A SIMPLE SOLUTION OF HILBERT’S FOURTEENTH PROBLEM
IN DIMENSION FIVE

BY

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Abstract. We give a short proof of a counterexample (due to Daigle and Freudenburg) to Hilbert’s fourteenth problem in dimension five.

Introduction. In 1900 at the International Congress of Mathematicians in Paris David Hilbert presented a list of 23 problems, intended to challenge the mathematicians of the new century. The fourteenth problem of this list can be stated as follows: let \( k \) be a field, \( k[x] := k[x_1, \ldots, x_n] \) the polynomial ring, \( k(x) \) its quotient field and \( L \) a subfield containing \( k \).

Is \( L \cap k[x] \) a finitely generated \( k \)-algebra?

A positive answer was given by Zariski ([7]) in case \( \text{trdeg}_k L \leq 2 \). However in 1958 Nagata ([5]) constructed a counterexample in dimension 32. Then in 1988 Roberts ([6]) found a new counterexample in dimension 7. Recently, in 1998 Freudenburg ([2]), studying Robert’s example, found a 6-dimensional counterexample, from which a 5-dimensional example was obtained in 1999 by Daigle and Freudenburg in [1]: they consider on \( B := k[X, S, T, U, V] \) the derivation \( D := X^3\partial_S + S\partial_T + T\partial_U + X^2\partial_V \) and show that \( B^D := \ker D : B \to B \) is not finitely generated over \( k \) (then the quotient field \( L \) of \( B^D \) is a counterexample to Hilbert fourteen, since \( L \cap B = B^D \)).

The main aim of this note is to give a short proof of this result, by substantially simplifying the arguments given in [1] and [2].

Finally, I would like to mention that recently S. Kuroda has constructed new counterexamples to Hilbert fourteen in the missing dimensions 4 and 3 ([3], [4]).

1. The main result. Throughout this paper we use the following notations: \( k \) is a field of characteristic zero,

\[
B := k[X, S, T, U, V], \quad D_0 := X^3\partial_S + S\partial_T + T\partial_U, \quad D := D_0 + X^2\partial_V.
\]

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Furthermore, 
\[ A := k[S, T, U], \quad D_1 := \partial_S + S\partial_T + T\partial_U. \]
Finally, for any \( f \in B \), \( \deg f \) denotes the usual degree of \( f \). We also use another grading on \( A \) given by a vector \( w \in \mathbb{N}^3 \) and we write \( \text{w-deg} \) to denote the degree with respect to this grading. The main aim of this note is to give a short proof of

**Theorem 1.1 (Daigle–Freudenburg).** \( B^D \) is not a finitely generated \( k \)-algebra.

The proof is based on the following result which will be proved in the next section.

**Proposition 1.2.** Let \( e : \mathbb{N} \to \mathbb{N} \) be defined by \( e(3l) = 2l \), \( e(3l + 1) = e(3l + 2) = 2l + 1 \) for all \( l \geq 0 \). There exist \( c_0 = 1, c_1, c_2, \ldots \) in \( A \) with \( D_1 c_i = c_{i-1} \) and \( \deg c_i \leq e(i) \) for all \( i \geq 1 \)

**Proof of Theorem 1.1.** (i) Define
\[ a_i := X^{2i+1} c_i \left( \frac{S}{X^3}, \frac{T}{X^3}, \frac{U}{X^3} \right) \text{ for } i \geq 0. \]
Then one easily verifies that \( D_0 a_i = X^2 a_{i-1} \) for all \( i \geq 1 \) and that
\[ F_n := \sum_{i=0}^{n} (-1)^i \frac{n!}{(n-i)!} a_i V^n i \in B^D \text{ for all } n \geq 1. \]

Suppose now that \( B^D \) is finitely generated by \( g_1, \ldots, g_s \) over \( k \). We may assume that \( g_i(0) = 0 \) for all \( i \). Write \( g_i = \sum g_{ij} V^j \) with \( g_{ij} \in k[X, S, T, U] \). By (ii) below we find that \( g_{ij} \in (X, S, T, U) \) for all \( i, j \). Let \( d \) denote the maximum of the \( V \)-degrees of all \( g_i \). Consider \( F_{d+1} = X V^{d+1} + \text{lower degree } V \)-terms as above. So \( F_{d+1} \in B^D = k[g_1, \ldots, g_s] \). Looking at the coefficient of \( V^{d+1} \), we deduce that \( X \in (X, S, T, U)^2 \), a contradiction.

(ii) To prove that \( g_{ij} \in (X, S, T, U) \) for all \( i, j \) it suffices to show that if \( g = \sum g_j V^j \in B^D \) satisfies \( g(0) = 0 \) then each \( g_j \in (X, S, T, U) \). First, clearly \( g_0 \in (X, S, T, U) \). So let \( j \geq 1 \). From \( Dg = 0 \) we get \( jg_j X^2 = D_0(-g_{j-1}) \in D_0(k[X, S, T, U]) \subset (X^3, S, T) \) for all \( j \geq 1 \). If \( g_j(0) \in k^* \), then \( X^2 \in (X^3, S, T, U X^2) \), contradiction. So \( g_j(0) = 0 \), i.e. \( g_j \in (X, S, T, U) \).

2. The proof of Proposition 1.2. Put
\[ T_1 := T - \frac{1}{2} S^2, \quad U_1 := U - ST + \frac{1}{3} S^3. \]
Then \( A = k[T_1, U_1][S] \). Since \( D_1 T_1 = D_1 U_1 = 0 \) and \( D_1 S = 1 \) we get \( A^D_1 = k[T_1, U_1] \). Consider on \( A \) the grading defined by \( w(S) = 1, w(T) = 2 \) and \( w(U) = 3 \). Then \( D_1(A_n) \subset A_{n-1} \) for all \( n \geq 1 \), where \( A_n \) is the \( k \)-span of all monomials of \( A \) of \( w \)-degree \( n \). By induction on \( n \) we construct \( c_n \in A \).
So assume that $c_n$ is already constructed. Write $c_n = \sum_{i=0}^{n} H_{n-i}S^i$ with $H_{n-i} \in A_{n-i} \cap A^{D_1}$ (this is possible since $A = A^{D_1}[S]$ and $c_n \in A_n$). Then

$$\tilde{c}_{n+1} := \sum_{i=0}^{n} \frac{1}{i+1} H_{n-i}S^{i+1} \in A_{n+1}$$

and $D_1(\tilde{c}_{n+1}) = c_n$. Finally, by Lemma 2.1 below, there exists $h \in A_{n+1} \cap A^{D_1}$ such that $\tilde{c}_{n+1} := c_{n+1} - h$ satisfies $\deg c_{n+1} \leq e(n+1)$.

**Lemma 2.1.** If $f \in A_{n+1}$ is such that $\deg D_1f \leq e(n)$, then there exists $h \in A_{n+1} \cap A^{D_1}$ such that $\deg(f - h) \leq e(n+1)$.

**Proof.** (i) Let $n = 3l$ (the cases $n = 3l + 1$ and $n = 3l + 2$ are treated similarly) and let $M$ be the $k$-span of all $f \in A_{n+1}$ such that $\deg D_1f \leq 2l$ ($= e(3l)$). Write $f = \sum \alpha_{ijk}S^iT^jU^k$ with $i + 2j + 3k = 3l + 1$ and $\alpha_{ijk} \in k$. Then

$$D_1f = \sum_{i+2j+3k=3l+1} (i\alpha_{ijk} + (j+1)\alpha_{i-2,j+1,k} + (k+1)\alpha_{i-1,j-1,k+1})S^{i-1}T^jU^k.$$

So

$$(*) \quad \deg D_1f \leq 2l \iff i\alpha_{ijk} + (j+1)\alpha_{i-1,j-1,k+1} + (k+1)\alpha_{i-2,j+1,k} = 0$$

for all $i, j, k$ satisfying $i + 2j + 3k = 3l + 1$ and $(i - 1) + j + k \geq 2l + 1$, i.e. $i + j + k \geq 2l + 2$. For such a triple we have $i > 0$. Hence by $(*)$ each $\alpha_{ijk}$ is a linear combination of certain $\alpha_{pqr}$’s with $p + q + r < i + j + k$. Consequently, each $\alpha_{ijk}$ is a linear combination of the $\alpha_{pqr}$’s satisfying $p + q + r = 2l + 2$. Since there are $\lfloor (l-1)/2 \rfloor + 1$ of them (just solve the equations $p + 2q + 3r = 0$ and $p + q + r = 2l + 2$) it follows that $\dim \pi(M) \leq \lfloor (l-1)/2 \rfloor + 1$, where for any $g \in A$, $\pi(g)$ denotes the sum of all monomials of $g$ of degree $\geq 2l + 2$.

(ii) Put $N := A^{D_1} \cap A_{n+1}$. Then $N$ is the $k$-span of all “monomials”

$$n_p := T_1^{3p+2}U_1^{l-(2p+1)}, \quad \text{where } 0 \leq p \leq \lfloor (l-1)/2 \rfloor.$$

**Claim.** The $\pi(n_p)$ are linearly independent over $k$.

It then follows from (i) and the inclusion $\pi(N) \subset \pi(M)$ that $\pi(N) = \pi(M)$, which proves the lemma.

(iii) To see the claim put

$$w_p := (-2)^{3p+2}3^{l-(2p+1)}\pi(n_p)|_{T=0, U=\frac{1}{3}S} = \pi((S^2)^{3p+2}(S + S^3)^{l-(2p+1)}).$$

Observe that

$$(S^2)^{3p+2}(S + S^3)^{l-(2p+1)} = \sum_{j=0}^{l-(2p+1)} \binom{l-(2p+1)}{j} S^{3l+1-2j}.$$
Since $3l + 1 - 2j \geq 2l + 2$ iff $0 \leq j \leq [(l - 1)/2]$ we get

$$w_p = \sum_{j=0}^{[(l-1)/2]} \binom{l - (2p + 1)}{j} S^{3l+1-2j}.$$ 

Then the linear independence of the $w_p$ (and hence of the $\pi(n_p)$) follows since

$$\det \left( \binom{l - (2p + 1)}{j} \right)_{0 \leq p,j \leq [(l-1)/2]} \neq 0.$$ 

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