Nilpotent Jacobians in dimension three

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Abstract. In this paper we completely classify all polynomial maps of the form $H = (u(x, y), v(x, y, z), h(u(x, y), v(x, y, z))$ with JH nilpotent.

AMS subject classification : 14R15, 14R10, 14E07.

Key Words. Jacobian Conjecture, Nilpotent Jacobian.

Running title: Nilpotent Jacobians.

Introduction

In [1] Bass, Connell and Wright showed that it suffices to investigate the Jacobian Conjecture for polynomial maps of the form $x+H$ with JH nilpotent (and H homogeneous of degree 3). Studying these maps led various authors to the following problem (see [4], [8], [9], [10]), where k is a field of characteristic zero.

(Homogeneous) Dependence Problem.
Let $H = (H_1, \dots, H_n) \in k[x_1, \dots, x_n]^n$ (homogeneous of degree $d \geq 1$) such that JH is nilpotent and $H(0) = 0$. Does it follow that $H_1, \dots, H_n$ are linearly dependent over k ?

It was shown in [1] that the answer is affirmative if rank$JH \leq 1$. In particular this implies that the dependence problem has an affirmative answer if $n = 2$. If H is homogeneous of degree 3 the case $n = 3$ was solved affirmatively by Wright in [11] and the case $n = 4$ by Hubbers in [7]. Then in [5] (see also [6], Theorem 7.1.7) the second author found the first counterexample in dimension three (see below). On the other hand, recently de Bondt and van den Essen showed in [2] that in case $n = 3$ and H homogeneous of arbitrary degree $d \geq 1$, the answer to the dependence problem is affirmative!
In this paper we study the inhomogeneous case in dimension three. More precisely we describe a large class of H with JH nilpotent and such that H_1, H_2, H_3 are linearly independent over k. The surprising result is that, apart from a linear coordinate change, all these examples are essentialy of the same form as the first counterexample (to the dependence problem) mentioned above.
Finally we would like to mention that very recently Michiel de Bondt [3] has constructed counterexamples to the homogeneous dependence problem for all dimensions $n \geq 5$! So only in dimension 4 the homogeneous dependence problem remains open.

1 Preliminaries on Nilpotent Jacobian Matrices

In this section we briefly recall some more or less known results on nilpotent Jacobian matrices. Throughout this paper, k denotes a field of characteristic 0 and $n \in \mathbf{Z}^+$.

It is well-known that a matrix $N \in M_n(k)$ is nilpotent if and only if for each $1 \le p \le n$ the sum of all $p \times p$ principal minors of N equals zero (a $p \times p$ principal minor of N is by definition the determinant of the submatrix of N obtained by deleting $n-p$ rows and $n-p$ columns with the same index).

Now let $H_1, \dots, H_n \in k[x] := k[x_1, \dots, x_n]$, the polynomial ring in n variables over k. Put $H := (H_1, \dots, H_n)$ and let JH denote the Jacobian matrix of H. The main problem in order to solve the Jacobian Conjecture is to describe the nilpotent Jacobian matrices JH and to show that for such H the corresponding polynomial map $F := x + H = (x_1 + H_1, \dots, x_n + H_n)$ is invertible over k. Obviously, if $\mathrm{rank}(JH) = 0$ (where $\mathrm{rank}(JH)$ is the rank of the matrix JH considered in $M_n(k(x))$), i.e. $JH = 0$, then each H_i belongs to k, which implies that $F = x + H$ is invertible over k. The following result is more involved (see Essen[5, Theorem 7.1.7]).

Proposition 1.1 *If JH is nilpotent and rank$(JH) \le 1$, then there exists $g \in k[x]$ such that $H_i \in k[g]$ for all i. Furthermore, if $H_i(0) = 0$ for all i, then there exist $c_1, \dots, c_n \in k$, not all zero, such that $c_1 H_1 + \cdots + c_n H_n = 0$.*

Using this proposition, the following result is proved in [5, Theorem 7.2.25].

Theorem 1.1 *Let A be a UFD of characteristic zero and $H = (H_1, H_2) \in A[x_1, x_2]^2$. Then $J_{x_1,x_2}(H)$ is nilpotent if and only if $H = (a_2 f(a_1 x_1 + a_2 x_2) + c_1, -a_1 f(a_1 x_1 + a_2 x_2) + c_2)$ for some $a_1, a_2, c_1, c_2 \in A$ and $f(t) \in A[t]$.*

Corollary 1.1 *Let $H = (H_1, H_2, H_3) \in k[x, y, z]^3$. Assume that $H(0) = 0$ and H_1, H_2, H_3 are linearly dependent over k. Then JH is nilpotent if and only if there exists $T \in Gl_3(k)$ such that*

$$THT^{-1} = (a_2(z) f(a_1(z)x + a_2(z)y) + c_1(z), -a_1(z) f(a_1(z)x + a_2(z)y) + c_2(z), 0)$$

for some $a_i(z), c_i(z) \in k[z]$ *and* $f(t) \in k[z][t]$*.*

Proof: Let $c_1H_1 + c_2H_2 + c_3H_3 = 0$ with $c_i \in k$ not all zero. We may assume that $c_3 \neq 0$. Then putting

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

the last component of TH equals $c_1H_1 + c_2H_2 + c_3H_3 = 0$. Hence $THT^{-1} = (h_1, h_2, 0)$ for some $h_i \in k[z][x, y]$. One easily verifies that JH is nilpotent if and only if $J(h_1, h_2, 0)$ is nilpotent if and only if $J_{x,y}(h_1, h_2)$ is nilpotent. Then the result follows from Theorem 1.1 applied to $A := k[z]$. □

Corollary 1.2 *Let* $H = (H_1, H_2, H_3) \in k[x, y, z]^3$ *with* $H(0) = 0$ *and rank*$(JH) \leq 1$*. If* JH *is nilpotent then* H *is of the form described in Corollary 1.1.*

Proof: This follows from Proposition 1.1 and Corollary 1.1. □

It was conjectured for several years that the result obtained in Proposition 1.1 (in the case rank$(JH) \leq 1$) would hold in general i.e. that JH is nilpotent together with $H(0) = 0$ would imply that $H_1, \ldots, H_n$ are linearly dependent over k. However, the following counterexample was found by the second author [6] (see also [5, Theorem 7.1.7]):

$$H = \big(y - x^2, z + 2x(y - x^2), -(y - x^2)^2\big)$$

Indeed, one easily verifies that JH is nilpotent, rank$(JH) = 2$ and H_1, H_2, H_3 are linearly independent over k. Looking more closely at the example one observes that it has the special form

$$H = (u(x, y), v(x, y, z), h(u(x, y)))$$

In the next section we describe completely which of these maps have a nilpotent Jacobian matrix.

2 Some Nilpotent Jacobians with Independent Rows

In this section we classify all the polynomial mappings of the form

$$H = (u(x,y), v(x,y,z), h(u(x,y)))$$

for which the Jacobian matrix JH is nilpotent. Before we do this we make some simple reductions. First we may assume that $H(0) = 0$ i.e. $u(0,0) = v(0,0,0) = 0$ and $h(0) = 0$. Also by Proposition 1.1, we may assume that the components of H are linearly independent over k, in particular $h \neq 0$. We may also assume that $h'(0) = 0$, for if $\rho := h'(0) \neq 0$, consider the linear map $T(x,y,z) = (x,y,z-\rho x)$ and put $\tilde{H} := THT^{-1}$. This implies $\tilde{H} = (u, v(x,y,z+\rho x), \tilde{h}(u))$, where $\tilde{h}(t) := h(t) - h'(0)t$, so $\tilde{h}'(0) = 0$ and one easily verifies that $J\tilde{H}$ is nilpotent if and only if JH is nilpotent. In summary, we may assume $h \neq 0$, $h(0) = 0$, $h'(0) = 0$ and $\deg_t h \geq 2$. In order to classify all mappings H of the above form it remains to prove the following result.

Theorem 2.1 *Let $H = (u(x,y), v(x,y,z), h(u(x,y)))$. Assume that $H(0) = 0$, $h'(0) = 0$ and that the components of H are linearly independent over k (hence $h \neq 0$ and $\deg_t h \geq 2$). Then the following two statements are equivalent:*

1. *JH is nilpotent.*
2. *There exist $v_1, \lambda \in k^*$, $b_1 \in k$ and $g(t) \in k[t]$ with $g(0) = 0$ and $\deg_t g(t) \geq 1$ such that $u = g(y + b(x))$, $v = v_1 z - b'(x) g(y + b(x))$ and $h = \lambda t^2$, where $b(x) := v_1 \lambda x^2 + b_1 x$.*

Proof:

First observe that the first and the third row of JH are linearly dependent, whence $\det JH = 0$. So by the remark at the beginning of the previous section, concerning the principal $p \times p$ minors of JH, we get that JH is nilpotent if and only if

$$u_x + v_y = 0 \qquad (1)$$

and

$$u_x v_y - u_y v_x = h'(u) u_y v_z \tag{2}$$

It is an easy exercise to verify that the formulas for u,v and h given in Statement 2 of Theorem 2.1 satisfy (1)–(2) which shows the implication Statement 2 $\rightarrow$ Statement 1 of Theorem 2.1. Thus it remains to show that the implication Statement 1 $\rightarrow$ Statement 2 holds.

Let $v = v_m z^m + \cdots + v_1 z + v_0$ with $v_i \in k[x,y]$ and $v_m \neq 0$. Observe that $m \geq 1$, for if $m = 0$ i.e. $v_z = 0$, then both u and v belong to $k[x,y]$, hence (1)–(2) show that $J_{x,y}(u,v)$ is nilpotent. So by Proposition 1.1, u and v are linearly dependent over k, a contradiction. This shows that $m \geq 1$. By (1) we get that $v_m = v_m(x), \ldots, v_1 = v_1(x)$ and $v_{0y} = -u_x$.

So $v_0 = p_x$ and $u = -p_y$ for some $p \in k[x,y]$ with $p(0) = 0$. Now look at the coefficient of z^m in (2). This gives $p_{yy} v'_m(x) = 0$. Observe that $u_y \neq 0$ (for otherwise by (2) $u_x v_y = 0$ which by (1) gives that also $u_x = 0$ and hence both u_y and u_x are zero, thus $u = 0$, a contradiction). So $p_{yy} = -u_y \neq 0$, whence $v'_m(x) = 0$ i.e. $v_m \in k^*$.

Now we show that $m = 1$. Namely, assume that $m \geq 2$. Looking at the coefficient of z^{m-1} in (2) gives that $-u_y v'_{m-1}(x) = h'(u) u_y m v_m$ i.e. $v'_{m-1}(x) = -h'(u) m v_m$. Since $\deg_t h \geq 2$ and u depends on y ($u_y \neq 0$) the righthandside of this equation depends on y, but the lefthandside does not, a contradiction. So $m = 1$. Summarizing we get

$$u = -p_y, \quad v = v_1 z + p_x, \quad \text{with } v_1 \in k^*, \ p \in k[x,y], \ p_{yy} \neq 0 \tag{3}$$

Substituting these formulas in (2) gives

$$-p_{xy}^2 + p_{yy} p_{xx} = -h'(-p_y) p_{yy} v_1$$

so if we put $G(t) := v_1 h'(-t)$ we get

$$p_{xy}^2 - p_{xx} p_{yy} = G(p_y) p_{yy} \tag{4}$$

Write $G = c_r t^r + \cdots + c_1 t + c_0$ with $c_i \in k$ and $c_r \neq 0$. Since $\deg_t h \geq 2$ it follows that $r \geq 1$. Now we will show that $r = 1$. Therefore assume that $r \geq 2$

and write $p = p_n(x)y^n + \cdots + p_0(x)$ with $p_n \neq 0$ and $p_i \in k[x]$ for all i. Since $p_{yy} \neq 0$, we have $n \geq 2$. Now look at the highest degree y term in (4). On the righthandside we get

$$c_r(np_n y^{n-1})^r n(n-1)p_n y^{n-2} = c_r n^{r+1}(n-1)p_n^{r+1} y^{r(n-1)+n-2} \tag{5}$$

On the lefthandside we get

$$(np'_n y^{n-1})^2 - p''_n y^n n(n-1)p_n y^{n-2} = (n^2(p'_n)^2 - n(n-1)p_n p''_n)y^{2n-2} \tag{6}$$

Looking at the y-degree of these equations we get $r(n-1)+n-2 \leq 2n-2$, so if $r \geq 3$, then $n \leq 3/2$, a contradiction since $n \geq 2$. Since we assumed that $r \geq 2$ it remains to exclude the case $r = 2$. Then $n \leq 2$, and our earlier restriction implies $n = 2$, so $r = n = 2$. Then (5)–(6) give

$$4(p'_2)^2 - 2p_2 p''_2 = 8c_2 p_2^3 \neq 0. \tag{7}$$

It follows that $p'_2 \neq 0$, so $d := \deg_x p_2(x) \geq 1$. Finally, comparing the x-degrees in (7) gives that $3d \leq 2(d-1)$ i.e. $d \leq -2$, a contradiction. So apparently $r = 1$. Hence $\deg_t h = 2$. Since $h(0) = h'(0) = 0$ we get $h = \lambda t^2$ for some $\lambda \in k^*$. So $G(t) = v_1 h'(-t) = -2v_1\lambda t$ and (4) becomes

$$p_{xy}^2 - p_{xx}p_{yy} = -2v_1\lambda p_y p_{yy} \tag{8}$$

To solve this equation we need

Lemma 2.1 *Let $\mu \in k$. Then $p \in k[x,y]$ with $p(0) = 0$ satisfies $p_{xy}^2 - p_{xx}p_{yy} = \mu p_y p_{yy}$ if and only if*

$$p(x,y) = f(a_1 x + a_2(y - \mu x^2/2)) + c_1 x + c_2(y - \mu x^2/2)$$

for some $a_i, c_i \in k$ and $f(t) \in k[t]$ with $f(0) = 0$.

Proof: Put $\tilde{p} := p(x, y + \mu x^2/2)$. Then by the Chain rule one finds that $p_{xy}^2 - p_{xx}p_{yy} = \mu p_y p_{yy}$ if and only if $\tilde{p}_{xy}^2 - \tilde{p}_{xx}\tilde{p}_{yy} = 0$. This last equation is equivalent to $J(\tilde{p}_y, -\tilde{p}_x)$ is nilpotent. By Theorem 1.1 it then follows that $\tilde{p}_y = a_2 f(a_1 x + a_2 y) + c_1$ and $-\tilde{p}_x = -a_1 f(a_1 x + a_2 y) + c_2$ for some $a_i,\ c_i \in k$

and $f(t) \in k[t]$ with $f(0) = 0$. Consequently $\tilde{p} = F(a_1x + a_2y) - c_2x + c_1y$, where $F'(t) = f(t)$ and $F(0) = 0$, which implies the lemma. $\square$

Proof of theorem 2.1(completed)

From (8) and Lemma 2.1 (with $\mu = -2v_1\lambda$) we get that $p = f(a_1x + a_2(y + v_1\lambda x^2)) + c_1x + c_2(y + v_1\lambda x^2)$, for some a_i, c_i in k and $f(t) \in k[t]$ with $f(0) = 0$. Since $u = -p_y$ and $u_y \neq 0$, it follows that $a_2 \neq 0$ and $f''(t) \neq 0$ i.e. $\deg_t f \geq 2$. So $u = -p_y = -a_2f'(a_2(y + v_1\lambda x^2 + \frac{a_1}{a_2}x)) - c_2$. Hence if we put $g(t) := -a_2f'(a_2t) - c_2$ and $b(x) := v_1\lambda x^2 + \frac{a_1}{a_2}x$, then $u = g(y + b(x))$ with $\deg_t g \geq 1$. Since $u(0,0) = 0$ we get $g(0) = 0$. Also $v = v_1z + v_0$ and $v_{0y} = -u_x = -b'(x)g'(y + b(x))$ whence $v_0 = -b'(x)g(y + b(x)) + c(x)$ for some $c(x) \in k[x]$. Substituting these formulas into (2) and using that $h = \lambda t^2$ we obtain that $c'(x) = 0$ i.e. $c \in k$. Hence $v(0,0,0) = 0$ together with $g(0) = 0$ imply that $c = 0$, so $v_0 = -b'(x)g(y + b(x))$. Consequently

$$H = (g(y + b(x)), v_1z - b'(x)g(y + b(x)), \lambda(g(y + b(x)))^2)$$

with $b(x) = v_1\lambda x^2 + b_1x$, $b_1 \in k$ and $\deg_t g \geq 1$ as desired. This completes the proof. $\square$

3 The Magic Equations and an Extension of Theorem 2.1

In the previous section we studied the case $H = (u(x,y), v(x,y,z), h(u(x,y)))$. In this case the equations (1)–(2) describing the nilpotency of JH are relatively simple. However, if we replace the third component of H by a polynomial in both u and v i.e. $h(u(x,y), v(x,y,z))$, then the equations describing the nilpotency become more involved. In particular, the equation which expresses that the sum of the 2×2 principal minors of JH is equal to zero is rather complicated.

The aim of this section is to replace these complicated equations by another pair of much nicer (and useful) equations, which we call the magic equations.

They play a crucial role throughout this paper. As a first application we show at the end of this section how they can be used to extend Theorem 2.1 to the case $H = (u(x,y), v(x,y,z), h(u(x,y), v(x,y,z)))$.

Throughout this section we have the following notations: $H = (u, v, h(u,v))$ where $u, v \in k[x,y,z]$, $h \in k[s,t]$ and none of these polynomials has a constant term. Instead of $h_s(u,v)$ and $h_t(u,v)$ we write h_u and h_v respectively.

Proposition 3.1 (Magic equations) *If JH is nilpotent, then*

$$(u_z A + v_z B)h_u = -(u_x A + v_x B) \tag{9}$$

$$(u_z A + v_z B)h_v = -(u_y A + v_y B) \tag{10}$$

where $A := v_x u_z - u_x v_z$ and $B := v_y u_z - u_y v_z$. Conversely, if $u_z A + v_z B \neq 0$ then (9)–(10) imply that JH is nilpotent.

Proof: Since the last row of JH is a linear combination of the first two rows, it follows from the remark in the beginning of the first section that JH is nilpotent if and only if both trace JH is zero and the sum of the 2×2 principal minors of JH is zero. Writing these two conditions explicitly yields

$$u_x + v_y + h_u u_z + h_v v_z = 0$$

and

$$(u_x v_y - u_y v_x) + u_x(h_u u_z + h_v v_z) - u_z(h_u u_x + h_v v_x) + v_y(h_u u_z + h_v v_z) - v_z(h_u u_y + h_v v_y) = 0.$$

Now consider both equations as linear equations in h_u and h_v and write them in matrix form. This gives

$$M \begin{pmatrix} h_u \\ h_v \end{pmatrix} = -\begin{pmatrix} u_x + v_y \\ u_x v_y - u_y v_x \end{pmatrix} \tag{11}$$

where

$$M = \begin{pmatrix} u_z & v_z \\ B & -A \end{pmatrix}.$$

Observe that $\det M = -(u_zA + v_zB)$. Then the proposition follows from Cramer's Rule.

□

So it remains to describe the situation when JH is nilpotent and $u_zA+v_zB = 0$. This is done in the next result.

Proposition 3.2 *If JH is nilpotent and $u_zA + v_zB = 0$ then rank$(JH) \leq 1$ (and hence by Proposition 1.1 Corollary 1.1 applies).*

Proof: The assumption $u_zA + v_zB = 0$ together with (9)–(10) imply that $u_xA + v_xB = 0$ and $u_yA + v_yB = 0$. So if not both A and B are zero it follows that all the 2×2 minors of $J(u, v)$ are zero which implies that this matrix has rank less than or equal to one. Since the last row of JH is a linear combination of the rows of $J(u, v)$, we deduce that $\text{rank}(JH) = \text{rank}(J(u, v)) \leq 1$. Finally, if both A and B are zero, then $u_xv_y - u_yv_x = 0$ (this follows from (11). So again all 2×2 minors of $J(u, v)$ are zero, which as above implies that $\text{rank}(JH) \leq 1$.
□

Corollary 3.1 *In the remainder of this paper we may assume that $u_zA+v_zB \neq 0$.*

To conclude this section we show how Proposition 3.1 can be used to extend Theorem 2.1. More precisely, we consider polynomial maps of the form

$$H = (u(x, y), v(x, y, z), h(u(x, y), v(x, y, z)))$$

As in the previous section we may assume that $H(0) = 0$, $h(0, 0) = 0$ and that the linear part of h is zero (if $h(u, v) = \lambda_1 u + \lambda_2 v+$ higher order terms, then consider the linear map $T(x, y, z) = (x, y, z - \lambda_1 x - \lambda_2 y)$ and replace H by $\tilde{H} := THT^{-1}$). Also by Corollary 1.1 we may assume that the components of H are linearly independent over k. Now we will show

Proposition 3.3 *Let $H = (u(x, y), v(x, y, z), h(u(x, y), v(x, y, z)))$. Assume that $H(0) = 0$, h has no linear part in u and v and the components of H*

are linearly independent over k. If JH is nilpotent, then $h_v = 0$ i.e. h depends only on u (and Theorem 2.1 applies).

Proof: Observe that $u_z A + v_z B \neq 0$ (for otherwise rank$(JH) \leq 1$ by Proposition 3.2 and hence the components of H are linearly dependent over k by Proposition 1.1, a contradiction). Since $u_z = 0$ this implies that $v_z \neq 0$ and $B \neq 0$. Furthermore $A = -u_x v_z$ and $B = -u_y v_z$, whence $u_y \neq 0$. Substituting these formulas into (9)–(10) and dividing (9) by v_z and (10) by $v_z u_y$ we get

$$-v_z u_y h_u = u_x^2 + v_x u_y \tag{12}$$

$$-v_z h_v = u_x + v_y \tag{13}$$

Now let $h(u,v) = h_n(u)v^n + \cdots + h_0(u)$ with $h_i \in k[u]$ and $h_n \neq 0$. We need to show that $n = 0$, so assume $n \geq 1$. Since $v_z \neq 0$ we have $v = v_d z^d + \cdots + v_0$ with $v_i \in k[x,y]$ for all i, $d \geq 1$, and $v_d \neq 0$. The highest z-degree term on the lefthandside of (13) equals $(-dv_d z^{d-1})(nh_n(u)(v_d z^d)^{(n-1)})$. The highest z-degree term on the righthandside of (13) equals $v_{dy} z^d$. So we get $(n-1)d \leq 1$. Hence there are two cases, namely $n = 2$ and $d = 1$ and the case $n = 1$.

For the case $n = 2$ and $d = 1$, let $h = h_2(u)v^2 + h_1(u)v + h_0(u)$ with $h_2 \neq 0$ and $v = v_1 z + v_0$ with $v_1 \neq 0$. Looking at the z-coefficient in (13) we get $-v_1^2 2h_2(u) = v_{1y}$, which gives a contradiction looking at the y-degrees.

For the case $n = 1$, we have $h = h_1(u)v + h_0(u)$ with $h_1 \neq 0$ and $v = v_d z^d + \cdots + v_0$ with $v_d \neq 0$ and $d \geq 1$. In (12) the highest degree z-term on the lefthandside equals $(-dv_d z^{d-1})u_y h_1'(u)(v_d z^d)$, while the highest degree z-term on the righthandside equals $v_{dx} z^d u_y$. If $d \geq 2$, then $2d - 1 > d$, so we get that $h_1'(u) = 0$ (since $u_y \neq 0$ and $v_d \neq 0$). Since $n = 1$, $h_1 \neq 0$ whence $h_1 \in k^*$. But then $h = h_1 v + h_0(u)$ has a non-trivial linear part, contradicting the hypotheses, so $d = 1$. Setting equal the z-coefficients in (12) we get $-v_1^2 u_y h_1'(u) = v_{1x} u_y$, whence $-v_1^2 h_1'(u) = v_{1x}$ (since $u_y \neq 0$). If $h_1'(u) \neq 0$ we get a contradiction by looking at the x-degrees. So $h_1'(u) = 0$ which again implies that $h_1 \in k^*$ and hence h is a non-trivial linear part, a contradiction. Thus the hypothesis $n \geq 1$ leads to a contradiction, hence $n = 0$ as desired.

$\square$

4 Some special conditions on H

In this section we study some special conditions on $H = (u, v, h(u, v))$ that enable us to describe all such H whose Jacobian matrix JH is nilpotent. By Proposition 3.3 we may assume that both $u_z \neq 0$ and $v_z \neq 0$. The following result may be viewed as another generalization of Theorem 2.1.

Proposition 4.1 *Let $H = (u(x, y, z), v(x, y, z), h(u(x, y, z), v(x, y, z)))$. Assume that $H(0) = 0$, h has no linear part in u or v and the components of H are linearly independent over k. If JH is nilpotent and* $\deg_z h_u, \ \deg_z h_v = 0$ *then there exists $T \in GL_3(k)$ such that THT^{-1} is of the form described in Theorem 2.1.*

Proof: Since both h_u and h_v do not depend on z, differentiation of both polynomials with respect to z gives

$$h_{uu}u_z + h_{uv}v_z = 0, \quad h_{vu}u_z + h_{vv}v_z = 0$$

Since not both u_z and v_z are zero it follows that $h_{uu}h_{vv} - h_{uv}^2 = 0$, so by Lemma 2.1 (with $\mu = 0$) we get that $h = f(a_1u + a_2v) + c_1u + c_2v$ for some $a_1, a_2, c_1, c_2 \in k$ and $f(t) \in k[t]$ with $f(0) = 0$. This implies $h_u = a_1f'(a_1u + a_2v) + c_1$ and $h_v = a_2f'(a_1u + a_2v) + c_2$. Since both h_u and h_v do not depend on z the same holds for $a_1f'(a_1u + a_2v)$ and $a_2f'(a_1u + a_2v)$. Since not both a_1 and a_2 are zero (otherwise h is linear, a contradiction) it follows that $f'(a_1u + a_2v)$ does not depend on z. Also f' is not constant (otherwise again h is linear), so $a_1u + a_2v \in k[x, y]$. If $a_2 = 0$ then $a_1 \neq 0$, $h = f(a_1u) + c_1u + c_2v$ and $u \in k[x, y]$. Since h has no linear part $c_2 = 0$ i.e. $h = f(a_1u) + c_1u$. Then we are in the situation of Theorem 2.1. If $a_2 \neq 0$ consider the invertible linear map $T(x, y, z) = (a_1x + a_2y, x, z)$, whence $TH = (a_1u + a_2v, u, f(a_1u + a_2v) + c_1u + c_2v)$. Using that $a_1u + a_2v \in k[x, y]$ we get that $THT^{-1} = (\tilde{u}(x, y), \tilde{v}(x, y, z), \tilde{h}(\tilde{u}(x, y) + c\tilde{v})$ for some $\tilde{u} \in k[x, y]$,

$v \in k[x,y,z]$, $c \in k$ and $\tilde{h} \in k[t]$ with $\tilde{h}(0) = 0$. Finally conjugating with one more invertible linear map (if necessary) we can remove the linear part of $\tilde{h}(\tilde{u}) + c\tilde{v}$ and arrive in the situation of Theorem 2.1, as desired. $\square$

Corollary 4.1 *Let $H = (u(x,y,z), v(x,y,z), h(u(x,y,z), v(x,y,z)))$ such that $H(0) = 0$, h has no linear part and the components of H are linearly independent over k. If JH is nilpotent and $\deg_z uA \neq \deg_z vB$ then there exists $T \in GL_3(k)$ such that THT^{-1} is of the form described in Theorem 2.1.*

Proof: Without loss of generality we may assume that $\deg_z uA > \deg_z vB$ i.e. $\deg_z u + \deg_z A > \deg_z v + \deg_z B$. Consequently $\deg_z u_z A > \deg_z v_z B$, whence $\deg_z(u_z A + v_z B) = \deg_z u_z A$. So by (9) we get

$$\deg_z h_u + \deg_z u + \deg_z A - 1 \leq \deg_z u + \deg_z A$$

whence $\deg_z h_u \leq 1$. Similarly, using (10), we get $\deg_z h_v \leq 1$.

Now assume that $\deg_z h_u = 1$. Then $\deg_z(u_z A + v_z B)h_u = \deg_z uA$. Since $\deg_z(u_x A + v_x B) \leq \deg_z uA$ and $\deg_z v_x B \leq \deg_z v_x + \deg_z B \leq \deg_z vB < \deg_z uA$ it follows from (9) that the highest degree z-term of $u_z A h_u$ equals the highest z-term of $-u_x A$. So if we write $u = u_d z^d + \cdots + u_0$ with $u_d \neq 0$, $d \geq 1$ and $u_i \in k[x,y]$ for all i, then we get $d u_d h_1 z^d = u_{dx} z^d$ i.e. $d u_d h_1 = u_{dx}$, which gives a contradiction by looking at the x-degrees. Consequently, $\deg_z h_u = 0$. Similarly, using (10) we get $\deg_z h_v = 0$. The result now follows from Theorem 4.1. $\square$

Corollary 4.2 *The study for which $u, v \in k[x,y,z]$ and $h \in k[s,t]$ with $u_z \neq 0$ and $v_z \neq 0$ the Jacobian matrix of the map $H = (u, v, h(u,v))$ is nilpotent, reduces to the case where $\deg_z uA = \deg_z vB$.*

Acknowledgement: MC is grateful for being supported by a Harris Fellowship from Grinnell College, a research leave which gave the time and financial resources to visit AE during the spring of 2002.

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