 4 to study coordinates in $R[t^{2m^3}]$, where $R[t^{2m}] := R[t^{m}, t^{m+1}, \ldots] = R[t^m, t^{m+1}, \ldots, t^{2m-1}]$. It is shown that if $f \in R[t^{2m}]$ is a coordinate in $R[t]/(t^m)$, then $f$ is a coordinate in $R[t^{2m}]$.

It is worth mentioning that there exists an equivalent formulation of the Jacobian Conjecture in terms of the automorphism group of $R_m[X]$ (see [3]). Study of the automorphism group of $R_m[X]$ as in this paper and the paper [7] can help in giving a good foundation for an attack on the Jacobian Conjecture.

2. CONSEQUENCES OF THE MAIN THEOREM

Let $A \subset B$ be rings. Considering a polynomial $f \in A[x_1, \ldots, x_n]$ is a coordinate over $B$ if for any $n \in \mathbb{N}$:

$$f \in A[x_1, \ldots, x_n] \text{ is a coordinate over } B \quad \Rightarrow \quad f \text{ is a coordinate over } A.$$

It is shown in [10] that $B = \mathbb{Z}[t]$ introduces new coordinates over $A = \mathbb{Z}[t^2, t^3, 3t]$, so this does occur. Of course, an interesting general question is: under what conditions does $B$ not introduce new coordinates over $A$? With

$$A = R[t^{2m}] = R[t^m, t^{m+1}, \ldots] = R[t^m, t^{m+1}, \ldots, t^{2m-1}] \subset B = R[t]$$

we get in section 4 the following

Corollary 2.2. Let $R$ be a ring containing $\mathbb{Q}$ and $m$ a positive integer. Then $R[t]$ does not introduce new coordinates over $R[t^{2m}]$. 

Let $f$ be a polynomial in $R[t^{2m}][x, Y] = R[t^{2m}][x, y_1, \ldots, y_n] = R[t^{2m}[n+1]]$ such that
\begin{itemize}
  \item $f \equiv x \mod t^m \cdot R[t][x, Y];$
  \item $f$ is a hyperplane over $R[t].$
\end{itemize}
Then $f$ is a hyperplane over $R[t^{2m}].$

The above theorem has consequences for the Vénéréau polynomial, $y + x[xz + y(yu + z^2)]$: at the end of [9] (see also [5]) the following question is asked:

**Question 2.4.** Is $A[y, z, u]/(y + x[xz + y(yu + z^2)] - c)$ isomorphic to $A[2]$ where $A = \mathbb{C}[x, c]/(x^2 - c^3)$?

An answer of “no” would settle this Vénéréau polynomial to not be a coordinate, proving that it is a counterexample to the Abhyankar-Sathaye conjecture. Let us shortly explain why this is the case, and see where this question came from:

Note that, when $g \in \mathbb{C}[n]$, “$f \in \mathbb{C}[n]$ is a $g$-coordinate” means that there exist $f_1, \ldots, f_n \in \mathbb{C}[n]$ such that $\mathbb{C}[f, g, f_1, \ldots, f_n] = \mathbb{C}[n]$. As shown in [9, 5, 6, 11] (and mentioned in the introduction), the “Vénéréau” polynomial $y + x[xz + y(yu + z^2)] \in \mathbb{C}[x][y, z, u]$ fulfills a bunch of necessary conditions to be a coordinate and even an $x$-coordinate of $\mathbb{C}[x][y, z, u]$ (e.g. it is an $x$-plane, meaning that $\mathbb{C}[x][y, z, u]/(f) \cong \mathbb{C}[x]$).

Corollary 2.5. $A[y, z, u]/(y + x[xz + y(yu + z^2)] - c) \cong A[2]$ where $A = \mathbb{C}[x, c]/(x^2 - c^3)$.

**Proof.** Let us change notation to match notations of theorem 2.3: replace $x, c, y, z, u$ by $t^2, t^3, x, y_1, y_2$, thus replace $\mathbb{C}[x, c]/(x^2 - c^3)$ by $\mathbb{C}[t^2, t^3] = \mathbb{C}[t^{2z}]$ and replace $y + x[xz + y(yu + z^2)] - c$ by $x + t^3[y_1 x + x(y_2 + y_1^2)] - t^2$. As required in theorem 2.3 (with $R = \mathbb{C}, Y = (y_1, y_2), m = 2$ and $f = x + t^3[y_1 x + x(y_2 + y_1^2)] - t^2$ ) one has $f \equiv x \mod t^2 \cdot \mathbb{C}[t][x, Y]$. The condition “$f$ is an hyperplane over $\mathbb{C}[t]$” is not obvious to check but can be retrieved from e.g. [6]. Then we get the desired conclusion that $f$ is a (hyper)plane of $\mathbb{C}[t^2, t^3][x, y_1, y_2].$}

3. **Surjectivity of the Special Automorphism Group**

The following notations and assumptions are fixed throughout the rest of the article:
• $R$ is a ring containing $\mathbb{Q}$ i.e. $R$ is a $\mathbb{Q}$-algebra;
• $m$ is a positive integer and $R_m := R[t]/(t^m)$;
• if $f \in R[t][X]$ then $f \mod t^m$ denotes the polynomial in $R_m[X]$ obtained from $f$ by reduction modulo $t^m$;
• if no confusion (about the value of $m$) is possible, we will denote this polynomial by $f$;
• similarly, if $F = (F_1, \ldots, F_n) \in R[t][X]^n$ we define $\bar{F} := (\bar{F}_1, \ldots, \bar{F}_n)$.

In this section we prove the

**Main Theorem.** The map

$$\text{SAut}_{R[t]} R[t][X] \longrightarrow \text{SAut}_{R_m} R_m[X]

F \longmapsto \bar{F}$$

is surjective.

Note that without the "$S$" in $\text{SAut}$ the corresponding map is not surjective anymore. The reason is that the map $R[t][X]^x \to R_m[X]^x$, where $^x$ denotes the set of invertible elements, is not surjective. For example there is no $p \in R[t][X]^x$ such that $\bar{p} = 1 + \bar{t} \in R_m[X]^x$ and consequently there is no $F \in \text{Aut}_{R[t]} R[t][X]$ such that $\bar{F} = (1 + \bar{t}x_1, x_2, \ldots, x_n) \in \text{Aut}_{R_m} R_m[X]$.

Before we can prove the Main Theorem we need some preparations.

**Lemma 3.1.** Let $H = (H_1, \ldots, H_n) \in R[X]^n$ and $G = (G_1, \ldots, G_n) \in R[X]^n$. Put $h := X + \epsilon^{-1}H$ and $g := X + \epsilon^{-1}G$, where $\epsilon = t \in R_m$. Then

1. $h \circ g = X + \epsilon^{-1}(H + G)$
2. $h \in \text{Aut}_{R_m} R_m[X]$ (with inverse $X - \epsilon^{-1}H$)
3. $h \in \text{SAut}_{R_m} R_m[X]$ iff $\frac{\partial}{\partial x_i} H_1 + \ldots + \frac{\partial}{\partial x_n} H_n = 0$

**Proof.** (1) and (2) are immediate. To see (3) just observe that $\det(\text{Jac}(h)) = 1 + \epsilon^{-1}(\frac{\partial}{\partial x_i} H_1 + \ldots + \frac{\partial}{\partial x_n} H_n)$. □

**Lemma 3.2.** Let $(H_1, H_2) \in R[x,y]^2$ and $f = (x,y)+\epsilon^{-1}(H_1, H_2) \in \text{SAut}_{R_m} R_m[x,y]$. Then there exists $G \in \text{SAut}_{R[t]} R[t][x,y]$ such that $\bar{G} = f$.

**Proof.** (i) Since $\det(\text{Jac}(f)) = 1$ we get $\frac{\partial}{\partial x} H_1 + \frac{\partial}{\partial y} H_2 = 0$ (by 3.1(3)). So there exists $P \in R[x,y]$ with $H_1 = P_y$ and $H_2 = P_x$ (see [2], 1.3.53). So $f = (x, y) + \epsilon^{-1}(P_y, -P_x)$. Since $P$ is a sum of monomials, it follows from 3.1(1) that we may assume that $P = rx^iy^j$ for some $r \in R$ and $i, j \geq 0$. It is well-known that each monomial $x^iy^j$ is a $Q$-linear combination of polynomials of the form $L^d$, where $d = i + j$ and $L = x + qy$ with $q \in Q$ (see for example Exercise 1, paragraph 5.2 in [2]). Therefore, again by 3.1(1), we may assume that $P = rL^d$ for some $r \in R$ and $q \in Q$.

(ii) Finally consider the derivation $D = t^{n-1}((rL^d)_y \partial_x - (rL^d)_x \partial_y)$ ($= t^{n-1}rL^{d-1}(q\partial_x - \partial_y)$). Then $D$ is a locally nilpotent derivation on $R[t][x, y]$. So $G = \exp(D) = (x, y) + t^{n-1}(P_y, -P_x) \in \text{Aut}_{R[t]} R[t][x,y]$. From the special form of $P$ it follows that $\det(\text{Jac}(G)) = 1$. So $G \in \text{SAut}_{R[t]} R[t][x, y]$. Since $H_1 = P_y$ and $H_2 = -P_x$ it follows that $\bar{G} = f$, as desired. □
Corollary 3.3. Let $F = X + t^{m-1}H \in \text{End}_{R[t]}R[t][X]$ with $H \in R[X]$. If $\tilde{F} \in \text{SAut}_{R[t]}R_{m}[X]$, then there exists $F_{*} \in \text{SAut}_{R[t]}R[t][X]$ such that $\tilde{F} = F_{*}$.

Proof. By induction on $n$. The case $n = 1$ is obvious, so let $n \geq 2$. Put $x := x_{1}, y := x_{2}$ and $A := R[x_{3}, \ldots, x_{n}]$. (So $A = R$ if $n = 2$.) Choose $K_{2} \in R[X] = A[x, y]$ such that $\frac{\partial}{\partial x}H_{1} + \frac{\partial}{\partial y}K_{2} = 0$. Then by 3.1(3), $(x, y) + e^{m-1}(H_{1}, K_{2})$ satisfies the hypothesis of 3.2 (with $A$ instead of $R$). So there exists $G \in \text{SAut}_{R[t]}R[t][x, y]$ with $\tilde{G} = (x, y) + e^{m-1}(H_{1}, K_{2})$. Obviously $G$ defines an element of $\text{SAut}_{R[t]}[x_{1}, \ldots, x_{n}]$, which we also denote by $G$. Then by 3.1(1) we get

$$G^{-1} \circ \tilde{F} = G^{-1} \circ F = X + e^{m-1}(0, H_{2}, \ldots, H_{n})$$

for some $\tilde{H}_{i} \in R[X]$. Now the result follows from the induction hypothesis. \hfill \n
4. Hyperplanes and Coordinates of $R[t^{2}, t^{3}][X, Y]$

In this section we will prove the results announced in section 2. Recall that $R[t^{2}]$ denotes $R[t^{m}, t^{m+1}, \ldots] = R[t^{m}, t^{m+1}, \ldots, t^{2m-1}]$.

Remark 4.1. One has

$$R[t^{\geq m}] \subset R[t] \twoheadrightarrow R_{m}$$

and $\forall a \in R[t]$, $a \in R[t^{\geq m}]$ if and only if $a \in R$.

The following two lemmas are well-known:

Lemma 4.2. Let $S$ be any ring. A polynomial $f = f(X) \in S[X]$ is a hyperplane with respect to a coordinate if and only if its canonical image in $(S/\nu)[X]$ is, where $\nu$ is the nilradical of $S$.

Lemma 4.3. Let $S$ be any ring. A polynomial $f \in S[X] = S^{[n]}$ is a coordinate if and only if $S[X] = S[f]$-isomorphic to $S[f]^{[n-1]}$ (the prefix ‘$S[f]$’ means ‘as $S[f]$-algebras’). This condition is equivalent to the following: $f - c$ is a $S[c]$-hyperplane of $S[c][X]$ where $c$ is an additional indeterminate (here again the prefix ‘$S[c]$’ means that the isomorphism required in the definition of ‘hyperplane’ is a $S[c]$-isomorphism: $S[c][X]/(f - c) \simeq S[c]^{[n-1]}$).

In the proof of theorem 2.3 we will need the following easy lemma.

Lemma 4.4. Let $Z = (Z_{1}, \ldots, Z_{n}) \in R[X]^{n} = R[x_{1}, \ldots, x_{n}]^{n}$ be such that

$$R[X]/(Z_{1}) = R[Z_{2}, \ldots, Z_{n}] \simeq R^{[n-1]}$$

where $\sim$ denotes the image by the canonical epimorphism $R[X] \twoheadrightarrow R[X]/(Z_{1})$. 
Then the jacobian determinant of $Z$ with respect to $X$, $\widetilde{j}_X(Z)$, i.e. the determinant of the jacobian matrix, $(\frac{\partial Z}{\partial X})$, is an invertible element of $R[X]/(Z_1)$.

Proof. By assumption
\[
R[X] = R[Z_2, \cdots, Z_n] + (Z_1) = R[Z_2, \cdots, Z_n] + Z_1 \cdot R[X]
\]
\[
= R[Z_2, \cdots, Z_n] + Z_1 \cdot (R[Z_2, \cdots, Z_n] + Z_1 \cdot R[X])
\]
\[
= R[Z] + (Z_1^2)
\]
hence there exists $P = (P_1, \cdots, P_n) \in R[X]^n$ such that
\[
X = P(Z) + Z_1^2 \cdot R[X]^n
\]
but then
\[
\text{Id} = \text{Jac}_X(X) = \text{Jac}_X(P(Z)) + Z_1 \cdot M \quad (\text{for some } M \in \text{Mat}_{n\times n}(R[X]))
\]
\[
\text{Id} = \text{Jac}_X(P)(Z) \cdot \text{Jac}_X(Z) + Z_1 \cdot M
\]
and the conclusion follows.

Now we can give the proof of 2.3. Notice that in this lemma, the Main Theorem is used.

Proof (of theorem 2.3). In view of lemma 4.2 one may assume that $R[t]$ and hence $R$ is reduced. By assumptions there exists $G = (G_1, \ldots, G_n) \in R[t][x, Y]^n$ such that
\[
\widetilde{R}[t][x, Y]/(f) = R[t][\tilde{G}_1, \ldots, \tilde{G}_n] \simeq R[t][n]
\]
where $\widetilde{\cdot}$ denotes the image by the canonical epimorphism: $R[t][x, Y] \rightarrow R[t][x, Y]/(f)$. By lemma 4.4,
\[
\widetilde{j}_{x, Y}(f, G) \in R[t][x, Y]/(f)^\times
\]
but since $R[t][x, Y]/(f) \simeq R[t][n]$ and $R[t][n]^\times = R^\times$ ($R$ is reduced!) we have
\[
\widetilde{j}_{x, Y}(f, G) \in R^\times.
\]
Up to multiplying $G_1$ by the inverse of this latter scalar (which does not modify $R[t][\tilde{G}_1, \ldots, \tilde{G}_n]$) one may therefore assume that
\[
\widetilde{j}_{x, Y}(f, G) = 1.
\]
Now remark that since $\widetilde{f} = \pi$ one may identify
\[
\widetilde{R}[t][x, Y]/(f) = R_m[x, Y]/(f) = R_m[x, Y]/(x)
\]
with $R_m[Y] = R_m^n$ by taking $\pi$ to 0. So we have
\[
R_m[\tilde{G}(0, Y)] = R_m[Y]
\]
with
\[
\tilde{1} = \widetilde{j}_{x, Y}(f, G) = j_{x, Y}(x, G) = j_Y(G) = j_Y(G(0, Y)).
\]

Hence $\tilde{G}(0, Y) \in S\text{Aut}_{R_m} R_m[Y]$ and by the Main Theorem there exists $H \in S\text{Aut}_{R[t]} R[t][Y]$ such that $H = \tilde{G}(0, Y)$. This automorphism extends naturally to
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$R[t, x][Y]$ and we get $R[t][\tilde{G}] = R[t][H^{-1}(\tilde{G})]$ with $H^{-1}(\tilde{G}) = \tilde{H}^{-1} \circ G(0, Y) = Y$.

Hence with $G' := H^{-1}(\tilde{G})$ instead of $G$ one has

$$R[t][x, Y]/(f) = R[t][\tilde{G}']$$

with $\tilde{G}' = Y \mod (\tilde{f}) = Y \mod (f)$. Up to adding to $G'$ some multiple of $f$, which does not affect $R[t][\tilde{G}']$ one can hence assume that $\tilde{G}' = Y$ and, in view of remark 4.1, $G' \in R[t^{\leq m}][x, Y]^n$. Let $(p, Q_1, \ldots, Q_n) = (p, Q) \in R[t][x_1, \ldots, x_n]^{n+1}$ be such that $\tilde{x} = p(\tilde{G}')$ and $\tilde{Y} = Q(\tilde{G}')$. One has then $0 = p(\tilde{G}'(0, Y)) = \tilde{Y}$ and $Y = \tilde{Q}(\tilde{G}'(0, Y)) = \tilde{Q}(Y)$ therefore $(p, Q) \in R[t^{\leq m}][X]^{n+1}$ (again by 4.1). Hence we have

$$R[t^{\leq m}][x, Y]/(f) = R[t^{\leq m}][\tilde{G}'] \simeq R[t^{\leq m}][c]$$
i.e. $f$ is a hyperplane of $R[t^{\leq m}][x, Y]$.

Now we prove corollary 2.2:

proof (of corollary 2.2). In order to fit the notations of theorem 2.3 we replace $Y$ by $x, Y$ which is harmless. Let $p \in R[t^{\leq m}][x, Y]$ be a variable over $R[t]$. We have to show that $p$ is a variable over $R[t^{\leq m}]$ i.e. a variable in $R[t^{\leq m}][x, Y]$. In view of lemma 4.3 this amounts to show that $p - c$ is a $R[t^{\leq m}][c]$-hyperplane of $R[t^{\leq m}][c][x, Y]$. By assumptions $p$ is a coordinate of $R[t][x, Y]$ hence $p - c$ is a $R[t, c]$-coordinate of $R[t, c][x, Y]$ and $p - c \mod (t)$ is a $R[c][x, Y]$ i.e. there exists $\alpha \in \text{Aut}_{R[c]}R[c][x, y]$ such that $\alpha(x) = p - c \mod (t)$ i.e. $\alpha^{-1}(p - c) = x \mod (t)$. This automorphism has a natural extension to $R[t^{\leq m}][c][x, Y]$. It is now sufficient to prove that $f := \alpha^{-1}(p - c)$ is an $R[t^{\leq m}][c]$-hyperplane of $R[t^{\leq m}][c][x, Y]$. We have $f = x \mod (t)$ i.e. $f - x \in (t)$ but $f$ and $x$ are in $R[t^{\leq m}][c][x, Y]$ hence $f - x \in (t) \cap R[t^{\leq m}][c][x, Y] = (t^m)$ i.e. $\tilde{f} = \tilde{x}$ and theorem 2.3 (with $R[c]$ instead of $R$) concludes.

\begin{thebibliography}{9}


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Author’s addresses

Arno van den Essen
Department of Mathematics
Radboud University Nijmegen
Toernooiveld, 6525ED Nijmegen
The Netherlands
essen@math.ru.nl

Stefan Maubach
Department of Mathematics
University of Texas at Brownsville
80 Fort Brown, Brownsville, TX 78520
USA
stefan.maubach@utb.edu

Stéphane Vénéreau
Mathematisches Institut
Universität Basel
Rheinsprung 21, CH-4051 Basel
Switzerland
stephane.venereau@unibas.ch