The version of the following full text has not yet been defined or was untraceable and may differ from the publisher's version.

For additional information about this publication click this link.
http://hdl.handle.net/2066/36159

Please be advised that this information was generated on 2019-03-28 and may be subject to change.
A commuting derivations theorem on UFD's

H.Derksen;
University of Michigan, MI, USA
hderksen@umich.edu

A.van den Essen
Radboud University, Nijmegen, The Netherlands
a.vandenessen@math.ru.nl

S.Maubach (corresponding author)
Radboud University, Nijmegen, The Netherlands
s.maubach@math.ru.nl

June 12, 2008

Abstract

Let $A$ be the polynomial ring over $k$ (a field of characteristic zero) in $n+1$ variables. The commuting derivations conjecture states that $n$ commuting locally nilpotent derivations on $A$, linearly independent over $A$, must satisfy $A^{D_1,...,D_n} = k[f]$ where $f$ is a coordinate. The conjecture can be formulated as stating that a $(G_m)^n$-action on $k^{n+1}$ must have invariant ring $k[f]$ where $f$ is a coordinate. In this paper we prove a statement (theorem 2.1) where we assume less on $A$ ($A$ is a UFD over $k$ of transcendence degree $n+1$ satisfying $A^* = k$) and prove less ($A/(f - \alpha)$ is a polynomial ring for all but finitely many $\alpha$). Under certain additional conditions (the $D_i$ are linearly independent modulo $(f - \alpha)$ for each $\alpha \in k$) we prove that $A$ is a polynomial ring itself and $f$ is a coordinate. This statement is proven even more generally by replacing “free unipotent action of dimension $n$” for “$G_a^n$-action”.

We make links with the (Abhyankar-)Sataye conjecture and give a new equivalent formulation of the Sataye conjecture.

---

*Funded by NSF CAREER grant, DMS 0349019, Invariant Theory, Algorithms and Applications.

†Funded by Veni-grant of council for the physical sciences, Netherlands Organisation for scientific research (NWO)
1 Preliminaries and introduction

Notations: $k$ will denote a field of characteristic zero. For a $k$-algebra $A$ we define $\text{LND}(A)$ as the set of all locally nilpotent derivations, and $\text{DER}(A)$ as the set of derivations. We will denote by $A^{D_1, ..., D_m} := \{a \in A; D_1(a) = \ldots = D_m(a) = 0\}$.

In the paper [7], the following conjecture is posed:

**Commuting Derivations Conjecture:** Let $A := k[X_1, \ldots, X_{n+1}]$, and let $D_1, \ldots, D_n \in \text{LND}(A)$ be commuting, linearly independent over $A$, locally nilpotent derivations. Then $A^{D_1, \ldots, D_n} = k[f]$ and $f$ is a coordinate.

**Geometric version:** Suppose we have a $G := (G_a)^n$-action on $k^{n+1}$. Then $k[X_1, \ldots, X_{n+1}]^G = k[f]$ and $f$ is a coordinate.

In the elegant paper [1], it is shown that this conjecture is equivalent to the following:

**Weak Abhyankar-Sataye Conjecture:** Let $A := k[X_1, \ldots, X_{n+1}]$, and let $f \in A$ be such that $k(f)[X_1, \ldots, X_n] \cong k(f)[Y_1, \ldots, Y_{n-1}]$. Then $f$ is a coordinate in $A$.

For completeness sake, let us state

**Abhyankar-Sataye Conjecture:** Let $A := k[X_1, \ldots, X_{n+1}]$, and let $f \in A$ be such that $A/(f) \cong k[Y_1, \ldots, Y_n]$. Then $f$ is a coordinate.

**Sataye Conjecture:** Let $A := k[X_1, \ldots, X_{n+1}]$, and let $f \in A$ be such that $A/(f - \alpha) \cong k[Y_1, \ldots, Y_n]$ for all $\alpha \in \mathbb{C}$. Then $f$ is a coordinate.

In [7], the Commuting Derivations Conjecture is proven for $n = 3$. But there is no indication that it might be true in higher dimensions. Even more, the Vénéreau polynomials (see[8]) (or similar objects), which are candidate counterexamples to the Abhyankar-Sataye conjecture, could very well spoil things for the Commuting Derivations Conjecture in higher dimensions. In any case, it seems like a proof is far away.

Therefore, it seems a good idea to be a little less ambitious. In this paper, we consider the weaker statement that $A$ is a UFD (instead of a polynomial ring). It turns out that the situation can be quite different and interesting. Let us consider a famous example:

**Example 1.1.** Let $A := \mathbb{C}[x, y, z, t] = \mathbb{C}[X, Y, Z, T]/(X^2Y + X + Z^2 + T^3)$ and let $D_1 := 2z \frac{\partial}{\partial y} - x \frac{\partial}{\partial z}$ and $D_2 := 3t^2 \frac{\partial}{\partial y} - x \frac{\partial}{\partial t}$. $A$ is a UFD of transcendence degree 3 which is not a polynomial ring (see [6], or use the fact that the commuting derivations conjecture in dimension 3 holds). $D_1$ and $D_2$ commute, and $A^{D_1, D_2} = \mathbb{C}[x]$. Now $A/(x - \alpha) \cong \mathbb{C}[Y_1, Y_2]$ except in the case that $\alpha = 0$. 


Also, $D_1 \mod (x - \alpha), D_2 \mod (x - \alpha)$ are independent over $A/(x - \alpha)$ if and only if $\alpha \neq 0$.

2 The UFD Commuting derivations theorem

The following theorem is the main result of this paper.

**Theorem 2.1.** Let $A$ be a UFD over $k$ with $\text{trdeg}_k Q(A) = n + 1 (\geq 1)$, $A^* = k^*$, and let $D_1, \ldots, D_n$ be commuting locally nilpotent derivations (linearly independent over $A$). Now $A^{D_1, \ldots, D_n} = k[f]$ for some $f \in A \setminus k$, and

1. If $D_1 \mod (f - \alpha), \ldots, D_n \mod (f - \alpha)$ are independent over $A/(f - \alpha)$, then $A/(f - \alpha) \cong \mathbb{C}[\alpha]$. There are only finitely many $\alpha \in \mathbb{C}$ for which $D_1 \mod (f - \alpha), \ldots, D_n \mod (f - \alpha)$ are dependent over $A/(f - \alpha)$.

2. In the case that $D_1 \mod (f - \alpha), \ldots, D_n \mod (f - \alpha)$ are independent over $A/(f - \alpha)$ for each $\alpha \in k$, then $A = k[s_1, \ldots, s_n, f]$, a polynomial ring in $n + 1$ variables.

**Geometric Version:** Let $V$ be a factorial affine surface over $k$ of dimension $n + 1$ such that $\mathcal{O}(V)^* = k^*$. Suppose there exists a $G := (G\alpha)^n$-action on $V$. Then $\mathcal{O}(V)^G = k[f]$ and

1. Suppose that the fiber $f = \alpha$ has a point with trivial stabilizer. Then the fiber $f = \alpha$ is isomorphic to $\mathbb{C}^n$. There are only finitely many $\alpha \in \mathbb{C}$ for which $f = \alpha$ has no point with trivial stabilizer.

2. Suppose that all fibers $f = \alpha$ have a point with trivial stabilizer. (Then, all points have trivial stabilizers.) Then $V \cong \mathbb{C}^{n+1}$ and the action $G \times V \rightarrow V$ is a translation on the first $n$ coordinates.

In the last section we will prove a more general geometric statement of part 2 for unipotent groups in stead of $G\alpha^n$-actions, but we will stick with this description for the moment, as this is the most interesting case for us, and has a simpler, direct, algebraic proof.

Before we give a proof of the above theorem, let us meditate on this a bit. The example 1.1 is a typical case of part 1 of the above theorem. But there is a connection with the Sataye Conjecture. Let us consider the following conjecture:

**Modified Sataye Conjecture:** Let $A := k[X_1, \ldots, X_{n+1}]$, and let $f \in A$ be such that $A/(f - \alpha) \cong k[Y_1, \ldots, Y_n]$ for all $\alpha \in \mathbb{C}$. Then there exist $n$ commuting locally nilpotent derivations $D_1, \ldots, D_n$ on $A$ such that $A^{D_1, \ldots, D_n} = \mathbb{C}[f]$ and the $D_i$ are linearly independent modulo $(f - \alpha)$ for each $\alpha \in \mathbb{C}$.

**Proposition 2.2.** The Modified Sataye Conjecture is equivalent to the Sataye Conjecture.
Proof. Let us abbreviate the conjectures by SC and MSC. Suppose we have proven the MSC. Then for any $f$ satisfying $A/(f - \alpha) \cong k[Y_1, \ldots, Y_n]$ for all $\alpha \in \mathbb{C}$, we can find commuting derivations as stated in the MSC. But using theorem 2.1 part 2 we get that $f$ is a coordinate in $A$. So the SC is true in that case.

Now suppose we have proven the SC. Let $f$ satisfy the requirements of the MSC, that is, $A/(f - \alpha) \cong k[Y_1, \ldots, Y_n]$ for all $\alpha \in \mathbb{C}$. Since $f$ satisfies the requirements of the SC, $f$ must then be a coordinate. So it has $n$ so-called mates: $\mathbb{C}[f, f_1, \ldots, f_n] = \mathbb{C}[X_1, \ldots, X_{n+1}]$. But then each of these $n+1$ polynomials $f, f_1, \ldots, f_n$ defines a locally nilpotent derivation, all of them commute, and the intersection of the last $n$ derivations is $\mathbb{C}[f]$; so the MSC holds.

But now it is time to stop daydreaming about big conjectures, and start doing some hard-core proofs. Since the following proof uses the tools of the next section, the reader is encouraged to read section 3 before reading the following proof in detail.

Proof. (of theorem 2.1) Using lemma 3.4 we have $p_i \in A$ such that $D_j(p_i) = 0$ if $i \neq j$, and $D_i(p_i) = q_i(f) \in \mathbb{C}[f]$ of lowest possible degree.

Part 1: $D_1, \ldots, D_n$ are independent over $A$, but they may become dependent modulo $(f - \alpha)$. Let us first consider the case where they are independent modulo $(f - \alpha)$: then $D_1, \ldots, D_n$ are linearly independent over $A/(f - \alpha)$. Then, by proposition 3.1 we have that $A/(f - \alpha) \cong k^{[n]}$.

So, left to prove is that $D_1, \ldots, D_n$ can only be linearly dependent modulo finitely many $(f - \alpha)$. But this follows directly from lemma 3.5, as there are only finitely many zeroes in $q_1q_2 \cdots q_n$.

Part 2: Lemma 3.5 tells us directly that for each $1 \leq i \leq n$ and $\alpha \in k$, we have $q_i(\alpha) \neq 0$. But this means that the $q_i \in k^*$, so the $p_i$ are in fact slices, and using 3.3 we are done. □

3 Tools

The tools proven in this section focus on the situation of theorem 2.1 part 1, and are interesting in their own respect.

In this section, $A$ is a $k$-domain, and $trdeg(A) = n + 1(\geq 1)$.

The following two propositions are proposition 3.2 and 3.4 in [7].

**Proposition 3.1.** Let $D_1, \ldots, D_{n+1}$ be commuting locally nilpotent $k$-derivations on $A$ which are linearly independent over $A$. Then

(i). There exist $s_i$ in $A$ such that $D_is_i = \delta_{ij}$ for all $i, j$ and

(ii). $A = k[s_1, \ldots, s_{n+1}]$ a polynomial ring in $n+1$ variables over $k$.

**Proposition 3.2.** Let $A$ be a UFD and let $A^* = k^*$. Let $D_1, \ldots, D_n$ be commuting locally nilpotent derivations, linearly independent over $A$. Then $A^{D_1, \ldots, D_n} = k[f]$ for some $f \in A\setminus k$, and $f - \alpha$ is irreducible for each $\alpha \in \mathbb{C}$. 
Proposition 3.3. Let $A$, $D_i$, $f$ as in proposition 3.2. Suppose there exist $s_1, \ldots, s_n$ such that $D_i(s_i) = 1$. Then $A = k[s_1, \ldots, s_n, f]$, a polynomial ring in $n + 1$ variables.

Proof. This is an easy consequence of the fact that, if $D \in \text{LND}(A)$ having an $s \in A$ such that $D(s) = 1$, then $A^D[s] = A$. □

Define the following abbreviation:

(S1:) Let $A$ be a uFD and let $A^* = k^*$. Let $D_1, \ldots, D_n$ be commuting locally nilpotent derivations, linearly independent over $A$.

Lemma 3.4. Assume (S1).

1. Then there exist $p_i \in A$ such that $D_j(p_i) = 0$ if $j \neq i$, and $D_i(p_i) \in k[f]\{0\}$. Furthermore, $k[p_1, \ldots, p_n, f] \subseteq A$ is algebraic.

2. Define $P_i := \{p_i \in A \mid D_j(p_i) = 0 \text{ if } i \neq j \text{ and } D_i(p_i) \in k[f]\}$. Then $D_i(P_i) = q_i(f)k[f]$ for some nonzero polynomial $q_i$. Taking $p_i$ such that $D_i(p_i)$ is of lowest possible degree yields $D_i(p_i) \in kq_i(f)$.

Proof. (1) We assume that all $n$ derivations commute, so $D_i(A^{D_j}) \subseteq A^{D_j}$, and therefore $D_i$ sends $A_i := A^{D_1, \ldots, D_{i-1}, D_{i+1}, \ldots, D_n}$ to itself. Taking some $a \in A_i \setminus k[f]$ nonzero, we use the fact that $D_j$ is locally nilpotent to find the lowest $m \in \mathbb{N}$ such that $D^m(a) = 0$. Now define $p_i := D^{m-2}(a)$ (indeed $m > 2$). The rest is easy.

(2) Take $p_i$ such that $D_i(p_i) = q_i(f) \neq 0$ has lowest possible degree. Let $\bar{p}_i \in P_i$, then $D_i(\bar{p}_i) = h_i(f)q_i(f) + r_i(f)$ where $\deg(r_i) < \deg(q_i)$. Now $D_i(\bar{p}_i - h_i(f)p_i) = r_i(f)$ so $r_i = 0$. So $D_i(\bar{p}_i) \in q_i(f)k[f]$. □

Lemma 3.5. Assume (S1). Choose $p_i$ such that $D_i(p_i) = q_i(f)$ as in lemma 3.4, where $q_i$ is of lowest possible degree. The $D_i$ are linearly dependent modulo $f - \alpha$ if and only if $q_i(\alpha) = 0$ for some $i$.

Proof. ($\Rightarrow$): Write “bars” for “modulo $f - \alpha$”. Suppose that $0 \neq D := g_1D_1 + \ldots + g_nD_n$ satisfies $\bar{D} = 0$ where $g_i \in A$, and not all $\bar{g}_i = 0$. Now $\bar{g}_iD_i(\bar{p}_i) = \bar{D}(\bar{p}_i) = 0$ for each $i$, so for each $i$, either $\bar{g}_i = 0$ or $q_i(f) = 0$ (as $f - \alpha$ is irreducible by proposition 3.2). Since not all $\bar{g}_i = 0$, at least one $q_i(f) = 0$. Since $f - \alpha$ is irreducible for each $\alpha$, we not only have $(f - \alpha)|q_i(f)$, but even $(X - \alpha)|q_i(X)$, so $q_i(\alpha) = 0$.

($\Leftarrow$): Assume $f - \alpha$ divides $q_i(f)$. We need to show that the $D_i$ mod $(f - \alpha)$ are linearly dependent over $A/(f - \alpha)$. Suppose the $\bar{D}_i$ are linearly independent over $A$. Then we have $n$ commuting, linearly independent LNDs on a domain of transcendence degree $n$, so we can use proposition 3.1 and conclude that $\overline{A^{D_1, \ldots, D_n}} = k$. This means, since $\overline{q_i(f)} = 0$, that $\overline{p_i} \in k$. So, $p_i = (f - \alpha)a + \lambda$ where $a \in A, \lambda \in k$. Now taking $a \in A$ we still have $D_j(a) = 0$ for all $j \neq i$, and $D_i(a) = q_i(f)(f - \alpha)^{-1} \in \mathbb{C}[f]$. This contradicts the assumption that $q_i$ was minimal, so our assumption that the $\bar{D}_i$ are linearly independent was incorrect. □
Now we want to point out the following phenomenon:

**Example 3.6.** Let \( D_1 = Z \partial X + \partial Y, D_2 = \partial Y \) on \( A = \mathbb{C}[X, Y, Z] \). Now \( A^{D_1, D_2} = \mathbb{C}[Z] \). The \( D_1, D_2 \) are linearly independent modulo \( Z - \alpha \) as long as \( \alpha \neq 0 \). But it is clear that a different set of derivations, namely \( E_1 = \partial X, E_2 = \partial Y \) commute, their \( \mathbb{C}[Z] \)-span contains \( D_1, D_2 \) and the \( E_i \) are linearly independent for more fibers \( f - \alpha \).

The \( E_i \) of the example are an improvement over the \( D_i \): all the same properties, but they are linearly independent for more \( f - \alpha \). Perhaps for your given space \( A \) and derivations \( D_i \) it is impossible to find \( E_i \) such that the \( E_i \) are independent modulo every \( f - \alpha \), giving more information on your ring \( A \). Before we elaborate on this, let us give a lemma that enables construction of the \( E_i \):

**Lemma 3.7.** Assume (S1). Define \( \mathcal{M} := k(f)D_1 + \ldots + k(f)D_n \cap \text{DER}(A) \). Then \( \mathcal{M} = k[f]E_1 \oplus \ldots \oplus k[f]E_n \) for some \( E_i \in \mathcal{M} \), and the \( E_i \) have all the properties that the \( D_i \) have (i.e. commuting locally nilpotent, linearly independent over \( A \)). Furthermore, if the \( D_i \) are linearly independent modulo \( f - \alpha \), then the \( E_i \) are too (but not necessary the other way around).

**Proof.** Use lemma 3.4 we find preslices \( p_i \) and \( D(p_i) = q_i(f) \) as stated there.

If \( D \in \mathcal{M} \) then \( D = g_1(f)D_1 + \ldots + g_n(f)D_n \) where \( g_i(f) \in k(f) \). Now since \( D \in \text{DER}(A) \) we have \( D(p_i) \in A \). Also \( D(p_i) = g_i(f)D(p_i) = g_i(f)q_i(f) \in k(f) \) thus \( D(p_i) \in A \cap k(f) \), which equals \( k[f] \) since \( A^* = k^* \).

Therefore the map \( \varphi : \mathcal{M} \rightarrow k[f]^n \) sending \( D \rightarrow (D(p_1), \ldots, D(p_n)) \) is well-defined. If \( 0 = \varphi(g_1(f)D_1 + \ldots + g_n(f)D_n) \) then \( g_i(f)D_i(p_i) = 0 \) and therefore \( g_i(f) = 0 \); thus \( \varphi \) is injective.

Since \( \varphi \) is an injective map, \( \mathcal{M} \) must be a free \( k[f] \)-module. Note that \( \mathcal{M} \) can only have dimension \( n \). Therefore we can find \( E_1, \ldots, E_n \) as required.

Any derivation in \( \mathcal{M} \) is locally nilpotent. Even more, any two derivations of \( \mathcal{M} \) commute! Next to that, the \( E_i \) are clearly independent over \( A \). \( \square \)

Note that the \( E_i \) can be constructively made, given the injective map \( \varphi \) in the above proof. This actually gives an interesting concept. Given the situation (S1), one can improve the derivations \( D_i \) (by replacing them by the \( E_i \)) and then they are linearly independent modulo as much as possible \( f - \alpha \). For every such \( \alpha \) we have that \( A/(f - \alpha) \) is a polynomial ring. The question is if the converse holds:

**Question:** Assume (S1). Additionally, assume \( k[f]D_1 + \ldots + k[f]D_n = (k(f)D_1 + \ldots + k(f)D_n) \cap \text{DER}(A) \). Is the set \( \{ \alpha \in \mathbb{C} \mid D_1, \ldots, D_n \text{ linearly dependent modulo } (f - \alpha) \} \) equal to the set \( \{ \alpha \in \mathbb{C} \mid A/(f - \alpha) \text{ is not a polynomial ring} \} \)? (One always has \( \geq \).) Or, if this equality does not hold, what type of rings \( A \) do have equality?

Note that the requirement “\( A \text{ UFD} \)” is absolutely necessary, as for a simple Danielewski surface \( \mathbb{C}[X, Y, Z]/(X^2Y - Z^2) \) we find a LND \( 2Z \partial Y + X^2 \partial Z \) which
is nonzero modulo each $X - \alpha$. (But $A/(f - \alpha)$ is not always a domain in this case, even.)

4 Unipotent actions

The authors would like to thank prof. Kraft for pointing out the generalization of theorem 2.1 part 2, which has become the below theorem 4.2.

**Proposition 4.1.** If $U \times V \rightarrow V$ is an action of a unipotent group $U$ on an affine variety $V$, then for each $u \in U$, the map $u^* : O(V) \rightarrow O(V)$ is an exponent of a locally nilpotent derivation.

For the proof we can refer to proposition 2.1.3 in [2], or ask the reader to verify that $u^* - Id$ is a locally nilpotent endomorphism, and that thus “log($u^*$)” can be defined, and is a derivation.

This proposition has some immediate consequences, like that the invariants of a unipotent group action are the intersection of kernels of locally nilpotent derivations. Since kernels of locally nilpotent derivations are factorially closed, their intersection is too, so the invariants of a unipotent group is factorially closed.

In the below theorem, $\mathbb{C}$ is a field of characteristic zero, which is algebraically closed.

**Theorem 4.2.** Let $U$ be a unipotent algebraic group of dimension $n$, acting freely on $X$, a factorial variety of dimension $n + 1$ satisfying $O(X)^* = \mathbb{C}^*$. Then $X$ is $U$-isomorphic to $U \times \mathbb{C}$. In particular, $X \simeq \mathbb{C}^{n+1}$.

**Proof.** The fact that $U$ acts free means that each $x \in X$ has trivial stabilizer: $U_x = \{u \in U; ux = x\} = \{id\}$. So, each orbit $Ux$ is of dimension $n$. This means that $X//U$ is of dimension 1. Also, as remarked above, $X^U$ is factorial. But then it is also normal, and smooth. So $X//U$ is a smooth, rational, affine curve, in other words, an open subvariety of $\mathbb{C}$. Now suppose that $X//U \neq T$, so $X//U = \mathbb{C} - \{p_1, \ldots, p_n\}$, then $O(X)^U = O(\mathbb{C} - \{p_1, \ldots, p_n\}) = \mathbb{C}[t, (t - p_1)^{-1}, \ldots, (t - p_n)^{-1}]$. This means that $O(X)$ contains invertible elements $(t - p_1)^{-1}$, giving a contradiction with the assumption $O(X)^* = \mathbb{C}^*$. Hence, $X//U \simeq \mathbb{C}$, so $O(X)^U = O(X//U) = O(\mathbb{C}) \simeq \mathbb{C}[f]$ for some $f$. Now every $f - \lambda (\lambda \in \mathbb{C})$ is irreducible, as otherwise any irreducible factor of $f - \lambda$ would be in $O(X)^U$ too.

Now consider the map $f : X \rightarrow \mathbb{C}$. This is in fact the map $X \rightarrow X//U$ (as it corresponds to the map $O(X) \leftarrow O(X)^U = \mathbb{C}[f]$) and thus surjective. Also note that the fibers $f^{-1}(\lambda)$ are invariant under $U$: they correspond to the function space $O(X)/(f - \lambda)$. By assumption, $U$ acts free on each fiber of $X \rightarrow X//U$, which means exactly that $U$ acts free on $f^{-1}(\lambda)$ for each $\lambda$. Let $x \in f^{-1}(\lambda)$. Then $Ux$ is of dimension $n$ (it is just a copy of $U$). Also, each orbit of a unipotent group is closed (see Satz 4 from [3]), and therefore the inclusion $Ux \subseteq f^{-1}(\lambda)$ is an equality. So orbits of $U$ are the same as fibers of $f$, i.e. we have an orbit fibration (or $U$-fibration).
$X_{\text{sing}}$ is closed and $U$-stable, hence a union of $U$-orbits, and so $\text{codim} X_{\text{sing}} = 1$ or $X_{\text{sing}}$ is empty. But $X$ is factorial, so in particular normal, which implies $\text{codim}(X_{\text{sing}}) \geq 2$. So $X_{\text{sing}}$ is empty, in other words: $X$ is smooth.

Now we claim that $f : X \to \mathbb{C}$ is smooth. To see this, first note that $\mathcal{O}(f^{-1}(\lambda)) = \mathcal{O}(X) / (f - \lambda)$ is reduced as $f - \lambda$ is irreducible, as seen before. And, as we already implied, the set of functions vanishing on $f^{-1}(\lambda)$ is the ideal $(f - \lambda)$. Now consider the tangent map $df_x : T_x X \to T_{\mathbb{C}} = \mathbb{C}$ where $x \in f^{-1}(\lambda)$. Using "Satz 2", page 269 in [3] we see that, $\text{ker} df \supseteq T_x f^{-1}(\lambda)$, but since $f^{-1}(\lambda)$ is reduced, we even have equality $\text{ker} df = T_x f^{-1}(\lambda)$. Now remember that the fiber $f^{-1}(\lambda)$ is an orbit, hence smooth (as any orbit is smooth!). This implies $\dim T_x f^{-1}(\lambda) = n$ and thus $\dim \text{ker} df = n$. Since $\dim T_x X = n + 1$ we have $\dim \text{Im}(df_x) = 1$, hence $df_x$ is surjective. A morphism between smooth varieties is smooth if and only if the differential is surjective. So we have shown that $f$ is smooth.

So: $f : X \to \mathbb{C}$ is surjective, and smooth. Let $K := \text{ker} df_x \subset T_x X$. Take some linear subspace $C$ such that $K \oplus C = T_x X$. Note that $C$ has dimension $1$. Seeing $X$ as a subset of some $\mathbb{C}^N$, we can find hyperplanes $H$ that contains $C$. We even want $H \cap T_x X = C$, so this means that $H \oplus T_x \subset \mathbb{C}^N$, so let us take a hyperplane $H$ of codimension $n$ such that $H \cap T_x X = C$. Now let $Z$ be an irreducible component of $H \cap X$ which contains $x$. Also, $\dim_x H \cap X \geq 1$, thus $\dim_x Z = 1$ and $Z$ is smooth at $x$. Now $Z$ and $\mathbb{C}$ are smooth, and the differential of $f|_Z : Z \to \mathbb{C}$ is an isomorphism at $x$ (implying surjective), thus we have that $f|_Z$ is smooth at $x$. Replacing $Z$, if necessary, by a (special) open subset $Z' \subset Z$, we have $f|_Z$ is étale.

Now look at the following diagram

$$
\begin{array}{ccc}
Z \times_{\mathbb{C}} X & \xrightarrow{p} & X \\
\downarrow f & & \downarrow f \\
Z & \xrightarrow{f|_Z} & \mathbb{C}
\end{array}
$$

where $Z \times_{\mathbb{C}} X = \{(x, z) \in X \times Z \mid f(x) = f|_Z(z)\}$ is the (schematic) fiber product. Since $f$ is smooth, the same holds for $f$ and so $Z \times_{\mathbb{C}} X$ is smooth. Moreover, $U$ acts on $Z \times_{\mathbb{C}} X$ by $u(z, x) = (z, ux)$ and $p(u(x, z)) = ux$ ($p$ is $U$-equivariant) and $f(u(x, z)) = z = f(x, z)$ ($f$ is $U$-invariant). The fibers of $\tilde{f}$ are $\tilde{f}^{-1}(z) = \{(x, z) \mid f(x) = f|_Z(z)\} = \{x \mid f(x) = \alpha\} = f^{-1}(\alpha)$ where $\alpha = f_Z(z)$. Now $\tilde{f}$ has a section $\sigma : Z \to Z \times_{\mathbb{C}} X$ given by $z \mapsto (z, z)$, i.e. $\tilde{f} \circ \sigma = \text{id}_Z$. Therefore, we can extend the diagram above

$$
\begin{array}{ccc}
U \times Z & \xrightarrow{q} & Z \times_{\mathbb{C}} X & \xrightarrow{p} & X \\
\downarrow \text{pr}_Z & & \downarrow \tilde{f} & & \downarrow f \\
Z & \xrightarrow{f|_Z} & \mathbb{C}
\end{array}
$$

where $q : U \times Z \to Z \times_{\mathbb{C}} X$ is given by $(u, z) \mapsto (z, uz)$. By construction, $q$ is bijective, hence an isomorphism, since the second variety is normal (see [4]
proposition 5.7). Note that the role of $x$ was arbitrary: for each $x$ we find a neighborhood $Z$ where $Z \times_{\mathbb{C}} X = Z \times_{\mathbb{C}} U$. This last statement exactly means that the map $f : X \to \mathbb{C}$ is a locally trivial principal $U$-bundle with respect to the étale topology: for every point $\lambda \in \mathbb{C}$ there is an étale map $Z \to \mathbb{C}$ such that $\lambda$ is in the image and the fiber product $Z \times_{\mathbb{C}} X$ is a trivial $U$-bundle, i.e. isomorphic to $U \times Z \overset{pr_2}{\longrightarrow} Z$.

In the paper [5] we now find a result that tells us that a principal $G$-bundle where $G$ is a unipotent group is trivial over any affine variety, and then we are done. 

□

References


[6] L. Makar-Limanov, On the hypersurface $x + x^2y + z^2 + t^3 = 0$ in $\mathbb{C}^4$ or a $\mathbb{C}^3$ -like threefold which is not $\mathbb{C}^3$, Israel J. Math., 96(1996), 419-429
