PAIRINGS AND ACTIONS FOR DYNAMICAL QUANTUM GROUPS

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Abstract. Dynamical quantum groups constructed from a FRST-construction using a solution of the quantum dynamical Yang-Baxter equation are equipped with a natural pairing. The interplay of the pairing with *-structures, corepresentations and dynamical representations is studied, and natural left and right actions are introduced. Explicit details for the elliptic $U(2)$ dynamical quantum group are given, and the pairing is calculated explicitly in terms of elliptic hypergeometric functions. Dynamical analogues of spherical and singular vectors for corepresentations are introduced.

1. Introduction

Dynamical quantum groups have been introduced by Etingof and Varchenko [9], [10] in order to provide an algebraic framework, in terms of $\hbar$-Hopf algebroids (see §2.1), for the study of the quantum dynamical Yang-Baxter equation (2.7) similar to the relation of quantum groups and (constant) solutions of the quantum Yang-Baxter equation. The construction is an analogue of the Faddeev-Reshetikhin-Sklyanin-Takhtajan (FRST) construction, which we recall in §2.2. The quantum dynamical Yang-Baxter equation arises in a natural way from the construction of correlation functions and corresponding fusion and exchange matrices in the study of the Knizhnik-Zamolodchikov-equations and its difference analogue, see [6], [8] and references given there. In particular, this construction attaches an algebraic framework to the elliptic $R$-matrix (2.11) related to $\mathfrak{sl}(2)$ involving both a spectral and a dynamical parameter. The corresponding algebraic framework has been studied in detail by Felder and Varchenko [11], which predates [9], [10], and we call this the elliptic $U(2)$ dynamical quantum group and this is the main example in this paper, see §2.2.

Another well-studied example is the dynamical quantum group associated to the rational $R$-matrix (2.17) for $\mathfrak{sl}(2)$ that can be obtained by a limit transition from the elliptic $R$-matrix. The corresponding algebraic structure is essentially simpler, since the dependence on the spectral parameter is removed. In [17] its corepresentation theory has been studied, and it turns out that there is a direct link to special functions, in particular to Askey-Wilson and $q$-Racah polynomials which occur in the description of matrix elements of irreducible corepresentations. These results have been partly extended to the elliptic $U(2)$ dynamical quantum group in [16], linking corepresentations to so-called elliptic hypergeometric series, originally introduced by Frenkel and Turaev [12] and which are recalled in §5.1.

An important ingredient for this paper is the fact that dynamical quantum groups arising from a FRST-construction for which the $R$-matrix is a solution to the quantum dynamical Yang-Baxter equation is self-dual in the sense that there is a natural pairing between the dynamical quantum group and its co-opposite dynamical quantum group, which is compatible with the algebraic structures, but which in general is not non-degenerate. The pairing is defined in terms of the $R$-matrix by Rosengren [26], and the example of the dynamical quantum group associated to the rational $R$-matrix for $\mathfrak{sl}(2)$ is completely worked out in [26]. A similar duality has been given in the context of weak Hopf...

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algebras by Etingof and Nikshych [7, §5] related to the case of a dynamical quantum group at a root of 1, in which case the pairing is non-degenerate. We recall Rosengren’s definitions in §3, and we extend it to include the case of a spectral parameter in the \( R \)-matrix and to include a *-structure. In particular, we work out the details of the pairing for the elliptic \( U(2) \) dynamical quantum group. The algebraic structures for dynamical quantum groups are \( h \)-Hopf algebroids, which are recalled in §2.1, and in general the pairing between two \( h \)-Hopf algebroids induces a natural action of one \( h \)-Hopf algebroid on the other, extending the results for ordinary Hopf algebras and which can be thought of a generalisation of the action \( (X \cdot f)(g) = \frac{d}{dt} \big|_{t=0} f(g \exp(tX)) \) for \( X \in \mathfrak{g} \), \( f \) a polynomial on the Lie group \( G \) with Lie algebra \( \mathfrak{g} \). This also gives the opportunity to go from corepresentations of one \( h \)-Hopf algebroid to dynamical representations of the other \( h \)-Hopf algebroid. In the case of the elliptic \( U(2) \) dynamical quantum group we work out the details, and in particular we calculate the pairing between matrix elements of irreducible corepresentations in terms of elliptic hypergeometric series. This then allows us to give a dynamical quantum group derivation of the quantum dynamical Yang-Baxter equation for the elliptic hypergeometric series, biorthogonality relations, the Bailey transform and the Jackson sum for elliptic hypergeometric series already obtained by Frenkel and Turaev [12], see [13, Ch. 11] for more information and references. For this we rely on a previous paper [16] in which a slightly different dynamical quantum group theoretic derivation of biorthogonality relations, the Bailey transform and the Jackson sum for elliptic hypergeometric series are given.

Starting with the work of Koornwinder [19] we also have the interpretation of Askey-Wilson polynomials as spherical functions for irreducible corepresentations of the standard quantum \( SU(2) \) group using so-called twisted primitive elements, see also [15], [24] for extensions to arbitrary matrix elements. This idea with the appropriate replacement of twisted primitive elements by co-ideals has turned out to be very fruitful, especially for quantum analogues of compact symmetric spaces and the corresponding spherical functions in terms of Askey-Wilson and Macdonald-Koornwinder polynomials, see Dijlkhuijzen and Noumi [5], Noumi [22], Noumi, Dijkhuizen and Sugitani [23], and Letzter [20], and for a non-compact quantum group and the Askey-Wilson functions see [18].

Since the interpretation of Askey-Wilson polynomials on the standard quantum \( SU(2) \) group and dynamical quantum group associated to the rational \( R \)-matrix for \( \mathfrak{sl}(2) \) is similar, a natural link between these algebras is to be expected and the precise relation has been given by Stokman [29]. A key ingredient is the twisted coboundary element of Babelon, Bernard and Billey [1] in the form discovered by Rosengren [25] as a universal element intertwining the standard Cartan element with the twisted primitive element of Koornwinder. This element is only known explicitly for the case \( \mathfrak{sl}(2) \), see Buffenoir and Roche [4] for conjectural forms of the twisted coboundary element for other simple \( \mathfrak{g} \), in particular \( \mathfrak{g} = \mathfrak{sl}(n) \). One may conjecture that the harmonic analysis on dynamical quantum analogues (for the rational \( R \)-matrices) of compact symmetric spaces may shed light on this matter. This is the motivation for the discussion in §6 in which we consider an application of the action to the notion of singular and spherical vectors in corepresentations for dynamical quantum groups that are equipped with a pairing. The general definitions have to be worked out for explicit examples, and we intend to do so in a future paper.

For compact quantum groups there is a natural analogue of the Haar functional, and there is an analogue of Schur’s orthogonality relations. For the dynamical quantum group associated to the rational \( R \)-matrix for \( \mathfrak{sl}(2) \) the Haar functional was introduced using the analogue of the Peter-Weyl theorem and the Clebsch-Gordan decomposition, see [17, §7]. This method is not generally applicable to dynamical quantum groups, and we expect that the pairing and the actions defined in this paper can give rise to an alternative definition. It remains open if the elliptic beta integral, see e.g. [13, Exerc. 11.29], can be given a dynamical quantum group theoretic interpretation in this way. See Böhm and Szlachányi [2] for integrals in the context of Takeuchi’s \( x_R \)-bialgebras with an antipode.
The organization of the paper is as follows. In section 2 we recall the algebraic definitions for dynamical quantum groups such as \( h \)-algebras, \( h \)-Hopf algebroids, etc, and we introduce notation. Moreover, we recall the FRST-construction in this setting and we consider the example of the elliptic \( U(2) \) dynamical quantum group. In section 3 we recall the definition of a pairing between two \( h \)-Hopf algebroids, and we extend this to include a \( * \)-structure and we work out the details for the main example. In section 4 we introduce actions of one \( h \)-Hopf algebroid on another one in case there exists a pairing. In section 5 we apply this to representations and corepresentations, and work out the details for the main example. In particular, we calculate the pairing of two matrix elements of irreducible corepresentations in terms of elliptic hypergeometric series. In section 6 we propose definitions of singular and spherical vectors, which we intend to apply in more elaborate examples such as analogues of dynamical symmetric spaces in future work.

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2. Preliminaries on dynamical quantum groups

The algebraic notion for dynamical quantum groups is in terms of \( h \)-Hopf algebroids. We recall these notions in §2.1. Note that in case the vector space \( h \) involved equals the trivial space \( \{0\} \), we are back in the case of Hopf algebras. In §2.2 we recall the basic FRST-construction for \( h \)-bialgebroids, and we discuss the \( h \)-Hopf algebroid extension for the explicit case of the elliptic \( U(2) \) dynamical quantum group. In §6 we also consider \( h \)-subalgebroids etc. for different vector spaces \( h \).

2.1. Algebraic notions. Dynamical quantum groups have been introduced by Etingof and Varchenko [9], see also the lecture notes by Etingof and Schiffmann [8]. For \( h \)-Hopf algebroids we follow the slightly different definition as in [17], see also [26], and we discuss shortly other approaches at the end of this subsection in Remark 2.1.

We denote by \( h^* \) a finite-dimensional complex vector space. The notation is influenced by a natural construction where \( h \) occurs as the Cartan subalgebra of a semisimple Lie algebra, see [8]. By \( M_{h^*} \) we denote the field of meromorphic functions on \( h^* \), and the function identically equal to \( 1 \) on \( h^* \) is denoted by \( 1 \in M_{h^*} \).

A \( h \)-prealgebra \( \mathcal{A} \) is a complex vector space equipped with a decomposition \( \mathcal{A} = \bigoplus_{\alpha, \beta \in h^*} \mathcal{A}_{\alpha, \beta} \) and two left actions \( \mu^\Lambda_\alpha, \mu^\Lambda_\beta : M_{h^*} \to \text{End}_C(\mathcal{A}) \), the left and right moment map, which preserve the decomposition and such that \( \mu^\Lambda_\alpha(f) \mu^\Lambda_\beta(g) = \mu^\Lambda_\beta(g) \mu^\Lambda_\alpha(f) \) for all \( f, g \in M_{h^*} \). A \( h \)-prealgebra homomorphism is a \( C \)-linear map preserving the moment maps and the decomposition.

A \( h \)-algebra \( \mathcal{A} \) is a \( h \)-prealgebra, which is also a unital associative algebra such that the decomposition is a bigrading for the algebra, i.e. \( m^\Lambda : \mathcal{A}_{\alpha, \beta} \times \mathcal{A}_{\gamma, \delta} \to \mathcal{A}_{\alpha + \gamma, \beta + \delta} \) where \( m^\Lambda \) denotes the multiplication of \( \mathcal{A} \), and such that the left and right moment map \( \mu^\Lambda_\alpha, \mu^\Lambda_\beta : M_{h^*} \to \mathcal{A}_{00} \) given by \( \mu^\Lambda_\alpha(f) = \mu^\Lambda_\alpha(f) 1_{\mathcal{A}}, \mu^\Lambda_\beta(f) = \mu^\Lambda_\beta(f) 1_{\mathcal{A}} \) are algebra embeddings, and such that the commutation relations

\[
\mu^\Lambda_\alpha(f) a = a \mu^\Lambda_\alpha(T_\alpha f), \quad \mu^\Lambda_\beta(f) a = a \mu^\Lambda_\beta(T_\beta f), \quad \forall a \in \mathcal{A}_{\alpha, \beta}, \forall f \in M_{h^*},
\]

hold, where \( T_\alpha \) is the automorphism of \( M_{h^*} \) defined by \( (T_\alpha f)(\lambda) = f(\lambda + \alpha) \). Note that in case \( \mathcal{A} = \mathcal{A}_{00} \) this is just an extension of scalars to \( M_{h^*} \). A \( h \)-algebra homomorphism \( \phi : \mathcal{A} \to \mathcal{B} \) of \( h \)-algebras is an algebra homomorphism which preserves the moment maps and the bigrading, i.e. \( \phi(\mu^\Lambda_\alpha(f)) = \mu^\Lambda_\alpha(f), \phi(\mu^\Lambda_\beta(f)) = \mu^\Lambda_\beta(f) \) and \( \phi(\mathcal{A}_{\alpha, \beta}) \subseteq \mathcal{B}_{\alpha, \beta} \).

An important example is the algebra \( D_{h^*} \) of finite difference operators \( \sum f_i T_{\alpha_i} \), on the space \( M_{h^*} \), of meromorphic functions on \( h^* \). Then \( D_{h^*} \) is a \( h \)-algebra. The grading is given by \( f T_{-\alpha} \in (D_{h^*})_{\alpha \alpha} \), so \( (D_{h^*})_{\alpha \beta} = \{0\} \) for \( \alpha \neq \beta \). The left and right moment map are equal to the natural embedding...
by viewing an element $f \in M_{b^r}$ as a multiplication operator by $f$. For later use we also note the identity

$$RS1 = (R1)(T_{-\beta}S1), \quad R \in (D_{b^r})_{ao}, S \in D_{b^r}. \quad (2.2)$$

Note that for a $h$-algebra $A$ we obtain the $h$-algebra $A^{lr}$ by interchanging the left and right moment maps. So $(A^{lr})_{ao} = A_{ao}, \mu^{lr}_f = \mu^A_f, \mu^{lr}_r = \mu^A_r$. Also, the opposite algebra $A^{opp}$, i.e. $m^{A^{opp}} = m^A \circ P$ with $P : A \times A \to A \times A, (a, b) \mapsto (b, a)$ the natural flip operator, is again a $h$-algebra with $\mu^{A^{opp}}_l = \mu^A_r, \mu^{A^{opp}}_r = \mu^A_l; (A^{opp})_{ao} = A_{-ao}$. The (matrix) tensor product $A \otimes B$ of two $h$-algebras $A$ and $B$ is a $h$-algebra with the following definitions of the bigrading, moment maps and multiplication;

$$\left( (A \otimes B)_{ao} \right) = \bigoplus_{\gamma} (A_{\alpha\gamma} \otimes M_{b^r}, B_{\gamma \beta}), \quad (A \otimes B)_o = A_o \otimes B_o, \quad (A \otimes B)_r = A_r \otimes B_r; \quad (2.3a)$$

$$\mu^A_{\alpha}(f) = \mu^A_{\alpha}(f) \otimes 1, \quad \mu^B_{\beta}(f) = 1 \otimes \mu^B_{\beta}(f), \quad (2.3b)$$

$$(a \otimes b)(c \otimes d) = (ac) \otimes (bd), \quad (2.3c)$$

where $\otimes_{M_{b^r}}$ denotes the tensor product modulo the relations

$$\mu^A_r(f)a \otimes b = a \otimes \mu^B_r(f)b, \quad a \in A, \ b \in B, \ f \in M_{b^r}. \quad (2.4)$$

It is straightforward to check that $A \otimes B$ is a $h$-algebra. Also note that we can define the (matrix) tensor product $A \otimes B$ of two $h$-prealgebras $A, B$ in the same way but only requiring $(2.3a), (2.3b)$. It can be checked that in this case $A \otimes B$ gives again a $h$-prealgebra.

For later use we note that, as $h$-prealgebras, $A \otimes D_{b^r} \cong A \cong D_{b^r} \otimes A$ by $a = a \otimes T_{-\beta} = T_{-\beta} \otimes a$ for $a \in A_{ao}$ and using (2.4) this implies

$$\mu^A_r(f)a = a \otimes fT_{-\beta}, \quad \mu^B_{\alpha}(f)a = fT_{-\alpha} \otimes a, \quad a \in A_{ao}. \quad (2.5)$$

This identification holds in particular for $h$-algebras, and for $A = D_{b^r}$ this gives the identification $D_{b^r} \otimes D_{b^r} \cong D_{b^r}, \; \mu^{D_{b^r}}(f) = fT_{-\alpha} \otimes \mu^{D_{b^r}}(f)$.

A $h$-coalgebroid is a $h$-prealgebra $A$ with two $h$-prealgebra homomorphisms $\Delta^A : A \to A \otimes A$, the comultiplication, $\varepsilon^A : A \to D_{b^*}$, the counit, satisfying the coassociativity condition $(\Delta^A \otimes \Id) \circ \Delta^A = (\Id \otimes \Delta^A) \circ \Delta^A$ and the counit condition $(\varepsilon^A \otimes \Id) \circ \Delta^A = \Id = (\Id \otimes \varepsilon^A) \circ \Delta^A$ using the identification (2.5).

We use Sweedler’s notation for the comultiplication, i.e. $\Delta^A(a) = \sum_{(a)} a(1) \otimes a(2)$ where the decomposition on the right hand side is with respect to the bigrading for a homogeneous element, so $a \in A_{ao}, \; a(1) \in A_{ao}, \; a(2) \in A_{ao}$. Then the condition for the counit can be rewritten as $\varepsilon^A(a(1) \otimes a(2)) = a = \sum_{(a)} \mu^A_r(\varepsilon^A(a))a(1)$. The counit is compatible with $A \otimes D_{b^r} \cong A \cong D_{b^r} \otimes A$ by $fT_{-\alpha} \otimes gT_{-\beta} = (fg)T_{-\alpha}$.

Note that the maps $\varepsilon^A \otimes \varepsilon^{D_{b^*}} : A \otimes D_{b^r} \to D_{b^r} \otimes D_{b^r}$ and $\varepsilon^{D_{b^*}} \otimes \varepsilon^A : D_{b^r} \otimes A \to D_{b^r} \otimes D_{b^r}$ are well-defined. So the counit axiom can be rewritten as $\sum_{(a)} \varepsilon^A(a(1))a(2) \cong a \cong \sum_{(a)} a(1) \otimes \varepsilon^A(a(2))$.

A $h$-bialgebroid is a $h$-algebra $A$ which is also a $h$-coalgebroid and such that the comultiplication $\Delta^A$ and the counit $\varepsilon^A$ are $h$-algebra homomorphisms. A $h$-bialgebroid homomorphism $\phi : A \to B$ is a $h$-algebra homomorphism preserving the comultiplication and counit, i.e. $(\phi \otimes \phi) \circ \Delta^A = \Delta^B \circ \phi$ and $\varepsilon^B \circ \phi = \varepsilon^A$. Note that $\phi \otimes \phi : A \otimes A \to B \otimes B$ is a well-defined operator.

We can make $D_{b^r}$ into a $h$-bialgebroid by setting the comultiplication $\Delta^{D_{b^*}} : D_{b^r} \to D_{b^r} \otimes D_{b^r}$ to be the canonical isomorphism, and $\varepsilon^{D_{b^*}}$ to be the identity.
A \( \h \)-Hopf algebroid is a \( \h \)-bialgebroid \( \mathcal{A} \) equipped with a \( \mathbb{C} \)-linear map, the antipode, \( S^A : \mathcal{A} \to \mathcal{A} \) satisfying \( S^A(a\eta^A(f)a) = S^A(a)\eta^A(f)S^A(a) \) for all \( a \in \mathcal{A} \) and \( f \in M_h \), and

\[
\begin{align*}
  &m^A \circ (\text{Id} \otimes S^A) \circ \Delta^A(a) = \eta^A(a)(1), &a \in \mathcal{A}, \\
  &m^A \circ (S^A \otimes \text{Id}) \circ \Delta^A(a) = \eta^A(T_a \varepsilon^A(a)(1)), &a \in \mathcal{A}_{\alpha, \beta}.
\end{align*}
\]  

(2.6)

This definition follows [17, §2], and it differs slightly from [9], see also [8]. It is straightforward to check that \( m^A \circ (S^A \otimes \text{Id}) \) in (2.6) is well-defined on \( \mathcal{A} \otimes \mathcal{A} \), and for \( m^A \circ (\text{Id} \otimes S^A) \) we note that for \( a \in \mathcal{A}_{\alpha, \gamma}, b \in \mathcal{A}_{\gamma, \beta}, f \in M_h \), we have \( m^A \circ (\text{Id} \otimes S^A)((\eta^A(f)a) \otimes b) = \eta^A(f)(S^A(b)\eta^A(f)) = (S^A(\eta^A(f)b) = m^A \circ (\text{Id} \otimes S^A)(a \otimes (\eta^A(f)b)) \) using \( S^A(b) \in \mathcal{A}_{\alpha, \gamma}, \mathcal{A}_{\beta, \gamma} \subseteq \mathcal{A}_{\alpha, \beta, 0} \) by (iii) below. With this definition it follows that antipode satisfies (i) \( S^A(ab) = S^A(b)S^A(a) \), (ii) \( \Delta^A \circ S^A = P \circ (S^A \otimes S^A) \circ \Delta^A \), (iii) \( S^A(\mathcal{A}_{\alpha, \beta}) \subseteq \mathcal{A}_{-\alpha, -\beta} \), (iv) \( S^A(\mu^A(f)) = S^A(\eta^A(f)) = \eta^A(f), \) \( S^A(1) = 1, \) \( \varepsilon^A \circ S^A = S^{D_h^*} \circ \varepsilon \) where \( P \) denotes the flip \( P(a \otimes b) = b \otimes a \), and \( S^{D_h^*} : D_h^* \to D_h^* \), \( fT_a \to T_{-\alpha} \circ f = T_{-\alpha} \circ (fT_a) \circ T_{-\alpha} \), is an algebra anti-isomorphism. See [17, Prop. 2.2].

With the antipode as defined in the previous paragraph the \( \h \)-bialgebroid \( D_h^* \) is a \( \h \)-Hopf algebroid.

The antipode on a \( \h \)-Hopf algebroid is compatible with \( \mathcal{A} \otimes D_h^* \cong \mathcal{A} \cong D_h^* \otimes \mathcal{A} \) by \( S(a \otimes fT_{-\beta}) = S^{D_h^*}(fT_{-\beta}) \otimes S^A(a) \) and \( S(fT_{-\alpha} \otimes a) = S^A(a) \otimes S^{D_h^*}(fT_{-\alpha}) \) using the notation of (2.5). Note that these maps are well-defined.

Note that for a \( \h \)-bialgebroid \( \mathcal{A} \) the opposite \( \h \)-algebra \( \mathcal{A}^{opp} \) is again a \( \h \)-bialgebroid with \( \Delta^{opp} = \Delta^A, \) \( \varepsilon^{opp} = S^{D_h^*} \circ \varepsilon^A. \) Similarly, we can construct the co-opposite \( \h \)-bialgebroid \( \mathcal{A}^{cop} \) which has the same algebra structure, but with interchanged moment maps \( \mu_{opp} = \mu^A, \) \( \mu_{cop} = \mu^A, \) and \( (\mathcal{A}^{cop})_{\alpha, \beta} = \mathcal{A}_{-\alpha, -\beta}. \) (So as an \( \h \)-algebra we have \( \mathcal{A}^{opp} = \mathcal{A}^{fr}. \)) Moreover, the comultiplication is defined by \( \Delta^{cop} = P \circ \Delta^A, \) and the counit \( \varepsilon^{cop} = \varepsilon^A. \) Furthermore, if \( \mathcal{A} \) is a \( \h \)-Hopf algebroid with invertible antipode, then \( \mathcal{A}^{opp} \) and \( \mathcal{A}^{cop} \) are \( \h \)-Hopf algebroids with \( S^{opp} = (S^A)^{-1} \) and \( S^{cop} = (S^A)^{-1}. \)

Assume that \( \h \) is equipped with a conjugation \( \lambda \mapsto \bar{\lambda}, \) or equivalently, a real form. Then this defines an operator on \( M_h \) by \( f(\bar{\lambda}) = \overline{f(\lambda)}. \) A *-operator on a \( \h \)-algebra \( \mathcal{A} \) is a \( \mathbb{C} \)-antilinear antimultiplicative involutive map \( a \mapsto a^*, \) such that \( \mu^A(f)* = \eta^A(f), \) \( \eta^A(f)* = \mu^A(f) \). This implies that \( (\mathcal{A}_{\alpha, \beta})^* = \mathcal{A}_{-\alpha, -\beta}. \) In this case we see that \( D_h^* \) has a *-structure given by \( (fT_a)^* = (T_{-\alpha}f)T_{-\alpha} = T_{-\alpha} \circ \bar{f}, \) viewing \( f \in D_h^* \) as the operator of multiplication by \( f. \) Note that the antipode and the *-operator in \( D_h^* \) commute, which is not true for a general \( \h \)-Hopf *-algebroid. If \( \mathcal{A} \) is a \( \h \)-bialgebroid we require the *-structure to satisfy \( (a \otimes b)^* = a^* \otimes (bT_{-\beta})^* \) and \( (fT_{-\alpha} \otimes a)^* = (fT_{-\alpha}a)^* \). Note that these maps are well-defined.

For \( \mathcal{A} \) a \( \h \)-Hopf *-algebroid with invertible antipode, \( \mathcal{A}^{opp} \) and \( \mathcal{A}^{cop} \) are \( \h \)-Hopf *-algebroids for the same *-structure.

**Remark 2.1.** Related structures motivated by the quantisation of Poisson-Lie groupoids have appeared before in the papers by Lu [21] and Xu [31], see also the discussion and motivation in Etingof and Schiffmann [8]. It has been proved by Brzeziński and Militaru [3] that on the bialgebroid level these structures are equivalent to Takeuchi’s [30] notion of \( \times \mathbb{R} \)-bialgebras. Since Etingof and Nikshych [7] have proved that weak Hopf algebras fit into the framework of Lu [21], this also includes weak bialgebras.

We can also view a \( \h \)-bialgebroid as a \( \times \mathbb{R} \)-bialgebra equipped with a grading. Indeed, using the notation as in Schauenburg [27, §2], we see that \( M_h \) and \( D_h^* \) play the roles of \( R \) and \( \text{End}(R) \), where
the action of $R$ is given by the left moment map and the action of $\bar{R} = R^{opp}(= R)$ is given by the right moment map. Moreover, the tensor product (2.4) corresponds to the notation $\int_{\mathcal{F}} \mathcal{A} \otimes \mathcal{B}$ and the weight space requirement (2.3a) in the tensor product corresponds to $\int \mathcal{A}_r \otimes \mathcal{B}_r$, so that Takeuchi’s notion $\mathcal{A} \times_R \mathcal{B}$ corresponds to the (matrix) tensor product. However, the antipode in the context of $\times_R$-bialgebras is much more involved, and it seems that this has not been settled yet, see Schauenburg [27, §3] for a discussion based on categorical concepts and Böhm and Szlachanyi [2] for a somewhat more restrictive but algebraic definition. See also the references in the papers [2], [27] for more information. Note that the antipode in $\mathfrak{h}$-Hopf algebroid is different, since it makes essential use of the bigrading.

2.2. FRST-construction and the elliptic $U(2)$ dynamical quantum group. We recall the FRST-construction for the case of an $R$-matrix with dynamical and spectral parameter, see Etingof and Varchenko [9]. The FRST-construction associates to a $R$-matrix a $\mathfrak{h}$-bialgebroid, which in many cases can be extended to a $\mathfrak{h}$-Hopf algebroid. In general, the $R$-matrix does not need to satisfy the quantum dynamical Yang-Baxter equation (2.7), but for the most interesting examples this is the case. We give the details for the elliptic $R$-matrix for $\mathfrak{sl}(2)$, and we refer to Felder and Varchenko [11] and [16] for an even more explicit description of the corresponding $\mathfrak{h}$-Hopf algebroid.

We start by recalling the definition of the quantum dynamical Yang-Baxter equation. Let $\mathfrak{h}$ be a finite dimensional complex vector space viewed as a commutative Lie algebra as before, and let $V = \bigoplus_{\alpha \in \mathfrak{h}^*} V_\alpha$ be a diagonalisable $\mathfrak{h}$-module. A meromorphic function $R: \mathfrak{h}^* \times \mathbb{C} \to \text{End}(V \otimes V)$ is a $R$-matrix if it is $\mathfrak{h}$-invariant, i.e. commutes with the $\mathfrak{h}$-action on $V \otimes V$, and if it satisfies the quantum dynamical Yang-Baxter equation (with spectral parameter);

$$R^{12}(\lambda - h^{(1)}, z_{12})R^{13}(\lambda, z_{13})R^{23}(\lambda - h^{(2)}, z_{23}) = R^{23}(\lambda, z_{23})R^{13}(\lambda - h^{(1)}, z_{13})R^{12}(\lambda, z_{12}).$$

(2.7)

Here $z_{ij} = z_i/z_j$, and e.g. $R^{12}(\lambda - h^{(3)}, z_{12})(u \otimes v \otimes w) = (R(\lambda - \mu, z_{12})(u \otimes v))(\otimes w)$ where $w \in V_\mu$.

Let $V = \bigoplus_{\alpha \in \mathfrak{h}^*} V_\alpha$ be a finite-dimensional diagonalizable $\mathfrak{h}$-module and $R: \mathfrak{h}^* \times \mathbb{C} \to \text{End}_\mathbb{C}(V \otimes V)$ a meromorphic function, so $R(\lambda, z)$ commutes with the $\mathfrak{h}$-action on $V \otimes V$. Let $\{e_x\}_{x \in X}$ be a homogeneous basis of $V$, where $X$ is an index set. Write $R_{xy}^{ab}(\lambda, z)$ for the matrix elements of $R$,

$$R(\lambda, z)(e_a \otimes e_b) = \sum_{x,y \in X} R_{xy}^{ab}(\lambda, z)e_x \otimes e_y,$$

and define $\omega: X \to \mathfrak{h}^*$ by $e_x \in V_{\omega(x)}$. Let $\mathcal{A}_R$ be the unital complex associative algebra generated by the elements $\{L_{xy}(z)\}_{x,y \in X}$, with $z \in \mathbb{C}$, together with two copies of $\mathbb{C}_{\mathfrak{h}^*}$, embedded as subalgebras. The elements of these two copies are denoted by $\mu_{\ell} A_R(f) = f(\lambda)$ and $\mu_{r} A_R(f) = f(\mu)$, respectively. The defining relations of $\mathcal{A}_R$ are $f(\lambda) g(\mu) = g(\mu) f(\lambda)$, $f(\lambda) L_{xy}(z) = L_{xy}(z) f(\lambda + \omega(x))$, $f(\mu) L_{xy}(z) = L_{xy}(z) f(\mu + \omega(y))$, (2.8)

for all $f, \, g \in M_{\mathfrak{h}^*}$, together with the RLL-relations

$$\sum_{x,y \in X} R_{xy}^{ac}(\lambda, z_1/z_2) L_{zb}(z_1) L_{yd}(z_2) = \sum_{x,y \in X} R_{xy}^{bd}(\mu, z_1/z_2) L_{cy}(z_2) L_{ax}(z_1),$$

(2.9)

for all $z_1, z_2 \in \mathbb{C}$ and $a, b, c, d \in X$.

The bigrading on $\mathcal{A}_R$ is defined by $L_{xy}(z) \in \mathcal{A}_{\omega(x)\omega(y)}$. The $\mathfrak{h}$-invariance of $R$ ensures that the bigrading is compatible with the RLL-relations (2.9). The counit and comultiplication defined by

$$\varepsilon_{\mathcal{A}_R}(L_{ab}(z)) = \delta_{ab} T_{-\omega(a)}, \quad \Delta_{\mathcal{A}_R}(L_{ab}(z)) = \sum_{x \in X} L_{ax}(z) \otimes L_{zb}(z)$$

(2.10)

make $\mathcal{A}_R$ into a $\mathfrak{h}$-bialgebroid, see [9].
Example 2.2. Take $h \cong h^* \cong \mathbb{C}$ and let $V$ the two-dimensional $h$-module $V = \mathbb{C}e_1 \oplus \mathbb{C}e_{-1}$. In the basis $e_1 \otimes e_1, e_1 \otimes e_{-1}, e_{-1} \otimes e_1, e_{-1} \otimes e_{-1}$ the $R$-matrix is given by

$$R(\lambda, z) = R(\lambda, z, p, q) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a(\lambda, z) & b(\lambda, z) & 0 \\ 0 & c(\lambda, z) & d(\lambda, z) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where we assume $0 < q < 1$ and

$$a(\lambda, z) = \frac{\theta(z, q^{2(\lambda+2)})}{\theta(q^2 z, q^{2(\lambda+1)})}, \quad b(\lambda, z) = \frac{\theta(q^2, q^{-2(\lambda+1)} z)}{\theta(q^2 z, q^{-2(\lambda+1)})},$$
$$c(\lambda, z) = \frac{\theta(q^2, q^{2(\lambda+1)} z)}{\theta(q^2 z, q^{2(\lambda+1)})}, \quad d(\lambda, z) = \frac{\theta(z, q^{-2\lambda})}{\theta(q^2 z, q^{-2(\lambda+1)})}.$$

Here the theta functions are normalised theta functions defined by

$$\theta(z) = (z, p/z; p)_\infty, \quad (z, p)_{k} = \prod_{i=1}^{k} (1 - ap^{i}), \quad (z, p; p) = \lim_{k \to \infty} (a; p)_{k}, \quad \prod_{i=1}^{r} (a_i; p)_{k}.$$  

For later use we note that theta functions satisfy $\theta(pz) = \theta(z^{-1}) = -z^{-1}\theta(z)$ and the following addition formula

$$\theta(xy, x/y, zw, z/w) = \theta(xw, x/w, zy, z/y) + (z/y)\theta(xz, x/z, yw, y/w).$$

The $R$-matrix defined by (2.11) satisfies the quantum dynamical Yang-Baxter equation (2.7), see e.g. [11], [14], [16], and references given there.

The four $L$-generators are denoted by $\alpha(z) = L_{1,1}(z), \beta(z) = L_{1,-1}(z), \gamma(z) = L_{-1,1}(z)$ and $\delta(z) = L_{-1,-1}(z)$. We do not give the relations for the generators arising from (2.9) explicitly, but we refer to [11], [16]. The corresponding $h$-bialgebroid contains a group-like central element, $\det(z) = \mu_{\mu}(F)\mu_{\mu}(F_1) [\alpha(z)\beta(q^2 z) - \gamma(z)\beta(q^2 z)],$ with $F(\lambda) = q^\lambda \theta(q^{-2(\lambda+1)})$, and adjoining the inverse, denoted by $\det^{-1}(z)$, gives a $h$-Hopf algebroid structure with antipode given by

$$S(\alpha(z)) = \mu_{\mu}(F)\mu_{\mu}(F_1)\det^{-1}(q^{-2}z)\beta(q^{-2}z), \quad S(\beta(z)) = -\mu_{\mu}(F)\mu_{\mu}(F_1)\det^{-1}(q^{-2}z)\beta(q^{-2}z),$$
$$S(\gamma(z)) = -\mu_{\mu}(F)\mu_{\mu}(F_1)\det^{-1}(q^{-2}z)\gamma(q^{-2}z), \quad S(\delta(z)) = \mu_{\mu}(F)\mu_{\mu}(F_1)\det^{-1}(q^{-2}z)\alpha(q^{-2}z),$$
$$S(\det^{-1}(z)) = \det(z).$$

We denote the corresponding $h$-Hopf algebroid by $\mathcal{E} = \bigoplus_{k \in \mathbb{Z}} \mathcal{E}_{kl}$. It can be made into a $h$-Hopf $*$-algebroid by defining $\det^{-1}(z)^* = \det^{-1}(q^{-2}/z), \alpha(z)^* = \delta(1/z), \beta(z)^* = -\gamma(1/z), \gamma(z)^* = -\beta(1/z), \delta(z)^* = \alpha(1/z)$, see [16]. We come back to this example throughout this paper.
Example 2.3. With the same convention for $\mathfrak{h}, \mathfrak{h}^*, V, V \otimes V$ as in Example 2.2 the $R$-matrix

$$
R(\lambda) = R(\lambda, q) = 
\begin{pmatrix}
q & 0 & 0 & 0 \\
0 & 1 & \frac{q^{-1} - q}{q^{2(\lambda + 1)} - 1} & 0 \\
0 & \frac{q^{-1} - q}{q^{-2(\lambda + 1)} - 1} & (q^{2(\lambda + 1)} - q^2)(q^{2(\lambda + 1)} - q^{-2}) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
$$

(2.17)

satisfies the quantum dynamical Yang-Baxter equation (2.7) and is also $\mathfrak{h}$-invariant. The corresponding $\mathfrak{h}$-algebra arising from the FRST-construction can also be made into a $\mathfrak{h}$-Hopf algebroid. This example can be found in [8, §2.2], and the corresponding harmonic analysis has been studied in [17]. It has been proved by Stokman [29] that this case can be obtained from the usual quantum $SL(2)$ group by use of a vertex IRF transform based on the twisted coboundary element introduced by Babelon, Bernard and Billey [1], see also [4], [25] for more information.

The $R$-matrix in (2.17) can be obtained by a suitable limit transition from the $R$-matrix (2.11). To see this we recall that the $R$-matrix in (2.11) is gauge equivalent to the elliptic $R$-matrix in [14, §3] by an explicit closed multiplicative 2-form. In the notation of [9, §§1-2], [8, §6.2] we only have to specify the multiplicative 2-form $\phi_{12}(\lambda) = \frac{1}{q(q-2(\lambda + 2), pq^2(\lambda + 1), p)\infty}$. The closedness follows from $\phi_{12}(\lambda) = \frac{f_2(\lambda)f_1(\lambda + 1)}{f_1(\lambda)f_2(\lambda - 1)}$ with $f_1(\lambda) = 1$, $f_2(\lambda) = q^{-\lambda}(q^{-2(\lambda + 2), pq^2(\lambda + 1); p)\infty}$. The limit transition from the $R$-matrix of [14] by taking $\lim_{p \to 0}$, $\lim_{z \to 0}$ gives $q$ times the $R$-matrix in (2.17) with $q$ replaced by $q^{-1}$. Then the map $L_{ab}(z) \mapsto \frac{\mu_r(f_b)}{\mu_l(f_a)} L_{ab}(z)$ defines a $\mathfrak{h}$-bialgebroid-isomorphism of the $\mathfrak{h}$-bialgebroid from the FRST-construction for the $R$-matrix (2.11) to the $\mathfrak{h}$-bialgebroid from the FRST-construction for the $R$-matrix in [14, §3]. Now the limit transition of the elliptic $R$-matrix in [14, §3] to (2.17) gives a corresponding formal limit transition at the level of the corresponding $\mathfrak{h}$-bialgebroids and $\mathfrak{h}$-Hopf algebroids. The harmonic analysis on the corresponding $\mathfrak{h}$-bialgebroid and $\mathfrak{h}$-Hopf algebroid is studied in [17], and the pairing for this case is studied extensively by Rosengren [26].

3. Pairings for dynamical quantum groups

We consider in this section a natural pairing between dynamical quantum group. For certain weak Hopf algebras this has been considered by Etingof and Nikshych [7, §5]. The pairing is studied by Rosengren [26] in the context of duals of $\mathfrak{h}$-bialgebroids and $\mathfrak{h}$-Hopf algebroids. Rosengren [26] shows in particular that the quantized universal enveloping algebra $U_q(\mathfrak{sl}(2))$ is contained in the $\mathfrak{h}$-Hopf algebroid associated to the $R$-matrix (2.17). We study the pairing for $\mathfrak{h}$-Hopf $*$-algebras and recall the explicit pairings for the $\mathfrak{h}$-bialgebroids arising from the FRST-construction.

The following definition has been introduced by Rosengren [26, §3]. This definition is different from the pairing for $\times_R$-bialgebras as introduced by Schauenburg [27, §5], cf. Remark 2.1.
Definition 3.1. A pairing for \( \mathcal{U} \)-bialgebroids \( \mathcal{U} \) and \( \mathcal{A} \) is a \( \mathbb{C} \)-bilinear map \( \langle \cdot, \cdot \rangle: \mathcal{U} \times \mathcal{A} \to D_{h^*} \) satisfying

\[
\begin{align*}
(\mu^\mathcal{U}_t(f), a) &= \langle X, \mu^\mathcal{A}_t(f) a \rangle = \langle X, a \mu^\mathcal{A}_t(f) \rangle = \langle X, a \rangle \circ f, \\
(X, ab) &= \sum_{(X)} \langle X(1), a \rangle T_\rho(Y(2), b), \quad \Delta^\mathcal{A}(a) = \sum_{(a)} a(1) \otimes a(2), \quad a(1) \in \mathcal{A}_{\gamma^\mathcal{U}}, \\
(X, 1) &= \varepsilon^\mathcal{U}(X), \quad \langle 1, a \rangle = \varepsilon^\mathcal{A}(a). 
\end{align*}
\]

If moreover, \( \mathcal{U} \) and \( \mathcal{A} \) are \( \mathfrak{h} \)-Hopf algebroids, then we require

\[
\langle SU(X), a \rangle = S_{D_{h^*}}(\langle X, SA(a) \rangle).
\]

Remark 3.2. (i) Note (3.1a) implies that \( \langle X, a \rangle = 0 \) whenever \( X \in \mathcal{U}_{\alpha, \beta}, a \in \mathcal{A}_{\gamma^\mathcal{U}} \) with \( \alpha + \delta \neq \beta + \gamma \).

(ii) In (3.1b) and (3.1c) we consider \( f \) as a multiplication operator in \( D_{h^*} \).

(iii) Requiring only (3.1a), (3.1b), and (3.1c) gives the definition of a pairing between \( \mathfrak{h} \)-algebras \( \mathcal{U} \) and \( \mathcal{A} \). Similarly, we can define a pairing between a \( \mathfrak{h} \)-algebra \( \mathcal{U} \) and \( \mathfrak{h} \)-coalgebroid \( \mathcal{A} \) by requiring (3.1a), (3.1b), (3.1c), with the convention that in the \( \mathfrak{h} \)-coalgebroid \( \mu^\mathcal{A}_t(f) = \mu_r(T_\delta f)a \) for \( a \in \mathcal{A}_{\gamma^\mathcal{U}}, \)

(iv) Note that (3.2) is self-dual, since the antipode of \( D_{h^*} \) is involutive. If a pairing exists for \( \mathcal{U} \) and \( \mathcal{A} \) and assuming the pairing is non-degenerate, and \( \mathcal{A} \) is \( \mathfrak{h} \)-Hopf algebroid we can use (3.2) to equip \( \mathcal{U} \) with an antipode. In the case \( \mathcal{U} \) and \( \mathcal{A} \) are \( \mathfrak{h} \)-Hopf algebroids, which are \( \mathfrak{h} \)-bialgebroids with a pairing, it follows from (3.1d), (3.1e) and the last equality of (3.1f).

For applications to dynamical quantum groups we are particularly interested in a specific type of pairings induced by \( R \)-matrices as studied previously in \([7],[26]\). For this we need the following definition, see \([26, \text{Def. 3.16}]\), and recall the definition of the cooposite \( \mathfrak{h} \)-bialgebroid as in \( \S 2.1 \).

Definition 3.3. A cobraiding on a \( \mathfrak{h} \)-bialgebroid \( \mathcal{A} \) is a pairing \( \langle \cdot, \cdot \rangle: \mathcal{A}_{\gamma^\mathcal{U}} \times \mathcal{A} \to D_{h^*} \) for \( \mathfrak{h} \)-bialgebroids satisfying

\[
\sum_{(a), (b)} \mu^\mathcal{A}_t((a(1), b(1))1) a(2) b(2) = \sum_{(a), (b)} \mu^\mathcal{A}_t((a(2), b(2))1) b(1) a(1),
\]

as an identity in \( \mathcal{A} \) and where \( \Delta^\mathcal{A}(a) = \sum_{(a)} a(1) \otimes a(2), \quad \Delta^\mathcal{A}(b) = \sum_{(b)} b(1) \otimes b(2). \)

Remark 3.4. (i) It suffices to check (3.3) for generators of the \( \mathfrak{h} \)-bialgebroid \( \mathcal{A} \), see \([26, \text{Lemma 3.17}]\). For this reason we need the cooposite \( \mathfrak{h} \)-algebra in the first leg of the pairing.

(ii) The relation (3.3) for \( a \) and \( b \) generators of a \( \mathfrak{h} \)-bialgebroid constructed by the FRST-construction as in \( \S 2.2 \) can be matched with the quadratic relations (2.9) using (2.10). Then the fact that this pairing defined on the generators in terms of the \( R \)-matrix by

\[
\langle L_{ij}(w), L_{kl}(z) \rangle = R^R_{ik}(\lambda, \frac{w}{z}) T_{-\omega(i)-\omega(k)}
\]

extends to a pairing on \( (\mathcal{A}_R)^{\gamma^\mathcal{U}} \times \mathcal{A}_R \) satisfying the conditions of Definition 3.1 is equivalent to the \( R \)-matrix being a solution to the quantum dynamical Yang-Baxter equation (2.7). Consequently, any solution of the quantum dynamical Yang-Baxter equation gives rise to a \( \mathfrak{h} \)-bialgebroid with a
cobraiding. See Rosengren [26, §3] for these results in case there is no spectral parameter, which can easily be adapted to include the case of an R-matrix with both dynamical parameters and a spectral parameter.

(iii) In general the cobraiding arising from an R-matrix is not non-degenerate. In case of a finite dimensional weak Hopf algebra the pairing is non-degenerate, see Etingof and Nikshych [7]. For the case of Example 2.3 the radical is non-trivial, and it has been determined explicitly by Rosengren [26, §5].

For later use we also need the relation between *-structures on h-Hopf algebroids and pairings. It turns out that we need this in two, closely related, variants, one of them well-suited for left corepresentations and the other well-suited for right corepresentations. We assume that one of the h-Hopf *-algebroids has a fixed *-operator, and for the other we use the notation * and † depending on the pairing. This definition has been communicated to us by Hjalmar Rosengren.

**Definition 3.5.** Let $\mathcal{U}, \mathcal{A}$ be h-Hopf algebroids with invertible antipodes being paired as h-Hopf algebroids. Assume that $\mathcal{A}$ is a h-Hopf *-algebroid. We say that $\mathcal{U}$ and $\mathcal{A}$ are paired as h-Hopf *-algebroids if moreover $\mathcal{U}$ is a h-Hopf *-algebroid and

$$\langle X^*, a \rangle = T_{-\gamma} \circ \langle (X, S^4 (a)^*) \rangle^* \circ T_{-\delta}, \quad \forall \ a \in \mathcal{A}_{\gamma\delta}, \ \forall \ X \in \mathcal{U},$$

or if $\mathcal{U}$ is a h-Hopf *-algebroid (with * denoted by †) and

$$\langle X^\dagger, a \rangle = T_{-\gamma} \circ \langle (X, S^4 (a^*)^*) \rangle^* \circ T_{-\delta}, \quad \forall \ a \in \mathcal{A}_{\gamma\delta}, \ \forall \ X \in \mathcal{U}. \quad (3.5)$$

It will be clear from the context and/or the notation in the sequel if we use (3.4) or (3.5) for paired h-Hopf *-algebroids. For pairings for Hopf algebras the standard choice is compatible with (3.4).

**Remark 3.6.** (i) It is not clear from (3.4) and (3.5) that * and † on $\mathcal{U}$ are compatible with the usual properties of a *-operator. This is the content of Proposition 3.7.

(ii) If $\mathcal{U}$ and $\mathcal{A}$ are paired as h-Hopf algebroids with invertible antipodes, and $\mathcal{A}$ is also a h-Hopf *-algebroid, it follows that $\langle (X^\dagger)^*, a \rangle = \langle X, S^4 (a) \rangle = \langle S^2 (X), a \rangle$, or the h-Hopf algebroid isomorphism $S^2 : \mathcal{U} \to \mathcal{U}$ interwines the two *-structures, $S^2 (X^\dagger) = X^*$. Note that this implies $S \circ \dagger = \cdot \circ S$. The choice for the *-structure on $\mathcal{U}$ is related to the notion of unitarisability for right or left corepresentations, see §5.

(iii) Assume that $\mathcal{U}$ and $\mathcal{A}$ are h-Hopf *-algebroids with invertible antipodes that are paired as in (3.4) or (3.5). Note that this definition is not obviously self-dual, but it is:

$$\langle X, a^* \rangle = T_{-\alpha} \circ \langle (S^U (X)^*, a) \rangle^* \circ T_{-\beta},$$

$$\langle X, a^* \rangle = T_{-\alpha} \circ \langle (S^U (X^\dagger)^*) \rangle^* \circ T_{-\beta}, \quad (3.6)$$

for $X \in \mathcal{U}_{\alpha\delta, \ a \in \mathcal{A}$ and assuming the pairing (3.4) for the first equation of (3.6) and the pairing (3.5) for the second equation of (3.6). We prove the first equation of (3.6), the second being proved more easily. The assumption that $\mathcal{A}$ and $\mathcal{U}$ have invertible antipodes implies $S \circ \cdot$ is an involution. For $X \in \mathcal{U}_{\alpha\beta, \ a \in \mathcal{A}_{\gamma\delta}$ we have $a^* \in \mathcal{A}_{-\gamma, -\delta}$, so that

$$\langle X, a^* \rangle = \langle (X^\dagger)^*, a^* \rangle = T_{-\gamma} \circ \langle (X^\dagger, S^4 (a^*)^*) \rangle^* \circ T_{-\delta} = T_{-\gamma} \circ \langle (X^\dagger, (S^4)^{-1} (a)) \rangle^* \circ T_{-\delta}$$

$$= T_{-\gamma} \circ \langle (S^D_{a^*} ((S^4)^{-1} (X^\dagger), a)) \rangle^* \circ T_{-\delta} = T_{-\gamma} \circ \langle (S^D_{a^*} ((S^U (X^\dagger)^*, a)) \rangle^* \circ T_{-\delta}$$

using (3.2) and $S^D_{a^*}$ being an involution as well. Write $\langle S^U (X)^*, a \rangle = f T_{-\beta, -\delta} = f T_{-\alpha, -\gamma}$ with $f = \langle S^U (X)^*, a \rangle 1$ for the moment, so

$$\langle X, a^* \rangle = T_{-\gamma} \circ f \circ T_{-\beta, -\delta} \circ T_{-\delta} = T_{-\alpha} \circ (f T_{-\alpha, -\gamma})^* \circ T_{-\beta} = T_{-\alpha} \circ \langle (S^U (X)^*, a) \rangle^* \circ T_{-\beta},$$

proving the first equality of (3.6).
Proposition 3.7. Assume that $\mathcal{U}$ and $\mathcal{A}$ are paired \(\mathfrak{h}\)-bialgebroids, and that $\mathcal{A}$ is a \(\mathfrak{h}\)-Hopf *-algebroid with invertible antipode. Define the pairing $(X^*, a)$ by (3.4), then for all $X, Y \in \mathcal{U}$, for all $a, b \in \mathcal{A}$, for all $c_1, c_2 \in \mathbb{C}$,

$$
\langle (c_1 X + c_2 Y)^*, a \rangle = c_1 \langle X^*, a \rangle + c_2 \langle Y^*, a \rangle, \quad (3.7a)
$$

$$
\langle (XY)^*, a \rangle = \langle Y^*X^*, a \rangle, \quad (3.7b)
$$

$$
\langle (X^*)^*, a \rangle = \langle X, a \rangle, \quad (3.7c)
$$

$$
\langle (\mu^A_{\mathfrak{h}}(f))^*, a \rangle = \langle \mu^A_{\mathfrak{h}}(f), a \rangle, \quad (3.7d)
$$

$$
\langle X^*, 1 \rangle = \langle \varepsilon_{\mathcal{U}}(X) \rangle^*, \quad (3.7e)
$$

$$
\langle X^*, ab \rangle = \sum_{(X)} \langle (X(1))^*, a_{(1)} \rangle T_{-\eta} \circ ((X(2))^*, b), \quad (3.7f)
$$

where we assume that $X \in \mathcal{U}_{\alpha, \beta}$ in (3.7f). The results in (3.7) remain valid upon replacing * by $\dagger$ in the left hand sides of all pairings and using (3.5).

Comparing with Definition 3.1, we see that we can look upon (3.7a), (3.7b) and (3.7c) as the weak formulation of the *-operator being a $\mathbb{C}$-antilinear antimultiplicative involution. Then (3.7d) is the weak formulation of being a *-operator on a \(\mathfrak{h}\)-algebra. Finally, (3.7e) is the weak formulation of $\varepsilon \circ * = * \circ \varepsilon$, and (3.7f) is the weak formulation of $(* \otimes *) \circ \Delta = \Delta \circ (*)$.

Proof. The $\mathbb{C}$-linearity of the pairing and (3.4) imply (3.7a). To prove (3.7b) assume $X \in \mathcal{U}_{\alpha, \beta}$, $Y \in \mathcal{U}_{\gamma, \tau}$ so that $XY \in \mathcal{U}_{\alpha + \sigma, \beta + \tau}$ and $a \in \mathcal{A}_{\gamma, \delta}$, so $\Delta^\mathcal{U}(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}$, $a_{(1)} \in \mathcal{A}_{\gamma, \eta}$, $a_{(2)} \in \mathcal{A}_{\eta, \delta}$. Using $(* \otimes *) \circ \Delta = \Delta \circ *$, $P \circ (S \otimes S) \circ \Delta = \Delta \circ S$ in a \(\mathfrak{h}\)-Hopf algebroid and (3.1d) we find

$$
\langle (XY)^*, a \rangle = T_{-\gamma} \circ ((X, S^A(a)^*)^*) \circ T_{-\delta}
$$

$$
= T_{-\gamma} \circ \left( \sum_{(a)} (X, S^A(a_{(2)})^*) T_{-\eta} (Y, S^A(a_{(1)})^*) \right)^* \circ T_{-\delta}
$$

$$
= \sum_{(a)} T_{-\gamma} \circ (Y, S^A(a_{(1)})^*)^* \circ T_{-\eta} \circ ((X, S^A(a_{(2)})^*)^* \circ T_{-\delta}
$$

$$
= \sum_{(a)} (Y^*, a_{(1)}) T_{-\eta} \circ T_{-\delta} \circ (X^*, a_{(2)}) = (Y^*X^*, a),
$$

since $S^A(a_{(2)})^* \in \mathcal{A}_{\delta, \eta}$. The proof of (3.7c) is an immediate consequence of the fact that $S^A \circ *$ is an involution in a \(\mathfrak{h}\)-Hopf algebroid with invertible antipode, see §2.1, and the definition of the *-operator in $D_{\mathfrak{h}}$. For (3.7d) we use for $a \in \mathcal{A}_{\gamma, \delta}$

$$
\langle (\mu^A_{\mathfrak{h}}(f))^*, a \rangle = T_{-\gamma} \circ ((\mu^A_{\mathfrak{h}}(f), S^A(a)^*)^*) \circ T_{-\delta} = T_{-\gamma} \circ (f \circ \varepsilon^A(S^A(a)^*)) \circ T_{-\delta}
$$

$$
= T_{-\gamma} \circ S^{D_{\mathfrak{h}}}(\varepsilon^A(a)) \circ f \circ T_{-\delta} = g \circ T_{-\gamma}.
$$

where we write $\varepsilon^A(a) = g T_{-\gamma}$. On the other hand $(\mu^A_{\mathfrak{h}}(f), a) = f \circ \varepsilon^A(a) = f \circ g T_{-\gamma}$. Since $\varepsilon^A(a) \in (D_{\mathfrak{h}})^*_{\gamma, \delta}$, both expressions are zero for $\gamma \neq \delta$, see Remark 3.2(i), and that for $\gamma = \delta$ they are equal. The statement for the right moment map in (3.7d) is proved similarly.
Finally, to prove (3.7f) we take $X \in \mathcal{U}_a$, so $\Delta^f(X) = \sum_{(X)} X_{(1)} \otimes X_{(2)}$, with $X_{(1)} \in \mathcal{U}_a$, $X_{(2)} \in \mathcal{U}_b$, and $a \in \mathcal{A}_a, b \in \mathcal{A}_b$, so that the left hand side of (3.7f) can be rewritten as

$$
\langle X^*, ab \rangle = T_{-\gamma-\sigma} \circ \left( \langle X, S^A(ab)^* \rangle^* \otimes T_{-\delta-\tau} = T_{-\gamma-\sigma} \circ \left( \langle X, S^A(a)^* S^A(b)^* \rangle^* \otimes T_{-\delta-\tau} 
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so that, using the explicit expressions (2.12) and elementary properties of the theta function,
\[ \langle \delta(w), \det(z) \rangle = \frac{1}{F(\lambda)} \left( d(\lambda, \frac{w}{z}) - b(\lambda, \frac{w}{z}) c(\lambda + 1, \frac{w}{q^{2}z}) \right) F(\lambda + 1) T_{1} \]
\[ = \frac{\theta_{1}(w, w, q^{2} \lambda, q^{2}(\lambda + 2)) - \theta(q^{2}, q^{-2}(\lambda + 2)) \theta(q^{2}, q^{2} \lambda, q^{2}(\lambda + 2))}{q \theta(q^{2}w^{2}, q^{-2}\lambda + 1, \frac{w}{z}, q^{2}(\lambda + 1))} T_{1} \]
and applying (2.15) (with \(x^{2}, y^{2}, w^{2}, z^{2}\) replaced by \(q^{-2}\lambda w/z, q^{2}\lambda w/z, q^{-2}(\lambda + 2) w/z, q^{2}(\lambda + 2) w/z\)) we see that the numerator simplifies to a product of 4 theta functions partially cancelling the theta functions in the numerator. This proves one of the eight statements in (3.9). All other statements are proved in this way, and only the pairings with the \(\delta(w), \delta(z)\) require the use of (2.15).

From (3.9) we see how to extend the pairing to \(E^{\text{cop}} \times E;\)
\[ \left( \begin{array}{cc} -1 & 0 \\ 0 & T_{1} \end{array} \right) = \left( \begin{array}{cc} \alpha(w) & \beta(w) \\ \gamma(w) & \delta(w) \end{array} \right), 1 = \left( \begin{array}{cc} \alpha(w) & \beta(w) \\ \gamma(w) & \delta(w) \end{array} \right), \det(z) \det^{-1}(z) \]
and next we use (3.1) and (3.9). E.g. for the pairing with \(\alpha(w)\) we get, using \(\Delta^{E^{\text{cop}}}(\alpha(w)) = \alpha(w) \otimes \alpha(w) + \gamma(w) \otimes \beta(w),\)
\[ T_{1} = \langle \alpha(w), \det(z) \rangle T_{1} \langle \alpha(w), \det^{-1}(z) \rangle + \langle \gamma(w), \det(z) \rangle T_{1} \langle \beta(w), \det^{-1}(z) \rangle \]
\[ = \frac{\theta(w/q^{2}z)}{\theta(w/z)} \langle \alpha(w), \det^{-1}(z) \rangle, \]
since the second term cancels. This proves the first statement in
\[ \langle \left( \begin{array}{cc} \alpha(w) & \beta(w) \\ \gamma(w) & \delta(w) \end{array} \right), \det^{-1}(z) \rangle = q^{-1} \frac{\theta(w/z)}{\theta(q^{-2}w/z)} \left( \begin{array}{cc} T_{1} & 0 \\ 0 & T_{1} \end{array} \right), \]
\[ \langle \det^{-1}(w), \left( \begin{array}{cc} \alpha(z) & \beta(z) \\ \gamma(z) & \delta(z) \end{array} \right) \rangle = q^{-1} \frac{\theta(q^{2}w/z)}{\theta(w/z)} \left( \begin{array}{cc} T_{1} & 0 \\ 0 & T_{1} \end{array} \right), \]
the other ones being proved similarly.

**Lemma 3.8.** The pairing on \(E^{\text{cop}} \times E \to D_{h}\), defined on the generators by (3.8) and (3.10) makes \(E^{\text{cop}} \) and \(E \) paired as \(h\)-Hopf \(*\)-algebroids using (3.4).

**Proof.** Since the \(R\)-matrix in (2.11) is a solution to the quantum dynamical Yang-Baxter equation (2.7), we already obtain the pairing on the level of \(h\)-bialgebroids by Remark (3.4)(ii). A straightforward check shows that (3.2) holds on the level of generators, including \(\det^{-1}(z)\), so that we obtain the pairing on the level of \(h\)-Hopf algebroids. Finally, it remains to be checked that the \(*\)-structures and the pairing are compatible using (3.4), and using Proposition 3.7 it suffices to check this for the generators. \(\square\)

**Remark 3.9.** For the case of the dynamical quantum group associated with the rational \(R\)-matrix (2.17) Rosengren [26, §4] has calculated the radical of the pairing, and given it an appropriate representation theoretic meaning. For the radical of the pairing for the elliptic \(U(2)\) dynamical quantum group this is not known. \(\square\)

### 4. Actions arising from pairings

In this section we consider two \(h\)-bialgebroids with a pairing. The pairing can be used to construct a natural action of one \(h\)-bialgebroid on the other. This is the content of Theorem 4.1, and we also consider the weak formulation of this action and its intertwining properties with the \(h\)-coalgebroid-structure.
Theorem 4.1. Let $\mathcal{U}, \mathcal{A}$ be $h$-bialgebroids with a pairing $\langle \cdot, \cdot \rangle : \mathcal{U} \times \mathcal{A} \to D_h$, as in Definition 3.1. For $X \in \mathcal{U}_{\alpha\beta}$ and $a \in \mathcal{A}$ the elements
\[ a \cdot X = (T_a(X, \cdot) \otimes \text{Id})\Delta^A(a) = \sum_{(a)} T_a(X, a_{(1)}) \otimes a_{(2)} = \sum_{(a)} \mu^A_{(a)}(T_a(X, a_{(1)})1) a_{(2)}, \tag{4.1a} \]
\[ X \cdot a = (\text{Id} \otimes (X, \cdot)T_\beta)\Delta^A(a) = \sum_{(a)} (X, a_{(2)})T_\beta = \sum_{(a)} \mu^A_{(a)}((X, a_{(2)})T_\beta1) a_{(1)}, \tag{4.1b} \]
are well-defined elements in $\mathcal{A}$, and for $X \in \mathcal{U}_{\alpha\beta}$ and $a \in \mathcal{A}_{\gamma\delta}$ we have $X \cdot a \in \mathcal{A}_{\gamma+\delta-\alpha-\beta}$ and $a \cdot X \in \mathcal{A}_{\gamma+\beta-\alpha-\delta}$. Then (4.1) defines a right and left action of $\mathcal{U}$ on $\mathcal{A}$:
\[ a \cdot (XY) = (a \cdot X) \cdot Y, \quad (XY) \cdot a = X \cdot (Y \cdot a), \quad \forall a \in \mathcal{A}, \forall X, Y \in \mathcal{U}. \]
The left and right action commute; $(X \cdot a) \cdot Y = X \cdot (a \cdot Y), \forall X, Y \in \mathcal{U}, \forall a \in \mathcal{A}$. Moreover, with $\Delta^U = \sum_{(X)} X_{(1)} \otimes X_{(2)}$, we have
\[ X \cdot (ab) = \sum_{(X)} (X_{(1)} \cdot a) (X_{(2)} \cdot b), \quad (ab) \cdot X = \sum_{(X)} (a \cdot X_{(1)}) (b \cdot X_{(2)}), \]
\[ X \cdot 1_A = \mu^A_{(a)}(\Delta^U(X)1), \quad 1_A \cdot X = \mu^A_{(a)}(T_{a\cdot\Delta^U(X)1}), \quad X \in \mathcal{U}_{\alpha\beta}. \]
If $\mathcal{U}$ and $\mathcal{A}$ are paired $h$-Hopf algebroids, then the action of $\mathcal{U}$ on $\mathcal{A}$ satisfies $S^A(a \cdot S^U(X)) = X \cdot S^A(a)$ for all $X \in \mathcal{U}$ and all $a \in \mathcal{A}$. If $\mathcal{U}$ and $\mathcal{A}$ are paired $h$-Hopf $*$-algebroids with invertible antipodes, then the action of $\mathcal{U}$ on $\mathcal{A}$ satisfies $X \cdot a^* = (S^U(X)^* \cdot a)^* = (S^U(X^* \cdot a)^*)^* \quad \text{and} \quad a^* \cdot X = ((a \cdot S^U(X)^*)^*)^*$. The action of Theorem 4.1 can be formulated in terms of the pairing. Because the pairings are in general not assumed to be non-degenerate, this is a weaker statement and this is given in Proposition 4.2. We postpone its proof to the end of the section.

Proposition 4.2. The action defined in (4.1) satisfies
\[ \langle Y, X \cdot a \rangle = \langle Y, X \rangle \circ T_\beta, \quad \langle Y, a \cdot X \rangle = T_\alpha \circ \langle XY, a \rangle \]
for all $X \in \mathcal{U}_{\alpha\beta}, \, Y \in \mathcal{U}$ and $a \in \mathcal{A}$.

Proof of Theorem 4.1. We prove the statements for the right action of $\mathcal{U}$ on $\mathcal{A}$, the results for the left action being proved analogously. We put $\Delta = \Delta^A, \mu = \mu^A, \mu_r = \mu^A, S = S^A$ for the proof.

Since the comultiplication $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is well-defined, we only have to show that $T_{\alpha}(X, \cdot) \otimes \text{Id}$ preserves the relation (2.4), i.e. for $f \in M_h^\gamma$, $a, b \in \mathcal{A}$ we need to check
\[ T_{\alpha}(X, \mu_r(f)a) \otimes b = T_{\alpha}(X, a) \otimes \mu(f)b. \]
Take $X \in \mathcal{U}_{\alpha\beta}, a \in \mathcal{A}_{\gamma\delta}$, then (3.1c) shows that the left hand side equals
\[ T_{\alpha}(X, \mu_r(f)a) \otimes b = T_{\alpha}(X, a_{(1)} \mu_r(T_\delta f)) \otimes b = T_{\alpha}(X, a) \otimes (T_\delta f) \otimes b = \mu(T_{\alpha}(X, a)T_\delta f1) b \]
and the right hand side equals $\mu(T_{\alpha}(X, a)1) \mu(f)b$ and the equality follows from (2.2), since by (3.1a) we have $T_{\alpha}(X, a) \in (D_{h^r})_{\delta, \beta+\gamma-\alpha}$.

Next, for $a \in \mathcal{A}_{\gamma\delta}$ we write $\Delta(a) = \sum(a) a_{(1)} \otimes a_{(2)}$ with $a_{(1)} \in \mathcal{A}_{\eta\gamma}, a_{(2)} \in \mathcal{A}_{\eta\delta}$. With $X \in \mathcal{U}_{\alpha\beta}$ we have $T_{\alpha}(X, a_{(1)}) \in (D_{h^r})_{\eta, \beta+\gamma-\alpha}$, so this is only non-zero for $\eta = \beta + \gamma - \alpha$. Hence, $a \cdot X \in \mathcal{A}_{\beta+\gamma-\alpha, \delta}$. So in particular, $\cdot : \mathcal{A} \to \mathcal{A}$ does not preserve the grading, but $(\mu_r(f)a) \cdot X = \mu_r(f)(a \cdot X)$ is
immediate from (4.1a) and (2.3b). For the left moment map and \( X \in \mathcal{U}_{\alpha\beta} \) we have

\[
(\mu_1(f)a) \cdot X = \sum_{(a)} \mu_1(T_\alpha\langle X, \mu_1(f)a(1) \rangle 1)a(2) = \sum_{(a)} \mu_1(T_\alpha \circ f \circ \langle X, a(1) \rangle 1)a(2)
\]

\[
= \mu_1(T_\alpha f) \sum_{(a)} \mu_1(T_\alpha\langle X, a(1) \rangle 1)a(2) = \mu_1(T_\alpha f)(a \cdot X)
\]

using (2.3b), (4.1a) and (3.1b).

To show that this defines an action we write \( \Delta(a) \) for \( a \in \mathcal{A}_{\alpha\beta} \) as above and we use the coassociativity

\[
(\text{Id} \otimes \Delta)\Delta(a) = (\Delta \otimes \text{Id})\Delta(a) = \sum_{(a)} a(1) \otimes a(2) \otimes a(3), \quad a(1) \in \mathcal{A}_{\gamma\eta}, \ a(2) \in \mathcal{A}_{\eta\rho}, \ a(3) \in \mathcal{A}_{\rho\delta}.
\]

With this convention we have for \( X \in \mathcal{U}_{\alpha\beta}, Y \in \mathcal{U}_{\sigma\tau} \), and hence \( XY \in \mathcal{U}_{\alpha+\sigma,\beta+\tau} \),

\[
a \cdot (XY) = \sum_{(a)} T_{\alpha+\sigma}\langle XY, a(1) \rangle \otimes a(2) = \sum_{(a)} T_{\alpha+\sigma}\langle X, a(1) \rangle T_\eta\langle Y, a(2) \rangle \otimes a(3)
\]

\[
= \sum_{(a)} \mu_1(T_{\sigma+\alpha}\langle X, a(1) \rangle T_\eta\langle Y, a(2) \rangle 1)a(3)
\]

using (3.1d). On the other hand, using the \( \mathbb{C} \)-linearity of the action and (4.2) we have

\[
(a \cdot X) \cdot Y = \left( \sum_{(a)} \mu_1(T_\alpha\langle X, a(1) \rangle 1)a(2) \right) \cdot Y = \sum_{(a)} \mu_1(T_\sigma T_\alpha\langle X, a(1) \rangle 1)(a(2) \cdot Y)
\]

\[
= \sum_{(a)} \mu_1(T_{\sigma+\alpha}\langle X, a(1) \rangle 1)\mu_1(T_\sigma\langle Y, a(2) \rangle 1)a(3)
\]

and since \( T_{\sigma+\alpha}\langle X, a(1) \rangle \in (D_{\mathbb{H}})_{\eta-\gamma+\beta-\alpha} \), we see that \( a \cdot (XY) = (a \cdot X) \cdot Y \) follows from (2.2) and \( (D_{\mathbb{H}})_{\alpha\beta} = \{0\} \) for \( \alpha \neq \beta \).

For the commutativity of the left and right action we take \( X \in \mathcal{U}_{\alpha\beta}, Y \in \mathcal{U}_{\sigma\tau} \), and using (4.3),

\[
(X \cdot a) \cdot Y = \left( \sum_{(a)} \mu_\tau(\langle X, a(2) \rangle T_\beta 1)a(1) \right) \cdot Y = \sum_{(a)} \mu_\tau(\langle X, a(2) \rangle T_\beta 1)(a(1) \cdot Y)
\]

\[
= \sum_{(a)} \mu_\tau(\langle X, a(3) \rangle T_\beta 1)\mu_\tau(\langle X, a(2) \rangle T_\beta 1)a(3)
\]

and similarly

\[
X \cdot (a \cdot Y) = X \cdot \left( \sum_{(a)} \mu_\tau(\langle X, a(1) \rangle T_\beta 1)a(2) \right) = \sum_{(a)} \mu_\tau(\langle X, a(1) \rangle T_\beta 1)(X \cdot a(2))
\]

\[
= \sum_{(a)} \mu_\tau(\langle X, a(1) \rangle T_\beta 1)\mu_\tau(\langle X, a(3) \rangle T_\beta 1)a(2),
\]

which proves the statement.

For \( X \in \mathcal{U}_{\alpha\beta} \) we see, using the \( \mathfrak{h} \)-algebra homomorphism property of \( \Delta \) and (3.1e),

\[
(ab) \cdot X = \sum_{(a)} \sum_{(b)} \mu_1(T_\alpha\langle X, a(1) b(1) \rangle 1)a(2)b(2)
\]

\[
= \sum_{(a)} \sum_{(b)} \mu_1(T_\alpha\sum_{(X)} \langle X(1), a(1) \rangle T_\eta\langle X(2), b(1) \rangle 1)a(2)b(2)
\]

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using $\Delta(X) = \sum (X) X(1) \otimes X(2)$ with $X(1) \in U_{\alpha\beta}$, $X(2) \in U_{\eta\beta}$. Writing $\Delta(a) = \sum (a) a(1) \otimes a(2)$, $a(1) \in A_{\gamma\beta}$, $a(2) \in A_{\delta\beta}$ we see that $T_\alpha(X(1), a(1)) \in (D_\eta)^{\rho, \gamma + \eta - \alpha}$ and by (2.2),

$$\mu(T_\alpha(X(1), a(1)) T_\eta(X(2), b(1))) = \mu(T_\alpha(X(1), a(1))) \mu(T_{\alpha - \gamma}(X(2), b(1))).$$

This gives

$$(ab) \cdot X = \sum (a) \sum (b) (X) \mu(T_\alpha(X(1), a(1)) 1) \mu(T_{\alpha - \gamma}(X(2), b(1))) a(2) b(2)$$

and since $(D_\eta)^{\rho, \gamma + \eta - \alpha} = \{0\}$ unless $\rho + \alpha - \gamma = \eta$ we find

$$(ab) \cdot X = \sum (X) \mu(T_\alpha(X(1), a(1)) 1) a(2) \mu(T_{\rho + \alpha - \gamma}(X(2), b(1))) b(2)$$

This proves the statements for the right action.

Assume now that $U$ and $A$ are paired $\mathfrak{h}$-Hopf algebroids. For the proof of $X \cdot S(a) = S(a \cdot S^L(X))$ we use $A \otimes D_{\mathfrak{h}^*} \cong A \cong D_{\mathfrak{h}^*} \otimes A$, see §2.1. We take $X \in U_{\alpha\beta}$, so

$$X \cdot S(a) = (\text{Id} \otimes \langle X, \cdot \rangle T_\beta) \Delta(S(a)) = (\text{Id} \otimes \langle X, \cdot \rangle T_\beta) \circ P \circ (S \otimes S) \Delta(a)$$

$$= P \circ ((X, \cdot) T_\beta \otimes \text{Id}) \circ (S \otimes S) \Delta(a) = \sum \mu(S(a(2))) \otimes \langle X, S(a(1)) \rangle T_\beta$$

$$= \sum \mu(S(a(2))) \otimes S^D_{\mathfrak{h}^*}(T_{\beta - \alpha}(S^L(X), a(1))) = S(T_{\beta - \alpha}(S^L(X), a(1))) \otimes a(2) = S(a \cdot S^L(X)),$$

since $S^L(X) \in U_{\beta - \alpha}$. Finally, if $U$ and $A$ are in paired $\mathfrak{h}$-Hopf $*$-algebroids, we have in the same way for $X \in U_{\alpha\beta}$

$$X \cdot a^* = \sum (a) a^*(1) \otimes \langle X, a^*(2) \rangle T_\beta = \sum (a) a^*(1) \otimes T_{-\alpha}(S^L(X)^*, a(2)))$$

$$= \left(\sum (a) a^*(1) \otimes S^L(X)^*, a(2) \right) T_{-\alpha} = (S^L(X)^* \cdot a)^*,$$

since $S^L(X)^* \in U_{\beta - \alpha}$. □

For later use we note, see also the proof of Theorem 4.1, that for $X \in U_{\alpha\beta}$, $a \in A_{\gamma\delta}$ we have

$$X \cdot (\mu^L(f)a) = \mu^L(f)(X \cdot a), \quad X \cdot (\mu^R(f)a) = \mu^R(f)(X \cdot a),$$

$$\langle \mu^L(f)a \rangle \cdot X = \mu^L(T_{\alpha}(f))(a \cdot X), \quad \langle \mu^R(f)a \rangle \cdot X = \mu^R(f)(a \cdot X),$$

and

$$\langle \mu^L(f)X \rangle \cdot a = \mu^L(f)(X \cdot a), \quad \langle \mu^R(f)X \rangle \cdot a = \mu^R(T_{\alpha - \beta - \delta}(f))(X \cdot a),$$

$$a \cdot (\mu^L(f)X) = \mu^L(T_{\alpha}(f))(a \cdot X), \quad a \cdot (\mu^R(f)X) = \mu^R(T_{\alpha - \gamma f})(a \cdot X).$$

In particular, taking $X = 1 \in U_{\delta 0}$ in (4.4) gives the left and right action of the left and right moment map on $A$. 

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**For later use we note, see also the proof of Theorem 4.1, that for $X \in U_{\alpha\beta}$, $a \in A_{\gamma\delta}$ we have**

$$X \cdot (\mu^L(f)a) = \mu^L(f)(X \cdot a), \quad X \cdot (\mu^R(f)a) = \mu^R(f)(X \cdot a),$$

$$\langle \mu^L(f)a \rangle \cdot X = \mu^L(T_{\alpha}(f))(a \cdot X), \quad \langle \mu^R(f)a \rangle \cdot X = \mu^R(f)(a \cdot X),$$

**and**

$$\langle \mu^L(f)X \rangle \cdot a = \mu^L(f)(X \cdot a), \quad \langle \mu^R(f)X \rangle \cdot a = \mu^R(T_{\alpha - \beta - \delta}(f))(X \cdot a),$$

$$a \cdot (\mu^L(f)X) = \mu^L(T_{\alpha}(f))(a \cdot X), \quad a \cdot (\mu^R(f)X) = \mu^R(T_{\alpha - \gamma f})(a \cdot X).$$

**In particular, taking $X = 1 \in U_{\delta 0}$ in (4.4) gives the left and right action of the left and right moment map on $A$.**
The following Proposition describes the interaction of the action and the \( h \)-coalgebroid structure of \( A \), and this is useful in constructing corepresentations using invariance properties in terms of the action defined in Theorem 4.1.

**Proposition 4.3.** With the notation and assumptions as in Theorem 4.1 we have

\[
\Delta^A(X \cdot a) = (\text{Id} \otimes X)\Delta^A(a), \quad \Delta^A(a \cdot X) = (X \otimes \text{Id})\Delta^A(a),
\]

\[
\varepsilon^A(X \cdot a) = \langle X, a \rangle \circ T\beta, \quad \varepsilon^A(a \cdot X) = T\alpha \circ \langle X, a \rangle,
\]

for \( X \in U_{\alpha\beta} \) and \( a \in A \).

**Proof.** By (4.3) it follows that \( a \otimes (X \cdot \mu^A_\tau(f)b) = a \otimes \mu^A_\tau(f)(X \cdot b) = \mu^A_\tau(f)a \otimes (X \cdot b) \) so that \( \text{Id} \otimes X : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \) is well-defined, cf. (2.4). Now for \( X \in U_{\alpha\beta} \)

\[
\Delta^A(X \cdot a) = \sum_{a(1)} \Delta^A(\langle X, a_{(2)} \rangle T\beta 1) a_{(1)} = \sum_{a(1)} (1 \otimes \mu^A_\tau(\langle X, a_{(2)} \rangle T\beta 1)) \Delta^A(a_{(1)})
\]

\[
= \sum_{a(1)} a_{(1)} \otimes \mu^A_\tau(\langle X, a_{(3)} \rangle T\beta 1) a_{(2)} = \sum_{a(1)} a_{(1)} \otimes (X \cdot a_{(2)})
\]

using \( \Delta^A(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)} \), \( (\Delta^A \otimes \text{Id})\Delta^A(a) = (\text{Id} \otimes \Delta^A)\Delta^A(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)} \otimes a_{(3)} \), so the result is a consequence of the coassociativity.

Now for \( X \in U_{\alpha\beta} \)

\[
\varepsilon^A(X \cdot a) = (\varepsilon^A \otimes (X, \cdot)T\beta)\Delta^A(a) = (\text{Id}_{D_{\alpha\beta}} \otimes (X, \cdot)T\beta)(\varepsilon^A \otimes \text{Id}_{\mathcal{A}})\Delta^A(a) = \langle X, a \rangle \circ T\beta
\]

using the counit axiom and the appropriate identifications as discussed in §2.1. The statements for the right action are proved similarly. \( \square \)

It remains to prove Proposition 4.2.

**Proof of Proposition 4.2.** We prove the first statement, the other one being proved analogously. Assume \( X \in U_{\alpha\beta} \), \( Y \in U_{\sigma\tau} \), \( a \in A_{\gamma\delta} \), \( \Delta^A(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)} \) with \( a_{(1)} \in A_{\gamma\rho}, a_{(2)} \in A_{\rho\delta} \),

\[
\langle Y, X \cdot a \rangle = \langle Y, \sum_{(a)} \mu^A_\tau(\langle X, a_{(2)} \rangle T\beta 1) a_{(1)} \rangle = \langle Y, \sum_{(a)} a_{(1)} \mu^A_\tau(T_\rho(X, a_{(2)}) T\beta 1) \rangle
\]

\[
= \sum_{(a)} \langle Y, a_{(1)} \rangle \circ (T_\rho(X, a_{(2)}) T\beta 1).
\]

Write \( \langle X, a_{(2)} \rangle = ft_{-\beta-\sigma} \in (D_{\beta\rho})_{\alpha+\delta, \beta+\rho}, \langle Y, a_{(1)} \rangle = gT_{-\gamma} \in (D_{\gamma\delta})_{\sigma+\rho, \tau+\gamma} \), so that the summand can be rewritten as \( g \circ (T_{-\gamma}) \circ T_{-\sigma-\rho} \in D_{\gamma\delta} \). On the other hand, it is straightforward that this is also equal to \( \langle Y, a_{(1)} \rangle \circ T_\rho \circ \langle X, a_{(2)} \rangle \circ T_{\beta} \), so that

\[
\langle Y, X \cdot a \rangle = \sum_{(a)} \langle Y, a_{(1)} \rangle \circ T_\rho \circ \langle X, a_{(2)} \rangle \circ T_{\beta} = \langle Y X, a \rangle \circ T_{\beta}.
\]

\( \square \)

**Remark 4.4.** Having paired \( h \)-Hopf algebroids \( U \) and \( A \) one might expect to have an analogue of the double construction of Drinfeld, see Lu [21] for the case of Hopf algebroids. However, if we want to mimic the classical construction we see that for elements \( X, Y \in U \), with \( \Delta^U(X) = \sum_{(X)} X_{(1)} \otimes X_{(2)} \), and \( a \in A \) neither the element \( \sum_{(X)} S^U(X_{(2)}) \cdot a \cdot X_{(1)} \) nor \( \sum_{(X)} X_{(1)} Y S^U(X_{(2)}) \), or any other similarly defined element, is well-defined, i.e. respects the relation (2.4) in \( U \otimes U \). For a double construction for \( \times_R \)-bialgebras using a different pairing, see [27, §6].
5. Representation and Corepresentation Theory

Given two \(h\)-bialgebroids with a pairing, we consider the relation between dynamical representations of the one \(h\)-bialgebroid and the corepresentations of the other \(h\)-bialgebroid. For \(h\)-Hopf \(\ast\)-algebroids we rephrase the notion of unitarisability introduced in [17] and consider it in the context of two \(h\)-Hopf \(\ast\)-algebroids equipped with a pairing, and we show that this extends the previously introduced notion of unitarisability. For the elliptic \(U(2)\) dynamical quantum group we make this explicit, and this enables us to calculate the pairing between matrix elements of irreducible corepresentations in terms of elliptic hypergeometric series. We start by recalling the notions of corepresentations and dynamical representations.

A \(h\)-space \(V\) is a vector space over \(M_h\), which is also a diagonalisable \(h\)-module, \(V = \bigoplus_{\alpha \in h^*} V_{\alpha}\), \(V_{\alpha} = \{v \in V \mid H \cdot v = \alpha(H)v, \forall H \in h\}\), with \(M_h V_{\alpha} \subseteq V_{\alpha}\) for all \(\alpha\). A morphism of \(h\)-spaces is a \(M_h\)-linear \(h\)-invariant map. In case we want to emphasize the multiplication of \(v \in V\) with \(f \in M_h\), we also write \(\mu_V(f)v = fv\). Note that the definition of a \(h\)-space implies that \(V\) has a weight space decomposition, and we can speak of a homogeneous vector \(v \in V_{\alpha}\) as a vector of weight \(\alpha \in h^*\).

For a \(h\)-algebra \(A\) and a \(h\)-space \(V\) we define \(A \widehat{\otimes} V = \bigoplus_{\alpha \beta} A_{\alpha \beta} \widehat{\otimes} M_{h} V_{\beta}\), where we mod out by the relations \(\mu_{\alpha}(f)a \otimes v = a \otimes \mu_{\alpha}(f)v\) for all \(f \in M_h\), \(a \in A\), \(v \in V\), cf. (2.4). We make \(A \widehat{\otimes} V\) into a \(h\)-space by the grading \(A_{\alpha \beta} \widehat{\otimes} M_{h} V_{\beta} \subseteq (A \widehat{\otimes} V)_{\alpha}\) for any \(\beta\) in \(h^*\), and by \(\mu_{A \widehat{\otimes} V}(f)(a \otimes v) = \mu_{A}(f)a \otimes v\). This is easily checked. This definition is compatible with the tensor product of \(h\)-algebras, see (2.3), in the sense \((A \widehat{\otimes} B) \widehat{\otimes} V = A \widehat{\otimes} (B \widehat{\otimes} V)\) as \(h\)-spaces for \(h\)-algebras \(A\) and \(B\) and a \(h\)-space \(V\).

Similarly we define \(V \widehat{\otimes} A = \bigoplus_{\alpha \beta} V_{\alpha} \widehat{\otimes} M_{h} A_{\alpha \beta}\), where we mod out by the relations \(v \otimes \mu_{\alpha}(f)a = fv \otimes a\) for all \(f \in M_h\), \(a \in A\), \(v \in V\), cf. (2.4). Then \(V \widehat{\otimes} A\) is a \(h\)-space with grading \(V_{\alpha} \widehat{\otimes} M_{h} A_{\alpha \beta} \subseteq (V \widehat{\otimes} A)_{\beta}\) for all \(\alpha\) and \(\mu_{V \widehat{\otimes} A}(f)(v \otimes a) = v \otimes \mu_{A}(f)a\). This time we have the compatibility relation \(V \widehat{\otimes} (A \widehat{\otimes} B) = (V \widehat{\otimes} A) \widehat{\otimes} B\) as \(h\)-spaces for \(h\)-algebras \(A\) and \(B\) and a \(h\)-space \(V\).

In the special case \(A = D_{h^\ast}\), we obtain \(V \widehat{\otimes} D_{h^\ast} \cong V \cong D_{h^\ast} \widehat{\otimes} V\) as \(h\)-spaces with the isomorphism given by \(v \otimes f T_{-\alpha} \cong fv \otimes f T_{-\alpha} \otimes v\) for \(f \in M_h\), \(v \in V_{\alpha}\).

Now a left corepresentation of a \(h\)-coalgebroid \(A\) on a \(h\)-space \(V\) is a \(h\)-space morphism \(L: V \rightarrow A \widehat{\otimes} V\) satisfying \((\Delta^A \otimes \Id) \circ L = (\Id \otimes L) \circ L\) and \((\varepsilon^A \otimes \Id) \circ L = \Id\). Note that for the first requirement we use \((A \widehat{\otimes} A) \widehat{\otimes} V = A \widehat{\otimes} (A \widehat{\otimes} V)\) and for the second we use \(V \cong D_{h^\ast} V\). Similarly, a right corepresentation of a \(h\)-coalgebroid \(A\) on a \(h\)-space \(V\) is a \(h\)-space morphism \(R: V \rightarrow V \widehat{\otimes} A\) satisfying \((R \otimes \Id) \circ R = (\Id \otimes \Delta^A) \circ R\) and \((\Id \otimes \varepsilon^A) \circ R = \Id\). An intertwiner of two left, respectively right, corepresentations \(L_1\) and \(L_2\), \(R_1\) and \(R_2\), in \(V_1\) and \(V_2\) is a \(h\)-space morphism \(\Phi: V_1 \rightarrow V_2\) such that \(L_2 \circ \Phi = (\Id \otimes \Phi) \circ L_1\), respectively \(R_2 \circ \Phi = (\Phi \otimes \Id) \circ R_1\). Note that \(\Id \otimes \Phi\), respectively \(\Phi \otimes \Id\), do factor to a map on \(A \widehat{\otimes} V_1\), respectively \(V_1 \widehat{\otimes} A\).

To a \(h\)-space \(V\) we can associate a \(h\)-algebra \(D_{h^\ast} \widehat{\otimes} V\) as follows. First, we define the graded sub-space \((D_{h^\ast} \widehat{\otimes} V)_{\alpha \beta}\) as the space of \(C\)-linear operators \(U: V \rightarrow V\) satisfying \(U(fv) = (T_{-\beta}f)(Uv)\) for all \(f \in M_h\), \(v \in V\), and \(U(V_{\alpha}) \subseteq V_{-\alpha + \beta}\). The moment maps \(\mu_1, \mu_\tau: M_h^\ast \rightarrow (D_{h^\ast} \widehat{\otimes} V)_{00}\) are defined by \(\mu_1(f)v = fv\), \(\mu_\tau(f)v = (T_{-\tau}f)v\) for \(v \in V_{\tau}\). One easily checks that this makes \((D_{h^\ast} \widehat{\otimes} V, U)\) into a \(h\)-algebra. A dynamical representation of a \(h\)-algebra \(A\) on a \(h\)-space \(V\) is a \(h\)-algebra-morphism \(\pi: A \rightarrow D_{h^\ast} \widehat{\otimes} V\). For \(\pi_i: A \rightarrow D_{h^\ast} \widehat{\otimes} V_i, i = 1, 2\), dynamical representations of a \(h\)-algebra \(A\) we say that \(\pi_1\) and \(\pi_2\) are \(\Phi\)-invariant if \(\Phi \circ \pi_1(a) = \pi_2(a) \circ \Phi\) for all \(a \in A\).

Recall from Remark 3.2 the pairing between \(h\)-algebras and \(h\)-coalgebroids. The applications of Proposition 5.1 are for \(h\)-bialgebroids, we formulate it in the slightly more general case.
Proposition 5.1. Let $\mathcal{U}$ be a $\h$-algebra and $\mathcal{A}$ be $\h$-coalgebroid equipped with a pairing, and let $V$ be a $\h$-space.

(i) Let $R: V \to V \hat{\otimes} A$ be a right corepresentation of the $\h$-coalgebroid $A$, then $\pi(X)v = (\text{Id} \otimes \langle X, \cdot \rangle T_\beta) R(v)$ for $X \in \mathcal{U}_{\alpha \beta}$, defines a $\h$-algebra morphism $\pi: \mathcal{U} \to (D^{l,r}_\h V)^{l,r}$ on $\mathcal{U}$ on $V$, hence $\pi: U_l^k \to D^{l,r}_\h V$ defines a dynamical representation of $U_l^k$ on $V$.

(ii) Let $L: V \to A \hat{\otimes} V$ be a left corepresentation of the $\h$-coalgebroid $A$, then $\pi(X)v = (T_\alpha(X, \cdot) \otimes \text{Id}) L(v)$ for $X \in \mathcal{U}_{\alpha \beta}$, defines a $\h$-algebra homomorphism $\pi: U_l^{opp} \to (D^{opp}_{\h} V)^{l,r}$. In particular, $\pi: (U_l^{opp})^r \to D^{l,r}_\h V$ defines a dynamical representation of $(U_l^{opp})^r$ on $V$. Moreover, if $U$ is $\h$-Hopf algebroid, then $X \mapsto \pi(SU(X))$ defines a dynamical representation of $U$ on $V$.

In case $\mathcal{A}$ is equipped with a cobraiding, see Definition 3.3, we can use $\mathcal{A}^{l,r} = \mathcal{A}^{opp}$ as $\h$-algebras. Also $(\mathcal{A}^{opp})^{opp} = (\mathcal{A}^{opp})^{opp}$, and this immediately implies the next result.

Corollary 5.2. Let $\mathcal{A}$ be a $\h$-bialgebroid equipped with a cobraiding.

(i) A right corepresentation of $\mathcal{A}$ on $V$ gives rise to a dynamical representation of $\mathcal{A}$ on $V$.

(ii) A left corepresentation of $\mathcal{A}$ on $V$ gives rise to a dynamical representation of $\mathcal{A}^{opp}$ on $V$.

Proof of Proposition 5.1. The proof is a minor variation on the proof of Theorem 4.1, so we do not give all details.

For $v \in V_\gamma$ write $Rv = \sum_\nu \nu(1) \otimes \alpha(2)$ with $\nu(1) \in V_\delta$, $\alpha(2) \in A_{\gamma, \gamma}$, so that for $X \in \mathcal{U}_{\alpha \beta}$

$$\pi(X)v = (\text{Id} \otimes \langle X, \cdot \rangle T_\beta) R(v) = \sum_{(v)} \mu_V(\langle X, \alpha(2) \rangle T_\beta 1) \nu(1),$$

and since $\langle X, \alpha(2) \rangle T_\beta \in (D^{l,r}_\h)^{\alpha+\gamma-\beta, \beta}$ we see that $\pi(X)v$ is well-defined and that we only find a contribution to the sum for $\delta = \gamma + \alpha - \beta$. So $\pi(X)V_\gamma \subseteq V_{\gamma+\alpha-\beta}$. Next, for $f \in M^{l,r}$ we find $R(fv) = f(Rv) = \sum_{(v)} \nu(1) \otimes \mu^A_{\gamma}(f) \alpha(2)$, so that

$$\pi(X)(fv) = \sum_{(v)} \mu_V(\langle X, \mu^A_{\gamma}(f) \alpha(2) \rangle T_\beta 1) \nu(1) = \sum_{(v)} \mu_V(\langle X, \alpha(2) \rangle (T_\gamma f) T_\beta 1) \nu(1)$$

$$= (T_{-\alpha} f) \sum_{(v)} \mu_V(\langle X, \alpha(2) \rangle T_\beta 1) \nu(1) = (T_{-\alpha} f) \pi(X)v$$

using (3.1c) and $\langle X, \alpha(2) \rangle \in (D^{l,r}_\h)^{\alpha+\gamma, \delta+\beta}$. This also shows the $C$-linearity, and hence $\pi(X) \in (D^{l,r}_\h)^{\alpha+\gamma, \delta+\beta}$. Furthermore, $\pi: U \to D^{l,r}_\h V$ is $C$-linear, and the relation $\pi(XY) = \pi(X) \pi(Y)$ is proved as the similar statement for the left action in Theorem 4.1. For the actions of the left and right moment maps we find

$$\pi(\mu^U_{l}(f))v = \sum_{(v)} \mu_V(\langle \mu^U_{l}(f), \alpha(2) \rangle T_\beta 1) \nu(1) = \mu_V(f \epsilon^A(\alpha(2)) T_\beta 1) \nu(1)$$

$$= \mu_V(f)(\text{Id} \otimes \epsilon^A) Rv = \mu_V(f)v,$$

$$\pi(\mu^U_{r}(f))v = \sum_{(v)} \mu_V(\langle \mu^U_{r}(f), \alpha(2) \rangle T_\beta 1) \nu(1) = \mu_V(\epsilon^A(\alpha(2)) f T_\beta 1) \nu(1)$$

$$= \mu_V(T_{-\gamma} f)(\text{Id} \otimes \epsilon^A) Rv = \mu_V(T_{-\gamma} f)v,$$

using (3.1b), (3.1f) and $\epsilon^A(\alpha(2)) \in (D^{l,r}_\h)^{\delta, \gamma}$. Hence, $\pi: U \to (D^{l,r}_\h V)^{l,r}$ is a $\h$-algebra-homomorphism, and consequently $\pi: U_l^k \to D^{l,r}_\h V$ defines a dynamical representation of $U_l^k$ on $V$.

For the second statement we put for $v \in V_\gamma L_{l} = \sum_{(v)} \alpha(1) \otimes \nu(2)$ with $\alpha(1) \in A_{\gamma, \delta}$, $\nu(2) \in V_\delta$, so that for $X \in \mathcal{U}_{\alpha \beta}$ we have $\pi(X)v = \sum_{(v)} \mu_V(\langle T_\alpha(X, \alpha(1)) \rangle) \nu(2)$. This implies $\pi(X)V_\gamma \subseteq V_{\gamma+\beta-\alpha}$.
and \( \pi(X)[fv] = \mu_v(T_\alpha f)\pi(X)v \), and hence \( \pi(X) \in (D_{h^*}, V)_{-\beta, -\alpha} \) for \( X \in U_{a\beta} \). We also obtain \( \pi(\mu^v(\Gamma))v = \mu_v(f)v \), \( \pi(\mu_\Gamma^v(\Gamma))v = \mu_v(T_{-\gamma} f)v \) using (3.1b), (3.1c), (3.1f) and \((\varepsilon \otimes \text{Id}) \circ L = \text{Id} \). The proof of \( \pi(XY) = \pi(Y)\pi(X) \) proceeds as the corresponding statement for the right action in Theorem 4.1. Since \( \pi((U^{opp})_{a\alpha}) = \pi(U^{opp}_{a\beta}, \tilde{\omega}_\beta) = \pi(D^*, y^{a\alpha})_\text{ga} = \pi(D^*, y\tilde{\omega}^{a\alpha}) \), we obtain the first statement in (ii), and the second follows immediately.

In case \( U \) is \( h \)-Hopf algebroid, put \( \tilde{\varepsilon}(X) = \tilde{\varepsilon}(S(U(X))) \). Then the antimultiplicativity of \( \varepsilon \) makes \( \varepsilon \) an algebra (over \( C \)) homomorphism satisfying \( \varepsilon(U_{a\beta}) \subseteq (D^*, y^{a\alpha})_{\text{ga}} = (D^*, y\tilde{\omega}^{a\alpha}) \). The proof of Lemma 5.3 is analogous to the proof of Proposition 4.3, and we leave the proof to the reader.

Next we want to consider unitary corepresentations of a \( h \)-bialgebroid \( A \) equipped with a \(*\)-operator. This means that \( h^* \) and \( M_{h^*} \) are equipped with a complex conjugation. For a \( h \)-space \( V \) a form is a \( M_{h^*} \)-sesquilinear form \( (\cdot, \cdot): V \times V \to M_{h^*} \) satisfying \((v,v) \neq 0 \) in \( M_{h^*} \) and \((v,w) = (w,v)^* \) for all \( v, w \in V \). We extend this form to \( A \otimes V \) by
\[
(a \otimes v, b \otimes w) = b^* \mu_A^v((v, w))a,
\]
and to \( V \otimes A \) by
\[
(v \otimes a, w \otimes b) = b^* \mu_A^v((v, w))a.
\]
We have to check that (5.1) and (5.2) are well-defined. For (5.2) we have \((f_v \otimes a, w \otimes b) = b^* \mu_A^v((f_v, w))a = b^* \mu_A^v((v, w))a = b^* \mu_A^v((v, w))a \) since the form is sesquilinear. For the well-definedness of (5.1) an analogous argument can be given.

The extended forms on \( A \otimes V \) and \( V \otimes A \) are no longer sesquilinear over \( M_{h^*} \). It follows directly that \( (v \otimes a, w \otimes b) = (w \otimes v, a \otimes v)^* \), and \( (a \otimes v, b \otimes w) = (b \otimes w, a \otimes v)^* \). For the forms on \( A \otimes V \) we find for \( a \otimes v \in (A \otimes V)_\gamma \), respectively \( v \otimes a \in (V \otimes A)_\gamma \),
\[
\langle f(a \otimes v), b \otimes w \rangle = (a \otimes v, b \otimes w)\mu_A^v(T_{\gamma} f), \quad \langle f(v \otimes a), w \otimes b \rangle = (v \otimes a, w \otimes b)\mu_A^v(T_{\gamma} f).
\]

Note that in particular such a form is extended to \( (\cdot, \cdot)_D: V \times V \to D_{h^*} \) by applying the above procedure to \( A = D_{h^*} \) and using the identifications \( V \otimes D_{h^*} \cong V \cong D_{h^*} \otimes V \). Since the left and right moment map for \( D_{h^*} \) coincide we find \( (\cdot, \cdot)_D: V \times V \to D_{h^*} \) is given by \( (v, w)_D = T_{\gamma} \circ (v, w) \circ T_{\gamma} \) for \( v \in V_{h^*}, \) \( w \in V_{h^*} \), where the original \( (v, w) \in M_{h^*} \) is considered as multiplication operator. Using this we can rewrite (5.1) and (5.2) as follows
\[
\langle a \otimes v, b \otimes w \rangle = b^* a \otimes \langle v, w \rangle_D, \quad \langle v \otimes a, w \otimes b \rangle = \langle v, w \rangle_D \otimes b^* a,
\] using the identification (2.5).
Definition 5.4. Let $A$ be a $\h$-Hopf $\star$-algebroid and $V$ a $\h$-space. A right corepresentation $R: V \to V \otimes A$ is unitarisable if there exists a form $\langle \cdot, \cdot \rangle: V \times V \to M_{\h^*}$ such that $\langle Rv, Rw \rangle = \mu^A_\h(\langle v, w \rangle_D 1)$ for all $v, w \in V$. A left corepresentation $L: V \to A \otimes V$ is unitarisable if there exists a form $\langle \cdot, \cdot \rangle: V \times V \to M_{\h^*}$ such that $\langle Lv, Lw \rangle = \mu^A_\h(\langle v, w \rangle_D 1)$ for all $v, w \in V$.

Unitarisable corepresentations have been introduced before in [17, Def. 3.11], and to see its relation to Definition 5.4, we choose a homogeneous basis $\{v_i\}_{i \in I}$ of $V$ as a vector space over $M_{\h^*}$. Let $\omega: I \to \h^*$ be given by $v_i \in V_{\omega(i)}$. We can write for the right corepresentation $R: V \to V \otimes A$ its matrix elements by $Rv_i = \sum_{k \in I} v_k \otimes R_{k,i}$, $R_{k,i} \in A_{\omega(k), \omega(i)}$, and the requirements $(R \otimes \text{Id}) \circ R = (\text{Id} \otimes \Delta^A) \circ R$ and $(\text{Id} \otimes \varepsilon^A) \circ R = \text{Id}$ translate into

$$\Delta^A(R_{ij}) = \sum_{k \in I} R_{ik} \otimes R_{kj}, \quad \varepsilon^A(R_{ij}) = \delta_{ij} T_{-\omega(i)}.$$

Note that (5.5) combined with the antipode axiom (2.6) implies

$$\sum_{k \in I} S^A(R_{ik}) R_{kj} = \delta_{ij} \varepsilon^A = \sum_{k \in I} R_{ik} S^A(R_{kj}).$$

Assume the existence of a form on $V$ making $R$ a unitarisable corepresentation such that a homogeneous basis is orthogonal, $\langle v_i, v_j \rangle = \delta_{ij} N_i$, $N_i \in M_{\h^*}$. Then $\mu^A_\h(\langle v_i, v_j \rangle_D 1) = \delta_{ij} \mu^A_\h(T_{\omega(i)} N_i)$, and

$$\langle Rv_i, Rv_j \rangle = \sum_{k, l \in I} \langle v_k \otimes R_{ki}, v_l \otimes R_{lj} \rangle = \sum_{k, l \in I} \langle v_k, v_l \rangle_D \otimes R^*_k R_{kj}$$

$$= \sum_{k \in I} T_{\omega(k)} \circ N_k \circ T_{-\omega(k)} \otimes R_{kj}^* R_{kj} = \sum_{k \in I} R_{k,j}^* \mu^A_\h(N_k) R_{k,j} = \delta_{ij} \mu^A_\h(T_{\omega(i)} N_i)$$

using the unitarisability in the last equation. Multiplying this last equation by $S^A(R_{ip})$ from the right, summing over $i \in I$ we obtain from (5.6)

$$R_{pj} \mu^A_\h(N_p) = \mu^A_\h(T_{\omega(j)} N_j) S^A(R_{ip}) \quad \Rightarrow \quad \mu^A_\h(N_p) R_{pj} = \mu^A_\h(N_j) S^A(R_{jp}) = S^A(\mu^A_\h(N_j) R_{jp}).$$

Now (5.7) corresponds to the definition of unitarisability of a matrix corepresentation as in [17, Def. 3.11]. Note that (5.7) implies $\omega(i) = \omega(i)$ for all $i \in I$, since $S^A(R_{jp}) \in A_{\omega(p), \omega(j)}$ and $R_{pj} \in A_{\omega(p), \omega(j)}$.

Similarly, for a unitarisable left corepresentation $Lv_i = \sum_{k \in I} L_{ik} \otimes v_k$ with $\langle v_i, v_j \rangle = \delta_{ij} N_i$ we find

$$\langle Lv_i, Lv_j \rangle = \sum_{k \in I} L^*_{jk} \mu^A_\h(N_k) L_{ik} = \delta_{ij} \mu^A_\h(T_{\omega(i)} N_i).$$

Applying $S^A$ to this identity, multiplying from the left by $L_{pi}$, summing over $i \in I$, we find from the analogue of (5.6) for the matrix elements of the left corepresentation,

$$S^A((\mu^A_\h(N_p) L_{jp})^*) = \mu^A_\h(N_p) S^A(L^*_{jp}) = \mu^A_\h(N_j) L_{pj}.$$

Note that we can use the standard Gram-Schmidt process to obtain from a set of linearly independent (over $M_{\h^*}$) vectors in $V$ a set of orthogonal vectors, but in general not orthonormal vectors.

Proposition 5.5. Assume $\mathcal{U}$ and $A$ are paired $\h$-Hopf algebroids having invertible antipodes. (i) Let $R$ be a unitarisable right corepresentation of $A$ in $V$ such that $\{v_i\}_{i \in I}$ is a homogeneous orthogonal basis for $V$ with $\langle v_i, v_j \rangle = \delta_{ij} N_i$. Let $Rv_i = \sum_{k \in I} v_k \otimes R_{ki}$, then

$$S^A(X^* \cdot (\mu^A_\h(N_j) R_{ji}))^* = (\mu^A_\h(N_i) R_{ij}) \cdot X, \quad \forall X \in \mathcal{U}.$$
Let $L$ be a unitarisable left corepresentation of $A$ in $V$ such that $\{v_i\}_{i\in I}$ is a homogeneous orthogonal basis for $V$ with $\langle v_i, v_j \rangle = \delta_{ij} N_j$. Let $L v_i = \sum_{k\in I} L_{ik} \otimes v_k$, then

$$X \cdot (\mu_r^A(N_j) L_{ij}) = S^A((\mu_r^A(N_j) L_{ji}) \cdot X^*), \quad \forall X \in \mathcal{U}. \quad (5.10)$$

Note that (5.9), respectively (5.10), reduces to (5.7), respectively (5.8), for $X = 1 \in \mathcal{U}$.

**Proof.** Start with $R_{ij} \cdot X = \sum_{k\in I} \alpha(X, R_{ik}) \otimes R_{kj}, \ X \in \mathcal{U}_a, \ i, j$ in which we use (5.7), (3.1b), (3.1c) and (3.4) to find the first equality in

$$R_{ij} \cdot X = \sum_{k\in I} \alpha(X, R_{ik}) \otimes N_k \otimes \mu_r^A(N_j) \mu_r^A(N_k^{-1}) S^A(R_{jk})^*$$

$$= (\sum_{k\in I} \alpha(X, R_{ik}) \otimes N_k \otimes \mu_r^A(N_j) \mu_r^A(N_k^{-1}) S^A(R_{jk})^*)^*$$

$$= S^A(\sum_{k\in I} \mu_r^A(N_j) \mu_r^A(N_k^{-1}) R_{jk} \otimes \alpha(X, R_{ik}) \otimes N_k \otimes \mu_r^A(N_k^{-1}) S^D h^* ((X^*, R_{kk}) \otimes N_k \otimes T_{-\omega(k)}).$$

Write $\langle X^*, R_{kk} \rangle = f T_{\alpha+w(i)}$, so $S^D h^* ((X^*, R_{kk})) = T_{-\alpha+w(i)} \circ f$, to find using (2.5)

$$R_{ij} \cdot X = S^A(\sum_{k\in I} \mu_r^A((T_{\alpha} N_k^{-1} f N_k) \mu_r^A(N_j) \mu_r^A(N_k^{-1}) R_{jk})^*$$

$$= \mu_r^A(T_{\alpha} N_k^{-1}) \mu_r^A(N_j) S^A(\sum_{k\in I} \mu_r^A((X^*, R_{kk}) T_{-\beta} 1) R_{jk})^*,$$

which we rewrite as $\mu_r^A(T_{\alpha} N_j)(R_{ij} \cdot X) = S^A(\mu_r^A(N_j)(X^* \cdot R_{jj}))^*$ and the result follows from (4.3).

The statement (ii) for the left corepresentation is proved analogously. □

**Proposition 5.6.** Assume $U$ and $A$ are paired $\mathfrak{h}$-Hopf $*$-algebroids having invertible antipodes.

(i) Let $R$ be a unitarisable right corepresentation of $A$ in $V$, and let $\pi(X), \ X \in \mathcal{U}_a, \ be$ defined as in Proposition 5.1(i), then $T_{\gamma} \circ \langle \pi(X) v, w \rangle_D = \langle v, \pi(X^*) w \rangle_D \circ T_{\gamma}.$

(ii) Let $L$ be a unitarisable left corepresentation of $A$ in $V$, and let $\pi(X), \ X \in \mathcal{U}_a, \ be$ defined as in Proposition 5.1(ii), then $\langle \pi(X) v, w \rangle_D \circ T_{\gamma} = T_{\gamma} \circ \langle v, \pi(X^*) w \rangle_D.$

**Proof.** For (i) note that both sides are not sesquilinear over $M_{\mathfrak{h}}^*$, but for $f \in M_{\mathfrak{h}}^*$ and $v \in V_\gamma$ we have $\langle f v, \pi(X^*) w \rangle_D \circ T_{\gamma} = \langle v, \pi(X^*) w \rangle_D \circ T_{\gamma} = \langle v, \pi(X^*) w \rangle_D \circ T_{\gamma} \circ (T_{\alpha} f)$, and on the other hand $T_{\gamma} \circ \langle \pi(X) f v, w \rangle_D = T_{\gamma} \circ \langle (T_{\alpha} f) \pi(X) v, w \rangle_D = T_{\gamma} \circ \langle \pi(X) v, w \rangle_D \circ (T_{\alpha} f)$ using $\pi(X)V_\gamma \subseteq V_{\gamma+\alpha-\beta}$.

Using $\langle v, w \rangle_D = \langle w, v \rangle_D$, or performing a similar calculation we find $T_{\gamma} \circ \langle \pi(X) f v, w \rangle_D = (T_{\delta} f) \circ T_{\gamma} \circ \langle \pi(X) v, w \rangle_D = \pi(X) f v, w \rangle_D \circ T_{\gamma}$.

This shows that we can restrict to proving the result for $v, w$ elements of a basis of $V$ over $M_{\mathfrak{h}}^*$.

Let $\{v_i\}_{i\in I}, \ v_i \in V_{\omega(i)}$, be a homogeneous basis for $V$ over $M_{\mathfrak{h}}^*$ with $\langle v_i, v_j \rangle = \delta_{ij} N_j, \ R v_i = \sum_{k\in I} R_{ik} \otimes v_k$. Then the unitariness of $R$ is expressed by (5.7). Now

$$\langle v_i, \pi(X^*) v_j \rangle_D = \langle v_i \otimes T_{-\omega(i)}, \sum_{l\in I} v_l \otimes \langle X^*, R_{lj} \rangle \otimes T_{-\omega(l)} = \sum_{l\in I} \langle v_i, v_l \rangle_D \otimes T_{-\omega(l)} \circ \langle X, S^A(R_{lj})^* \rangle \otimes T_{-\omega(l)}$$

and using (5.7) and (3.1b), (3.1c), and the orthogonality of the basis we find

$$\langle v_i, \pi(X^*) v_j \rangle_D = T_{\omega(i)} N_i \circ T_{-\omega(i)} \otimes T_{-\omega(i)} \circ N_j \otimes \langle X, R_{ji} \rangle \circ (T_{\omega(i)} N_i^{-1}) \circ T_{-\omega(i)} \circ T_{-\omega(i)}$$

$$= (T_{\omega(i)} N_i) \circ T_{\omega(i)} N_j \otimes \langle X, R_{ji} \rangle \circ (T_{\omega(i)} N_i^{-1}) = T_{\omega(i)} N_i \circ N_j \otimes \langle X, R_{ji} \rangle.$$
using \(\omega(i) = \omega(i)\) and \(T_\beta \omega) \circ N_j \circ (X, R_{j\beta})\) being a multiplication operator as well. Comparing this with

\[
\langle T(X)v_i, v_j \rangle = \sum_{k \in I} \langle v_k, v_j \rangle_D \otimes T_{\omega(j)} \circ (X, R_{k\beta}) \circ T_\beta = T_{\omega(j)} N_j \otimes T_{\omega(j)} \circ (X, R_{j\beta}) \circ T_\beta
\]

\[
= (T_{\omega(j)} N_j) \otimes T_{\omega(j)} \circ (X, R_{j\beta}) \circ T_\beta = T_{\omega(j)} \circ N_j \circ (X, R_{j\beta}) \circ T_\beta
\]

proves the first statement. Part (ii) is proved along the same lines. □

5.1. Elliptic \(U(2)\) dynamical quantum group. In this subsection we calculate the pairing between matrix elements of irreducible corepresentations of the elliptic \(U(2)\) dynamical quantum group using the pairing on the generators given by the \(R\)-matrix of (2.11) as in Remark 3.4(ii). It turns out that the pairing is described in terms of elliptic hypergeometric series, whose definition we recall later. Such series have been considered for the first time by Frenkel and Turaev [12], and this paper has triggered a lot of activity, see [13, Ch. 11] and references given there. We show how the link between the pairing of matrix elements can be used to rederive the results of Frenkel and Turaev [12], giving a somewhat alternative quantum group theoretic derivation of these results for elliptic hypergeometric series as in [16].

Before doing so we recall the corepresentations of the \(h\)-Hopf \(*\)-algebroid \(E\) as constructed in [16]. Let \(N \in \mathbb{N}\) and put

\[
v_k(z) = \gamma(z) \gamma(q^{2z}) \cdots \gamma(q^{2(N-k-1)z}) \alpha(q^{2(N-k)z}) \cdots \alpha(q^{2(N-1)z}), \quad k \in \{0, 1, \ldots, N\},
\]

and let \(V^N_{2k-N} = \mu^E(M_{v^k}) v_k(z), V^N = \bigoplus_{k=0}^N V^N_{2k-N}\). Then \(V^N\) is an \(h\)-space with \(\mu_{V^N}\) given by the left moment map \(\mu_{v^k}\). Then \(V^N\) is a left corepresentation of \(E\) given by the restriction of the comultiplication;

\[
L: V^N \rightarrow E \otimes V^N, \quad L = \Delta^E|_{V^N}, \quad \Delta^E(v_k(z)) = \sum_{i=0}^N t_{kj}^N(z) \otimes v_j(z).
\]

The matrix elements can be expressed as follows, see [16, Thm. 3.4] for the first expression and its proof for the second expression;

\[
t_{kj}^N(z) = \sum_{l = \max(0, j + k - N)}^{\min(k, j)} \left[ \begin{array}{c} k \\ l \end{array} \right] \left[ \begin{array}{c} N - k \\ j - l \end{array} \right] \mu_{v^k}(q^{2(k-l+l+2j+2l)}) \frac{(q^{2(N-j-k-l+2j+2l)})_{j-l}}{(q^{2(N-j-k-l+2j+2l)})_{j-l}}
\]

\[
\times \gamma(q^{2(N-k-1)z}) \cdots \gamma(q^{2(N-j-1)z}) \delta(q^{2(N-j-k-l+1)z}) \cdots \delta(z)
\]

\[
\times \alpha(q^{2(N-1)z}) \cdots \alpha(q^{2(N-l-1)z}) \beta(q^{2(N-l-1)z}) \cdots \beta(q^{2(N-1)z})
\]

\[
(5.13)
\]

where the elliptic binomial coefficient is defined by \(\left[ \begin{array}{c} k \\ l \end{array} \right] = \prod_{i=1}^l \frac{\theta(q^{2-k+l+1})}{\theta(q^{2i})}\), and the elliptic shifted factorials are

\[
(a)_n = \prod_{i=0}^{n-1} \theta(aq^{2i}), \quad (a_1, \ldots, a_k)_n = \prod_{i=1}^k (a_i)_n.
\]

It then follows that \(t_{kj}^N(z) \in E_{2k-N,2j-N}, \varepsilon^E(t_{kj}^N(z)) = \delta_{kj}T_{N-2k}, \Delta^E(t_{kj}^N(z)) = \sum_{p=0}^N t_{kp}^N(z) \otimes t_{pj}^N(z)\).

We first need to calculate the pairing of the generators with the matrix elements.
Theorem 5.7. The cobrading $\mathcal{E}^\text{cop} \times \mathcal{E} \rightarrow D_b^\ast$ satisfies

$$
\langle \alpha(w), t^N_{kj}(z) \rangle = \delta_{k,j} \frac{\theta(q^{2(1-N+k)}w/z, q^{2(\lambda+\alpha-k+1)})}{\theta(q^2w/z, q^{2(\lambda+1)})} T_{N-2k-1},
$$

$$
\langle \beta(w), t^N_{kj}(z) \rangle = \delta_{k,j-1} \frac{\theta(q^{2(N-k)}, q^{2(1-N+k)}w/z)}{\theta(q^2w/z, q^{2(\lambda+1)})} T_{N-2k-1},
$$

$$
\langle \gamma(w), t^N_{kj}(z) \rangle = \delta_{j,k+1} \frac{\theta(q^{2k}, q^{2(\lambda-k+2)}w/z)}{\theta(q^2w/z, q^{2(\lambda+1)})} T_{N-2k+1},
$$

$$
\langle \delta(w), t^N_{kj}(z) \rangle = \delta_{k,j} \frac{\theta(q^{2(k-\lambda-1)}, q^{2(1-k)}w/z)}{\theta(q^2w/z, q^{2(\lambda+1)})} T_{N-2k+1},
$$

$$
\langle \text{det}^{-1}(w), t^N_{kj}(z) \rangle = \delta_{k,j} q^{-N} \frac{\theta(q^2w/z)}{\theta(q^{2(1-N)}w/z)} T_{N-2k},
$$

where we consider the functions in $\lambda$ on the right hand sides as elements of $M_{b^\ast}$.

Proof. First observe that $t^N_{kj}(z) \in (\mathcal{E})_{2k-N,2j-N}$, and $\alpha(w) \in \mathcal{E}_{11} = (\mathcal{E}^\text{cop})_{11}$, $\beta(w) \in \mathcal{E}_{1,1} = (\mathcal{E}^\text{cop})_{1,1}$, $\gamma(w) \in \mathcal{E}_{-1,1} = (\mathcal{E}^\text{cop})_{-1,1}$, and $\delta(w) \in \mathcal{E}_{-1,-1} = (\mathcal{E}^\text{cop})_{-1,-1}$. So $(D_b^\ast)_{\alpha} = \{0\}$ for $\alpha \neq \beta$ and (3.1a) imply $\langle \alpha(w), t^N_{kj}(z) \rangle = 0$ unless $1 + 2j - N = 1 + 2k - N$ or $k = j$, similarly $\langle \beta(w), t^N_{kj}(z) \rangle = 0$ unless $j - k = 1$, $\langle \gamma(w), t^N_{kj}(z) \rangle = 0$ unless $k - j = 1$, and $\langle \delta(w), t^N_{kj}(z) \rangle = 0$ unless $k = j$.

From the explicit expression (5.13) we see that $t^N_{kj}(z)$ is composed of elements of the form $(k \geq 1)$

$$
t^k_{00}(z) = \delta(q^{2(k-1)}z) \cdots \delta(z), \quad t^k_{0k}(z) = \gamma(q^{2(k-1)}z) \cdots \gamma(z),
$$

$$
t^k_{kk}(z) = \beta(q^{2(k-1)}z) \cdots \beta(z), \quad t^k_{kk}(z) = \alpha(q^{2(k-1)}z) \cdots \alpha(z).
$$

Using (3.1e), the comultiplication for the generators defined by (2.10) (see [11], [16] for explicit formulas), (2.12), (3.8) we prove by induction on $k$ that the only non-zero pairings between generators and matrix elements of the form (5.15) are given by

$$
\langle \alpha(w), t^k_{00}(z) \rangle = \frac{\theta(q^{2(1-k)}w/z, q^{2(\lambda+1+k)})}{\theta(q^2w/z, q^{2(\lambda+1)})} T_{k-1}, \quad \langle \alpha(w), t^k_{kk}(z) \rangle = T_{k+1},
$$

$$
\langle \delta(w), t^k_{kk}(z) \rangle = \frac{\theta(q^{-2(1-k)}w/z, q^{-2(\lambda-k+1)})}{\theta(q^2w/z, q^{2(\lambda+1)})} T_{1-k}, \quad \langle \delta(w), t^k_{00}(z) \rangle = T_{k+1},
$$

$$
\langle \beta(w), t^k_{0k}(z) \rangle = \begin{cases} b(\lambda, w/z) T_0, & k = 1, \\ 0, & k > 1, \end{cases} \quad \langle \gamma(w), t^k_{0k}(z) \rangle = \begin{cases} c(\lambda, w/z) T_0, & k = 1, \\ 0, & k > 1. \end{cases}
$$

This proves the theorem in this particular case.

Next we treat the pairing with $t^N_{k,k}(z)$. Now $\langle \beta(w), t^N_{k,k}(z) \rangle = 0$ for all $j$ by the weight considerations in the first paragraph of the proof, and the non-zero pairings with $\alpha(w), \delta(w)$ are already contained in (5.16) for $j = N$. The only non-zero case is $\langle \gamma(w), t^N_{N-1}(z) \rangle$, and from (5.13) we see that $t^N_{N-1}(z)$ is a product of $\mu(\lambda)(CN)$ for an explicit function $CN$ times $N - 1$ $\alpha$'s and one $\beta$. Bringing the function to the right, using (3.1c), and next using the comultiplication $\Delta \gamma(w) = \alpha(w) \otimes \gamma(w) + \gamma(w) \otimes \delta(w)$, we calculate the pairing from (5.16) and (3.8), where we write the functions in $M_{b^\ast}$ as functions of
\[ \langle \gamma(w), t_{N,N-1}^N(z) \rangle = \langle \gamma(w), \alpha(q^{2(N-1)} z) \cdots \alpha(q^2 z) \beta(z) \rangle \circ (T_{N-2} C_N)(\lambda) \]
\[ = \left( \langle \alpha(w), \alpha(q^{2(N-1)} z) \cdots \alpha(q^2 z) \rangle T_1 \langle \gamma(w), \beta(z) \rangle + \langle \gamma(w), \alpha(q^{2(N-1)} z) \cdots \alpha(q^2 z) \rangle T_{-1} \langle \delta(w), \beta(z) \rangle \right) \circ C_N(\lambda + N - 2) \]
\[ = T_{-(N-1)-1} \circ T_1 \circ c(\lambda, \frac{w}{z}) C_N(\lambda + N - 2) = \frac{\theta(q^{2N}, q^{2(N+2)w/z})}{\theta(q^{2w/z}, q^{2(\lambda+1)})} T_{-N+1}. \]
Similarly, we can establish the pairing of a generator with \( t_{0j}^N(z) \). Now \( \langle \gamma(w), t_{0j}^N(z) \rangle = 0 \) for all \( j \), and the non-zero pairings with \( \alpha(w), \delta(w) \) are already contained in (5.16) for \( j = 0 \). The only non-zero case is \( \langle \beta(w), t_{0j}^N(z) \rangle \), and we prove similarly
\[ \langle \beta(w), t_{0j}^N(z) \rangle = \frac{\theta(q^{2N}, q^{-2(\lambda+N)w/z})}{\theta(q^{2w/z}, q^{2(\lambda+1)})} T_{N-1}. \]

After these preparations we can use the second expression of (5.13) to calculate the pairing of a generator with an arbitrary matrix element. We give the details for the pairing \( \langle \delta(w), t_{kj}^N(z) \rangle \).

\[ \langle \delta(w), t_{kj}^N(z) \rangle = \sum_{l=\min(k,j)}^{\min(k,j)} \langle \delta(w), t_{0j-l}^N(z) \rangle T_1 \langle \gamma(w), t_{kk}^k(q^{2(N-k)} z) \rangle + \sum_{l=\max(0,j+k-N)}^{\min(k,j)} \langle \delta(w), t_{0j-k}^N(z) \rangle T_{-1} \langle \delta(w), t_{kk}^k(q^{2(N-k)} z) \rangle. \]

The summand in the first sum is non-zero if and only if \( j - l = 1 \) and \( l = k - 1 \), and the summand in the second sum is non-zero if and only if \( j = l \) and \( k = l \), so we are left with only two non-zero terms in the case \( k = j \) and no non-zero terms in case \( k \neq j \). For the pairing with \( \alpha(w), \beta(w) \) and \( \gamma(w) \) instead of \( \delta(w) \) we only get at most one non-zero term, so that these case are simpler. So we obtain \( \langle \delta(w), t_{kj}^N(z) \rangle \)

\[ = \delta_{kj} \left( \langle \beta(w), t_{0j-k}^N(z) \rangle T_1 \langle \gamma(w), t_{kk}^k(q^{2(N-k)} z) \rangle + \delta(w), t_{00}^N(z) \rangle T_{-1} \langle \delta(w), t_{kk}^k(q^{2(N-k)} z) \rangle \right) \]
\[ = \delta_{kj} \left( \frac{\theta(q^{2N-k}, q^{-2(\lambda+k)w/z})}{\theta(q^{2w/z}, q^{-2(\lambda+1)})} T_{-N+k-1} \frac{\theta(q^{2k}, q^{2(\lambda-k+1)w/z})}{\theta(q^{2(1+k-N)w/z}, q^{2(\lambda+1)})} T_{-k+1} \right. \]
\[ + T_{N-k+1} T_{-1} \frac{\theta(q^{2N-k}, q^{-2(\lambda+k)w/z})}{\theta(q^{2w/z}, q^{-2(\lambda+1)})} \]
\[ \left. \frac{\theta(q^{2(k-1)}, q^{-2(\lambda+k-1)w/z})}{\theta(q^{2(1+k-N)w/z}, q^{2(\lambda+1-k+1)})} \right) T_{N-k+2}. \]

It remains to calculate the term in parentheses, and this is done using the addition theorem (2.15) with \((x,y,z,w) \) replaced by \((q^{-\lambda} \sqrt{w/z}, q^{2(N-k)+\lambda} \sqrt{z/w}, q^{\lambda+2} \sqrt{w/z}, q^{2k-\lambda-2} \sqrt{z/w}) \) to rewrite the numerator of the first quotient. A straightforward calculation gives the required result. We note that the calculation for the pairing with \( \alpha(w), \beta(w) \) and \( \gamma(w) \) instead of \( \delta(w) \) does not require the addition formula (2.15).

Having the pairings with the generators \( \alpha(w), \beta(w), \gamma(w), \delta(w) \), we can calculate the pairing with \( \det(w) \) and from this derive the pairing with \( \det^{-1}(w) \) as in §3.1 using (2.15) again. \( \square \)
Now we combine Theorem 5.7 with Proposition 5.1 for the corresponding left corepresentation \( V^N \) as in (5.11), (5.12). This yields the dynamical representation of \( \mathcal{E}^{opp} \) on \( V^N \) given by
\[
\pi(\alpha(w))(\mu_{V^N}(f)\nu_k(z)) = \mu_{V^N} \left( \frac{\theta(q^{2(1-N+k)w/z}, q^2(\lambda+N-k+2))}{\theta(q^{2w/z}, q^2(\lambda+2))} (T_1 f) \right) \nu_k(z),
\]
\[
\pi(\beta(w))(\mu_{V^N}(f)\nu_k(z)) = \mu_{V^N} \left( \frac{\theta(q^{2(N-k)}, q^{-2(\lambda-1+N-k)w/z})}{\theta(q^{2w/z}, q^{-2\lambda})} (T_{-1} f) \right) \nu_{k+1}(z),
\]
\[
\pi(\gamma(w))(\mu_{V^N}(f)\nu_k(z)) = \mu_{V^N} \left( \frac{\theta(q^{2k}, q^{2(\lambda-k+3)w/z})}{\theta(q^{2w/z}, q^{2(\lambda+2)})} (T_1 f) \right) \nu_{k-1}(z),
\]
\[
\pi(\delta(w))(\mu_{V^N}(f)\nu_k(z)) = \mu_{V^N} \left( \frac{\theta(q^{2(k-\lambda)}, q^{2(1-k)w/z})}{\theta(q^{2w/z}, q^{2\lambda})} (T_{-1} f) \right) \nu_k(z),
\]
\[
\pi(\text{det}^{-1}(w))(\mu_{V^N}(f)\nu_k(z)) = q^{-N} \frac{\theta(q^{2w/z})}{\theta(q^{2(1-N)w/z})} \mu_{V^N}(f)\nu_k(z).
\]

We now proceed as follows to determine the pairing of matrix elements. First we use the dynamical representation of \( \mathcal{E}^{opp} \) in (5.17) and the explicit expression (5.13) to calculate \( \pi(t^{M}_{rs}(w))\nu_k(z) \) explicitly as a multiple of \( \nu_{k-s+r}(z) \), whereas from the definition in Proposition 5.1 we have
\[
\pi(t^{M}_{rs}(w))\nu_k(z) = \sum_{j=0}^{N} T_{2s-M}(t^{M}_{rs}(w), t^{N}_{kj}(z)) \otimes \nu_j(z) = \sum_{j=0}^{N} \mu_{V} \left( T_{2s-M}(t^{M}_{rs}(w), t^{N}_{kj}(z)) \right) \nu_j(z),
\]
and upon comparing these expression we obtain the desired result in Theorem 5.8. In order to state the result we need some notations from special functions in a special case. The (very-well-poised) elliptic hypergeometric series is defined by
\[
\eta_{r+1}(a_1; a_6, \ldots, a_{r+1}) = \sum_{n=0}^{\infty} \frac{\theta(a_1 q^n)}{\theta(a_1)} \frac{q^{2n}(a_1, a_6, \ldots, a_{r+1})}{(q^2, q^2 a_1, q^2 a_2, \ldots, q^2 a_{r+1})/(q^2 a_6, \ldots, q^2 a_{r+1})},
\]
assuming the elliptic balanced condition \((a_6 a_7 \ldots a_{r+1})^2 q^4 = (a_1 q^2)^{r-5}\) and where we have used the notation (5.14). This is not the most general definition, but it suffices for our purposes, see [13, §11.2], [28] for a discussion and references. Compared to these references we have switched from base \( q \) to base \( q^2 \), and we have specialised \( z = 1 \). In this paper all elliptic hypergeometric series are terminating series.

**Theorem 5.8.** The pairing between matrix elements of the irreducible corepresentations as in (5.13) are given by
\[
\langle t^{M}_{rs}(w), t^{N}_{kj}(z) \rangle = \delta_{s+j,r+k} C \eta_{r+1}(q^{2(\lambda+M-2s-r+1)}; q^{-2r}, q^{-2s}, q^{2(\lambda+s+1)}, q^{2(\lambda+M+N-k-s-r+2)}, q^{2(\lambda+M-k-s-r+1)}, q^{2(k-s)} w/z, q^{2(M-N+k+s+1)} w/z) T_{N+M-2s-2j},
\]
\[
C = (-1)^{M-r-s} q^{(M-r-s)(\lambda-s+1)} \frac{(q^{2(M-s-r+1)}, q^{2(k-s+1)}, q^{2(\lambda+s-k-r+2)} w/z)}{(q^2, q^{2(\lambda+M-2s-r+2)}, q^{2(M-r-s+1)} w/z)}
\times \frac{q^{2(\lambda-M+2s)} q^{-2(\lambda-N+k+s)} w/z}{(q^2(\lambda-M-2s), q^{2(M-r+1)} w/z)}
\times \frac{q^{2(k+s+r-\lambda-M)} q^{2(s-k+1)} w/z}{(q^{2(\lambda+1)}, q^{2w/z})_{M-r-s}.
\]

Before going into the proof of Theorem 5.8, we show how the quantum dynamical Yang-Baxter equation, the biorthogonality relations, Bailey's transformation formula, and it is well known that a corollary to Bailey's transformation is Jackson's summation formula, are implied by Theorem 5.8. Since we re-obtain the well-known properties of the elliptic hypergeometric series already obtained by Frenkel and Turaev [12], we only sketch the derivation.

First, by taking \( a = t_{rs}^M(w), b = t_{kj}^N(z) \) in the cobrading property (3.3) we obtain an identity in \( \mathcal{E} \), pairing this identity with an element \( t_{rs}^M(u) \in \mathcal{E}^{\text{cop}} \) gives an identity in \( D^* \). By Theorem 5.8 the resulting identity is trivial unless \( r + k + a = s + j + b \). In the latter case, it gives an identity of the form

\[
\sum_{\text{single}} \chi_{12} V_{11} V_{12} V_{11} = \sum_{\text{single}} \chi_{12} V_{11} V_{12} V_{11},
\]

which can be rephrased as an \( R \)-matrix satisfying the quantum dynamical Yang-Baxter equation (2.7), the case \( L = M = N = 1 \) corresponding to the \( R \)-matrix (2.11), see [12], or as an elliptic analogue of Wigner's symmetry for the \( 9j \)-symbols.

The proof of the biorthogonality relations for the elliptic hypergeometric series is essentially the same as in [16], i.e. we pair \( \sum_{k=0}^N t_{jk}^N(z) S^A(t_{kN}^N(z)) = \delta_{ij} 1_\mathcal{E} \) with an arbitrary \( t_{rs}^M(w) \). For this we also need the pairing \( (t_{rs}^M(w), S^A(t_{kj}^N(z))) \), which can be calculated using the unitarity of the corepresentation \( t^N \) as in [16]. Also, Bailey's transform can be obtained from the unitarity of the corepresentation \( t^N \), see [16].

**Proof of Theorem 5.8.** Observe that \( T_{2s-M}(t_{rs}^M(w), t_{kj}^N(z)) \in (D^*)_{2j-N,2r+2k-N-2s} \) so that there is only a non-zero contribution for \( j + s = k + r \). As noted previously, it suffices to calculate \( \pi(t_{rs}^M(w)) v_k(z) \) explicitly as a multiple of \( v_{k-s+r}(z) \). Now By Proposition 5.1(ii) \( \pi \) is an antimultiplicative representation of \( \mathcal{E}^{\text{cop}} \), so that (5.13) gives \( \pi(t_{rs}^M(w)) v_k(z) \)

\[
= \sum_{l=\max(0,r+s-M)}^{\min(r,s)} \pi(\beta(q^{2(M-r)}w)) \cdots \pi(\beta(q^{2(M-l-1)}w)) \pi(\alpha(q^{2(M-l)}w)) \cdots \pi(\alpha(q^{2(M-l)}w)) \\
\times \pi(\delta(w)) \cdots \pi(\delta(q^{2(M-s-r+l+1)}w)) \pi(\gamma(q^{2(M-s-r+l+1)}w)) \cdots \pi(\gamma(q^{2(M-s-l)}w)) \\
\times \pi(\mu_i^j \left( \frac{(q^{2(-M+r-2s+2l+2)})_i}{(q^{2(-M-2s+2l+2)})_s-i} \right) v_k(z)).
\]

It is now a tedious, but straightforward verification using (5.17) that we can rewrite (5.20) as an elliptic hypergeometric series of the type (5.18) times \( v_{k-s+r}(z) \). Since (5.20) is also equal to \( \sum_{j=0}^N T_{2s-M}(t_{rs}^M(w), t_{kj}^N(z)) \otimes v_j(z) \), we find the result. \( \square \)

6. **Singular and spherical vectors**

From the previous sections it is clear that we can calculate all pairings for the case of the elliptic \( U(2) \) dynamical quantum group in detail. In order to deal with more general cases we want to have the dynamical analogue of notions as spherical functions on symmetric spaces which we want to realize as subalgebras satisfying certain invariance conditions stated in terms of the actions defined in Theorem 4.1. The purpose of this section is to start a description for these notions in the setting of dynamical quantum groups, and to give the details for the case of the elliptic \( U(2) \) dynamical quantum group. We expect to deal with more involved examples in the future for which the notions in §6.1 are required.

6.1. **Subalgebras of \( \mathfrak{h} \)-algebras.** Using the notions of §2.1 we say that the \( \mathfrak{h} \)-prealgebra \( B \) is a \( \mathfrak{h} \)-subalgebra of the \( \mathfrak{h} \)-algebra \( A \) if there exists injective \( \mathfrak{h} \)-prealgebra homomorphism \( \iota : B \to A \), and similarly a \( \mathfrak{h} \)-algebra \( B \) is a \( \mathfrak{h} \)-subalgebra of the \( \mathfrak{h} \)-algebra \( A \) if there exists an injective unital \( \mathfrak{h} \)-algebra homomorphism \( \iota : B \to A \).
For later use we need the notion of subalgebras of $\mathfrak{h}$-prealgebras and $\mathfrak{h}$-algebras with not necessarily the same Lie algebra $\mathfrak{h}$. We do not need this for the particular example of the elliptic $U(2)$ dynamical quantum group, but we introduce the notion for use in future work. Assume $\tau$ is a complex vector space that is a subspace of the complex vector space $\mathfrak{h}$. Let $i : \tau \to \mathfrak{h}$ be the corresponding injection. Then the restriction $r : \mathfrak{h}^* \to \tau^*$ is given by $r(\alpha)(X) = \alpha(i(X))$ for $\alpha \in \mathfrak{h}^*$, $X \in \tau$. Note $r$ is a surjective linear map. For a function $f$ on $\tau^*$ we define the function $j(f)$ on $\mathfrak{h}^*$ by $j(f)(\alpha) = f(r(\alpha))$. It follows that $j : M^*_{\tau} \to M^*_{\mathfrak{h}}$. We assume this situation in the remainder of this section.

For a $\mathfrak{h}$-prealgebra $A$ we define the weights as $w(A) = \{ \alpha \in \mathfrak{h}^* \mid \exists \beta \in \mathfrak{h}^* \text{ such that } A_{\alpha \beta} \neq \{0\} \text{ or } A_{\beta \alpha} \neq \{0\} \} \subset \mathfrak{h}^*$.

A $\tau$-prealgebra $A$ is a $(\tau, \mathfrak{h})$-presubalgebra of the $\mathfrak{h}$-prealgebra $A$ if $\tau$ is a subspace of $\mathfrak{h}$, and such that there exists a map $s : w(A) \to w(A)$ with $r \circ s = \text{Id}_{w(A)}$ and an injective $\C$-linear map $i : B \to A$ such that $s(B_{\sigma, \tau}) \subseteq A_{s(\sigma), s(\tau)}$ for all $\sigma, \tau \in w(B)$, and such that $\mu_{B}(f) b = \mu_{A}(j(f)) i(b)$ and $\mu_{B}(f) b = \mu_{A}(j(f)) i(b)$ for all $f \in M^*_{\tau}$ and all $b \in B$.

A $\tau$-algebra $B$ is a $(\tau, \mathfrak{h})$-subalgebra of the $\mathfrak{h}$-algebra $A$ if $B$ is a $(\tau, \mathfrak{h})$-presubalgebra of $A$ such that $i$ is an injective unital algebra homomorphism.

In particular, this definition makes that a part $B_{s(\sigma), s(\tau)}$ of the decomposition for $B$ ends up in exactly one part $A_{s(\sigma), s(\tau)}$ of the decomposition of $A$. Requiring only the weaker condition $s(B_{\sigma, \tau}) \subseteq A_{\alpha \beta}$ does not imply this property. Note that in case $\tau = \mathfrak{h}$ we take $s$ to be the identity, and we obtain the notions of $\mathfrak{h}$-subprealgebra and $\mathfrak{h}$-subalgebra.

Using the map $s$ in the definition of $B$ being a $(\mathfrak{h}, \tau)$-subprealgebra of $A$ we extend the map $j : M^*_{\tau} \to M^*_{\mathfrak{h}}$ to shift operators of the form $\Delta_{\tau}^{\mathfrak{h}}(f)$ with $\alpha \in w(B)$ by putting $\Delta_{\tau}^{\mathfrak{h}}(f) = j(f)^{\tau} \circ i(f)$ for all $f \in M^*_{\tau}$ and all $b \in B$. A $\tau$-algebra $B$ is a $(\mathfrak{h}, \tau)$-subalgebra of the $\mathfrak{h}$-algebra $A$ if $B$ is a $(\mathfrak{h}, \tau)$-presubalgebra of $A$ such that $i$ is an injective unital algebra homomorphism.

6.2. Singular vectors. We assume that $U$ and $A$ are paired $\mathfrak{h}$-bialgebroids and that $\iota : W \to U$ makes the $\tau$-prealgebra $W$ into a $(\tau, \mathfrak{h})$-subprealgebra of $U$.

**Definition 6.1.** (i) Let $R : V \to V \otimes A$ be a right corepresentation of $A$ in the $\mathfrak{h}$-space $V$. A vector $v \in V$ is a $W$-singular vector if $\pi(\mu(Y))v = 0$ for all $Y \in W$ with $\pi(X) = (\text{Id} \otimes (X, \cdot))T_{\beta} \circ R$, $X \in U_{\alpha \beta}$, as in Proposition 5.1(i).

(ii) Let $L : V \to A \otimes V$ be a left corepresentation of $A$ in the $\mathfrak{h}$-space $V$. A vector $v \in V$ is a $W$-singular vector if $\pi(\imath(Y))v = 0$ for all $Y \in W$ with $\pi(X) = (T_{\alpha}(X, \cdot) \otimes \text{Id}) \circ L$, $X \in U_{\alpha \beta}$, as in Proposition 5.1(ii).

**Remark 6.2.** (i) By decomposing a $W$-singular vector into homogeneous components $v = \sum_{\gamma} v_{\gamma}$, $v_{\gamma} \in V_{\gamma}$, of the $\mathfrak{h}$-space $V$, we obtain that each of the homogeneous components is a $W$-singular vector. So we can assume without loss of generality that the $W$-singular vector is homogeneous.

(ii) Note that the space of $W$-singular vectors is a vector space over $M_{\mathfrak{h}}$. For the case of a right corepresentation $R$ we see that for $Y \in W_{\tau \sigma}$ we have $\pi(Y)(\mu_{A}(Y))v = 0$.

(iii) Assume that $R : V \to V \otimes A$ is a right corepresentation of $A$ in the $\mathfrak{h}$-space $V$, then for $Y \in W_{\tau \sigma}$ we have from Proposition 5.1(i) $\pi(\mu_{A}(Y))V_{\gamma} \subseteq V_{\gamma+s(\sigma)-s(\tau)}$. Hence $V_{\gamma}$ consists of $W$-singular vectors in case the weight $\gamma + s(\sigma) - s(\tau)$ does not occur in $V$ for all $\sigma, \tau \in w(W)$. A similar remark applies to singular vectors in left corepresentations.

**Example 6.3.** Let $\mathcal{W}$ be the $\mathfrak{h}$-prealgebra generated by $\beta(z)$ viewed as subalgebra of $\mathcal{E}^\text{opp}$. Consider the left corepresentation $L$ of $\mathcal{E}$ on $V^{N}$ defined by (5.12). It follows from (5.17) that $v_{N}(z)$ is a $W$-singular vector. Similarly, $v_{0}(z)$ is a $W$-singular vector for $W'$ be the $\mathfrak{h}$-prealgebra generated by


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Y(z) viewed as subalgebra of \( E^{\op} \). Combining this with the terminology of Example 6.6, \( v_0(z) \) is lowest weight vector and \( v_N(z) \) is highest weight vector with the weight of \( v_0(z) \), respectively \( v_N(z) \), being \(-N\) respectively \( N\).

6.3. Spherical vectors and spherical corepresentations. We assume that \( \mathcal{U} \) and \( \mathcal{A} \) are paired \( \mathfrak{h} \)-bialgebroids, and that \( \iota: \mathcal{W} \rightarrow \mathcal{U} \) makes the \( r \)-prealgebra \( \mathcal{W} \) into a \((r, \mathfrak{h})\)-subprealgebra of \( \mathcal{U} \).

Definition 6.4. (i) Let \( R: V \rightarrow V \otimes \mathcal{A} \) be a right corepresentation of \( \mathcal{A} \) in the \( \mathfrak{h} \)-space \( V \). A vector \( v \in V \) is a \( \mathcal{W} \)-spherical vector if \( \pi(\iota(Y))v = \mu_V(\varepsilon^U(\iota(Y))1)v \) for all \( Y \in \mathcal{W} \) with \( \pi(X) = (\text{Id} \otimes \langle X, \cdot \rangle)T_{\beta} \circ R \), \( X \in \mathcal{U}_{\alpha\beta} \), as in Proposition 5.1.(i).

(ii) Let \( L: V \rightarrow \mathcal{A} \otimes V \) be a left corepresentation of \( \mathcal{A} \) in the \( \mathfrak{h} \)-space \( V \). A vector \( v \in V \) is a \( \mathcal{W} \)-spherical vector if \( \pi(\iota(Y))v = \mu_V(\varepsilon^U(\iota(Y))1)v \) for all \( Y \in \mathcal{W} \) with \( \pi(X) = (\text{Id} \otimes \langle X, \cdot \rangle)T_{\gamma} \circ L \), \( X \in \mathcal{U}_{\alpha\beta} \), as in Proposition 5.1.(ii).

Note that Definition 6.4 coincides with Definition 6.1 in case \( \varepsilon^U(\iota(Y))1 = 0 \) for all \( Y \in \mathcal{W} \).

Remark 6.5. (i) By decomposing a \( \mathcal{W} \)-spherical vector into homogeneous components \( v = \sum_{\gamma} v_{\gamma} \), \( v_{\gamma} \in V_{\gamma} \), of the \( \mathfrak{h} \)-space \( V \), we obtain that each of the homogeneous components is a \( \mathcal{W} \)-spherical vector. So we can assume without loss of generality that the \( \mathcal{W} \)-spherical vector is homogeneous.

(ii) Assume then that \( v \in V_{\gamma} \) is \( \mathcal{W} \)-spherical for \( \gamma \) a right or left corepresentation of \( \mathcal{A} \), then for \( g \in M_{r^*} \) we have \( \mu_V(j(g))v = \mu_V(\varepsilon^U(\mu_{r^*}(g))1)v = \pi(\mu_{r^*}(j(g))v) = \mu_V(\varepsilon^U(\iota(Y))1)v, \) see Proposition 5.1. So \( j(g) = T_{\gamma, r^*}(g) \) for all \( g \in M_{r^*} \), or \( T_{-\gamma, r^*}(g) = g \), so we obtain \( r(\gamma) = 0 \).

(iii) Note that the space of \( \mathcal{W} \)-spherical vectors is in general not a vector space over \( M_{r^*} \). For \( Y \in \mathcal{W} \otimes \mathcal{T} \) we have \( \mu_V(\varepsilon^U(\iota(Y))1)v = \pi(\mu_{r^*}(j(g))v) = \mu_V(T_{\gamma, r^*}(g))v \) for all \( f, g \in M_{r^*} \), so that for \( \mathcal{W} \)-spherical vector \( v \) in a right corepresentation we have \( \pi(\iota(Y))(\mu_V(f)v) = \mu_V(T_{\gamma, r^*}(f))v \) unless \( s(\sigma) = 0 \) or \( \varepsilon^U(\iota(Y))1 = 0 \). For Example 6.6 we are in the first case. Put \( M_{r^*}(w(\mathcal{W})) \) to be the \( \mathbb{C} \)-linear space of functions \( f \in M_{r^*} \) satisfying \( T_{-\sigma, r^*}(f) = f \) for all \( \sigma \in w(\mathcal{W}) \) such that there exists a \( Y \in \mathcal{W}_{\sigma^*} \) for some \( \tau \in r^* \) with \( \varepsilon^U(\iota(Y))1 \neq 0 \) in \( M_{r^*} \). Then we see that in general the space of \( \mathcal{W} \)-spherical functions is a vector space over \( M_{r^*}(w(\mathcal{W})) \), which is a space of meromorphic functions on \( \mathfrak{h}^* \) with specific periodicity properties.

Example 6.6. Put \( I_{r^*} = M_{r^*} \otimes M_{r^*} \), and define the bigrading by \( I_{r^*} = (I_{r^*})_{00} \) and the left and right moment map by \( \mu_{l_{r^*}}(f) = f \cdot 1, \mu_{r_{r^*}}(g) = 1 \cdot g \). Then \( I_{r^*} \) is a \( \mathfrak{h} \)-algebra, see Rosengren [26] for more properties of \( I_{r^*} \). For any \( \mathfrak{h} \)-algebra \( \mathcal{U} \) the map \( \iota: I_{r^*} \rightarrow \mathcal{U}, \iota(f \otimes g) = \mu_{l_{r^*}}(f)\mu_{r_{r^*}}(g) \), shows that \( I_{r^*} \) is a \( \mathfrak{h} \)-subalgebra of \( \mathcal{U} \). With the notation of Definition 6.4 we find that a spherical vector \( v \in V \) in a right corepresentation space satisfies \( \mu_V(fg)v = \mu_V(\varepsilon^U(\iota(f \otimes g))1)v = \pi(\mu_{l_{r^*}}(f)\mu_{r_{r^*}}(g))v = \mu_V(f(T_{\gamma, r^*}(g))v \) for all \( f, g \in M_{r^*} \), so that \( v \) is an \( I_{r^*} \)-spherical vector if and only if \( v \in V_0 \), and \( V_0 \) is the space of \( I_{r^*} \)-spherical vectors. In this case \( M_{r^*}(w(I_{r^*})) = M_{r^*} \).

Note that we can extend this to define the weight of a vector \( v \) in the representation space \( V \) as \( \sum_{\gamma} \pi(\iota(f \otimes g))v = \mu_V(f(T_{\gamma, r^*}(g))v \) so this characterises \( v \in V_\gamma \).

In particular for the left corepresentation \( L \) of \( \mathcal{E} \) on \( V^N \) defined by (5.12), we see that there is a spherical vector only for \( N \in 2\mathbb{N} \), and then \( v_{N/2}(z) \) is the spherical vector.

We are interested in the matrix elements of a left or right corepresentation corresponding to a spherical vector, and in particular in their invariance properties. Assume that \( \mathcal{U} \) and \( \mathcal{A} \) are paired \( \mathfrak{h} \)-bialgebroids, that \( R \) is a right corepresentation of \( \mathcal{A} \), and that \( \mathcal{W} \) is a \((\mathfrak{h}, r)\)-subprealgebra of \( \mathcal{U} \).
Assume we have a basis \( \{v_i\}_{i \in I} \) of the \( \mathfrak{h} \)-space \( V \) such that \( v_0 \) is a \( \mathcal{W} \)-spherical vector, and let
Proposition 6.7. Assume that \( \mathcal{U} \) and \( \mathcal{A} \) are paired \( \mathfrak{h} \)-Hopf *-algebroids, and that \( \mathcal{W} \) is a \( (\mathfrak{h}, \tau) \)-subprealgebra of \( \mathcal{U} \) given by \( \iota : \mathcal{W} \rightarrow \mathcal{U} \).

(i) Let \( \{v_i\}_{i \in I} \) be a homogeneous orthogonal basis for the right unitarisable corepresentation \( R : \mathcal{V} \rightarrow \mathcal{V} \otimes \mathcal{A} \), \( Rv_i = \sum_{k \in I} v_k \otimes R_{ki} \) such that \( \langle v_i, v_j \rangle = \delta_{ij} N_j \). Assume that \( v_0 \in \mathcal{V} \) is a \( \mathcal{W} \)-spherical vector, then

\[
\iota(Y) \cdot (\mu^R_0(N_0)) = \mu^R_0(\varepsilon^U(\iota(Y))1)(\mu^R_0(N_0)),
\]

\[
(\mu^R_0(N_0)) \cdot \iota(Y)^* = \mu^R_0(\varepsilon^U(\iota(Y)^*)1)(\mu^R_0(N_0)).
\]

(ii) Let \( \{v_i\}_{i \in I} \) be a homogeneous orthogonal basis for the left unitarisable corepresentation \( L : \mathcal{V} \rightarrow \mathcal{A} \otimes \mathcal{V} \), \( Lv_i = \sum_{k \in I} L_{ik} \otimes v_k \) such that \( \langle v_i, v_j \rangle = \delta_{ij} N_j \). Assume that \( v_0 \in \mathcal{V} \) is a \( \mathcal{W} \)-spherical vector, then

\[
(\mu^L_0(N_0)L_00) \cdot \iota(Y) = \mu^L_0(\varepsilon^U(\iota(Y))1)(\mu^L_0(N_0)L_00),
\]

\[
\iota(Y)^\dagger \cdot (\mu^L_0(N_0)L_00) = \mu^L_0(\varepsilon^U(\iota(Y)^\dagger)1)(\mu^L_0(N_0)L_00).
\]

Proof. For the first statement, the first equality has been derived in more generality and (4.3) gives the first equation of (6.1). For the second we use the first in combination with (5.9) to find

\[
(\mu^R_0(N_0)) \cdot \iota(Y)^* = S^A(\iota(Y) \cdot (\mu^R_0(N_j)R_{j0}))^* = S^A(\mu^R_0(\varepsilon^U(\iota(Y))1)\mu^R_0(N_j)R_{j0}))^*
\]

\[
= \mu^R_0(N_j)\mu^R_0(\varepsilon^U(\iota(Y)^*1)) S^A(R_{j0})^*
\]

\[
= \mu^R_0(N_j)\mu^R_0(\varepsilon^U(\iota(Y)^*1)) \mu^A_\iota(N_0)\mu^R_0(N_j^{-1})R_{j0}
\]

\[
= \mu^R_0(\varepsilon^U(\iota(Y)^*1)) \mu^R_0(N_0)R_{j0}
\]

using (5.9) again and taking \( j = 0 \) gives the result. The second statement follows similarly. \( \square \)

Example 6.8. In the case \( \mathcal{W} = I_{\mathfrak{h}^*} \) as in Example 6.6, Proposition 6.7 says that the spherical functions \( \mu^R_0(N_0)) \) and \( \mu^L_0(N_0) \) are elements of \( \mathcal{A}_{00} \).

Because of Proposition 6.7 it is natural to consider the spaces

\[
\mathcal{A}^{\mathcal{W}} = \{ a \in \mathcal{A} | \iota(Y) \cdot a = \mu^R_0(\varepsilon^U(\iota(Y))1)a, \forall Y \in \mathcal{W} \},
\]

\[
\mathcal{W} \mathcal{A} = \{ a \in \mathcal{A} | \iota(Y)^\dagger \cdot a = \mu^R_0(\varepsilon^U(\iota(Y)^\dagger)1)a, \forall Y \in \mathcal{W} \}
\]

for a right or left unitarisable corepresentation, but in general we need more information on \( \mathcal{W} \) in order to be able to say more on \( \mathcal{A}^{\mathcal{W}} \) and \( \mathcal{W} \mathcal{A} \), except being \( \mathbb{C} \)-linear spaces. E.g. in case \( \varepsilon^U \circ \iota : \mathcal{W} \rightarrow D_{\mathfrak{h}^*} \) is zero, we see that \( \mathcal{A}^{\mathcal{W}} \) and \( \mathcal{W} \mathcal{A} \) are invariant under multiplication by \( \mu^R_0(f), \mu^L_0(f) \) for any \( f \in M_{\mathfrak{h}^*} \), and if moreover \( \Delta^U(\iota(W)) \subset \mathcal{U} \otimes \mathcal{U} + \iota(W) \otimes \mathcal{U} \) we see that \( \mathcal{A}^{\mathcal{W}} \) and \( \mathcal{W} \mathcal{A} \) are \( \mathfrak{h} \)-subalgebras of \( \mathcal{A} \) by Theorem 4.1.

So in this context it is natural to use the notion of \( \mathfrak{h} \)-coideal as introduced by Rosengren [26, Def. 4.5].

Definition 6.9. A \( \mathbb{C} \)-linear subspace \( I \) of a \( \mathfrak{h} \)-bialgebroid \( \mathcal{U} \) is a \( \mathfrak{h} \)-coideal if
(i) \( I \) is a 2-sided ideal in \( \mathcal{U} \) as an associative algebra,

(ii) \( (\mathcal{U}_{\alpha\beta} + I) \cap (\mathcal{U}_{\beta\alpha} + I) = I \) for \((\alpha, \beta) \neq (\gamma, \delta)\),

(iii) \( \Delta^I(X) \in \mathcal{U} \otimes I + I \otimes \mathcal{U} \) for all \( X \in I \),

(iv) \( \varepsilon^I(X) = 0 \) for all \( X \in I \).

Assuming that \( \mathcal{U} \) and \( \mathcal{A} \) are paired \( h \)-bialgebroids and that \( I \subset \mathcal{U} \) is a \( h \)-coideal, we have the following Proposition, whose proof is left to the reader.

**Proposition 6.10.** Let \( \mathcal{U} \) and \( \mathcal{A} \) be paired \( h \)-bialgebroids and assume that \( I \subset \mathcal{U} \) is a \( h \)-coideal. Put \( \mathcal{A}^I = \{ a \in A \mid X \cdot a = 0 = a \cdot X, \forall X \in I \} \), then \( \mathcal{A}^I \) is a \( h \)-subalgebra of \( A \). Moreover, in case \( \mathcal{U} \) and \( \mathcal{A} \) are paired \( h \)-Hopf \( * \)-algebroids and \( I \) is \( * \circ S^U \)-invariant, then \( \mathcal{A}^I \) is \( * \)-invariant.

**Example 6.11.** In order to give an example of a \( h \)-coideal, we first consider group-like elements in a \( h \)-coalgebroid or in a \( h \)-bialgebroid. We say that a homogeneous element \( X \in \mathcal{U}_{\alpha\beta} \) is a group-like element if \( \Delta^U(X) = X \otimes X \) and \( e^U(X) = T_{-\alpha} \). Note that this forces \( \beta = \alpha \). For two group-like elements \( X, Y \in \mathcal{U}_{\alpha\beta} \) we have

\[
\Delta^U(X - Y) = (X - Y) \otimes X + Y \otimes (X - Y), \quad e^U(X - Y) = 0,
\]

so the 2-sided ideal generated by differences of group like elements satisfies Definition 6.9(i), (iii), (iv), and Definition 6.9(ii) has to be checked in specific examples.

Consider an invertible element \( f \in M_{h^*} \), and let \( \mu^U_f(f)\mu^U_{f^{-1}}(f^{-1}) \in U_{00} \) then \( \varepsilon^U(\mu^U_f(f)\mu^U_{f^{-1}}(f^{-1})) = ff^{-1} = 1 \in D_{h^*} \) and

\[
\Delta^U(\mu^U_f(f)\mu^U_{f^{-1}}(f^{-1})) = (\mu^U_f(f)) \otimes (\mu^U_{f^{-1}}(f^{-1})) = (\mu^U_f(f)) \otimes (\mu^U_{f^{-1}}(f^{-1})) \otimes (\mu^U_f(f)) \mu^U_{f^{-1}}(f^{-1})
\]

using (2.4) in the last equality. So \( \mu^U_f(f)\mu^U_{f^{-1}}(f^{-1}) \) is a group-like element for any invertible \( f \in M_{h^*} \). Let \( I \) be the 2-sided ideal generated by \( \mu^U_f(f)\mu^U_{f^{-1}}(f^{-1}) - \mu^U_g(g)\mu^U_{g^{-1}}(g^{-1}) \), \( f, g \in M_{h^*} \) invertible. Then it is easily checked that \( I \) is a \( h \)-Hopf coideal, which is \( S^U \)-invariant in case \( \mathcal{U} \) is a \( h \)-Hopf algebroid and moreover \( * \)-invariant if \( \mathcal{U} \) is \( h \)-Hopf \( * \)-algebroid. In case \( \mathcal{U} \) and \( \mathcal{A} \) are paired \( h \)-Hopf algebroids, we have \( \mathcal{A}^I = \mathcal{A}_{00} \).

**References**


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