Rings of constants of the form $\mathbf{k[f]}^0$

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Abstract

Let $k[X]$ be the algebra of polynomials in n variables over a field k of characteristic zero, and let $f \in k[X] \setminus k$. We present a construction of a derivation d of $k[X]$ whose ring of constants is equal to the integral closure of $k[f]$ in $k[X]$. A similar construction for fields of rational functions is also given.

1 Introduction

Let k be a field of characteristic zero and let $k[X] := k[x_1, \dots, x_n]$ be the polynomial ring in n variables over k. If $d : k[X] \to k[X]$ is a derivation of $k[X]$, then we denote by $k[X]^d$ the ring of constants of d, that is, $k[X]^d = \{w \in k[X];\ d(w) = 0\}$.

Rings of constants appear in various classical problems. For example the Cancellation Problem asks if the ring of constants of a locally nilpotent derivation on a polynomial ring having a slice is a polynomial ring, Hilbert's fourteenth problem asks if the ring of constants of a derivation on a polynomial ring over a field k is a finitely generated k-algebra and the Jacobian Problem asks if the ring of constants associated to a Jacobian derivation of the form $\frac{\partial}{\partial F_n}$ is a polynomial ring generated by $F_1, \dots, F_{n-1}$, when $\det JF \in k^*$ (for more details we refer to [4]).

It is well known that every k-algebra B of the form $k[X]^d$, where d is a derivation of $k[X]$, is integrally closed in $k[X]$ and $B_0 \cap k[X] = B$ (where B_0 denotes the quotient field of B). In [9] (or [10]) the third author proved that every k-subalgebra B of $k[X]$ which is integrally closed in $k[X]$ with $B_0 \cap k[X] = B$ can be realized as the ring of constants of some derivation of $k[X]$. However his proof of this fact is not effective. For a given subalgebra B of $k[X]$ satisfying the above conditions it is not easy to construct a derivation d of $k[X]$ such that $k[X]^d = B$. We know only that such a derivation exists.

In this paper we discuss an effective counterpart of this result for k-subalgebras B generated over k by a one element. More precisely, for a given polynomial $f \in k[X] \setminus k$ we present (in Section 3) a construction of an explicit derivation d of $k[X]$ whose ring of constants is equal to the integral closure of the ring $k[f]$ in $k[X]$. A similar construction we present also for a given rational function $\varphi \in k(X) \setminus k$.

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Note that there exists an algorithm, given by J. Brennan and W. Vasconcelos in [2] (see also [15]), to compute the integral closure of finitely generated k-domains. This algorithm is based on the theory of Gröbner basis and Rees algebras. In our case we know, by Zaks' theorem (see [16] or [4], Theorem 1.2.26), that the integral closure of $k[f]$ in $k[X]$ is of the form $k[g]$ for some $g \in k[X] \setminus k$, so in our case the general algorithm of Brennan and Vasconcelos has a simpler form.

It is well known that if d is a derivation of $k[X]$, then the ring $k[X]^d$ coincides with the k-algebra of all polynomial first integrals of the following system of ordinary differential equations:

$$\frac{dx_i}{dt} = f_i(x_1(t), \dots, x_n(t)), \quad i = 1, \dots, n,$$

where $f_1 = d(x_1), \dots, f_n = d(x_n)$. The field of constants $k(X)^d$ coincides with the field of all rational first integrals of this system. Hence, for a given polynomial $f \in k[X] \setminus k$ we are ready to construct a system of ordinary differential equations such that its k-algebra of all polynomial first integrals is equal to the integral closure of $k[f]$ in $k[X]$. A similar construction we have for a given rational function $\varphi \in k(X) \smallsetminus k$.

Note also that our considerations are quite obvious for $n \leq 2$. If $n = 1$ and $f \in k[x_1] \setminus k$, then the integral closure of $k[f]$ in $k[x_1]$ is equal to $k[x_1]$. So, in this case, only the zero derivation of $k[x_1]$ satisfies the mentioned conditions. If $n = 2$ and $f \in k[x_1, x_2] \setminus k$, then the jacobian derivation $d = \frac{\partial f}{\partial x_2}\frac{\partial}{\partial x_1} - \frac{\partial f}{\partial x_1}\frac{\partial}{\partial x_2}$ satisfies our conditions. In this case the ring $k[x_1, x_2]^d$ is equal to the integral closure of $k[f]$ in $k[x_1, x_2]$ (see [7]).

2 Preliminaries

Throughout this paper all rings and algebras are commutative, k denotes a field of characteristic zero, $k[X] := k[x_1, \dots, x_n]$ is the polynomial ring in n variables over k, and $k(X) := k(x_1, \dots, x_n)$ is the field of quotients of $k[X]$. If A is a domain, then we denote by A_0 the field of quotients of A.

In this section we present some preparatory facts which will be important in the next section. Moreover, we present here examples of derivations with trivial fields of constants.

Let us start from the following lemma which is a special case of a more general fact (see, for example, [1] p.296 or [4] Proposition D.1.7). We present a proof because, in our case, this proof is easy.

Lemma 2.1. *If $h \in k[X]$, then $k[h]_0 \cap k[X] = k[h]$.*

Proof. Assume that $u = u(X) \in k[h]_0 \cap k[X]$. Then $u = \frac{p(h)}{q(h)}$, where $p(t), q(t)$ are relatively prime polynomials belonging to $k[t]$, the ring of polynomials in the one variable t over k. There exist polynomials $\alpha(t), \beta(t) \in k[t]$ such that $1 = \alpha(t)p(t) + \beta(t)q(t)$. Hence, in the ring $k[X]$ we have: $1 = \alpha(h)p(h) + \beta(h)q(h) = \alpha(h)u(X)q(h) + \beta(h)q(h) = (\alpha(h)u(X) + \beta(h))\, q(h)$ and this implies that the polynomial $q(h)$ is invertible in $k[X]$. So, $q(h) \in k$, that is, $u = \frac{p(h)}{q(h)} \in k[h]$. Therefore, $k[h]_0 \cap k[X] \subseteq k[h]$. The opposite inclusion is obvious. $\square$

Let us recall the following well known theorem (see [16] or [4] Theorem 1.2.26).

Theorem 2.2 (Zaks). *If R is a Dedekind subring of $k[X]$ containing k, then there exists a polynomial $f \in k[X]$ such that $R = k[f]$.*

Consider now the following family $\mathcal{M}$ of k-subalgebras of $k[X]$:

$$\mathcal{M} = \{k[h];\ h \in k[X] \smallsetminus k\}.$$

If $k[h_1] \subsetneq k[h_2]$, for some $h_1, h_2 \in k[X] \smallsetminus k$, then $\deg h_2 < \deg h_1$ and hence, in the family $\mathcal{M}$ there exist maximal elements. As a consequence of Theorem 2.2 we obtain the following lemma (see [11] Lemma 3.1 or [10] Proposition 5.2.1, for details).

Lemma 2.3. *If $h \in k[X] \smallsetminus k$, then $k[h]$ is a maximal element in the family $\mathcal{M}$ if and only if the algebra $k[h]$ is integrally closed in $k[X]$. In particular, if $f \in k[X] \smallsetminus k$, then the integral closure of $k[f]$ in $k[X]$ is of the form $k[g]$, for some $g \in k[X] \smallsetminus k$.*

Note also the following obvious lemma.

Lemma 2.4. *Let A be a k-subalgebra of $k[X]$. If A is integrally closed in $k[X]$, then the field A_0 is algebraically closed in $k(X)$.*

If d is a derivation of a ring R, then we denote by R^d the kernel of d, that is, $R^d = \{r \in R;\ d(r) = 0\}$. Note that R^d is a subring of R. If R is a field, then R^d is a subfield of R. The next lemma is a modification of Lemma 4 in [14].

Lemma 2.5. *Let $M \subset K \subset L$ be fields (of characteristic zero) such that the extension $K \subset L$ is algebraic. Assume that $d : K \to K$ is an M-derivation such that $K^d = M$ and let $\overline{d} : L \to L$ be the derivation which is the unique extension of d to L. If M is algebraically closed in L, then $L^{\overline{d}} = M$.*

Proof. Let $u \in L$, $\overline{d}(u) = 0$. Since u is algebraic over K, there exists a minimal $m \geq 1$ such that $u^m + a_{m-1}u^{m-1} + \cdots + a_1 u_0 = 0$, for some $a_0, \ldots, a_{m-1} \in K$. Applying $\overline{d}$ and using that $\overline{d}(u) = 0$ we get that $d(a_{m-1})u^{m-1} + \cdots + d(a_1)u + d(a_0) = 0$. ¿From the minimality of m it follows that $d(a_i) = 0$ for all i, that is, $a_0, \ldots, a_{m-1} \in K^d = M$. Hence, u is algebraic over M. But M is algebraically closed in L, so $u \in M$. Therefore, $L^{\overline{d}} = M$. $\square$

Let δ be a derivation of $k[X]$. We denote by $\overline{\delta}$ the unique extension of δ to $k(X)$. The field $k(X)^{\overline{\delta}}$ is called *the field of constants of* δ. If $k(X)^{\overline{\delta}} = k$, then we say that the field of constants of δ is *trivial*. A collection of examples of derivations of $k[X]$ with trivial field of constants can be found, for instance, in [10]. Now we recall some of these examples.

Let $\delta_1, \ldots, \delta_5$ be derivations of $k[X] := k[x_1, \ldots, x_n]$ (where $n \geqslant 2$) defined as follows:

$$\begin{aligned}
\delta_1 &= \partial_1 + (x_1x_2 + 1)\partial_2 + (x_2x_3 + 1)\partial_3 + \cdots + (x_{n-1}x_n + 1)\partial_n,\\
\delta_2 &= (x_1x_2\cdots x_{n-1})\left(\partial_1 + \tfrac{1}{x_1}\partial_2 + \tfrac{1}{x_2}\partial_3 + \cdots + \tfrac{1}{x_{n-1}}\partial_n\right),\\
\delta_3 &= \partial_1 + x_2\partial_2 + x_2x_3\partial_3 + \cdots + x_2x_3\cdots x_n\partial_n,\\
\delta_4 &= x_2^s\partial_1 + x_3^s\partial_2 + \cdots + x_n^s\partial_{n-1} + x_1^s\partial_n, \quad \text{for } \ n \geq 3,\ s \geq 2,\\
\delta_5 &= x_1x_2\partial_1 + x_2x_3\partial_2 + \cdots + x_{n-1}x_n\partial_{n-1} + x_nx_1\partial_n,
\end{aligned}$$

where $\partial_i = \frac{\partial}{\partial x_i}$, for $i = 1, \dots, n$. It is known that every derivation δ_j, for $j \in \{1, \dots, 5\}$, has a trivial field of constants. If $j = 1$, then it is a consequence of Shamsuddin's result [12] (see [10] Example 13.4.3). For $j = 2$ see S. Suzuki [14]. For $j = 3$ see H. Derksen [3]. The derivation δ_4 is called a Jouanolou derivation. A proof that $k(X)^{\overline{\delta_4}} = k$ is due to H. Żołądek [17] (in [5] is a proof in the case when n is prime). For $j = 5$ see [6] (or [10]).

Let δ_6 be a derivation of $k[x_1, x_2]$ defined by

$$\delta_6 = ax_1\partial_1 + (ax_2 + x_1)\partial_2,$$

where $0 \neq a \in k$. This derivation has also a trivial field of constants (see [8] or [10]).

3 Constructions

Let $f \in k[X] \smallsetminus k$ and $\varphi \in k(X) \smallsetminus k[X]$. In this section we present a construction of a derivation d of $k[X]$ whose ring of constants $k[X]^d$ is equal to the integral closure of $k[f]$ in $k[X]$. Moreover, we present also a construction of a derivation d_0 of $k[X]$ whose field of constants $k(X)^{\overline{d_0}}$ is equal to the algebraic closure of $k(\varphi)$ in $k(X)$. We already know (see Introduction) that such constructions are clear for $n \leq 2$. Hence, we assume that $n \geq 3$.

Since $f \notin k$, there exists an $i \in \{1, \dots, n\}$ such that $\frac{\partial f}{\partial x_i} \neq 0$. Let us assume (for simplicity) that $i = n$, that is, $\frac{\partial f}{\partial x_n} \neq 0$.

Consider now a derivation δ of the polynomial ring $k[x_1, \dots, x_{n-1}]$ with a trivial field of constants, that is, $k(x_1, \dots, x_{n-1})^{\overline{\delta}} = k$. We presented a list of examples of such derivations in the previous section. Denote by Δ the derivation of $k[X] = k[x_1, \dots, x_{n-1}][x_n]$ given by

$$\Delta(x_1) = \delta(x_1), \ \dots, \ \Delta(x_{n-1}) = \delta(x_{n-1}) \ \text{ and } \ \Delta(x_n) = 0,$$

and let $d : k[X] \to k[X]$ be the derivation defined by

$$d = -\Delta(f)\frac{\partial}{\partial x_n} + \frac{\partial f}{\partial x_n}\Delta. \tag{1}$$

It is clear that $k[f] \subseteq k[X]^d$.

Now let $\varphi = \frac{u}{v} \in k(X) \smallsetminus k[X]$, where $u, v \in k[X]$ and $v \neq 0$. Assume that $\frac{\partial \varphi}{\partial x_n} \neq 0$, and let $\overline{\Delta}$ be the extension of the above derivation Δ to $k(X)$. Then the elements $v^2\overline{\Delta}(\varphi)$ and $v^2\frac{\partial \varphi}{\partial x_n}$ belong to $k[X]$. Put

$$d_0 = -v^2\overline{\Delta}(\varphi)\frac{\partial}{\partial x_n} + v^2\frac{\partial \varphi}{\partial x_n}\Delta. \tag{2}$$

Then d_0 is a derivation of $k[X]$ and it is clear that $k(\varphi) \subseteq k(X)^{\overline{d_0}}$.

Theorem 3.1. *If d is the derivation defined by* (1)*, then the ring $k[X]^d$ is equal to the integral closure of $k[f]$ in $k[X]$, and the field $k(X)^{\overline{d}}$ is equal to the algebraic closure of $k(f)$ in $k(X)$.*

Proof. Denote by A the integral closure of $k[f]$ in $k[X]$ and put $M := A_0$. Then (see Lemma 2.4) M is the algebraic closure of $k(f)$ in $k(X)$. Note that $k(X)^{\overline{d}} \cap k[X] = k[X]^d$.

Let $D := w^{-1}\overline{d}$, where $w = \frac{\partial f}{\partial x_n}$ (recall that $w \neq 0$). Then it is obvious that D is an M-derivation of $k(X)$.

Since $\frac{\partial f}{\partial x_n} \neq 0$, the polynomials $x_1, \dots, x_{n-1}, f$ are algebraically independent over k, and this implies that the field extension $M(x_1, \dots, x_{n-1}) \subseteq k(X)$ is algebraic. Observe that $D(x_i) = \delta(x_i)$ for all $i = 1, \dots, n-1$. Hence, the restriction of D to the polynomial ring $M[x_1, \dots, x_{n-1}]$ is an M-derivation with a trivial field of constants. But the field M is, by Lemma 2.4, algebraically closed in $k(X)$. So, Lemma 2.5 implies that $k(X)^D = M$. Hence, $k(X)^{\overline{d}} = M$, because $D = w^{-1}\overline{d}$, and hence $k(X)^{\overline{d}}$ equals to the algebraic closure of $k(f)$ in $k(X)$. Moreover, we have: $k[X]^d = k(X)^{\overline{d}} \cap k[X] = M \cap k[X]$. But $M = A_0$ and A is (by Lemma 2.3) of the form $k[g]$ for some $g \in k[X] \smallsetminus k$. Therefore, by Lemma 2.1, $M \cap k[X] = A$, that is, $k[X]^d = A$. $\square$

Repeating the arguments from the above proof and using small modifications we obtain the following theorem.

Theorem 3.2. *If d_0 is the derivation defined by (2), then the field $k(X)^{\overline{d_0}}$ is equal to the algebraic closure of $k(\varphi)$ in $k(X)$.*

Proof. Denote by M the algebraic closure of $k(\varphi)$ in $k(X)$. Let $D := w^{-1}\overline{d_0}$, where $w = \frac{\partial \varphi}{\partial x_n}$ (recall that $w \neq 0$). Then it is obvious that D is an M-derivation of $k(X)$.

Since $\frac{\partial \varphi}{\partial x_n} \neq 0$, the elements $x_1, \dots, x_{n-1}, \varphi$ are algebraically independent over k, and this implies that the field extension $M(x_1, \dots, x_{n-1}) \subseteq k(X)$ is algebraic. Observe that $D(x_i) = v^2\delta(x_i)$ for all $i = 1, \dots, n-1$. Hence, the restriction of D to the polynomial ring $M[x_1, \dots, x_{n-1}]$ is an M-derivation with trivial field of constants. But the field M is algebraically closed in $k(X)$. So, Lemma 2.5 implies that $k(X)^D = M$. Hence, $k(X)^{\overline{d_0}} = k(X)^{w^{-1}\overline{d_0}} = k(X)^D = M$. $\square$

Using the above constructions and the derivations $\delta_1, \dots, \delta_6$, defined in the previous section, we obtain the following examples.

Example 3.3. *Let d_1, d_2, d_3 be derivations of $k[x, y, z]$ defined as follows:*

$$\begin{aligned} d_1 &= z\partial_x + z(xy+1)\partial_y - (xy^2 + x + y)\partial_z, \\ d_2 &= z\partial_x + yz\partial_y - (x^2 + y)\partial_z, \\ d_3 &= xz\partial_x + z(x+y)\partial_y - (x^2 + xy + y^2)\partial_z. \end{aligned}$$

Then $k[x, y, z]^{d_i} = k[x^2 + y^2 + z^2]$ and $k(x, y, z)^{\overline{d_i}} = k(x^2 + y^2 + z^2)$, for $i = 1, 2, 3$. $\square$

Example 3.4. *Let d_1, d_2, d_3 be derivations of $k[x, y, z]$ defined as follows:*

$$\begin{aligned} d_1 &= xy\partial_x + xy(xy+1)\partial_y - z(x^2y + x + y)\partial_z, \\ d_2 &= x\partial_x + xy\partial_y - z(x+1)\partial_z, \\ d_3 &= xy\partial_x + y(x+y)\partial_y - (x + 2y)\partial_z. \end{aligned}$$

Then $k[x, y, z]^{d_i} = k[xyz]$ and $k(x, y, z)^{\overline{d_i}} = k(xyz)$, for $i = 1, 2, 3$. $\square$

Example 3.5. *Let $d_1, \ldots, d_5$ be derivations of $k[x,y,z,u]$ defined as follows:*

$$\begin{aligned}
d_1 &= u\partial_x + u(xy+1)\partial_y + u(yz+1)\partial_z - (xy^2+yz^2+x+y+z)\partial_u,\\
d_2 &= xyu\partial_x + yu\partial_y + xu\partial_z - (x^2y+xz+y^2)\partial_u,\\
d_3 &= u\partial_x + yu\partial_y + yzu\partial_z - (yz^2+y^2+x)\partial_u,\\
d_4 &= y^2u\partial_x + z^2u\partial_y + x^2u\partial_z - (x^2z+xy^2+yz^2)\partial_u,\\
d_5 &= xyu\partial_x + yzu\partial_y + xzu\partial_z - (x^2y+y^2z+z^2x)\partial_u.
\end{aligned}$$

Then $k[x,y,z,u]^{d_i} = k[x^2+y^2+z^2+u^2]$ and $k(x,y,z,u)^{\overline{d_i}} = k(x^2+y^2+z^2+u^2)$, for $i = 1, \ldots, 5$. $\square$

Example 3.6. *Let $d_1, \ldots, d_5$ be derivations of $k[x,y,z,u]$ defined as follows:*

$$\begin{aligned}
d_1 &= xyz\partial_x + xyz(xy+1)\partial_y + xyz(yz+1)\partial_z + u(x^2yz - xy^2z - xy + xz + yz)\partial_u,\\
d_2 &= xy\partial_x + yz\partial_y + xz\partial_z - u(x-y-z)\partial_u,\\
d_3 &= x\partial_x + xy\partial_y + xyz\partial_z - u(xy-x-1)\partial_u,\\
d_4 &= xyz\partial_x + yz\partial_y + xz\partial_z + u(yz-x+z)\partial_u,\\
d_5 &= xy^3z\partial_x + xyz^3\partial_y + x^3yz\partial_z - u(x^3y - xz^3 - y^3z)\partial_u.
\end{aligned}$$

Then $k(x,y,z,u)^{\overline{d_i}} = k\left(\frac{xy}{zu}\right)$, for $i = 1, \ldots, 5$. $\square$

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