A commuting derivations theorem on UFD's

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Abstract

Let A be the polynomial ring over k (a field of characteristic zero) in $n+1$ variables. The commuting derivations conjecture states that n commuting locally nilpotent derivations on A, linearly independent over A, must satisfy $A^{D_1,\ldots,D_m} = k[f]$ where f is a coordinate. The conjecture can be formulated as stating that a $(G_m)^n$-action on k^{n+1} must have invariant ring $k[f]$ where f is a coordinate. In this paper we prove a statement (theorem 2.1) where we assume less on A (A is a UFD over k of transcendence degree $n+1$ satisfying $A^* = k$) and prove less ($A/(f-\alpha)$ is a polynomial ring for all but finitely many α). Under certain additional conditions (the D_i are linearly independent modulo $(f-\alpha)$ for each $\alpha \in k$) we prove that A is a polynomial ring itself and f is a coordinate. This statement is proven even more generally by replacing "free unipotent action of dimension n" for "G_a^n-action".

We make links with the (Abhyankar-)Sataye conjecture and give a new equivalent formulation of the Sataye conjecture.

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1 Preliminaries and introduction

Notations: k will denote a field of characteristic zero. For a k-algebra A we define LND(A) as the set of all locally nilpotent derivations, and DER(A) as the set of derivations. We will denote by $A^{D_1,\ldots,D_m} := \{a \in A; D_1(a) = \ldots = D_m(a) = 0\}$.

In the paper [7], the following conjecture is posed:

Commuting Derivations Conjecture: Let $A := k[X_1, \ldots, X_{n+1}]$, and let $D_1, \ldots, D_n \in LND(A)$ be commuting, linearly independent over A, locally nilpotent derivations. Then $A^{D_1,\ldots,D_n} = k[f]$ and f is a coordinate.
Geometric version: Suppose we have a $\mathcal{G} := (\mathcal{G}_a)^n$ -action on k^{n+1}. Then $k[X_1, \ldots, X_{n+1}]^{\mathcal{G}} = k[f]$ and f is a coordinate.

In the elegant paper [1], it is shown that this conjecture is equivalent to the following:

Weak Abhyankar-Sataye Conjecture: Let $A := k[X_1, \ldots, X_{n+1}]$, and let $f \in A$ be such that $k(f)[X_1, \ldots, X_n] \cong_{k(f)} k(f)[Y_1, \ldots, Y_{n-1}]$. Then f is a coordinate in A.

For completeness sake, let us state

Abhyankar-Sataye Conjecture: Let $A := k[X_1, \ldots, X_{n+1}]$, and let $f \in A$ be such that $A/(f) \cong k[Y_1, \ldots, Y_n]$. Then f is a coordinate.

Sataye Conjecture: Let $A := k[X_1, \ldots, X_{n+1}]$, and let $f \in A$ be such that $A/(f - \alpha) \cong k[Y_1, \ldots, Y_n]$ for all $\alpha \in \mathbb{C}$. Then f is a coordinate.

In [7], the Commuting Derivations Conjecture is proven for $n = 3$. But there is no indication that it might be true in higher dimensions. Even more, the Vénéreau polynomials (see[8]) (or similar objects), which are candidate counterexamples to the Abhyankar-Sataye conjecture, could very well spoil things for the Commuting Derivations Conjecture in higher dimensions. In any case, it seems like a proof is far away.

Therefore, it seems a good idea to be a little less ambitious. in this paper, we consider the weaker statement that A is a UFD (in stead of a polynomial ring). It turns out that the situation can be quite different and interesting. Let us consider a famous example:

Example 1.1. Let $A := \mathbb{C}[x, y, z, t] = \mathbb{C}[X, Y, Z, T]/(X^2Y + X + Z^2 + T^3)$ and let $D_1 := 2z\frac{\partial}{\partial y} - x^2\frac{\partial}{\partial z}$ and $D_2 := 3t^2\frac{\partial}{\partial y} - x^2\frac{\partial}{\partial t}$. A is a UFD of transcendence degree 3 which is not a polynomial ring (see [6], or use the fact that the commuting derivations conjecture in dimension 3 holds). D_1 and D_2 commute, and $A^{D_1,D_2} = \mathbb{C}[x]$. Now $A/(x - \alpha) \cong \mathbb{C}[Y_1, Y_2]$ except in the case that $\alpha = 0$.

Also, $D_1 \mod (x-\alpha), D_2 \mod (x-\alpha)$ are independent over $A/(x-\alpha)$ if and only if $\alpha \neq 0$.

2 The UFD Commuting derivations theorem

The following theorem is the main result of this paper.

Theorem 2.1. *Let A be a UFD over k with $trdeg_k Q(A) = n+1 (\geq 1)$, $A^* = k^*$, and let $D_1, \ldots, D_n$ be commuting locally nilpotent derivations (linearly independent over A). Now $A^{D_1,\ldots,D_n} = k[f]$ for some $f \in A \backslash k$, and*

1. *If $D_1 \mod (f-\alpha), \ldots, D_n \mod (f-\alpha)$ are independent over $A/(f-\alpha)$, then $A/(f-\alpha) \cong \mathbb{C}^{[n]}$. There are only finitely many $\alpha \in \mathbb{C}$ for which $D_1 \mod (f-\alpha), \ldots, D_n \mod (f-\alpha)$ are dependent over $A/(f-\alpha)$.*
2. *In the case that $D_1 \mod (f-\alpha), \ldots, D_n \mod (f-\alpha)$ are independent over $A/(f-\alpha)$ for each $\alpha \in k$, then $A = k[s_1, \ldots, s_n, f]$, a polynomial ring in $n+1$ variables.*

Geometric Version: *Let V be a factorial affine surface over k of dimension $n+1$ such that $\mathcal{O}(V)^* = k^*$. Suppose there exists a $\mathcal{G} := (\mathcal{G}_a)^n$-action on V. Then $\mathcal{O}(V)^{\mathcal{G}} = k[f]$ and*

1. *Suppose that the fiber $f = \alpha$ has a point with trivial stabilizer. Then the fiber $f = \alpha$ is isomorphic to $\mathbb{C}^n$. There are only finitely many α for which $f = \alpha$ has no point with trivial stabilizer.*
2. *Suppose that all fibers $f = \alpha$ have a point with trivial stabilizer. (Then,* all *points have trivial stabilizers.) Then $V \cong \mathbb{C}^{n+1}$ and the action $\mathcal{G} \times V \longrightarrow V$ is a translation on the first n coordinates.*

In the last section we will prove a more general geometric statement of part 2 for unipotent groups in stead of $\mathcal{G}_a^n$-actions, but we will stick with this description for the moment, as this is the most interesting case for us, and has a simpler, direct, algebraic proof.

Before we give a proof of the above theorem, let us meditate on this a bit. The example 1.1 is a typical case of part 1 of the above theorem. But there is a connection with the Sataye Conjecture. Let us consider the following conjecture:

Modified Sataye Conjecture: Let $A := k[X_1, \ldots, X_{n+1}]$, and let $f \in A$ be such that $A/(f-\alpha) \cong k[Y_1, \ldots, Y_n]$ for all $\alpha \in \mathbb{C}$. Then there exist n commuting locally nilpotent derivations $D_1, \ldots, D_n$ on A such that $A^{D_1,\ldots,D_n} = \mathbb{C}[f]$ and the D_i are linearly independent modulo $(f-\alpha)$ for each $\alpha \in \mathbb{C}$.

Proposition 2.2. *The Modified Sataye Conjecture is equivalent to the Sataye Conjecture.*

Proof. Let us abbreviate the conjectures by SC and MSC. Suppose we have proven the MSC. Then for any f satisfying "$A/(f-\alpha) \cong k[Y_1, \ldots, Y_n]$ for all $\alpha \in \mathbb{C}$" we can find commuting derivations as stated in the MSC. But using theorem 2.1 part 2 we get that f is a coordinate in A. So the SC is true in that case.

Now suppose we have proven the SC. Let f satisfy the requirements of the MSC , that is, "$A/(f-\alpha) \cong k[Y_1, \ldots, Y_n]$ for all $\alpha \in \mathbb{C}$". Since f satisfies the requirements of the SC, f then must be a coordinate. So it has n so-called mates: $\mathbb{C}[f, f_1, \ldots, f_n] = \mathbb{C}[X_1, \ldots, X_{n+1}]$. But then each of these $n+1$ polynomials $f, f_1, \ldots, f_n$ defines a locally nilpotent derivation, all of them commute, and the intersection of the last n derivations is $\mathbb{C}[f]$; so the MSC holds. □

But now it is time to stop daydreaming about big conjectures, and start doing some hard-core proofs. Since the following proof uses the tools of the next section, the reader is encouraged to read section 3 before reading the following proof in detail.

Proof. (of theorem 2.1) Using lemma 3.4 we have $p_i \in A$ such that $D_j(p_i) = 0$ if $i \neq j$, and $D_i(p_i) = q_i(f) \in \mathbb{C}[f]$ of lowest possible degree.
Part 1: $D_1, \ldots, D_n$ are independent over A, but they may become dependent modulo $(f-\alpha)$. Let us first consider the case where they are independent modulo $(f-\alpha)$: then $\bar{D}_1, \ldots, \bar{D}_n$ are linearly independent over $A/(f-\alpha)$. Then, by proposition 3.1 we have that $A/(f-\alpha) \cong k^{[n]}$.

So, left to prove is that $D_1, \ldots, D_n$ can only be linearly dependent modulo finitely many $(f-\alpha)$. But this follows directly from lemma 3.5, as there are only finitely many zeroes in $q_1 q_2 \cdots q_n$.

Part 2: Lemma 3.5 tells us directly that for each $1 \leq i \leq n$ and $\alpha \in k$, we have $q_i(\alpha) \neq 0$. But this means that the $q_i \in k^*$, so the p_i are in fact slices, and using 3.3 we are done. □

3 Tools

The tools proven in this section focus on the situation of theorem 2.1 part 1, and are interesting in their own respect.

In this section, A is a k-domain, and $trdeg(A) = n+1 (\geq 1)$.

The following two propositions are proposition 3.2 and 3.4 in [7].

Proposition 3.1. *Let $D_1, \ldots, D_{n+1}$ be commuting locally nilpotent k-derivations on A which are linearly independent over A. Then*

(i). There exist s_i in A such that $D_i s_i = \delta_{ij}$ for all i, j and

(ii). $A = k[s_1, \ldots, s_{n+1}]$ a polynomial ring in $n+1$ variables over k.

Proposition 3.2. *Let A be a* UFD *and let $A^* = k^*$. Let $D_1, \ldots, D_n$ be commuting locally nilpotent derivations, linearly independent over A. Then $A^{D_1, \ldots, D_n} = k[f]$ for some $f \in A \backslash k$, and $f - \alpha$ is irreducible for each $\alpha \in \mathbb{C}$.*

Proposition 3.3. *Let A, D_i, f as in proposition 3.2. Suppose there exist $s_1, \ldots, s_n$ such that $D_i(s_i) = 1$. Then $A = k[s_1, \ldots, s_n, f]$, a polynomial ring in $n+1$ variables.*

Proof. This is an easy consequence of the fact that, if $D \in \mathrm{LND}(A)$ having an $s \in A$ such that $D(s) = 1$, then $A^D[s] = A$. □

Define the following abbreviation:

(S1:) Let A be a UFD and let $A^* = k^*$. Let $D_1, \ldots, D_n$ be commuting locally nilpotent derivations, linearly independent over A.

Lemma 3.4. *Assume (S1).*
(1) Then there exist $p_i \in A$ such that $D_j(p_i) = 0$ if $j \neq i$, and $D_i(p_i) \in k[f] \backslash \{0\}$. Furthermore, $k[p_1, \ldots, p_n, f] \subseteq A$ is algebraic.
(2) Define $\mathcal{P}_i := \{p_i \in A \mid D_j(p_i) = 0 \text{ if } i \neq j \text{ and } D_i(p_i) \in k[f]\}$. then $D_i(\mathcal{P}_i) = q_i(f)k[f]$ for some nonzero polynomial q_i. Taking p_i such that $D_i(p_i)$ is of lowest possible degree yields $D_i(p_i) \in kq_i(f)$.

Proof. (1) We assume that all n derivations commute, so $D_i(A^{D_j}) \subseteq A^{D_j}$. and therefore D_i sends $A_i := A^{D_1, \ldots, D_{i-1}, D_{i+1}, \ldots, D_n}$ to itself. Taking some $a \in A_i \backslash \mathbb{C}[f]$ nonzero, we use the fact that D_i is locally nilpotent to find the lowest $m \in \mathbb{N}$ such that $D^m(a) = 0$. Now define $p_i := D^{m-2}(a)$ (indeed $m \geq 2$). The rest is easy.
(2) Take p_i such that $D_i(p_i) = q_i(f) \neq 0$ has lowest possible degree. Let $\tilde{p}_i \in \mathcal{P}_i$. then $D_i(\tilde{p}_i) = h_i(f)q_i(f) + r_i(f)$ where $deg(r_i) < deg(q_i)$. Now $D_i(\tilde{p}_i - h_i(f)p_i) = r_i(f)$ so $r_i = 0$. So $D_i(\tilde{p}_i) \in q_i(f)\mathbb{C}[f]$. □

Lemma 3.5. *Assume (S1). Choose p_i such that $D_i(p_i) = q_i(f)$ as in lemma 3.4, where q_i is of lowest possible degree. The D_i are linearly dependent modulo $f - \alpha$ if and only if $q_i(\alpha) = 0$ for some i.*

Proof. ($\Rightarrow$): Write "$\overline{\text{bars}}$" for "modulo $f - \alpha$". Suppose that $0 \neq D := g_1 D_1 + \ldots + g_n D_n$ satisfies $\overline{D} = 0$ where $g_i \in A$, and not all $\overline{g}_i = \bar{0}$. Now $\overline{g}_i \overline{D}_i(\overline{p}_i) = \overline{D}(\overline{p}_i) = \bar{0}$ for each i, so for each i, either $\overline{g}_i = \bar{0}$ or $\overline{q_i(f)} = \bar{0}$ (as $f - \alpha$ is irreducible by proposition 3.2). Since not all $\overline{g}_i = \bar{0}$, at least one $\overline{q_i(f)} = \bar{0}$. Since $f - \alpha$ is irreducible for each α, we not only have $(f-\alpha)|q_i(f)$, but even $(X - \alpha)|q_i(X)$, so $q_i(\alpha) = 0$.
($\Leftarrow$): Assume $f - \alpha$ divides $q_i(f)$. We need to show that the $D_i \mod (f - \alpha)$ are linearly dependent over $A/(f - \alpha)$. Suppose the $\overline{D}_i$ are linearly independent over $\overline{A}$. Then we have n commuting, linearly independent LNDs on a domain of transcendence degree n, so we can use proposition 3.1 and conclude that $\overline{A}^{\overline{D}_1, \ldots, \overline{D}_n} = k$. This means, since $\overline{q_i(f)} = 0$, that $\overline{p}_i \in k$. So, $p_i = (f - \alpha)a + \lambda$ where $a \in A, \lambda \in k$. Now taking $a \in A$ we still have $D_j(a) = 0$ for all $j \neq i$, and $D_i(a) = q_i(f)(f - \alpha)^{-1} \in \mathbb{C}[f]$. This contradicts the assumption that q_i was minimal, so our assumption that the $\overline{D}_i$ are linearly independent was incorrect. □

Now we want to point out the following phenomenon:

Example 3.6. Let $D_1 = Z\partial_X + \partial_Y, D_2 = \partial_Y$ on $A = \mathbb{C}[X, Y, Z]$. Now $A^{D_1, D_2} = \mathbb{C}[Z]$. The D_1, D_2 are linearly independent modulo $Z - \alpha$ as long as $\alpha \neq 0$. But it is clear that a different set of derivations, namely $E_1 = \partial_X, E_2 = \partial_Y$ commute, their $\mathbb{C}[Z]$-span contains D_1, D_2 and the E_i are linearly independent for more fibers $f - \alpha$.

The E_i of the example are an improvement over the D_i: all the same properties, but they are linearly independent for more $f - \alpha$. Perhaps for your given space A and derivations D_i it is impossible to find E_i such that the E_i are independent modulo every $f - \alpha$, giving more information on your ring A. Before we elaborate on this, let us give a lemma that enables construction of the E_i:

Lemma 3.7. *Assume (S1). Define $\mathcal{M} := k(f)D_1 + \ldots + k(f)D_n \cap \mathrm{DER}(A)$. Then $\mathcal{M} = k[f]E_1 \oplus \ldots \oplus k[f]E_n$ for some $E_i \in \mathcal{M}$, and the E_i have all the properties that the D_i have (i.e. commuting locally nilpotent, linearly independent over A). Furthermore, if the D_i are linearly independent modulo $(f - \alpha)$, then the E_i are too (but not necessary the other way around).*

Proof. Use lemma 3.4 we find preslices p_i and $D(p_i) = q_i(f)$ as stated there.

If $D \in \mathcal{M}$ then $D = g_1(f)D_1 + \ldots + g_n(f)D_n$ where $g_i(f) \in k(f)$. Now since $D \in \mathrm{DER}(A)$ we have $D(p_i) \in A$. Also $D(p_i) = g_i(f)D_i(p_i) = g_i(f)q_i(f) \in k(f)$ thus $D(p_i) \in A \cap k(f)$, which equals $k[f]$ since $A^* = k^*$.

Therefore the map $\varphi : \mathcal{M} \longrightarrow k[f]^n$ sending $D \longrightarrow (D(p_1), \ldots, D(p_n))$ is well-defined. If $0 = \varphi(g_1(f)D_1 + \ldots + g_n(f)D_n)$ then $g_i(f)D_i(p_i) = 0$ and therefore $g_i(f) = 0$; thus φ is injective.

Since φ is an injective map, $\mathcal{M}$ must be a free $k[f]$-module. Note that $\mathcal{M}$ can only have dimension n. Therefore we can find $E_1, \ldots, E_n$ as required.

Any derivation in $\mathcal{M}$ is locally nilpotent. Even more, any two derivations of $\mathcal{M}$ commute! Next to that, the E_i are clearly independent over A. □

Note that the E_i can be constructively made, given the injective map φ in the above proof. This actually gives an interesting concept. Given the situation (S1), one can improve the derivations D_i (by replacing them by the E_i) and then they are linearly independent modulo as much as possible $f - \alpha$. For every such α we have that $A/(f - \alpha)$ is a polynomial ring. The question is if the converse holds:

Question: Assume (S1). Additionally, assume $k[f]D_1 + \ldots + k[f]D_n = (k(f)D_1 + \ldots + k(f)D_n) \cap \mathrm{DER}(A)$. Is the set $\{\alpha \in \mathbb{C} \mid D_1, \ldots, D_n$ linearly dependent modulo $(f - \alpha)\}$ equal to the set $\{\alpha \in \mathbb{C} \mid A/(f - \alpha)$ is not a polynomial ring$\}$? (One always has $\supseteq$.) Or, if this equality does not hold, what type of rings A do have equality?

Note that the requirement "A UFD" is absolutely necessary, as for a simple Danielewski surface $\mathbb{C}[X, Y, Z]/(X^2Y - Z^2)$ we find a LND $2Z\partial_Y + X^2\partial_Z$ which is nonzero modulo each $X-\alpha$.(But $A/(f-\alpha)$ is not always a domain in this case, even.)

4 Unipotent actions

The authors would like to thank prof. Kraft for pointing out the generalization of theorem 2.1 part 2, which has become the below theorem 4.2.

Proposition 4.1. *If $U \times V \longrightarrow V$ is an action of a unipotent group U on an affine variety V, then for each $u \in U$, the map $u^* : O(V) \longrightarrow O(V)$ is an exponent of a locally nilpotent derivation.*

For the proof we can refer to proposition 2.1.3 in [2], or ask the reader to verify that $u^* - Id$ is a locally nilpotent endomorphism, and that thus "$log(u^*)$" can be defined, and is a derivation.

This proposition has some immediate consequences, like that the invariants of a unipotent group action are the intersection of kernels of locally nilpotent derivations. Since kernels of locally nilpotent derivations are factorially closed, their intersection is too, so the invariants of a unipotent group is factorially closed.

In the below theorem, $\mathbb{C}$ is a field of characteristic zero, which is algebraically closed.

Theorem 4.2. *Let U be a unipotent algebraic group of dimension n, acting freely on X, a factorial variety of dimension $n+1$ satisfying $\mathcal{O}(X)^* = \mathbb{C}^*$. Then X is U-isomorphic to $U \times \mathbb{C}$. In particular, $X \simeq \mathbb{C}^{n+1}$.*

Proof. The fact that U acts free means that each $x \in X$ has trivial stabilizer: $U_x = \{u \in U; ux = x\} = \{\text{id}\}$. So, each orbit Ux is of dimension n. This means that $X/\!\!/U$ is of dimension 1. Also, as remarked above, X^U is factorial. But then it is also normal, and smooth. So $X/\!\!/U$ is a smooth, rational, affine curve, in other words, an open subvariety of $\mathbb{C}$. Now suppose that $X/\!\!/U \not\cong \mathbb{C}$, so $X/\!\!/U = \mathbb{C} - \{p_1, \ldots, p_n\}$, then $\mathcal{O}(X)^U = \mathcal{O}(\mathbb{C} - \{p_1, \ldots, p_n\}) = \mathbb{C}[t, (t-p_1)^{-1}, \ldots, (t-p_n)^{-1}]$. This means that $\mathcal{O}(X)$ contains invertible elements $(t-p_1)^{-1}$, giving a contradiction with the assumption $\mathcal{O}(X)^* = \mathbb{C}^*$. Hence, $X/\!\!/U \simeq \mathbb{C}$, so $\mathcal{O}(X)^U = \mathcal{O}(X/\!\!/U) = \mathcal{O}(\mathbb{C}) \cong \mathbb{C}[f]$ for some f. Now every $f - \lambda$ ($\lambda \in \mathbb{C}$) is irreducible, as otherwise any irreducible factor of $f - \lambda$ would be in $\mathcal{O}(X)^U$ too.
Now consider the map $f : X \longrightarrow \mathbb{C}$. This is in fact the map $X \longrightarrow X/\!\!/U$ (as it corresponds to the map $\mathcal{O}(X) \longleftarrow \mathcal{O}(X)^U = \mathbb{C}[f]$) and thus surjective. Also note that the fibers $f^{-1}(\lambda)$ are invariant under U: they correspond to the function space $\mathcal{O}(X)/(f-\lambda)$. By assumption, U acts free on each fiber of $X \longrightarrow X/\!\!/U$, which means exactly that U acts free on $f^{-1}(\lambda)$ for each λ. Let $x \in f^{-1}(\lambda)$. Then Ux is of dimension n (it is just a copy of U). Also, each orbit of a unipotent group is closed (see Satz 4 from [3]), and therefore the inclusion $Ux \subseteq f^{-1}(\lambda)$ is an equality. So orbits of U are the same as fibers of f, i.e. we have an orbit fibration (or U-fibration).

X_{sing} is closed and U-stable, hence a union of U-orbits, and so codim $X_{\text{sing}} = 1$ or X_{sing} is empty. But X is factorial, so in particular normal, which implies $codim(X_{sing}) \geq 2$. So X_{sing} is empty, in other words: X is smooth.

Now we claim that $f : X \longrightarrow \mathbb{C}$ is smooth. To see this, first note that $\mathcal{O}(f^{-1}(\lambda)) = \mathcal{O}(X)/(f-\lambda)$ is reduced as $f-\lambda$ is irreducible, as seen before. And, as we already implied, the set of functions vanishing on $f^{-1}(\lambda)$ is the ideal $(f-\lambda)$. Now consider the tangent map $df_x \colon T_xX \to T_\lambda\mathbb{C} = \mathbb{C}$ where $x \in f^{-1}(\lambda)$. Using "Satz 2", page 269 in [3] we see that, $kerdf \supseteq T_xf^{-1}(\lambda)$, but since $f^{-1}(\lambda)$ is reduced, we even have equality $kerdf = T_xf^{-1}(\lambda)$. Now remember that the fiber $f^{-1}(\lambda)$ is an orbit, hence smooth (as any orbit is smooth!). This implies $\dim T_xf^{-1}(\lambda) = n$ and thus $\dim kerdf = n$. Since $\dim T_xX = n+1$ we have $\dim Im(df_x) = 1$, hence df_x is surjective. A morphism between smooth varieties is smooth if and only if the differential is surjective. So we have shown that f is smooth.

So: $f : X \longrightarrow \mathbb{C}$ is surjective, and smooth. Let $K := kerdf|_x \subset T_xX$. Take some linear subspace C such that $K \oplus C = T_xX$. Note that C has dimension 1. Seeing X as a subset of some $\mathbb{C}^N$, we can find hyperplanes H that contains C. We even want $H \cap T_xX = C$, so this means that $H \oplus T_x \subseteq \mathbb{C}^N$, so let us take a hyperplane H of codimension n such that $H \cap T_xX = C$. Now let Z be an irreducible component of $H \cap X$ which contains x. Also, $dim_xH \cap X \geq 1$, thus $\dim_x Z = 1$ and Z is smooth at x. Now Z and $\mathbb{C}$ are smooth, and the differential of $f\|_Z : Z \longrightarrow \mathbb{C}$ is an isomorphism at x (implying surjective), thus we have that $f\|_Z$ is smooth at x. Replacing Z, if necessary, by a (special) open subset $Z' \subset Z$, we have $f\|_Z$ is étale.

Now look at the following diagram

$$\begin{array}{ccc} Z \times_{\mathbb{C}} X & \xrightarrow{p} & X \\ \downarrow{\scriptstyle \bar{f}} & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{f|_Z} & \mathbb{C} \end{array}$$

where $Z \times_{\mathbb{C}} X = \{(x,z) \in X \times Z \mid f(x) = f|_Z(z)\}$ is the (schematic) fiber product. Since f is smooth, the same holds for $\bar{f}$ and so $Z \times_{\mathbb{C}} X$ is smooth. Moreover, U acts on $Z \times_{\mathbb{C}} X$ by $u(z,x) = (z,ux)$ and $p(u(x,z)) = ux$ (p is U-equivariant) and $f(u(x,z)) = z = f(x,z)$ (f is U-invariant). The fibers of $\bar{f}$ are $\bar{f}^{-1}(z) = \{(x,z) \mid f(x) = f|_Z(z)\} = \{x \mid f(x) = \alpha\} = f^{-1}(\alpha)$ where $\alpha = f_Z(z)$. Now $\bar{f}$ has a section $\sigma \colon Z \to Z \times_{\mathbb{C}} X$ given by $z \mapsto (z,z)$, i.e. $\bar{f} \circ \sigma = \mathrm{id}_Z$. Therefore, we can extend the diagram above

$$\begin{array}{ccccc} U \times Z & \xrightarrow{q} & Z \times_{\mathbb{C}} X & \xrightarrow{p} & X \\ \downarrow{\scriptstyle \mathrm{pr}_Z} & & \downarrow{\scriptstyle \bar{f}} & & \downarrow{\scriptstyle f} \\ Z & = & Z & \xrightarrow{f|_Z} & \mathbb{C} \end{array}$$

where $q \colon U \times Z \to Z \times_{\mathbb{C}} X$ is given by $(u,z) \mapsto (z,uz)$. By construction, q is bijective, hence an isomorphism, since the second variety is normal (see [4] proposition 5.7). Note that the role of x was arbitrary: for each x we find a neighborhood Z where $Z \times_{\mathbb{C}} X = Z \times_{\mathbb{C}} U$. This last statement exactly means that the map $f\colon X \to \mathbb{C}$ is a locally trivial principal U-bundle with respect to the étale topology: for every point $\lambda \in \mathbb{C}$ there is an étale map $Z \to \mathbb{C}$ such that λ is in the image and the fiber product $Z \times_{\mathbb{C}} X$ is a trivial U-bundle, i.e. isomorphic to $U \times Z \xrightarrow{\mathrm{pr}_Z} Z$.

In the paper [5] we now find a result that tells us that a principal G-bundle where G is a unipotent group is trivial over any affine variety, and then we are done.

□

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